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The Drazin inverse for perturbed block-operator matrices

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Abstract. We present new formulas of Drazin inverses for anti-triangular block-operator matrices. If $B^{\pi}A^DB = 0, B^{\pi}AB^D = 0$ and $B^{\pi}ABA^{\pi} = 0$, the explicit representation of the Drazin inverse of a block-operator anti- triangular matrix $\begin{pmatrix} A & I \\ B & 0 \end{pmatrix}$ is given. Thus a generalization of [A note on the Drazin inverse for an anti-triangular matrix, Linear Algebra Appl., **431**(2009), 1910–1922] is obtained. Some applications to full block-operator matrices are thereby considered.

1. Introduction

Lex X and Y be Banach spaces. Denote by $\mathcal{B}(X, Y)$ the set of all bounded linear operators from X to Y. Let $\mathcal{B}(X)$ denote the set of all bounded linear operators from X to itself. A bounded linear operator $A \in \mathcal{B}(X)$ has Drazin inverse $X \in \mathcal{B}(X)$ if it is the solution of the following equation system:

$$AX = XA, X = XAX \text{ and } A^n = A^{n+1}X$$

for some $n \in \mathbb{N}$. If such X exists, it is unique, and we denote it by A^D. The smallest n in the preceding equations is called the Drazin index of A and denote by $i(A)$. Let $\mathcal{B}(X)^D$ denote the set of all Drazin invertible bounded linear operators in $\mathcal{B}(X)$. Let $A, B \in \mathcal{B}(X)^D$ and I be the identity matrix over a Banach space X. It is attractive to investigate the Drazin inverse of the block-operator matrix $M = \begin{pmatrix} A & I \\ B & 0 \end{pmatrix}$. The relationship of computing the Drazin inverse of M to second order differential equations was observed by Campbell (see [2]). The application of Drazin inverse to singular differential equations was also found in [1]. Recently, the Drazin inverse of such anti-triangular block matrices is extensively studied by many authors (see [5, 6, 11, 14, 15, 17, 18]).

The additive property of Drazin inverse is interesting. It was studied from many different views, e.g., [3, 4, 9, 12, 13]. Let $T \in \mathcal{B}(X)^D$. We use T^{π} to stand for the spectral idempotent operator $I - TT^D$. In [3, Theorem 2.5], Castro-González obtained the representation of Drazin inverse of $A + B$ under the conditions $A^DB = 0, AB^D = 0$ and $B^{\pi}ABA^{\pi} = 0$ for square complex matrices A and B. The motivation of this paper is to present formulae for the Drazin inverse of M under the same conditions.

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In this paper, we present exact representations of the Drazin inverse of M. If $B^{\pi}A^{D}B = 0, B^{\pi}AB^{D} = 0$ and $B^{\pi}ABA^{\pi} = 0$, the formula of the Drazin inverse of a block anti-triangular matrix $\begin{pmatrix} A & I \\ B & 0 \end{pmatrix}$ is given. Evidently, we solve a wider kind of singular differential equations posed by Campbell (see [2]).

As applications, we explore the Drazin invertibility of a block operator matrix $N = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A \in \mathcal{B}(X)^{D}, B \in \mathcal{B}(X,Y), C \in \mathcal{B}(Y,X)$ and $D \in \mathcal{B}(Y)^{D}$. Here, N is a bounded linear operator on $X \oplus Y$. A new additive property of the Drazin invertibility for bounded linear operators is provided and we then establish new perturbed conditions under which the full block-operator matrix N has Drazin inverse.

Throughout the paper, all operators are bounded linear operators over a Banach space. Let $\mathbb{C}$ be the field of all complex numbers. $\mathbb{N}$ stands for the set of all natural numbers. $C^{n\times n}$ denotes the Banach algebra of all $n \times n$ complex matrices.

2. Key lemmas

To prove the main results, some lemmas are needed. The following result over complex fields was given in [8]. Similarly, it can be extended to bounded linear operators over a Banach space.

Lemma 2.1. *Let $P, Q \in \mathcal{B}(X)^{D}$. If $PQ = 0$, then*

$$(P+Q)^{D} = \sum_{i=0}^{t-1} Q^{i}Q^{\pi}(P^{D})^{i+1} + \sum_{i=0}^{t-1}(Q^{D})^{i+1}P^{i}P^{\pi},$$

where $t = \max\{i(P), i(Q)\}$.

