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Left and right Browder operator matrices on a Banach space

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Abstract. Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces, and upper triangular operator matrices acting on $\mathcal{X} \oplus \mathcal{Y}$ are studied. Given bounded operators A, B, we obtain several equivalent conditions for $M_X = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix}$ to be a left Browder, a right Browder and a Browder operator for some bounded unknown operator X. Finally, an example is presented to illustrate the main conclusion.

1. Introduction

Throughout this paper, let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be Banach spaces. If T is a bounded linear operator from $\mathcal{X}$ to $\mathcal{Y}$, we write $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and, if $\mathcal{X} = \mathcal{Y}$, write $\mathcal{B}(\mathcal{X})$ instead of $\mathcal{B}(\mathcal{X}, \mathcal{X})$. For $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, the range and the kernel of T are, respectively, denoted by $\mathcal{R}(T)$ and $\mathcal{N}(T)$; write $\alpha(T) := \dim \mathcal{N}(T)$ and $\beta(T) := \dim \mathcal{Y}/\mathcal{R}(T)$. Now let $T \in \mathcal{B}(\mathcal{X})$. The sets of all left and right Fredholm operators are, respectively, defined by

$$\Phi_l(X) := \{T \in \mathcal{B}(\mathcal{X}) : \alpha(T) < \infty, \mathcal{R}(T) \text{ is closed and complemented in } \mathcal{X}\},$$
$$\Phi_r(X) := \{T \in \mathcal{B}(\mathcal{X}) : \beta(T) < \infty, \mathcal{N}(T) \text{ is complemented in } \mathcal{X}\};$$

the set of all Fredholm operators is defined by

$$\Phi(\mathcal{X}) := \Phi_l(\mathcal{X}) \cap \Phi_r(\mathcal{X}).$$

The ascent and the descent of T are defined by

$$\mathrm{asc}(T) := \min\{k \in \mathbb{N} : \mathcal{N}(T^k) = N(T^{k+1})\},$$
$$\mathrm{des}(T) := \min\{k \in \mathbb{N} : \mathcal{R}(T^k) = \mathcal{R}(T^{k+1})\},$$

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respectively. Note that such minimums may not exist, in which case the corresponding asc(T) or des(T) will be designated as ∞; if asc(T) and des(T) are both finite, then they are equal (see [1, 14]). The sets of all left Browder, right Browder and Browder operators on $\mathcal{X}$ are, respectively, denoted by

$$B_l(X) := \{T \in \Phi_l(X) : \mathrm{asc}(T) < \infty\},$$
$$B_r(X) := \{T \in \Phi_r(X) : \mathrm{des}(T) < \infty\},$$
$$B(X) := \{T \in \Phi(X) : \mathrm{asc}(T) = \mathrm{des}(T) < \infty\}.$$

We say that $T \in \mathcal{B}(\mathcal{X})$ is relatively regular or simply regular if there exists $S \in \mathcal{B}(\mathcal{X})$ such that $TST = T$. Here S is called an inner generalized inverse of T. Obviously, the classes of left or right invertible, invertible, left or right Fredholm and Fredholm operators are all regular. If $\mathcal{M}$ is a closed subspace in Banach space $\mathcal{X}$, then $\mathcal{M}$ is said to be topologically complemented or simply complemented if there exists another closed subspace $\mathcal{N}$ of $\mathcal{X}$ such that $\mathcal{M} \cap \mathcal{N} = \{0\}$ and $\mathcal{X} = \mathcal{M} + \mathcal{N}$; in this case, we write $\mathcal{X} = \mathcal{M} \oplus \mathcal{N}$. As is well known, T is relatively regular if and only if $\mathcal{R}(T)$ and $\mathcal{N}(T)$ are closed and complemented subspaces of $\mathcal{X}$. Denote by $\mathcal{P}_T$ and $\mathcal{Q}_T$ the complementary subspaces with $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively.

For given $A \in \mathcal{B}(\mathcal{X})$, $B \in \mathcal{B}(\mathcal{Y})$, $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$, define

$$M_X := \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} \in \mathcal{B}(\mathcal{X} \oplus \mathcal{Y}), \ \ M := \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \in \mathcal{B}(\mathcal{X} \oplus \mathcal{Y}), \tag{1}$$

where $X \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ is an unknown element. The spectrum and its various subdivisions of M_X are considered in many papers such as [2–9, 11–13, 15–18] and the references therein. Although most of these papers worked in the context of Hilbert spaces, some results on the invertibility and Fredholm theory (such as left (right) spectrum, left (right) essential spectrum, Weyl spectrum, Browder spectrum, Drazin spectrum and generalized Drazin spectrum) of operator matrices were established in Banach spaces [6, 9, 12, 13, 16–18]. In this note, we investigate upper triangular left and right Browder operator matrices on a Banach space. Our main tools are the ghost of an index theorem, and the left and right Browder operator and their equivalent forms which are closely related to space decomposition technique.

2. Preliminaries

This section is devoted to collecting some basic results. Although most of them are well known standard results on Fredholm operators, we list it here for convenience of later proofs.

Lemma 2.1 (see [1]). *Let $T \in \mathcal{B}(\mathcal{X})$.*
(i) If asc(T) $< \infty$, then $\alpha(T) \le \beta(T)$;
(ii) If des(T) $< \infty$, then $\beta(T) \le \alpha(T)$;
(iii) If asc(T) $=$ des(T) $< \infty$, then $\alpha(T) = \beta(T)$;
(iv) If $\alpha(T) = \beta(T) < \infty$ and if either asc(T) or des(T) is finite, then asc(T) $=$ des(T).

Lemma 2.2 (see [14]). *Let M be defined as in* (1). *Then*
(i) asc(A) $\le$ asc(M) $\le$ asc(A) + asc(B);
(ii) des(B) $\le$ des(M) $\le$ des(A) + des(B);
(iii) $\alpha(A) \le \alpha(M) \le \alpha(A) + \alpha(B)$;
(iv) $\beta(B) \le \beta(M) \le \beta(A) + \beta(B)$.

Lemma 2.3 (see [18]). *Let M be defined as in* (1).
(i) If any two of operators A, B and M are invertible (resp., Fredholm, Weyl, Browder, Drazin inverible), then so is the third;
(ii) If A is Browder, then B is left Browder if and only if so is M;
(iii) If B is Browder, then A is right Browder if and only if so is M.

