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EP matrix and the solution of matrix equation

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Abstract. This paper mainly introduces some equivalent conditions for SEP matrix, specifically by constructing some specific matrix equations and discussing whether these matrix equations have solutions in given set to determine whether a group invertible matrix is a *SEP* matrix.

1. Introduction

Throughout this paper, $\mathbb{C}^{n\times n}$ stands for the set of all $n \times n$ complex matrices. Let $A \in \mathbb{C}^{n\times n}$. Denotes the conjugate transpose matrix of A by A^H. A is called a group invertible matrix if there exists $X \in \mathbb{C}^{n\times n}$ such that

$$AXA = A, XAX = X, AX = XA.$$

If such X exists, then it is unique, denoted by $A^\#$, and is called the group inverse of A [3].

A is said to be $Moore-Penrose$ invertible if there exists $X \in \mathbb{C}^{n\times n}$ such that

$$AXA = A, XAX = X, (AX)^H = AX, (XA)^H = XA.$$

Such X always exists uniquely by [1, 2], denoted by A^+, and is called the $Moore-Penrose$ inverse of A.

Let $A \in \mathbb{C}^{n\times n}$ is a group invertible matrix. We write

$$\chi_A = \{A, A^\#, A^+, A^H, (A^\#)^H, (A^+)^H\}.$$

A is called partial isometry (or *PI*) if $A = AA^HA$. Clearly A is *PI* if and only if $A^+ = A^H$; A is called *EP* [8] if $A^\#$ exists and $A^\# = A^+$; A is called *SEP* if $A^\#$ exists and $A^\# = A^+ = A^H$. Evidently, A is *SEP* if and only if A is *EP* and *PI*. In a ring with involution, *SEP* elements have been studied in [4, 6, 9–11], and in matrix theory, by [7].

In this paper, we continue to study *SEP* matrices. In Section 2, we discuss some properties of *SEP* matrices. In Section 3, we research the relationship between the consistency of matrix equations and *SEP* matrices. In Section 4, with the help of group invertible matrices and *EP* matrices, we discuss the form of the general solution to certain equation.

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2. Some properties of *SEP* matrices

Lemma 2.1. *Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then A is a SEP matrix if and only if $A^HA^HA = AA^\#A^HAA^\#$.*

Proof. ($\Longrightarrow$) Assume that A is *SEP*. Then $A^\# = A^+ = A^H$, this gives

$$AA^\#A^HAA^\# = AA^\#A^\#AA^\# = A^\# = A^\#A^\#A = A^HA^HA.$$

($\Longleftarrow$) If $A^HA^HA = AA^\#A^HAA^\#$, then

$$AA^+A^HA^HA = AA^+(AA^\#A^HAA^\#) = AA^\#A^HAA^\# = A^HA^HA.$$

Post-multiplying the equality by $A^+(A^\#)^H(A^\#)^HA^+$, one has $AA^+A^+ = A^+$. Hence, A is *EP*, it follows $A^HA^HA = AA^\#A^HAA^\# = A^\#AA^HAA^\# = A^+AA^HAA^+ = A^H$, this gives $A = A^HA^2$. Hence, A is *SEP* by [6, Theorem 1.5.3]. □

Theorem 2.2. *Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then*
1) A^HA^HA is an EP matrix with $(A^HA^HA)^+ = A^+(A^\#)^H(A^+)^H$.
2) $(AA^\#A^HAA^\#)^+ = A^+A(A^+)^HAA^+$.
3) $AA^\#A^HAA^\#$ is a group invertible matrix with $(AA^\#A^HAA^\#)^\# = (A^+)^H$.