Let $A, B \in \mathcal{B}(X)^{D}$. We are ready to prove:

Lemma 2.2. *Suppose $G = \begin{pmatrix} AB^{\pi} & B^{\pi} \\ BB^{\pi} & 0 \end{pmatrix}$ has Drazin inverse. If $B^{\pi}AB^{D} = 0$, then*

$$\begin{pmatrix} A & I \\ B & 0 \end{pmatrix}^{D} = G^{D} + \sum_{i=0}^{i(G)-1} \begin{pmatrix} 0 & B^{D} \\ BB^{D} & -AB^{D} \end{pmatrix}^{i+1} G^{i}G^{\pi}.$$

Proof. We check that $M = G + H$, where

$$G = \begin{pmatrix} AB^{\pi} & B^{\pi} \\ BB^{\pi} & 0 \end{pmatrix}, H = \begin{pmatrix} ABB^{D} & BB^{D} \\ B^{2}B^{D} & 0 \end{pmatrix}.$$

Since $B^{\pi}AB^{D} = 0$, we see that $GH = 0$. One directly verifies that

$$\begin{aligned} & \begin{pmatrix} 0 & B^{D} \\ BB^{D} & -AB^{D} \end{pmatrix} H = \begin{pmatrix} BB^{D} & 0 \\ 0 & BB^{D} \end{pmatrix} \\ = \ & H \begin{pmatrix} 0 & B^{D} \\ BB^{D} & -AB^{D} \end{pmatrix}, \\ & H[I - H \begin{pmatrix} 0 & B^{D} \\ BB^{D} & -AB^{D} \end{pmatrix}] \\ = \ & [I - H \begin{pmatrix} 0 & B^{D} \\ BB^{D} & -ABB^{D} \end{pmatrix}] \begin{pmatrix} 0 & B^{D} \\ BB^{D} & -AB^{D} \end{pmatrix} \\ = \ & 0. \end{aligned}$$

Therefore

$$H^{D} = \begin{pmatrix} 0 & B^{D} \\ BB^{D} & -AB^{D} \end{pmatrix}, H^{\pi} = \begin{pmatrix} B^{\pi} & 0 \\ 0 & B^{\pi} \end{pmatrix}.$$

Let $t = i(G)$. Using Lemma 2.1,

$$M^D = G^D + \sum_{i=0}^{t-1} (H^D)^{i+1} G^i G^\pi,$$

as asserted. □

The following lemma is known as the Cline's formula in matrix and operator theory (see [10, Corollary 3.3]).

Lemma 2.3. *Let $P \in \mathcal{B}(X,Y), Q \in \mathcal{B}(Y,X)$. If $PQ \in \mathcal{B}(X)^D$, then $QP \in \mathcal{B}(Y)^D$. In this case,*

$$(QP)^D = Q[(PQ)^D]^2 P.$$

Lemma 2.4. *(see [11, Lemma 3.2] and [14, Lemma 2.3]) Let $A, B \in \mathcal{B}(X)$. If A and B are nilpotent and $AB = 0$, then $\begin{pmatrix} A & B \\ I & 0 \end{pmatrix}$ is nilpotent.*

Lemma 2.5. *If $A^D B = 0, ABA^\pi = 0$ and B is nilpotent, then*

$$\begin{pmatrix} A & I \\ B & 0 \end{pmatrix}^D = \sum_{i=0}^{t-1} \begin{pmatrix} AA^\pi & A^\pi \\ BA^\pi & 0 \end{pmatrix}^i \begin{pmatrix} (A^D)^{i+1} & (A^D)^{i+2} \\ B(A^D)^{i+2} & B(A^D)^{i+3} \end{pmatrix},$$

$t = i\begin{pmatrix} AA^\pi & A^\pi \\ BA^\pi & 0 \end{pmatrix}$.

Proof. Clearly, we have $M = P + Q$, where

$$P = \begin{pmatrix} A^2A^D & AA^D \\ BAA^D & 0 \end{pmatrix}, Q = \begin{pmatrix} AA^\pi & A^\pi \\ BA^\pi & 0 \end{pmatrix}.$$

We easily see that

$$\begin{pmatrix} A^2A^D & 0 \\ I & 0 \end{pmatrix}^D = \begin{pmatrix} A^D & 0 \\ (A^D)^2 & 0 \end{pmatrix}.$$

We observe that

$$\begin{pmatrix} A^2A^D & AA^D \\ BAA^D & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & BAA^D \end{pmatrix} \begin{pmatrix} A^2A^D & AA^D \\ I & 0 \end{pmatrix},$$
$$\begin{pmatrix} A^2A^D & 0 \\ I & 0 \end{pmatrix} = \begin{pmatrix} A^2A^D & AA^D \\ I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & BAA^D \end{pmatrix}.$$