Lemma 2.4 (see [16]). *For $T \in \mathcal{B}(\mathcal{X})$, T is left Browder if and only if T can be decomposed into the form*

$$T = \begin{bmatrix} T_1 & T_{12} \\ 0 & T_2 \end{bmatrix}$$

with respect to space decomposition $\mathcal{X} = \mathcal{N}(T^p) \oplus \mathcal{P}_{T^p}$, where $p = asc(T) < \infty$, $\alpha(T^p) < \infty$, T_1 is nilpotent, and T_2 is left invertible.

Lemma 2.5 (see [16]). *For $T \in \mathcal{B}(\mathcal{X})$, T is right Browder if and only if T can be decomposed into the form*

$$T = \begin{bmatrix} T_1 & T_{12} \\ 0 & T_2 \end{bmatrix}$$

with respect to space decomposition $\mathcal{X} = \mathcal{R}(T^q) \oplus \mathcal{Q}_{T^q}$, where $q = des(T) < \infty$, $\beta(T^q) < \infty$, T_1 is right invertible, and T_2 is nilpotent.

Lemma 2.6 (see [9]). *Let M_X be defined as in* (1). *Then the following conditions are equivalent:*
(i) M_X is invertible for some $X \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$;
(ii) A is left invertible, B is right invertible, and $N(B) \cong \mathcal{X}/\mathcal{R}(A)$.

The following lemma is obvious.

Lemma 2.7. *Let M_X be defined as in* (1). *If A and B are, respectively, left and right invertible, then M_X is left invertible for some $X \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ if and only if $N(B) \preceq \mathcal{X}/\mathcal{R}(A)$.*

Lemma 2.8 (see [6]). *Let M_X be defined as in* (1). *Then the following conditions are equivalent:*
(i) M_X is Weyl for some $X \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$;
(ii) A is left Fredholm, B is right Fredholm, and $\mathcal{N}(A) \oplus \mathcal{N}(B) \cong \mathcal{X}/\overline{\mathcal{R}(A)} \oplus \mathcal{Y}/\overline{\mathcal{R}(B)}$.

Lemma 2.9 (see [10]). *If $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, $S \in \mathcal{B}(\mathcal{Y}, \mathcal{Z})$ and $ST \in \mathcal{B}(\mathcal{X}, \mathcal{Z})$ are regular, then*

$$N(T) \oplus N(S) \oplus \mathcal{Z}/\mathcal{R}(ST) \cong N(ST) \oplus Y/\mathcal{R}(T) \oplus \mathcal{Z}/\mathcal{R}(S)$$

.

3. Main results and proofs

First, we establish the left Browder, right Browder and Browder results of M_X, defined as in (1).

Theorem 3.1. *Let M_X be defined as in* (1). *Then there exists $X \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ such that M_X is left Browder if and only if*
(i) A is left Browder; and
(ii) There exists $J \in \mathcal{B}(\mathcal{Y}, \mathcal{P}_{A^p})$ such that $asc(\left[\begin{smallmatrix} A_2 & J \\ 0 & B \end{smallmatrix}\right]) < \infty$, and the column operator $\left[\begin{smallmatrix} P_{\mathcal{Q}} J \\ B \end{smallmatrix}\right]$ is left Fredholm, where $p = asc(A)$, $A_2 = P_{\mathcal{P}_{A^p}} A|_{\mathcal{P}_{A^p}}$, $\mathcal{Q} \subseteq \mathcal{P}_{A^p}$ with $\mathcal{P}_{A^p} = \mathcal{R}(A_2) \oplus \mathcal{Q}$, and $P_{\mathcal{P}_{A^p}}$ ($P_{\mathcal{Q}}$) is the projection onto $\mathcal{P}_{A^p}$ ($\mathcal{Q}$) along $\mathcal{N}(A^p)$ ($\mathcal{R}(A_2)$).

Proof. Sufficiency. Since A is left Browder, according to Lemma 2.4, $\mathcal{X}$ has the following decomposition

$$\mathcal{X} = \mathcal{N}(A^p) \oplus \mathcal{P}_{A^p}. \tag{2}$$

Then A can be correspondingly written as

$$A = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix} : \mathcal{N}(A^p) \oplus \mathcal{P}_{A^p} \to \mathcal{N}(A^p) \oplus \mathcal{P}_{A^p}, \tag{3}$$

where $A_1 = P_{\mathcal{N}(A^p)}A|_{\mathcal{N}(A^p)}$ is nilpotent, and A_2 is left invertible. Further, A_2 has the matrix form

$$A_2 = \begin{bmatrix} A_{21} \\ 0 \end{bmatrix} : \mathcal{P}_{A^p} \to \mathcal{R}(A_2) \oplus \mathcal{Q},$$

where $A_{21} : \mathcal{P}_{A^p} \to \mathcal{R}(A_2)$ is invertible. From the assumption, $\begin{bmatrix} P_{Q}J \\ B \end{bmatrix}$ is left Fredholm operator, it follows that

$$\begin{bmatrix} A_{21} & 0 \\ 0 & P_{Q}J \\ 0 & B \end{bmatrix} : \mathcal{P}_{A^p} \oplus \mathcal{Y} \to \mathcal{R}(A_2) \oplus \mathcal{Q} \oplus \mathcal{Y}$$

is a left Fredholm operator. Consequently,

$$\begin{bmatrix} A_2 & J \\ 0 & B \end{bmatrix} : \mathcal{P}_{A^p} \oplus \mathcal{Y} \to \mathcal{P}_{A^p} \oplus \mathcal{Y}$$

is left Fredholm operator. This together with the assumption $\mathrm{asc}(\begin{bmatrix} A_2 & J \\ 0 & B \end{bmatrix}) < \infty$ implies that $\begin{bmatrix} A_2 & J \\ 0 & B \end{bmatrix}$ is left Browder.