Proof. 1) Since

$$(A^HA^HA)(A^+(A^\#)^H(A^+)^H) = A^HA^H(A^\#)^H(A^+)^H = A^H(A^+)^H = A^+A,$$

$$(A^HA^HA)(A^+(A^\#)^H(A^+)^H)(A^HA^HA) = A^+A(A^HA^HA) = A^HA^HA$$

and we have

$$((A^HA^HA)(A^+(A^\#)^H(A^+)^H))^H = (A^+A)^H = A^+A = (A^HA^HA)(A^+(A^\#)^H(A^+)^H).$$

Since we get

$$(A^+(A^\#)^H(A^+)^H)(A^HA^HA) = A^+((A^\#)^H(A^+)^HA^H)A^HA = A^+(A^\#)^HA^HA = A^+A,$$

$$(A^+(A^\#)^H(A^+)^H)(A^HA^HA)(A^+(A^\#)^H(A^+)^H) = A^+A(A^+(A^\#)^H(A^+)^H) = A^+(A^\#)^H(A^+)^H$$

and

$$((A^+(A^\#)^H(A^+)^H)(A^HA^HA))^H = (A^+A)^H = (A^+(A^\#)^H(A^+)^H)(A^HA^HA).$$

Hence, $(A^HA^HA)^+ = A^+(A^\#)^H(A^+)^H$.
Noting that $(A^+(A^\#)^H(A^+)^H)(A^HA^HA) = (A^HA^HA)(A^+(A^\#)^H(A^+)^H)$.
Then $(A^HA^HA)^\# = A^+(A^\#)^H(A^+)^H = (A^HA^HA)^+$.
Therefore A^HA^HA is EP.
2) and 3) can be shown similarly. □

Corollary 2.3. *Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then*
1) A is an EP matrix if and only if $(A^HA^HA)^+ = A^+(A^+)^H(A^\#)^H$.
2) A is an EP matrix if and only if $AA^\#A^HAA^\#$ is an EP matrix.
3) A is a SEP matrix if and only if $AA^\#A^HAA^\#$ is a SEP matrix.

Proof. 1)($\Longrightarrow$) If A is EP, then $(A^\#)^H(A^+)^H = (A^\#)^H(A^\#)^H = (A^+)^H(A^\#)^H$.
By Theorem 2.2, we have $(A^HA^HA)^+ = A^+(A^+)^H(A^\#)^H$.
($\Longleftarrow$) If $(A^HA^HA)^+ = A^+(A^+)^H(A^\#)^H$, then, by Theorem 2.2, we have

$$A^+(A^\#)^H(A^+)^H = A^+(A^+)^H(A^\#)^H.$$

Pre-multiplying the last equality by A^HA^HA, one has $A^+A = (AA^\#)^H$. Hence, A is *EP*.
2)($\Longrightarrow$) Assume that A is *EP*. Then $AA^+ = A^+A$, it follows from Theorem 2.2 that

$$(AA^\#A^HAA^\#)^+ = A^+A(A^+)^HAA^+ = AA^+(A^+)^HA^+A = (A^+)^H = (AA^\#A^HAA^\#)^\#.$$

Hence, $AA^\#A^HAA^\#$ is *EP*.
($\Longleftarrow$) If $AA^\#A^HAA^\#$ is *EP*, then $(AA^\#A^HAA^\#)^+ = (AA^\#A^HAA^\#)^\#$.
By Theorem 2.2, one has

$$A^+A(A^+)^HAA^+ = (A^+)^H.$$

Multiplying the equality on the left by $AA^\#$, one has

$$(A^+)^H = (A^+)^HAA^+.$$

Applying the involution on the equality, one has $A^+ = AA^+A^+$. Hence, A is *EP*.
3) ($\Longrightarrow$) If A is a *SEP* matrix, then $AA^\#A^HAA^\#$ is an *EP* matrix by 2), and by Lemma 2.1, we have

$$(AA^\#A^HAA^\#)^H = (A^HA^HA)^H = A^HA^2 = A = (A^+)^H = (AA^\#A^HAA^\#)^\# = (AA^\#A^HAA^\#)^+.$$

Hence $AA^\#A^HAA^\#$ is *SEP*.
($\Longleftarrow$) If $AA^\#A^HAA^\#$ is a *SEP* matrix, then A is an *EP* matrix by 2) and $(AA^\#A^HAA^\#)^\# = (AA^\#A^HAA^\#)^H$. By Theorem 2.2, one obtains

$$(A^+)^H = (AA^\#A^HAA^\#)^H.$$

Hence, $A^+ = AA^\#A^HAA^\#$. Noting that A is *EP*. Then $AA^\#A^HAA^\# = A^H$, so $A^+ = A^H$. Thus A is *SEP*. □