By using Lemma 2.3, we get

$$\begin{aligned} P^D &= \begin{pmatrix} I & 0 \\ 0 & BAA^D \end{pmatrix} \Big[\begin{pmatrix} A^2A^D & 0 \\ I & 0 \end{pmatrix}^D \Big]^2 \begin{pmatrix} A^2A^D & AA^D \\ I & 0 \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ 0 & BAA^D \end{pmatrix} \begin{pmatrix} (A^D)^2 & 0 \\ (A^D)^3 & 0 \end{pmatrix} \begin{pmatrix} A^2A^D & AA^D \\ I & 0 \end{pmatrix} \\ &= \begin{pmatrix} A^D & (A^D)^2 \\ B(A^D)^2 & B(A^D)^3 \end{pmatrix}. \end{aligned}$$

By induction, we have

$$\begin{aligned}(P^D)^i &= \begin{pmatrix} I & 0 \\ 0 & BAA^D \end{pmatrix}\Big[\begin{pmatrix} A^2A^D & 0 \\ I & 0 \end{pmatrix}^D\Big]^{i+1}\begin{pmatrix} A^2A^D & AA^D \\ I & 0 \end{pmatrix}\\ &= \begin{pmatrix} I & 0 \\ 0 & BAA^D \end{pmatrix}\begin{pmatrix} A^D & 0 \\ (A^D)^2 & 0 \end{pmatrix}^{i+1}\begin{pmatrix} A^2A^D & AA^D \\ I & 0 \end{pmatrix}\\ &= \begin{pmatrix} I & 0 \\ 0 & BAA^D \end{pmatrix}\begin{pmatrix} (A^D)^{i+1} & 0 \\ (A^D)^{i+2} & 0 \end{pmatrix}\begin{pmatrix} A^2A^D & AA^D \\ I & 0 \end{pmatrix}\\ &= \begin{pmatrix} (A^D)^i & (A^D)^{i+1} \\ B(A^D)^{i+1} & B(A^D)^{i+2} \end{pmatrix}.\end{aligned}$$

Clearly, we have

$$\begin{pmatrix} AA^\pi & BA^\pi \\ I & 0 \end{pmatrix} = \begin{pmatrix} AA^\pi & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & BA^\pi \\ I & 0 \end{pmatrix},$$

$$\begin{pmatrix} AA^\pi & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & BA^\pi \\ I & 0 \end{pmatrix} = 0, \begin{pmatrix} 0 & BA^\pi \\ I & 0 \end{pmatrix}^2 = \begin{pmatrix} BA^\pi & 0 \\ 0 & BA^\pi \end{pmatrix}.$$

Since $A^\pi B = B$ is nilpotent then BA^π is nilpotent. By virtue of Lemma 2.4, $\begin{pmatrix} AA^\pi & BA^\pi \\ I & 0 \end{pmatrix}$ is nilpotent. Observing that

$$\begin{aligned}\begin{pmatrix} AA^\pi & A^\pi \\ BA^\pi & 0 \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 0 & BA^\pi \end{pmatrix}\begin{pmatrix} AA^\pi & A^\pi \\ I & 0 \end{pmatrix},\\ \begin{pmatrix} AA^\pi & BA^\pi \\ I & 0 \end{pmatrix} &= \begin{pmatrix} AA^\pi & A^\pi \\ I & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & BA^\pi \end{pmatrix}.\end{aligned}$$

According to [10, Corollary 3.3], Q is nilpotent.

Since $A^DB = 0$, we have $PQ = 0$. In view of Lemma 2.1, we get

$$\begin{aligned}\begin{pmatrix} A & I \\ B & 0 \end{pmatrix}^D &= \sum_{i=0}^{t-1} Q^iQ^\pi(P^D)^{i+1} + \sum_{i=0}^{t-1}(Q^D)^{i+1}P^iP^\pi\\ &= \sum_{i=0}^{t-1} Q^i(P^D)^{i+1},\end{aligned}$$

where $t = i(Q)$. This completes the proof. □

3. Main results

We now come to our main result.

Theorem 3.1. *If $B^\pi A^DB = 0, B^\pi AB^D = 0$ and $B^\pi ABA^\pi = 0$, then*

$$\begin{pmatrix} A & I \\ B & 0 \end{pmatrix}^D = G^D + \sum_{i=0}^{t-1}\begin{pmatrix} 0 & B^D \\ BB^D & -AB^D \end{pmatrix}^{i+1} G^iG^\pi,$$

where

$$G = \begin{pmatrix} AB^\pi & B^\pi \\ BB^\pi & 0 \end{pmatrix}, G^D = \begin{pmatrix} A & I \\ B & 0 \end{pmatrix}\begin{pmatrix} \Lambda & \Sigma \\ \Gamma & \Delta \end{pmatrix}^2\begin{pmatrix} B^\pi & 0 \\ 0 & B^\pi \end{pmatrix},$$