Define

$$X = \begin{bmatrix} 0 \\ J \end{bmatrix} : \mathcal{Y} \to \mathcal{N}(A^p) \oplus \mathcal{P}_{A^p}. \tag{4}$$

With respect to the decomposition $\mathcal{X} \oplus \mathcal{Y} = \mathcal{N}(A^p) \oplus \mathcal{P}_{A^p} \oplus \mathcal{Y}$, M_X can be decomposed into the following form

$$M_X = \begin{bmatrix} A_1 & A_{12} & 0 \\ 0 & A_2 & J \\ 0 & 0 & B \end{bmatrix}. \tag{5}$$

Note that A_1 is a nilpotent operator on the finite dimensional space $\mathcal{N}(A^p)$ and hence is a Browder operator. Using Lemma 2.3, we conclude from the left Browderness of $\begin{bmatrix} A_2 & J \\ 0 & B \end{bmatrix}$ that M_X is a left Browder operator.

Necessity. Let us say that M_X is left Browder; namely, M_X is left Fredholm and $\mathrm{asc}(M_X) < \infty$. Obviously, A is left Fredholm, and it follows from Lemma 2.2 that $p = \mathrm{asc}(A) \le \mathrm{acs}(M_X) < \infty$, which mean that A is left Browder, (i) is proven. At this point, the decomposition (3) of A still holds. As an operator on $\mathcal{N}(A^p) \oplus \mathcal{P}_{A^p} \oplus \mathcal{Y}$, M_X further has the matrix form

$$M_X = \begin{bmatrix} A_1 & A_{12} & X_1 \\ 0 & A_2 & X_2 \\ 0 & 0 & B \end{bmatrix}. \tag{6}$$

Note that A_1 is Browder (shown in the sufficiency part). From Lemma 2.4, it follows that

$$\begin{bmatrix} A_2 & X_2 \\ 0 & B \end{bmatrix} : \mathcal{P}_{A^p} \oplus \mathcal{Y} \to \mathcal{P}_{A^p} \oplus \mathcal{Y}$$

is left Browder. It is clear that $\mathrm{asc}(\begin{bmatrix} A_2 & X_2 \\ 0 & B \end{bmatrix}) < \infty$. Furthermore, $\begin{bmatrix} A_2 & X_2 \\ 0 & B \end{bmatrix}$ can be decomposed into the form

$$\begin{bmatrix} A_{21} & X_{21} \\ 0 & X_{22} \\ 0 & B \end{bmatrix} : \mathcal{P}_{A^p} \oplus \mathcal{Y} \to \mathcal{R}(A_2) \oplus \mathcal{Q} \oplus \mathcal{Y},$$

which together with the invertibility of $A_{21} : \mathcal{P}_{A^p} \to \mathcal{R}(A_2)$ implies that $\begin{bmatrix} X_{22} \\ B \end{bmatrix}$ is left Fredholm. Setting $J = X_2 \in \mathcal{B}(\mathcal{Y}, \mathcal{P}_{A^p})$, we have $P_Q J = X_{22}$, and hence $\begin{bmatrix} A_2 & J \\ 0 & B \end{bmatrix} = \begin{bmatrix} A_2 & X_2 \\ 0 & B \end{bmatrix}$, $\begin{bmatrix} P_Q J \\ B \end{bmatrix} = \begin{bmatrix} X_{22} \\ B \end{bmatrix}$ satisfy the desired conditions in (ii). □

Corollary 3.2. *Let M_X be defined as in* (1)*. If A is left Browder, asc(B) $< \infty$, and there exists $J \in \mathcal{B}(\mathcal{Y}, \mathcal{P}_{A^p})$ such that $\begin{bmatrix} P_Q J \\ B \end{bmatrix}$ is left Fredholm, then there exists $X \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ such that M_X is left Browder, where $p = asc(A)$ and P_Q is defined as in Theorem 3.1.*

Proof. We follow the notations in Theorem 3.1 and its proof. Here, it suffices to note that $\operatorname{asc}(B) < \infty$ implies $\operatorname{asc}(\begin{bmatrix} A_2 & J \\ 0 & B \end{bmatrix}) < \infty$. In fact, since A_2 is left invertible, we have $\operatorname{asc}(A_2) < \infty$; by Lemma 2.2, $\operatorname{asc}(\begin{bmatrix} A_2 & J \\ 0 & B \end{bmatrix}) \leq \operatorname{asc}(A_2) + \operatorname{asc}(B) < \infty$.

Theorem 3.3. *Let M_X be defined as in* (1)*. Then there exists $X \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ such that M_X is right Browder if and only if*

(i) B is right Browder; and

(ii)There exists $S \in \mathcal{B}(\mathcal{R}(B^q), \mathcal{X})$ such that $des(\begin{bmatrix} A & S \\ 0 & B_1 \end{bmatrix}) < \infty$, and the row operator $[A \quad S|_{\mathcal{N}(B_1)}]$ is right Fredholm, where $q = des(B)$, $B_1 = P_{\mathcal{R}(B^q)} B|_{\mathcal{R}(B^q)}$, and $P_{\mathcal{R}(B^q)}$ is the projection onto $\mathcal{R}(B^q)$ along $\mathcal{Q}_{B^q}$.

Proof. Sufficiency. Since B is right Browder, by Lemma 2.5, $\mathcal{Y}$ has the decomposition

$$\mathcal{Y} = \mathcal{R}(B^q) \oplus \mathcal{Q}_{B^q} \tag{7}$$

With respect to the decomposition (7), B can be written as

$$B = \begin{bmatrix} B_1 & B_{12} \\ 0 & B_2 \end{bmatrix} : \mathcal{R}(B^q) \oplus \mathcal{Q}_{B^q} \to \mathcal{R}(B^q) \oplus \mathcal{Q}_{B^q}, \tag{8}$$

where $B_1 = P_{\mathcal{R}(B^q)} B|_{\mathcal{R}(B^q)}$ is right invertible, and $B_2 = P_{\mathcal{Q}_{B^q}} B|_{\mathcal{Q}_{B^q}}$ is nilpotent. Obviously, B_1 further has the matrix form

$$B_1 = [0 \; B_{11}] : \mathcal{N}(B_1) \oplus \mathcal{P} \to \mathcal{R}(B^q), \tag{9}$$

