Filomat 38:8 (2024), 2773–2783
https://doi.org/10.2298/FIL2408773R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The first nonlinear mixed Jordan triple derivation on ∗-algebras

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Abstract. Let $\mathcal{A}$ be a unital ∗-algebra containing a non-trivial projection. We prove that if a map $\Pi : \mathcal{A} \to \mathcal{A}$ such that $\Pi(S \bullet T \circ U) = \Pi(S) \bullet T \circ U + S \bullet \Pi(T) \circ U + S \bullet T \circ \Pi(U)$ for all $S, T, U \in \mathcal{A}$, then Π is an additive ∗-derivation.

1. Introduction

Let $\mathcal{A}$ be an ∗-algebra over the complex field $\mathbb{C}$. For $S, T \in \mathcal{A}$, $S \bullet T = ST + TS^*$ and $S \circ T = ST + TS$ denotes the Jordan ∗- product and Jordan product of S and T respectively. The Jordan and Jordan ∗-product is gaining importance across a number of research fields, and various authors have been interested in its investigation (see [1–7, 9]). An additive map $\Pi : \mathcal{A} \to \mathcal{A}$ is called an additive derivation if $\Pi(ST) = \Pi(S)T + S\Pi(T)$ for all $S, T \in \mathcal{A}$. If $\Pi(S^*) = \Pi(S)^*$, for all $S \in \mathcal{A}$, then Π is an additive ∗-derivation. Let $\Pi : \mathcal{A} \to \mathcal{A}$ be a mapping (without the additivity assumption). We say Π is a nonlinear Jordan ∗- derivation if

$$\Pi(S \bullet T) = \Pi(S) \bullet T + S \bullet \Pi(T)$$

holds for all $S, T \in \mathcal{A}$. The mixed Jordan triple products can now be considered. For each $S, T, U \in \mathcal{A}$, there exists two cases of mixed Jordan triple products: $S \bullet T \circ U$ and $S \circ T \bullet U$. In order to distinguish them, we call $S \bullet T \circ U$ and $S \circ T \bullet U$ are first mixed Jordan triple product and second mixed Jordan triple product respectively. Accordingly, a mapping $\Pi : \mathcal{A} \to \mathcal{A}$ is called the first mixed Jordan triple derivation if

$$\Pi(S \bullet T \circ U) = \Pi(S) \bullet T \circ U + S \bullet \Pi(T) \circ U + S \bullet T \circ \Pi(U)$$

for any $S, T, U \in \mathcal{A}$. A mapping $\Pi : \mathcal{A} \to \mathcal{A}$ is called the second mixed Jordan triple derivation if

$$\Pi(S \circ T \bullet U) = \Pi(S) \circ T \bullet U + S \circ \Pi(T) \bullet U + S \circ T \bullet \Pi(U)$$

for any $S, T, U \in \mathcal{A}$. In recent years, the examination of mixed Lie and Jordan triple products has received significant attention from authors. In [8], Zhou et al. proved that every nonlinear mixed Lie triple derivation

2020 *Mathematics Subject Classification*. Primary 47B47 mandatory; Secondary 46J10, 46L10.
Keywords. First mixed Jordan triple derivation, ∗-derivation, ∗- algebra.
Received: 04 April 2023; Revised: 02 August 2023; Accepted: 02 October 2023
Communicated by Dijana Mosić
Research supported by the Council of Scientific and Industrial Research (CSIR-HRDG), India, Grant No. 25(0306)/20/EMR-II.
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on prime $*$-algebra is an additive $*$-derivation. Motivated by the above results, we combine the product of Jordan and Jordan $*$-product and give the characterization of first nonlinear mixed Jordan triple derivations. We prove that the first nonlinear mixed Jordan triple derivations is an additive $*$-derivation. In particular, we apply the aforementioned result to standard operator algebras, von Neumann algebras, and prime $*$-algebras.

2. Main Result

Theorem 2.1. *Let $\mathcal{A}$ be a unital $*$-algebra with unity I containing a non-trivial projection P satisfies*

$$X\mathcal{A}P = 0 \implies X = 0 \tag{$\blacktriangle$}$$

and

$$X\mathcal{A}(I-P) = 0 \implies X = 0. \tag{$\blacktriangledown$}$$

Define a map $\Pi : \mathcal{A} \to \mathcal{A}$ such that

$$\Pi(S \bullet T \circ U) = \Pi(S) \bullet T \circ U + S \bullet \Pi(T) \circ U + S \bullet T \circ \Pi(U)$$

for all $S, T, U \in \mathcal{A}$. Then Π is an additive $*$-derivation.