Theorem 2.4. *Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then A is a SEP matrix if and only if $A^HA^HA = AA^\#A^HA^+A$.*

Proof. ($\Longrightarrow$) Assume that A is *SEP*. Then $A^+A = AA^+ = AA^\#$ and $A^HA^HA = AA^\#A^HAA^\#$ by Lemma 2.1. Hence $A^HA^HA = AA^\#A^HA^+A$.
($\Longleftarrow$) If $A^HA^HA = AA^\#A^HA^+A$, then

$$AA^\#A^HA^+A = A^HA^HA = A^+AA^HA^HA = A^+A(AA^\#A^HA^+A) = A^HA^+A.$$

Post-multiplying the last equality by $(AA^\#)^H$, one gets $AA^\#A^H = A^H$, this infers A is *EP* by [6, Theorem 1.2.1]. Hence $A^HA^HA = AA^\#A^HA^+A = AA^\#A^HAA^\#$, by Lemma 2.1, A is *SEP*.
□

3. Compatibility of matrix equation

Observing the equality appeared in Theorem 2.4, we can construct the following equation:

$$A^HXA = AA^\#XA^+A. \tag{1}$$

In [6, Theorem 1.5.3], it is shown that a matrix A is *SEP* if and only if $A^HA^+ = A^\#A^+$. Inspired by this, we can give the following theorem.

Theorem 3.1. *Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then A is a SEP matrix if and only if Eq.(3.1) has at least one solution in $\chi_A = \left\{A, A^\#, A^+, A^H, (A^\#)^H, (A^+)^H\right\}$.*

Proof. $\Rightarrow$ Assume that A is *SEP*. Then $A^\# = A^+ = A^H$. It follows $A^HA^2 = A^\#A^2 = A = AA^\#AA^\#A = AA^\#AA^+A$. Hence, $X = A$ is a solution.
$\Leftarrow$ 1) If $X = A$, then $A^HA^2 = AA^\#AA^+A = A$. Hence, A is *SEP* by [6, Theorem 1.5.3].
2) If $X = A^\#$, then $A^HA^\#A = AA^\#A^\#A^+A = A^\#$. It follows that $A = A^\#A^2 = A^HA^\#AA^2 = A^HA^2$. Hence, A is *SEP* by [6, Theorem 1.5.3].
3) If $X = A^+$, then $A^HA^+A = AA^\#A^+A^+A$. Pre-multiplying the equality by $E_n - AA^+$, one gets

$$(E_n - AA^+)A^HA^+A = 0.$$

Post-multiplying the last equality by $(A^+A^\#A)^HA^+$, one has $(E_n - AA^+)A^+ = 0$, this infers A is EP. Hence, $X = A^+ = A^\#$ is a solution, so A is *SEP* by 2).
4) If $X = A^H$, then $A^HA^HA = AA^\#A^HA^+A$. Hence, A is *SEP* by Theorem 2.4.
5) If $X = (A^+)^H$, then $A^H(A^+)^HA = AA^\#(A^+)^HA^+A$, e.g. $A^+A^2 = (A^+)^H$. This gives

$$(A^+)^H = AA^\#(A^+)^H = AA^\#A^+A^2 = A$$

and so $A = (A^+)^H = A^+A^2$. Hence, A is *SEP*.
6) If $X = (A^\#)^H$, then $A^H(A^\#)^HA = AA^\#(A^\#)^HA^+A$. Pre-multiplying the equality by $E_n - AA^+$, one yields $(E_n - AA^+)A^H(A^\#)^HA = 0$. Post-multiplying the last equality by A^+A^+, one obtains $(E_n - AA^+)A^+ = 0$. Hence, A is *EP*. It follows that $X = (A^\#)^H = (A^+)^H$ is a solution. Thus A is *SEP* by 5). □

The proof of Theorem 3.1 implies the following corollary.