where $\mathcal{P} \subseteq \mathcal{R}(B^q)$ with $\mathcal{N}(B_1) \oplus \mathcal{P} = \mathcal{R}(B^q)$, and $B_{11} : \mathcal{P} \to \mathcal{R}(B^q)$ is invertible. Since $[A \quad S|_{\mathcal{N}(B_1)}]$ is right Fredholm,

$$\begin{bmatrix} A & S|_{\mathcal{N}(B_1)} & 0 \\ 0 & 0 & B_{11} \end{bmatrix} : \mathcal{X} \oplus \mathcal{N}(B_1) \oplus \mathcal{P} \to \mathcal{X} \oplus \mathcal{R}(B^q)$$

is also right Fredholm. Consequently,

$$\begin{bmatrix} A & S \\ 0 & B_1 \end{bmatrix} : \mathcal{X} \oplus \mathcal{R}(B^q) \to \mathcal{X} \oplus \mathcal{R}(B^q)$$

is a right Fredholm operator, which together with the assumption $\operatorname{des}(\begin{bmatrix} A & S \\ 0 & B_1 \end{bmatrix}) < \infty$ shows that $\begin{bmatrix} A & S \\ 0 & B_1 \end{bmatrix}$ is right Browder. Note that B_2 is nilpotent and $q = \operatorname{des}(B) < \infty$, and hence B_2 is Browder. According to Lemma2.3,

$$M_X = \begin{bmatrix} A & S & 0 \\ 0 & B_1 & B_{12} \\ 0 & 0 & B_2 \end{bmatrix} : \mathcal{X} \oplus \mathcal{R}(B^q) \oplus \mathcal{Q}_{B^q} \to \mathcal{X} \oplus \mathcal{R}(B^q) \oplus \mathcal{Q}_{B^q} \tag{10}$$

is right Browder, and

$$X = \begin{bmatrix} S & 0 \end{bmatrix} : \mathcal{R}(B^q) \oplus \mathcal{Q}_{B^q} \to \mathcal{X} \tag{11}$$

is the required operator in $\mathcal{B}(\mathcal{Y}, \mathcal{X})$.

Necessity. Suppose that M_X is right Browder for some $X \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$; namely, M_X is right Fredholm operator and $\operatorname{des}(M_X) < \infty$. Obviously, B is clearly right Fredholm, and $q = \operatorname{des}(B) \leq \operatorname{des}(M_X) < \infty$ by

Lemma 2.2, which show that B is a right Browder operator, the condition (i). Then we still have the decomposition (8) of B. It is clear that M_X has the decomposition

$$M_X = \begin{bmatrix} A & X_1 & X_2 \\ 0 & B_1 & B_{12} \\ 0 & 0 & B_2 \end{bmatrix} : \mathcal{X} \oplus \mathcal{R}(B^q) \oplus \mathcal{Q}_{B^q} \to \mathcal{X} \oplus \mathcal{R}(B^q) \oplus \mathcal{Q}_{B^q}. \tag{12}$$

Since B_2 is Browder, it follows from Lemma 2.5 that

$$\begin{bmatrix} A & X_1 \\ 0 & B_1 \end{bmatrix} : \mathcal{X} \oplus \mathcal{R}(B^p) \to \mathcal{X} \oplus \mathcal{R}(B^p)$$

is right Browder, and hence des$(\left[\begin{smallmatrix} A & X_1 \\ 0 & B_1 \end{smallmatrix}\right]) < \infty$. We now further decompose $\left[\begin{smallmatrix} A & X_1 \\ 0 & B_1 \end{smallmatrix}\right]$ into the form

$$\begin{bmatrix} A & X_{11} & X_{12} \\ 0 & 0 & B_{11} \end{bmatrix} : \mathcal{X} \oplus \mathcal{N}(B_1) \oplus \mathcal{P} \to \mathcal{X} \oplus \mathcal{R}(B^q),$$

where $\mathcal{P}$ and B_{11} are defined as in (9). This together with the invertibility of B_{11} gives that $[A \;\; X_{11}]$ is right Fredholm. Taking $S = X_1$, we have $S|_{\mathcal{N}(B_1)} = X_{11}$, and hence $\left[\begin{smallmatrix} A & S \\ 0 & B_1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} A & X_1 \\ 0 & B_1 \end{smallmatrix}\right]$, $[A \;\; S|_{\mathcal{N}(B_1)}] = [A \;\; X_{11}]$ satisfy the corresponding conditions in (ii). □

Corollary 3.4. *Let M_X be defined as in* (1). *If B is right Browder, des$(A) < \infty$, and there exists $S \in \mathcal{B}(\mathcal{R}(B^q), \mathcal{X})$ such that $[A \;\; S|_{\mathcal{N}(B_1)}]$ is right Fredholm, then there exists $X \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ such that M_X is right Browder, where $q =$ des(B) and B_1 is defined as in Theorem 3.3.*

Proof. We proceeds on the basis of Theorem 3.3 and its proof. So it suffices to note that des$(A) < \infty$ implies des$(\left[\begin{smallmatrix} A & X_1 \\ 0 & B_1 \end{smallmatrix}\right]) < \infty$. In fact, since B_1 is right invertible, we get des$(B_1) < \infty$; by Lemma 2.2, des$(\left[\begin{smallmatrix} A & X_1 \\ 0 & B_1 \end{smallmatrix}\right]) \leq$ des(A) + des$(B_1) < \infty$. □

Theorem 3.5. *Let M_X be defined as in* (1). *Then there exists $X \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ such that M_X is a Browder operator if and only if*

(i) A is left Browder, and B is right Browder; and

(ii) There exist $J \in \mathcal{B}(\mathcal{Y}, \mathcal{P}_{A^p})$ and $S \in \mathcal{B}(\mathcal{R}(B^q), \mathcal{X})$ such that asc$(\left[\begin{smallmatrix} A_2 & J \\ 0 & B \end{smallmatrix}\right]) < \infty$, $\left[\begin{smallmatrix} P_{\mathcal{Q}}J \\ B \end{smallmatrix}\right]$ is left Fredholm operator, des$(\left[\begin{smallmatrix} A & S \\ 0 & B_1 \end{smallmatrix}\right]) < \infty$, $[A \;\; S|_{\mathcal{N}(B_1)}]$ is right Fredholm operator, and $P_{\mathcal{P}_{A^p}} S = J|_{\mathcal{R}(B^q)}$, where p, q, A_2, B_1, $\mathcal{Q}$, $P_{\mathcal{P}_{A^p}}$, $P_{\mathcal{Q}}$ and $P_{\mathcal{R}(B^q)}$ are defined as in Theorem 3.1 and Theorem 3.3.