$$\begin{array}{rcl}\Lambda &=& B^{\pi}A^{D}+\sum_{i=1}^{s-1}\varepsilon_iB^{\pi}(A^D)^{i+1}+\sum_{i=1}^{s-1}\zeta_iBB^{\pi}(A^D)^{i+2}\\ \Sigma &=& B^{\pi}(A^{D})^2+\sum_{i=1}^{s-1}\varepsilon_iB^{\pi}(A^D)^{i+2}+\sum_{i=1}^{s-1}\zeta_iBB^{\pi}(A^D)^{i+3}\\ \Gamma &=& BB^{\pi}(A^{D})^2+\sum_{i=1}^{s-1}\eta_iB^{\pi}(A^D)^{i+1}+\sum_{i=1}^{s-1}\theta_iBB^{\pi}(A^D)^{i+2}\\ \Delta &=& BB^{\pi}(A^{D})^3+\sum_{i=1}^{s-1}\eta_iB^{\pi}(A^D)^{i+2}+\sum_{i=1}^{s-1}\theta_iBB^{\pi}(A^D)^{i+3}.\end{array}$$

$$\begin{array}{rcl}\varepsilon_1 &=& B^{\pi}AA^{\pi},\\ \zeta_1 &=& I-B^{\pi}AA^{D},\\ \eta_1 &=& BB^{\pi}A^{\pi},\\ \theta_1 &=& 0,\end{array}$$

$$\begin{array}{rcl}\varepsilon_{i+1} &=& B^{\pi}A^{\pi}A\varepsilon_i+(I-B^{\pi}AA^{D})\eta_i,\\ \zeta_{i+1} &=& B^{\pi}A^{\pi}A\zeta_i+(I-B^{\pi}AA^{D})\theta_i,\\ \eta_{i+1} &=& BB^{\pi}A^{\pi}\varepsilon_i,\\ \theta_{i+1} &=& BB^{\pi}A^{\pi}\theta_i,\end{array}$$

$$s=max\{i(A),i(B)\}, t=i(G), i\in\mathbb{N}.$$

Proof. Set $s=max\{i(A),i(B)\}$. By hypothesis, we check that

$$\begin{array}{rcl}(B^{\pi}A)(B^{\pi}A^D) &=& B^{\pi}AA^D=B^{\pi}A^DA=(B^{\pi}A^D)(B^{\pi}A),\\ (B^{\pi}A^D)^2(B^{\pi}A) &=& B^{\pi}A^D(B^{\pi}A^DB^{\pi}A)=B^{\pi}A^D,\\ (B^{\pi}A^D)(B^{\pi}A)^{s+1}-(B^{\pi}A)^s &=& B^{\pi}(A^DA^{s+1}-A^s)=0.\end{array}$$

Hence, $(B^{\pi}A)^D=B^{\pi}A^D$, and so $(B^{\pi}A)^{\pi}=I-B^{\pi}AB^{\pi}A^D=I-B^{\pi}AA^D$. It is easy to verify that

$$\begin{array}{c}(B^{\pi}A)^D(B^{\pi}B)=B^{\pi}A^DBB^{\pi}=0, (B^{\pi}A)(B^{\pi}B)^D=0,\\ (B^{\pi}B)^{\pi}(B^{\pi}A)(B^{\pi}B)(B^{\pi}A)^{\pi}=B^{\pi}ABA^{\pi}=0.\end{array}$$

Since BB^{π} is nilpotent, it follows by Lemma 2.5 that

$$\begin{array}{cl} & \begin{pmatrix} B^{\pi}A & I\\ B^{\pi}B & 0\end{pmatrix}^D\\ = & \sum_{i=0}^{t-1}\begin{pmatrix} B^{\pi}A(B^{\pi}A)^{\pi} & (B^{\pi}A)^{\pi}\\ BB^{\pi}(B^{\pi}A)^{\pi} & 0\end{pmatrix}^i\begin{pmatrix}(B^{\pi}A^D)^{i+1} & (B^{\pi}A^D)^{i+2}\\ BB^{\pi}(B^{\pi}A^D)^{i+2} & BB^{\pi}(B^{\pi}A^D)^{i+3}\end{pmatrix}\\ = & \sum_{i=0}^{t-1}\begin{pmatrix} B^{\pi}AA^{\pi} & I-B^{\pi}AA^D\\ B^{\pi}BA^{\pi} & 0\end{pmatrix}^i\begin{pmatrix}B^{\pi}(A^D)^{i+1} & B^{\pi}(A^D)^{i+2}\\ BB^{\pi}(A^D)^{i+2} & BB^{\pi}(A^D)^{i+3}\end{pmatrix},\end{array}$$

where $t=i\begin{pmatrix} B^{\pi}AA^{\pi} & I-B^{\pi}AA^D\\ B^{\pi}BA^{\pi} & 0\end{pmatrix}$.