Proof. Let $P = P_1$ be a non-trivial projection in $\mathcal{A}$ and $P_2 = I - P_1$, where I is the unity of this algebra. Then by Peirce decomposition of $\mathcal{A}$, we have $\mathcal{A} = P_1\mathcal{A}P_1 \oplus P_1\mathcal{A}P_2 \oplus P_2\mathcal{A}P_1 \oplus P_2\mathcal{A}P_2$ and, denote $\mathcal{A}_{11} = P_1\mathcal{A}P_1, \mathcal{A}_{12} = P_1\mathcal{A}P_2, \mathcal{A}_{21} = P_2\mathcal{A}P_1$ and $\mathcal{A}_{22} = P_2\mathcal{A}P_2$. Note that any $S \in \mathcal{A}$ can be written as $S = S_{11} + S_{12} + S_{21} + S_{22}$, where $S_{ij} \in \mathcal{A}_{ij}$ and $S_{ij}^* \in \mathcal{A}_{ji}$ for $i, j = 1, 2$. $\square$

We use various lemmas to demonstrate the above theorem.

Lemma 2.2. $\Pi(0) = 0$.

Proof. It is easy to prove $\Pi(0) = 0$. $\square$

Lemma 2.3. *Let $S_{12} \in \mathcal{A}_{12}$ and $S_{21} \in \mathcal{A}_{21}$. Then $\Pi(S_{12} + S_{21}) = \Pi(S_{12}) + \Pi(S_{21})$.*

Proof. Let $M = \Pi(S_{12} + S_{21}) - \Pi(S_{12}) - \Pi(S_{21})$. Since $S_{21} \bullet P_2 \circ P_1 = 0$ and using Lemma 2.2, we have

$$\begin{aligned}
\Pi((S_{12} + S_{21}) \bullet P_2 \circ P_1) &= \Pi(S_{12} \bullet P_2 \circ P_1) + \Pi(S_{21} \bullet P_2 \circ P_1) \\
&= \Pi(S_{12}) \bullet P_2 \circ P_1 + S_{12} \bullet \Pi(P_2) \circ P_1 + S_{12} \bullet P_2 \circ \Pi(P_1) \\
&\quad + \Pi(S_{21}) \bullet P_2 \circ P_1 + S_{21} \bullet \Pi(P_2) \circ P_1 + S_{21} \bullet P_2 \circ \Pi(P_1).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\Pi((S_{12} + S_{21}) \bullet P_2 \circ P_1) &= \Pi(S_{12} + S_{21}) \bullet P_2 \circ P_1 + (S_{12} + S_{21}) \bullet \Pi(P_2) \circ P_1 \\
&\quad + (S_{12} + S_{21}) \bullet P_2 \circ \Pi(P_1).
\end{aligned}$$

In the view of above two equations, we get $M \bullet P_2 \circ P_1 = 0$. That means $P_2M^*P_1 + P_1MP_2 = 0$. By left multiplication of P_1 both sides yields $P_1MP_2 = 0$. Similarly, one can show that $P_2MP_1 = 0$.
Now, again $(P_1 - P_2) \bullet I \circ S_{21} = 0$ and using Lemma 2.2, we have

$$\begin{aligned}
\Pi((P_1 - P_2) \bullet I \circ (S_{12} + S_{21})) &= \Pi((P_1 - P_2) \bullet I \circ S_{12}) + \Pi((P_1 - P_2) \bullet I \circ S_{21}) \\
&= \Pi(P_1 - P_2) \bullet I \circ S_{12} + (P_1 - P_2) \bullet \Pi(I) \circ S_{12} \\
&\quad + (P_1 - P_2) \bullet I \circ \Pi(S_{12}) + \Pi(P_1 - P_2) \bullet I \circ S_{21} \\
&\quad + (P_1 - P_2) \bullet \Pi(I) \circ S_{21} + (P_1 - P_2) \bullet I \circ \Pi(S_{21}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\Pi((P_1 - P_2) \bullet I \circ (S_{12} + S_{21})) &= \Pi(P_1 - P_2) \bullet I \circ (S_{12} + S_{21}) \\ &+ (P_1 - P_2) \bullet \Pi(I) \circ (S_{12} + S_{21}) \\ &+ (P_1 - P_2) \bullet I \circ \Pi(S_{12} + S_{21}).\end{aligned}$$