Proof. Sufficiency. Write $\Delta = P_{\mathcal{P}_{A^p}} S = J|_{\mathcal{R}(B^q)}$. Since A is left Browder and B is right Browder, the decompositions (3) and (8) still hold. From the corresponding proofs of Theorem 3.1 and Theorem 3.3, we see that $\left[\begin{smallmatrix} A_2 & J \\ 0 & B \end{smallmatrix}\right] : \mathcal{P}_{A^p} \oplus \mathcal{Y} \to \mathcal{P}_{A^p} \oplus \mathcal{Y}$ is left Browder, and $\left[\begin{smallmatrix} A & S \\ 0 & B_1 \end{smallmatrix}\right] : \mathcal{X} \oplus \mathcal{R}(B^p) \to \mathcal{X} \oplus \mathcal{R}(B^p)$ is right Browder. Note that $S = \left[\begin{smallmatrix} S_1 \\ \Delta \end{smallmatrix}\right] : \mathcal{R}(B^p) \to \mathcal{N}(A^p) \oplus \mathcal{P}_{A^p}$ and $J = [\Delta \;\; J_1] : \mathcal{R}(B^q) \oplus \mathcal{Q}_{B^q} \to \mathcal{P}_{A^p}$. Taking

$$X = \begin{bmatrix} S_1 & 0 \\ \Delta & J_1 \end{bmatrix} : \mathcal{R}(B^q) \oplus \mathcal{Q}_{B^q} \to \mathcal{N}(A^p) \oplus \mathcal{P}_{A^p}. \tag{13}$$

we have

$$M_X = \begin{bmatrix} A_1 & A_{12} & S_1 & 0 \\ 0 & A_2 & \Delta & J_1 \\ 0 & 0 & B_1 & B_{12} \\ 0 & 0 & 0 & B_2 \end{bmatrix}, \tag{14}$$

an operator on $\mathcal{N}(A^p) \oplus \mathcal{P}_{A^p} \oplus \mathcal{R}(B^q) \oplus \mathcal{Q}_{B^q}$.

Obviously, $\begin{bmatrix} A_2 & \Delta & J_1 \\ 0 & B_1 & B_{12} \\ 0 & 0 & B_2 \end{bmatrix}$ is left Browder, and $\begin{bmatrix} A_1 & A_{12} & S_1 \\ 0 & A_2 & \Delta \\ 0 & 0 & B_1 \end{bmatrix}$ is right Browder. Note that A_1 and B_2 are Browder operators. Using Lemma 2.3, we can easily know that M_X is a Browder operator.

Necessity. Since M_X is a Browder operator for some $X \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$, A and B are, respectively, left and right Browder, i.e., (i) holds, and hence they have the decompositions (3) and (8). Then, as an operator on $\mathcal{N}(A^p) \oplus \mathcal{P}_{A^p} \oplus \mathcal{R}(B^q) \oplus \mathcal{Q}_{B^q}$,

$$M_X = \begin{bmatrix} A_1 & A_{12} & X_1 & X_2 \\ 0 & A_2 & X_3 & X_4 \\ 0 & 0 & B_1 & B_{12} \\ 0 & 0 & 0 & B_2 \end{bmatrix}, \tag{15}$$

where A_1 and B_2 are Browder operators. From Lemma 2.2 and the Browderness of M_X, it follows that

$$\tilde{M}_{X_{34}} := \begin{bmatrix} A_2 & X_3 & X_4 \\ 0 & B_1 & B_{12} \\ 0 & 0 & B_2 \end{bmatrix}$$

is left Browder, and

$$\tilde{M}_{X_{13}} := \begin{bmatrix} A_1 & A_{12} & X_1 \\ 0 & A_2 & X_3 \\ 0 & 0 & B_1 \end{bmatrix}$$

is right Browder. Because A_2 is left invertible and B_1 is right invertible, we have the space decompositions

$$\mathcal{P}_{A^p} = \mathcal{R}(A_2) \oplus \mathcal{Q}, \ \ \mathcal{R}(B^q) = \mathcal{N}(B_1) \oplus \mathcal{P}.$$

Thus, as an operator from $\mathcal{P}_{A^p} \oplus \mathcal{Y}$ to $\mathcal{R}(A_2) \oplus \mathcal{Q} \oplus \mathcal{Y}$,

$$\tilde{M}_{X_{34}} = \begin{bmatrix} A_{21} & X_{34,1} \\ 0 & X_{34,2} \\ 0 & B \end{bmatrix}$$

with A_{21} invertible, and $\begin{bmatrix} X_{34,2} \\ B \end{bmatrix}$ is clearly left Fredholm; as an operator from $\mathcal{X} \oplus \mathcal{N}(B_1) \oplus \mathcal{P}$ to $\mathcal{X} \oplus \mathcal{R}(B^q)$,

$$\tilde{M}_{X_{13}} = \begin{bmatrix} A & X_{13,1} & X_{13,2} \\ 0 & 0 & B_{11} \end{bmatrix}$$

with B_{11} invertible, and hence $[A \ \ X_{13,1}]$ is right Fredholm.

Define $S = \begin{bmatrix} X_1 \\ X_3 \end{bmatrix}$ and $J = [X_3 \ \ X_4]$. Then $\begin{bmatrix} A_2 & J \\ 0 & B \end{bmatrix} = \tilde{M}_{X_{34}}$, $\begin{bmatrix} A & S \\ 0 & B_1 \end{bmatrix} = \tilde{M}_{X_{13}}$, $P_{\mathcal{P}_{A^p}} S = X_3 = J|_{\mathcal{R}(B^q)}$, and, obviously, the condition (ii) is valid. □

In [16, Theorem 2.9], the Browderness of upper triangular operator matrices is characterized as follows. We will use our descriptions (Theorem (3.5)) to show this theorem.