Write $\begin{pmatrix} B^{\pi}AA^{\pi} & I-B^{\pi}AA^D\\ B^{\pi}BA^{\pi} & 0\end{pmatrix}^i=\begin{pmatrix}\varepsilon_i & \zeta_i\\ \eta_i & \theta_i\end{pmatrix}$. Then

$$\begin{array}{rcl}\varepsilon_1 &=& B^{\pi}AA^{\pi},\\ \zeta_1 &=& I-B^{\pi}AA^{D},\\ \eta_1 &=& BB^{\pi}A^{\pi},\\ \theta_1 &=& 0.\end{array}$$

For each $i\geq 1$, we have

$$\begin{array}{rcl}\varepsilon_{i+1} &=& B^{\pi}A^{\pi}A\varepsilon_i+(I-B^{\pi}AA^{D})\eta_i,\\ \zeta_{i+1} &=& B^{\pi}A^{\pi}A\zeta_i+(I-B^{\pi}AA^{D})\theta_i,\\ \eta_{i+1} &=& BB^{\pi}A^{\pi}\varepsilon_i,\\ \theta_{i+1} &=& BB^{\pi}A^{\pi}\theta_i.\end{array}$$

Hence

$$\begin{pmatrix} B^{\pi}A & I \\ B^{\pi}B & 0 \end{pmatrix}^{D} = \begin{pmatrix} \Lambda & \Sigma \\ \Gamma & \Delta \end{pmatrix},$$

where

$$\begin{aligned}
\Lambda &= B^{\pi}A^{D} + \sum_{i=1}^{s-1} \varepsilon_i B^{\pi}(A^{D})^{i+1} + \sum_{i=1}^{s-1} \zeta_i BB^{\pi}(A^{D})^{i+2} \\
\Sigma &= B^{\pi}(A^{D})^{2} + \sum_{i=1}^{s-1} \varepsilon_i B^{\pi}(A^{D})^{i+2} + \sum_{i=1}^{s-1} \zeta_i BB^{\pi}(A^{D})^{i+3} \\
\Gamma &= BB^{\pi}(A^{D})^{2} + \sum_{i=1}^{s-1} \eta_i B^{\pi}(A^{D})^{i+1} + \sum_{i=1}^{s-1} \theta_i BB^{\pi}(A^{D})^{i+2} \\
\Delta &= BB^{\pi}(A^{D})^{3} + \sum_{i=1}^{s-1} \eta_i B^{\pi}(A^{D})^{i+2} + \sum_{i=1}^{s-1} \theta_i BB^{\pi}(A^{D})^{i+3}.
\end{aligned}$$

Let $G = \begin{pmatrix} AB^{\pi} & B^{\pi} \\ BB^{\pi} & 0 \end{pmatrix}$ and $H = \begin{pmatrix} B^{\pi}A & B^{\pi} \\ BB^{\pi} & 0 \end{pmatrix}$. Then we see that

$$H = \begin{pmatrix} B^{\pi}A & I \\ BB^{\pi} & 0 \end{pmatrix}\begin{pmatrix} B^{\pi} & 0 \\ 0 & B^{\pi} \end{pmatrix}.$$

Using [18, Lemma 1.4], we get

$$H^{D} = \begin{pmatrix} B^{\pi}A & I \\ BB^{\pi} & 0 \end{pmatrix}^{D}\begin{pmatrix} B^{\pi} & 0 \\ 0 & B^{\pi} \end{pmatrix}.$$

It is easy to verify that

$$\begin{aligned}
G &= \begin{pmatrix} A & I \\ B & 0 \end{pmatrix}\begin{pmatrix} B^{\pi} & 0 \\ 0 & B^{\pi} \end{pmatrix}, \\
H &= \begin{pmatrix} B^{\pi} & 0 \\ 0 & B^{\pi} \end{pmatrix}\begin{pmatrix} A & I \\ B & 0 \end{pmatrix}.
\end{aligned}$$

By using Lemma 2.3 again, we have

$$\begin{aligned}
G^{D} &= \begin{pmatrix} A & I \\ B & 0 \end{pmatrix}(H^{D})^{2}\begin{pmatrix} B^{\pi} & 0 \\ 0 & B^{\pi} \end{pmatrix} \\
&= M\begin{pmatrix} \Lambda & \Sigma \\ \Gamma & \Delta \end{pmatrix}^{2}\begin{pmatrix} B^{\pi} & 0 \\ 0 & B^{\pi} \end{pmatrix}.
\end{aligned}$$