Corollary 3.2. *Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then A is SEP if and only if $(A^+)^H = A^+A^2$.*

Now we can change Eq.(3.1) as follows

$$A^HXY = AA^\#XA^+Y. \tag{2}$$

Lemma 3.3. *[11, Corollary 2.10] Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then the followings are equivalent:*
1) A is a PI matrix;
2) $A^HA^+ = A^HA^H$;
3) $A^HA^+ = A^+A^+$.

Theorem 3.4. *Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then A is a SEP matrix if and only if Eq.(3.2) has at least one solution in $\chi_A^2 = \{(x, y)|x, y \in \chi_A\}$.*

Proof. "$\Rightarrow$" Assume that A is *SEP*. Then $(X, Y) = (A, A)$ is a solution.
"$\Leftarrow$" I) If $Y = A$, then we have the following equation

$$A^HXA = AA^\#XA^+A. \tag{3}$$

By Theorem 3.1, A is *SEP*;
II) If $Y = A^\#$, then we have the following equation

$$A^HXA^\# = AA^\#XA^+A^\#. \tag{4}$$

Post-multiplying Eq.(3.4) by A^2, we obtain Eq.(3.1). Hence, A is *SEP* by Theorem 3.1;
III) If $Y = A^+$, then we have the following equation

$$A^HXA^+ = AA^\#XA^+A^+. \tag{5}$$

1) If $X = A$, then $A^HAA^+ = AA^\#AA^+A^+$, e.g. $A^H = AA^+A^+$. Pre-multiplying the equality by $(AA^\#)^H$, one obtains $A^H = A^+$, it follows that $A^+ = A^H = AA^+A^+$. Thus A is *EP* and so A is *SEP*;
2) If $X = A^\#$, then $A^HA^\#A^+ = AA^\#A^\#A^+A^+ = A^\#A^+A^+$, this gives

$$(E_n - A^+A)A^\#A^+A^+ = (E_n - A^+A)A^HA^\#A^+ = 0.$$

Post-multiplying the last equality by $A(AA^{\#})^{H}A^{3}$, one obtains $(E_n - A^{+}A)A = 0$, this infers A is EP. Hence, $Y = A^{+} = A^{\#}$, it follows from II) that A is SEP;
3) If $X = A^{+}$, then $A^{H}A^{+}A^{+} = AA^{\#}A^{+}A^{+}A^{+}$. By [11, Lemma 2.11], we have $A^{H}A^{+} = AA^{\#}A^{+}A^{+}$, that is, $A^{H}AA^{+}A^{+} = AA^{\#}A^{+}A^{+}$. Again by [11, Lemma 2.11], we gets $A^{H} = AA^{\#}A^{+}$, and $A^{H}A^{2} = AA^{\#}A^{+}A^{2} = A$. Hence, A is SEP by [6, Theorem 1.5.3];
4) If $X = A^{H}$, then $A^{H}A^{H}A^{+} = AA^{\#}A^{H}A^{+}A^{+}$. Post-multiplying the equality by $A(AA^{\#})^{H}$, one has

$$A^{H}A^{H} = AA^{\#}A^{H}A^{+},$$

this gives

$$(E_n - AA^{+})A^{H}A^{H} = (E_n - AA^{+})AA^{\#}A^{H}A^{+} = 0.$$

Post-mutiplying the last equality by $(A^{\#}A^{\#})^{H}A^{+}$, one gets $(E_n - AA^{+})A^{+} = 0$, this implies A is EP. Hence, $A^{H}A^{H} = AA^{\#}A^{H}A^{+} = A^{\#}AA^{H}A^{+} = A^{+}AA^{H}A^{+} = A^{H}A^{+}$, this implies A is PI by Lemma 3.3. Thus A is SEP;
5) If $X = (A^{+})^{H}$, then $A^{H}(A^{+})^{H}A^{+} = AA^{\#}(A^{+})^{H}A^{+}A^{+}$, e.g. $A^{+} = (A^{+})^{H}A^{+}A^{+}$. This gives $A^{H}A^{+} = A^{H}(A^{+})^{H}A^{+}A^{+} = A^{+}A^{+}$, so A is PI by Lemma 3.3. Noting that $A^{+} = (A^{+})^{H}A^{+}A^{+} = AA^{+}A^{+}$. Then A is EP and so A is SEP;
6) If $X = (A^{\#})^{H}$, then $A^{H}(A^{\#})^{H}A^{+} = AA^{\#}(A^{\#})^{H}A^{+}A^{+}$, e.g. $A^{+} = AA^{\#}(A^{\#})^{H}A^{+}A^{+}$. Pre-multiplying the equality by $E_n - AA^{+}$, one yields