Corollary 3.6. *Let M_X be defined as in* (1)*. Then there exists $X \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ such that M_X is a Browder operator if and only if*

(i) A and B are left and right Browder, respectively; and

(ii) $\mathcal{N}(A) \oplus \mathcal{N}(B) \cong \mathcal{X}/\mathcal{R}(A) \oplus \mathcal{Y}/\mathcal{R}(B)$.

Proof. We adopt here the notations of Theorem 3.1, Theorem 3.3 and their proofs. Let M_X be a Browder operator for some $X \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. Then M_X is a Fredholm operator, and $\mathrm{asc}(M_X) = \mathrm{des}(M_X) < \infty$, which implies that $\alpha(M_X) = \beta(M_X)$ by Lemma 2.1. It is clear that M_X is a Weyl operator. From Lemma 2.8, it follows that A and B are, respectively, left and right Fredholm, and $\mathcal{N}(A) \oplus \mathcal{N}(B) \cong \mathcal{X}/\mathcal{R}(A) \oplus \mathcal{Y}/\mathcal{R}(B)$.

Furthermore, Because of $\mathrm{asc}(A) \le \mathrm{asc}(M_X) < \infty$ and $\mathrm{des}(B) \le \mathrm{des}(M_X) < \infty$, A and B are left and right Browder, respectively. This proves the necessity.

We now establish the sufficiency. Since A is left Browder, A can be expressed as the form (3). Due to

$$A = \begin{bmatrix} I & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} A_1 & A_{12} \\ 0 & I \end{bmatrix},$$

applying Lemma 2.9 yields

$$\mathcal{N}(\left[\begin{smallmatrix} A_1 & A_{12} \\ 0 & I \end{smallmatrix}\right]) \oplus \mathcal{N}(\left[\begin{smallmatrix} I & 0 \\ 0 & A_2 \end{smallmatrix}\right]) \oplus \mathcal{X}/\mathcal{R}(A) \cong \mathcal{N}(A) \oplus \mathcal{X}/\mathcal{R}(\left[\begin{smallmatrix} A_1 & A_{12} \\ 0 & I \end{smallmatrix}\right]) \oplus \mathcal{X}/\mathcal{R}(\left[\begin{smallmatrix} I & 0 \\ 0 & A_2 \end{smallmatrix}\right]).$$

From the left invertibility of A_2, it follows that

$$\mathcal{N}(\left[\begin{smallmatrix} A_1 & A_{12} \\ 0 & I \end{smallmatrix}\right]) \oplus \mathcal{X}/\mathcal{R}(A) \cong \mathcal{N}(A) \oplus \mathcal{X}/\mathcal{R}(\left[\begin{smallmatrix} I & 0 \\ 0 & A_2 \end{smallmatrix}\right]) \oplus \mathcal{X}/\mathcal{R}(\left[\begin{smallmatrix} A_1 & A_{12} \\ 0 & I \end{smallmatrix}\right]).$$

Since A_1 is Browder, we know from Lemma 2.3 that $\left[\begin{smallmatrix} A_1 & A_{12} \\ 0 & I \end{smallmatrix}\right]$ is Browder and hence

$$\alpha(\left[\begin{smallmatrix} A_1 & A_{12} \\ 0 & I \end{smallmatrix}\right]) = \beta(\left[\begin{smallmatrix} A_1 & A_{12} \\ 0 & I \end{smallmatrix}\right]) < \infty,$$

which implies

$$\mathcal{X}/\mathcal{R}(A) \cong \mathcal{N}(A) \oplus \mathcal{X}/\mathcal{R}(\left[\begin{smallmatrix} I & 0 \\ 0 & A_2 \end{smallmatrix}\right]),$$

i.e.,

$$\mathcal{X}/\mathcal{R}(A) \cong \mathcal{N}(A) \oplus \mathcal{Q} \tag{16}$$

At the same time, B is right Browder, and can be expressed as the form (8). From

$$B = \begin{bmatrix} I & B_{12} \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} B_1 & 0 \\ 0 & I \end{bmatrix}$$

and Lemma 2.9, we infer

$$\mathcal{N}(\left[\begin{smallmatrix} I & B_{12} \\ 0 & B_2 \end{smallmatrix}\right]) \oplus \mathcal{N}(\left[\begin{smallmatrix} B_1 & 0 \\ 0 & I \end{smallmatrix}\right]) \oplus \mathcal{Y}/\mathcal{R}(B) \cong \mathcal{N}(B) \oplus \mathcal{Y}/\mathcal{R}(\left[\begin{smallmatrix} I & B_{12} \\ 0 & B_2 \end{smallmatrix}\right]) \oplus \mathcal{Y}/\mathcal{R}(\left[\begin{smallmatrix} B_1 & 0 \\ 0 & I \end{smallmatrix}\right]).$$

Since B_1 is right invertible, it is reduced to

$$\mathcal{N}(\left[\begin{smallmatrix} I & B_{12} \\ 0 & B_2 \end{smallmatrix}\right]) \oplus \mathcal{N}(\left[\begin{smallmatrix} B_1 & 0 \\ 0 & I \end{smallmatrix}\right]) \oplus \mathcal{Y}/\mathcal{R}(B) \cong \mathcal{N}(B) \oplus \mathcal{Y}/\mathcal{R}(\left[\begin{smallmatrix} I & B_{12} \\ 0 & B_2 \end{smallmatrix}\right]).$$

By virtue of Lemma 2.3, the fact that B_2 is Browder means $\left[\begin{smallmatrix} I & B_{12} \\ 0 & B_2 \end{smallmatrix}\right]$ is Browder and thus

$$\alpha(\left[\begin{smallmatrix} I & B_{12} \\ 0 & B_2 \end{smallmatrix}\right]) = \beta(\left[\begin{smallmatrix} I & B_{12} \\ 0 & B_2 \end{smallmatrix}\right]) < \infty,$$

which implies

$$\mathcal{N}(B) \cong \mathcal{N}(B_1) \oplus \mathcal{Y}/\mathcal{R}(B). \tag{17}$$

Combining (16) and (17) with the assumption $\mathcal{N}(A) \oplus \mathcal{N}(B) \cong \mathcal{X}/\mathcal{R}(A) \oplus \mathcal{Y}/\mathcal{R}(B)$, we have

$$\mathcal{N}(B_1) \cong \mathcal{Q}$$

since $\alpha(A)$ and $\beta(B)$ are finite.