According to Lemma 2.2,

$$\begin{pmatrix} A & I \\ B & 0 \end{pmatrix}^{D} = G^{D} + \sum_{i=0}^{t-1}\begin{pmatrix} 0 & B^{D} \\ BB^{D} & -AB^{D} \end{pmatrix}^{i+1} G^{i}G^{\pi},$$

where $t = i(G)$. This completes the proof. □

Corollary 3.2. *If $A^{D}B = 0, AB^{D} = 0$ and $B^{\pi}ABA^{\pi} = 0$, then*

$$\begin{pmatrix} A & I \\ B & 0 \end{pmatrix}^{D} = G^{D} + \sum_{i=0}^{t-1}\begin{pmatrix} 0 & B^{D} \\ BB^{D} & 0 \end{pmatrix}^{i+1} G^{i}G^{\pi},$$

where

$$G = \begin{pmatrix} A & B^{\pi} \\ BB^{\pi} & 0 \end{pmatrix}, G^{D} = \begin{pmatrix} A & I \\ B & 0 \end{pmatrix}\begin{pmatrix} \Lambda & \Sigma \\ \Gamma & \Delta \end{pmatrix}^{2},$$

$$\begin{array}{rcl}\Lambda & = & B^\pi A^D+\sum\limits_{i=1}^{s-1}\varepsilon_iB^\pi(A^D)^{i+1}+\sum\limits_{i=1}^{s-1}\zeta_iBB^\pi(A^D)^{i+2}\\ \Sigma & = & B^\pi(A^D)^2+\sum\limits_{i=1}^{s-1}\varepsilon_iB^\pi(A^D)^{i+2}+\sum\limits_{i=1}^{s-1}\zeta_iBB^\pi(A^D)^{i+3}\\ \Gamma & = & BB^\pi(A^D)^2+\sum\limits_{i=1}^{s-1}\eta_iB^\pi(A^D)^{i+1}+\sum\limits_{i=1}^{s-1}\theta_iBB^\pi(A^D)^{i+2}\\ \Delta & = & BB^\pi(A^D)^3+\sum\limits_{i=1}^{s-1}\eta_iB^\pi(A^D)^{i+2}+\sum\limits_{i=1}^{s-1}\theta_iBB^\pi(A^D)^{i+3}.\end{array}$$

$$\begin{array}{rcl}\varepsilon_1 & = & B^\pi AA^\pi,\\ \zeta_1 & = & I-B^\pi AA^D,\\ \eta_1 & = & BB^\pi A^\pi,\\ \theta_1 & = & 0,\end{array}$$

$$\begin{array}{rcl}\varepsilon_{i+1} & = & B^\pi A^\pi A\varepsilon_i+(I-B^\pi AA^D)\eta_i,\\ \zeta_{i+1} & = & B^\pi A^\pi A\zeta_i+(I-B^\pi AA^D)\theta_i,\\ \eta_{i+1} & = & BB^\pi A^\pi\varepsilon_i,\\ \theta_{i+1} & = & BB^\pi A^\pi\theta_i,\end{array}$$

$$s=max\{i(A),i(B)\},t=i(G),i\in\mathbb{N}.$$

Proof. Since $AB^D=0$, we have $A^DB^D=(A^D)^2(AB^D)=0$, and so $A^DB^\pi=A^D$. Therefore we obtain the result by Theorem 3.1. □

Corollary 3.3. *If $A^DB=0$ and $ABA^\pi=0$, then*

$$\begin{pmatrix} A & I\\ B & 0\end{pmatrix}^D=G^D+\sum_{i=0}^{t-1}\begin{pmatrix} 0 & B^D\\ BB^D & 0\end{pmatrix}^{i+1}G^iG^\pi,$$

where G is constructed as in Corollary 3.2.

Proof. Since $A^DB=0$ and $ABA^\pi=0$, we have

$$AB^2=ABA^\pi B+ABAA^DB=0,$$

and then $AB^D=0$. Setting G as in Corollary 3.2, we have

$$\begin{pmatrix} A & I\\ B & 0\end{pmatrix}^D=\begin{pmatrix} B^\pi & 0\\ 0 & B^\pi\end{pmatrix}G^D+\sum_{i=0}^{t-1}\begin{pmatrix} 0 & B^D\\ BB^D & 0\end{pmatrix}^{i+1}G^iG^\pi,$$

as desired. □

Corollary 3.4. *If $AB^D=0$ and $B^\pi AB=0$, then*

$$\begin{pmatrix} A & I\\ B & 0\end{pmatrix}^D=G^D+\sum_{i=0}^{t-1}\begin{pmatrix} 0 & B^D\\ BB^D & 0\end{pmatrix}^{i+1}G^iG^\pi,$$

where G is constructed as in Corollary 3.2.