$$A^{+} = AA^{+}A^{+}.$$

Hence A is EP, it follows that $x = (A^{\#})^{H} = (A^{+})^{H}$. Thus A is SEP by 5).
IV) If $Y = A^{H}$, then we have the following equation

$$A^{H}XA^{H} = AA^{\#}XA^{+}A^{H}. \tag{6}$$

Post-mutiplying Eq.(3.6) by $(A^{\#})^{H}A^{+}$, one gets Eq.(3.5). Hence, A is SEP by III);
V) If $Y = (A^{+})^{H}$, then we have the following equation

$$A^{H}X(A^{+})^{H} = AA^{\#}XA^{+}(A^{+})^{H}. \tag{7}$$

Post-mutiplying Eq.(3.7) by $A^{H}A$, we get Eq.(3.3). Hence, A is SEP by I);
VI) If $Y = (A^{\#})^{H}$, then we have the following equation

$$A^{H}X(A^{\#})^{H} = AA^{\#}XA^{+}(A^{\#})^{H}. \tag{8}$$

Post-mutiplying the Eq.(3.8) by $A^{H}A^{H}$, we obtain Eq.(3.6) Hence, A is SEP by IV). □

Corollary 3.5. *Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then the followings are equivalent:*
1) A is a SEP matrix;
2) $A^{+} = (A^{+})^{H}A^{+}A^{+}$;
3) $A^{+} = A^{+}A^{+}(A^{+})^{H}$.

Now we changes Eq.(3.1) as follows

$$A^{H}XA = AA^{\#}XA^{\#}A. \tag{9}$$

Similar to Theorem 3.1, we can give the following theorem.

Theorem 3.6. *Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then A is a SEP matrix if and only if Eq.(3.9) has at least one solution in χ_A.*

The following lemma is interesting which proof is routine.

Lemma 3.7. *Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then*
(1) $A^H XA$ is a EP matrix with $(A^H XA)^+ = (A^H XA)^\# = A^+X^\#(A^+)^H$ for each $X \in \chi_A$;
(2) $(AA^\# XAA^\#)^+ = A^+AX^+AA^+$ for each $X \in \chi_A$;
(3) $AA^\# XAA^\#$ is a group invertible matrix with $(AA^\# XAA^\#)^\# = AA^\# X^+AA^\#$ for each $X \in \chi_A$.

Theorem 3.6 and Lemma 3.7 imply the following theorem.

Theorem 3.8. *Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then the followings are equivalent:*
(1) A is a SEP matrix;
(2) $A^+X^\#(A^+)^H = A^+AX^+AA^+$ for each $X \in \chi_A$;
(3) $A^+X^\#(A^+)^H = AA^\# X^+AA^\#$ for each $X \in \chi_A$.

4. The general solution of related equations

We now generalize Eq.(3.1) as follows.

$$A^H XA = AA^\# YA^+A. \tag{10}$$

Theorem 4.1. *Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then the general solution of Eq.(4.1) is given by*

$$\begin{cases} X = (A^+)^H A^+P + U - AA^+UAA^+ \\ Y = A^+PA + V - A^+AVA^+A \end{cases}, \quad \text{where } P,\ U,\ V \in \mathbb{C}^{n\times n} \text{ with } A^+P = AA^+A^+P. \tag{11}$$