Note that A_2 is left invertible, and B_1 is right invertible. From Lemma 2.6, it follows that there exists some $\Delta \in \mathcal{B}(\mathcal{R}(B^q), \mathcal{P}_{A^p})$ such that $\begin{bmatrix} A_2 & \Delta \\ 0 & B_1 \end{bmatrix}$ is invertible. Define

$$X = \begin{bmatrix} 0 & 0 \\ \Delta & 0 \end{bmatrix} : \mathcal{R}(B^q) \oplus \mathcal{Q}_{B^q} \to \mathcal{N}(A^p) \oplus \mathcal{P}_{A^p}. \tag{18}$$

Then, as an operator on $\mathcal{N}(A^p) \oplus \mathcal{P}_{A^p} \oplus \mathcal{R}(B^q) \oplus \mathcal{Q}_{B^q}$, M_X can be written as

$$M_X = \begin{bmatrix} A_1 & A_{12} & 0 & 0 \\ 0 & A_2 & \Delta & 0 \\ 0 & 0 & B_1 & B_{12} \\ 0 & 0 & 0 & B_2 \end{bmatrix}. \tag{19}$$

Since A_1 and B_2 are Browder, and $\begin{bmatrix} A_2 & \Delta \\ 0 & B_1 \end{bmatrix}$ is invertible, it follows from Lemma 2.3 that $\tilde{M}_{X_{34}} = \begin{bmatrix} A_2 & \Delta & 0 \\ 0 & B_1 & B_{12} \\ 0 & 0 & B_2 \end{bmatrix}$ is left Browder, and $\tilde{M}_{X_{13}} = \begin{bmatrix} A_1 & A_{12} & 0 \\ 0 & A_2 & \Delta \\ 0 & 0 & B_1 \end{bmatrix}$ is right Browder. Also, we further have that

$$\tilde{M}_{X_{34}} = \begin{bmatrix} A_{21} & P_{\mathcal{R}(A_2)}\Delta & 0 \\ 0 & P_{\mathcal{Q}}\Delta & 0 \\ 0 & B_1 & B_{12} \\ 0 & 0 & B_2 \end{bmatrix} : \mathcal{P}_{A^p} \oplus \mathcal{R}(B^q) \oplus \mathcal{Q}_{B^q} \to \mathcal{R}(A_2) \oplus \mathcal{Q} \oplus \mathcal{R}(B^q) \oplus \mathcal{Q}_{B^q},$$

$$\tilde{M}_{X_{13}} = \begin{bmatrix} A_1 & A_{12} & 0 & 0 \\ 0 & A_2 & \Delta|_{\mathcal{N}(B_1)} & \Delta|_{\mathcal{P}} \\ 0 & 0 & 0 & B_{11} \end{bmatrix} : \mathcal{N}(A^p) \oplus \mathcal{P}_{A^p} \oplus \mathcal{N}(B_1) \oplus \mathcal{P} \to \mathcal{P}_{A^p} \oplus \mathcal{R}(B^q) \oplus \mathcal{Q}_{B^q}.$$

Note that A_{21} and B_{11} are invertible operators. It is clear that

$$\begin{bmatrix} P_{\mathcal{Q}}\Delta & 0 \\ B_1 & B_{12} \\ 0 & B_2 \end{bmatrix}, \quad \begin{bmatrix} A_1 & A_{12} & 0 \\ 0 & A_2 & \Delta|_{\mathcal{N}(B_1)} \end{bmatrix}$$

are left and right Fredholm operators, respectively. Set

$$J = [\Delta \ \ 0] : \mathcal{R}(B^q) \oplus \mathcal{Q}_{B^q} \to \mathcal{P}_{A^p}, \quad S = \begin{bmatrix} 0 \\ \Delta \end{bmatrix} : \mathcal{R}(B^q) \to \mathcal{N}(A^p) \oplus \mathcal{P}_{A^p}.$$

Clearly, $P_{\mathcal{P}_{A^p}} S = \Delta = J|_{\mathcal{R}(B^q)}$; $\begin{bmatrix} P_{\mathcal{Q}}J \\ B \end{bmatrix}$ and $[A \quad S|_{\mathcal{N}(B_1)}]$ are, respectively, left and right Fredholm operators; $\begin{bmatrix} A_2 & J \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} A & S \\ 0 & B_1 \end{bmatrix}$ are, respectively, left and right Browder operators, which imply that $\mathrm{asc}(\begin{bmatrix} A_2 & J \\ 0 & B \end{bmatrix}) < \infty$ and $\mathrm{des}(\begin{bmatrix} A & S \\ 0 & B_1 \end{bmatrix}) < \infty$. By applying Theorem 3.5, the sufficiency is get proved. □

From the proof of Corollary 3.6, it is actually shown that

Corollary 3.7. *Let M_X be defined as in* (1). *Then there exists $X \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ such that M_X is a Browder operator if and only if*

(i) A and B are left and right Browder operators, respectively; and

(ii) $\mathcal{N}(B_1) \cong \mathcal{Q}$, where B_1 and $\mathcal{Q}$ are defined as in Theorem 3.3 and Theorem 3.1, respectively.

Based on the embedded relationship of certain spaces, the sufficient conditions under which the operator M_X of the form (1) is left or right Browder are given.

Definition 3.8 [6, Definition 4.2]). *For two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, we say that $\mathcal{X}$ can be embedded in $\mathcal{Y}$ and write $\mathcal{X} \preceq \mathcal{Y}$ if there exists a left invertible operator $J : \mathcal{X} \to \mathcal{Y}$. Note that $\mathcal{X} \preceq \mathcal{Y}$ if and only if there exists a right invertible operator $S : \mathcal{Y} \to \mathcal{X}$. In particular, $\mathcal{X} \cong \mathcal{Y}$ if and only if $\mathcal{X} \preceq \mathcal{Y}$ and $\mathcal{Y} \preceq \mathcal{X}$.*

Corollary 3.9. *Let M_X be defined as in* (1)*, and let B be right Browder. Then there exists $X \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ such that M_X is left Browder, if*

(i) A is left Browder; and

(ii) $\mathcal{N}(A) \oplus \mathcal{N}(B) \preceq \mathcal{X}/\mathcal{R}(A) \oplus \mathcal{Y}/\mathcal{R}(B)$.