Proof. Since $AB^D=0$ and $B^\pi AB=0$, we have

$$A^2B=AB^\pi AB+ABB^DAB=0;$$

hence, $A^DB=0$. Setting G as in Corollary 3.2, we derive

$$\begin{pmatrix} A & I\\ B & 0\end{pmatrix}^D=\begin{pmatrix} B^\pi & 0\\ 0 & B^\pi\end{pmatrix}G^D+\sum_{i=0}^{t-1}\begin{pmatrix} 0 & B^D\\ BB^D & 0\end{pmatrix}^{i+1}G^iG^\pi,$$

as required. □

We present a numerical example to demonstrate Corollary 3.3 which should be contrast to [16, Theorem 2.3].

Example 3.5. *Let* $M = \begin{pmatrix} A & I \\ B & 0 \end{pmatrix}$*, where*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \in \mathbb{C}^{4\times 4}.$$

We easily check that

$$A^D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A^\pi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$A^D B = 0, ABA^\pi = 0, \text{while } BAB^\pi \neq 0.$$

One directly verifies that

$$\begin{pmatrix} AA^\pi & A^\pi \\ BA^\pi & 0 \end{pmatrix}^i = 0 \quad \text{for every } i \geq 3.$$

Since B is nilpotent, by using the formula in Corollary 3.3, we have

$$\begin{aligned} M^D &= \sum_{i=0}^{2} \begin{pmatrix} AA^\pi & A^\pi \\ BA^\pi & 0 \end{pmatrix}^i \begin{pmatrix} (A^D)^{i+1} & (A^D)^{i+2} \\ B(A^D)^{i+2} & B(A^D)^{i+3} \end{pmatrix} \\ &= \begin{pmatrix} A^D & (A^D)^2 \\ B(A^D)^2 & B(A^D)^3 \end{pmatrix} + \begin{pmatrix} AA^\pi & A^\pi \\ BA^\pi & 0 \end{pmatrix} \\ & \begin{pmatrix} (A^D)^2 & (A^D)^3 \\ B(A^D)^3 & B(A^D)^4 \end{pmatrix} + \begin{pmatrix} AA^\pi & A^\pi \\ BA^\pi & 0 \end{pmatrix}^2 \begin{pmatrix} (A^D)^3 & (A^D)^4 \\ B(A^D)^4 & B(A^D)^5 \end{pmatrix}, \end{aligned}$$

Since A^D *is an idempotent and* $A^\pi A^D = 0$*, we obtain*

$$\begin{aligned} M^D &= \begin{pmatrix} A^D & A^D \\ BA^D & BA^D \end{pmatrix} + \begin{pmatrix} A^\pi BA^D & A^\pi BA^D \\ 0 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} A^\pi BA^D & A^\pi BA^D \\ BA^\pi BA^D & BA^\pi BA^D \end{pmatrix}. \end{aligned}$$

Obviously, $A^\pi BA^D = BA^D$ *and* $BA^\pi BA^D = BA^\pi$*. Therefore*

$$\begin{aligned} M^D &= \begin{pmatrix} A^D + A^\pi BA^D + BA^D & A^D + A^\pi BA^D + BA^D \\ BA^D + BA^\pi & BA^D + BA^\pi \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

4. Applications

The aim of this section is to develop the Drazin invertibility of the full block-operator matrix

$$N = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A \in \mathcal{B}(X)^D, B \in \mathcal{B}(X,Y), C \in \mathcal{B}(Y,X)$ and $D \in \mathcal{B}(Y)^D$. For the detailed formula of N^D, we leave to the readers as they can be derived by the straightforward computation according to our proof. For further use, we apply Corollary 3.3 to establish a new additive property for the Drazin inverse of bounded linear operators.

Lemma 4.1. *Let $P, Q, PQ \in \mathcal{B}(X)^D$. If $PQ^2 = 0, P^DQ = 0$ and $P^2QP^\pi = 0$, then $P + Q \in \mathcal{A}^D$.*

Proof. Clearly, $P + Q = (I, Q)\begin{pmatrix} P \\ I \end{pmatrix}$. Using Lemma 2.3, it suffices to prove

$$W = \begin{pmatrix} P \\ I \end{pmatrix}(I, Q) = \begin{pmatrix} P & PQ \\ I & Q \end{pmatrix}$$

has Drazin inverse. Write $M = K + L$, where

$$K = \begin{pmatrix} P & PQ \\ I & 0 \end{pmatrix}, L = \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}.$$

Let $H = \begin{pmatrix} P & I \\ PQ & 0 \end{pmatrix}$. According to Corollary 3.3, H has Drazin inverse. Clearly,

$$H = \begin{pmatrix} I & 0 \\ 0 & PQ \end{pmatrix}\begin{pmatrix} P & I \\ I & 0 \end{pmatrix}, K = \begin{pmatrix} P & I \\ I & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & PQ \end{pmatrix}.$$