Proof. Frist, we have the formula (4.2) is the solution of Eq.(4.1).
In fact,

$$\begin{aligned} A^H((A^+)^H)A^+P + U - AA^+UAA^+)A &= A^+PA = AA^+A^+PA = AA^\# AA^+A^+PA \\ &= AA^\# A^+PA = AA^\#(A^+PA + V - A^+AVA^+A)A^+A. \end{aligned}$$

Next, let

$$\begin{cases} X = X_0 \\ Y = Y_0 \end{cases} \tag{12}$$

be a solution of Eq.(4.1). Then

$$A^H X_0 A = AA^\# Y_0 A^+A.$$

Choose $P = AY_0A^+$, $U = X_0$, $V = Y_0$. Then

$$\begin{aligned} A^+P = A^+AY_0A^+ &= (A^+A)(AA^\# Y_0A^+A)A^+ = A^+A(A^H X_0A)A^+ = A^H X_0AA^+ \\ &= AA^\# Y_0A^+AA^+ = AA^\# Y_0A^+. \end{aligned}$$

So

$$AA^+A^+P = AA^+(AA^\# Y_0A^+) = AA^\# Y_0A^+.$$

Hence, $A^+P = AA^+A^+P$.
Noting that

$$\begin{aligned} (A^+)^H A^+P &= (A^+)^H AA^\# Y_0A^+ = (A^+)^H(AA^\# Y_0A^+A)A^+ \\ &= (A^+)^H A^H X_0AA^+ = AA^+X_0AA^+. \end{aligned}$$

Then

$$X_0 = AA^+X_0AA^+ + X_0 - AA^+X_0AA^+ = (A^+)^H A^+P + U - AA^+UAA^+.$$

Also

$$A^+AY_0A^+A = A^+(AY_0A^+)A = A^+PA,$$

it follows that

$$Y_0 = A^+AY_0A^+A + Y_0 - A^+AY_0A^+A = A^+PA + V - A^+AVA^+A.$$

Hence, the general solution of Eq.(4.1) is given by the formula (4.2). □

Theorem 4.2. *Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then A is a SEP matrix if and only if the general solution of Eq.(4.1) is given by*

$$\begin{cases} X = (A^+)^H A^+ P + U - AA^+ UAA^+ \\ Y = A^\# P(A^+)^H + V - A^+ AVA^+ A \end{cases}, \text{ where } P,\ U,\ V \in \mathbb{C}^{n\times n}. \tag{13}$$

Proof. $\Rightarrow$ If A is *SEP*, then $A^+ = A^\#$ and $A = (A^+)^H$. And $AA^+A^+P = A^+P$ for all $P \in \mathbb{C}^{n\times n}$.

Hence, the formula (4.2) is same as the formula (4.4), it follows from Theorem 4.1 that the general solution of Eq.(4.1) is given by the formula (4.4).

$\Leftarrow$ If the general solution of Eq.(4.1) is given by the formula (4.4), then

$$A^H((A^+)^H A^+ P + U - AA^+ UAA^+)A = AA^\#(A^\# P(A^+)^H + V - A^+ AVA^+ A)A^+ A.$$

By simple computation, we have

$$A^+ PA = A^\# P(A^+)^H.$$

Choose $P = A$. Then $A^+A^2 = A^\# A(A^+)^H = (A^+)^H$.
Hence A is *SEP* by Corollary 3.2. □

Now we construct the following equation.

$$A(AA^\#)^H AA^H XAA^+ = A^2 YA^H. \tag{14}$$

Similar to Theorem 4.1, we have the following theorem.

Theorem 4.3. *Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then the general solution of Eq.(4.5) is given by*

$$\begin{cases} X = (A^+)^H A^+ P + U - AA^+ UAA^+ \\ Y = A^\# P(A^+)^H + V - A^+ AVA^+ A \end{cases}, \text{ where } P,\ U,\ V \in \mathbb{C}^{n\times n} \text{ with } A^+P = A^+A^+AP. \tag{15}$$

Combining Theorem 4.2 with Theorem 4.3, we have the following theorem.

Theorem 4.4. *Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then A is a SEP matrix if and only Eq.(4.5) has the same solution as Eq.(4.1).*

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