Proof. Since A and B are, respectively, left and right Browder, it follows from the proof of Corollary 3.6 that the relations (16) and (17) are valid. From $\mathcal{N}(A) \oplus \mathcal{N}(B) \preceq \mathcal{X}/\mathcal{R}(A) \oplus \mathcal{Y}/\mathcal{R}(B)$, we then have

$$\mathcal{N}(B_1) \preceq \mathcal{Q}.$$

Note that A_2 is left invertible, and B_1 is right invertible. Using Lemma 2.7, one can find $\Delta \in \mathcal{B}(\mathcal{R}(B^q), \mathcal{P}_{A^p})$ such that $\begin{bmatrix} A_2 & \Delta \\ 0 & B_1 \end{bmatrix}$ is left invertible. Taking the operator X of the form (18), we have the representation (19) of M_X. From the Browderness of A_1, the left invertibility of $\begin{bmatrix} A_2 & \Delta \\ 0 & B_1 \end{bmatrix}$ and Lemma 2.3, we see that $\tilde{M}_{X_{34}} = \begin{bmatrix} A_2 & \Delta & 0 \\ 0 & B_1 & B_{12} \\ 0 & 0 & B_2 \end{bmatrix} \in \mathcal{B}(\mathcal{P}_{A^p} \oplus \mathcal{R}(B^q) \oplus \mathcal{Q}_{B^q})$ is left Browder. Further,

$$\tilde{M}_{X_{34}} = \begin{bmatrix} A_{21} & P_{\mathcal{R}(A_2)}\Delta & 0 \\ 0 & P_{\mathcal{Q}}\Delta & 0 \\ 0 & B_1 & B_{12} \\ 0 & 0 & B_2 \end{bmatrix} : \mathcal{P}_{A^p} \oplus \mathcal{R}(B^q) \oplus \mathcal{Q}_{B^q} \to \mathcal{R}(A_2) \oplus \mathcal{Q} \oplus \mathcal{R}(B^q) \oplus \mathcal{Q}_{B^q}.$$

Since A_{21} is invertible, $\begin{bmatrix} P_{\mathcal{Q}}\Delta & 0 \\ B_1 & B_{12} \\ 0 & B_2 \end{bmatrix}$ is left Fredholm. Define

$$J = [\Delta \ \ 0] : \mathcal{R}(B^q) \oplus \mathcal{Q}_{B^q} \to \mathcal{P}_{A^p}.$$

By comparing with the above arguments, $\begin{bmatrix} P_{\mathcal{Q}}J \\ B \end{bmatrix}$ is left Fredholm, $\begin{bmatrix} A_2 & J \\ 0 & B \end{bmatrix}$ is left Browder and hence $\mathrm{asc}(\begin{bmatrix} A_2 & J \\ 0 & B \end{bmatrix}) < \infty$. Applying Theorem 3.1 gives the desired result. □

Corollary 3.10. *Let M_X be defined as in* (1)*, and let B be right Browder. Then there exists $X \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ such that M_X is left Browder, if*

(i) A is left Browder; and

(ii) $\mathcal{N}(B_1) \preceq \mathcal{Q}$, where B_1 and $\mathcal{Q}$ are defined as in Theorem 3.3 and Theorem 3.1, respectively.

Finally, we have the following dual results.

Corollary 3.11. *Let M_X be defined as in* (1)*, and let A be left Browder. Then there exists $X \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ such that M_X is right Browder, if*

(i) B is right Browder; and

(ii) $\mathcal{X}/\mathcal{R}(A) \oplus \mathcal{Y}/\mathcal{R}(B) \preceq \mathcal{N}(A) \oplus \mathcal{N}(B)$.

Corollary 3.12. *Let M_X be defined as in* (1)*, and let A be left Browder. Then there exists $X \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ such that M_X is right Browder, if*

(i) B is right Browder; and

(ii) $\mathcal{Q} \preceq \mathcal{N}(B_1)$, where B_1 and $\mathcal{Q}$ are defined as in Theorem 3.3 and Theorem 3.1, respectively.

We end this section with the following illustrating example.

Example 3.13. *Let $\mathcal{X} = l^2 = \mathcal{Y}$, and define the operators $A, B \in \mathcal{B}(l^2)$ by*

$$A(x_1, x_2, x_3, x_4, x_5, x_6, x_7, ...) = (0, x_1, 0, x_4, 0, x_5, 0, x_6, 0, x_7, ...),$$
$$B(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, ...) = (0, y_2, 0, y_4, y_6, y_8, ...)$$

for $(x_1, x_2, x_3, ...) \in l^2$. Then there exists $X \in \mathcal{B}(l^2)$ such that $M_X = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix}$ is left Browder.

In the following, we still use the notations of Theorem 3.1 and Theorem 3.3. Direct computations reveal that A is a left Browder operator with $p = \mathrm{asc}(A) = 2$, and B is a right Browder operator with $q = \mathrm{des}(B) = 1$. Also, $A_2 = P_{\mathcal{P}_{A^2}} A|\mathcal{P}_{A^2}$ and $B_1 = P_{\mathcal{R}(B)} B|\mathcal{R}(B)$ are explicitly given by

$$A_2 : (0, 0, 0, x_4, x_5, x_6, ...) \mapsto (0, 0, 0, x_4, 0, x_5, 0, x_6, ...),$$
$$B_1 : (0, y_2, 0, y_4, y_6, y_8, y_{10}, ...) \mapsto (0, y_2, 0, y_4, y_8, y_{12}, ...),$$

and hence we can choose

$$\mathcal{N}(B_1) = \{(0, 0, 0, 0, c_5, 0, c_7, 0, c_9, 0, ...) \in l^2\} = Q$$

satisfying $\mathcal{N}(B_1) \preceq Q$ naturally. According to Corollary 3.10, there exists $X \in \mathcal{B}(l^2)$ such that $M_X = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix}$ is left Browder.

Remark 3.14. *The example 3.13, in fact, can be used to illustrate Corollary 3.7 and Corollary 3.12.*

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