By using Lemma 2.3, K has Drazin inverse. Since $PQ^2 = 0$, we have $KL = 0$. In light of Lemma 2.1, W has Drazin inverse. Therefore $P + Q$ has Drazin inverse. □

Theorem 4.2. *If $A^DB = 0, D^DC = 0, ABC = 0, DCB = 0, A^2BD^\pi = 0$ and $D^2CA^\pi = 0$, then N has Drazin inverse.*

Proof. Write $N = P + Q$, where

$$P = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, Q = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

Then

$$P^D = \begin{pmatrix} A^D & 0 \\ 0 & D^D \end{pmatrix}, P^\pi = \begin{pmatrix} A^\pi & 0 \\ 0 & D^\pi \end{pmatrix}.$$

We compute that

$$\begin{aligned} P^DQ &= \begin{pmatrix} A^D & 0 \\ 0 & D^D \end{pmatrix}\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}) \\ &= \begin{pmatrix} 0 & A^DB \\ D^DC & 0 \end{pmatrix} \\ &= 0, \\ PQ^2 &= \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} BC & 0 \\ 0 & CB \end{pmatrix} \\ &= \begin{pmatrix} ABC & 0 \\ 0 & DCB \end{pmatrix} \\ &= 0. \end{aligned}$$

Moreover, we check that

$$\begin{aligned} P^2QP^{\pi} &= \begin{pmatrix} A^2 & 0 \\ 0 & D^2 \end{pmatrix}\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\begin{pmatrix} A^{\pi} & 0 \\ 0 & D^{\pi} \end{pmatrix} \\ &= \begin{pmatrix} 0 & A^2BD^{\pi} \\ D^2CA^{\pi} & 0 \end{pmatrix} \\ &= 0. \end{aligned}$$

The result follows by Lemma 4.1. □

As an immediate consequence, we derive the following.

Corollary 4.3. *If $A^2B = 0, D^2C = 0, ABC = 0$ and $DCB = 0$, then N has Drazin inverse.*

Consider the block-operator matrix, whose generalized Shur complement is equal to zero, that is

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, D = CA^DB \qquad (*),$$

where $A \in \mathcal{B}(X)^D, B \in \mathcal{B}(X,Y), C \in \mathcal{B}(Y,X)$ and $D \in \mathcal{B}(Y)^D$.

Theorem 4.4. *Let $A \in \mathcal{B}(X)^D$ and S be defined in $(*)$. If $A^DBC = 0, ABCA^{\pi} = 0, BDC = 0$ and $BD^2 = 0$, then S has Drazin inverse.*

Proof. Clearly, we have

$$\begin{aligned} A^DBD &= (A^D)^2ABD = (A^D)^2AB(CA^DB) \\ &= AA^D(A^DBC)A^DB = 0. \end{aligned}$$

Write $S = P + Q$, where

$$P = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix}.$$

Since $A^DBC = 0$, we see that CA^DB is nilpotent. Obviously, P and Q have Drazin inverses. Moreover, we have

$$P^D = \begin{pmatrix} A^D & (A^D)^2B \\ 0 & 0 \end{pmatrix}, P^{\pi} = \begin{pmatrix} A^{\pi} & -A^DB \\ 0 & I_n \end{pmatrix};$$
$$Q^D = 0, Q^{\pi} = I.$$

We compute that

$$\begin{aligned} P^DQ &= \begin{pmatrix} A^D & (A^D)^2B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix} \\ &= \begin{pmatrix} (A^D)^2BC & (A^D)^2BD \\ 0 & 0 \end{pmatrix} \\ &= 0, \\ PQ^2 &= \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ DC & D^2 \end{pmatrix} \\ &= \begin{pmatrix} BDC & BD^2 \\ 0 & 0 \end{pmatrix} \\ &= 0. \end{aligned}$$

Moreover, we check that

$$\begin{aligned} P^2QP^{\pi} &= \begin{pmatrix} A^2 & AB \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix}\begin{pmatrix} A^{\pi} & -A^DB \\ 0 & I_n \end{pmatrix} \\ &= \begin{pmatrix} ABCA^{\pi} & ABD - ABCA^DB \\ 0 & 0 \end{pmatrix} \\ &= 0. \end{aligned}$$

By using Lemma 4.1, $S = P + Q$ has Drazin inverse, as asserted. □

Corollary 4.5. *If $A \in \mathcal{B}(X)^D$ and S be defined in $(*)$. If $A^D B = 0, CA^\pi = 0$ and $BD = 0$, then S has Drazin inverse.*

Proof. This is immediate from Theorem 4.4. □

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