By comparing the above two expressions, we find that $(P_1 - P_2) \bullet I \circ M = 0$. That means $2P_1M - 2P_2M + 2MP_1 - 2MP_2 = 0$. On pre and post multiplication by P_1, we get $P_1MP_1 = 0$. Similarly, multiplying by P_2 from left and right on both sides, we get $P_2MP_2 = 0$. Hence, $M = 0$ i.e., $\Pi(S_{12} + S_{21}) = \Pi(S_{12}) + \Pi(S_{21})$. □

Lemma 2.4. *For every $S_{ij} \in \mathcal{A}_{ij}, 1 \le i \ne j \le 2$, we have*

$$\Pi(S_{ii} + S_{ij} + S_{ji}) = \Pi(S_{ii}) + \Pi(S_{ij}) + \Pi(S_{ji}).$$

Proof. Let $M = \Pi(S_{ii} + S_{ij} + S_{ji}) - \Pi(S_{ii}) - \Pi(S_{ij}) - \Pi(S_{ji})$. On the one hand, we have

$$\begin{aligned}\Pi((S_{ii} + S_{ij} + S_{ji}) \bullet P_i \circ P_j) &= \Pi(S_{ii} + S_{ij} + S_{ji}) \bullet P_i \circ P_j \\ &+ (S_{ii} + S_{ij} + S_{ji}) \bullet \Pi(P_i) \circ P_j \\ &+ (S_{ii} + S_{ij} + S_{ji}) \bullet P_i \circ \Pi(P_j).\end{aligned}$$

On the other hand, using Lemma 2.3 and $S_{ii} \bullet P_i \circ P_j = 0$, we have

$$\begin{aligned}\Pi((S_{ii} + S_{ij} + S_{ji}) \bullet P_i \circ P_j) &= \Pi(S_{ii} \bullet P_i \circ P_j) + \Pi(S_{ij} \bullet P_i \circ P_j) + \Pi(S_{ji} \bullet P_i \circ P_j) \\ &= \Pi(S_{ii}) \bullet P_i \circ P_j + S_{ii} \bullet \Pi(P_i) \circ P_j + S_{ii} \bullet P_i \circ \Pi(P_j) \\ &+ \Pi(S_{ij}) \bullet P_i \circ P_j + S_{ij} \bullet \Pi(P_i) \circ P_j + S_{ij} \bullet P_i \circ \Pi(P_j) \\ &+ \Pi(S_{ji}) \bullet P_i \circ P_j + S_{ji} \bullet \Pi(P_i) \circ P_j + S_{ji} \bullet P_i \circ \Pi(P_j).\end{aligned}$$

From last two equations, we get $T \bullet P_i \circ P_j = 0$. This gives $P_iT^*P_j + P_jTP_i = 0$. Multiplying above equation by P_i from right, we get $P_jTP_i = 0$.
Since

$$\frac{I}{2} \bullet (P_i - P_j) \circ S_{ij} = \frac{I}{2} \bullet (P_i - P_j) \circ S_{ji} = 0.$$

From Lemma 2.2, we have

$$\begin{aligned}\Pi(\frac{I}{2} \bullet (P_i - P_j) \circ S_{ii} + S_{ij} + S_{ji}) &= \Pi(\frac{I}{2} \bullet (P_i - P_j) \circ S_{ii}) + \Pi(\frac{I}{2} \bullet (P_i - P_j) \circ S_{ij}) \\ &+ \Pi(\frac{I}{2} \bullet (P_i - P_j) \circ S_{ji}) \\ &= \Pi(\frac{I}{2}) \bullet (P_i - P_j) \circ S_{ii} + \frac{I}{2} \bullet \Pi(P_i - P_j) \circ S_{ii} \\ &+ \frac{I}{2} \bullet (P_i - P_j) \circ \Pi(S_{ii}) + \Pi(\frac{I}{2}) \bullet (P_i - P_j) \circ S_{ij} \\ &+ \frac{I}{2} \bullet \Pi(P_i - P_j) \circ S_{ij} + \frac{I}{2} \bullet (P_i - P_j) \circ \Pi(S_{ij}) \\ &+ \Pi(\frac{I}{2}) \bullet (P_i - P_j) \circ S_{ji} + \frac{I}{2} \bullet \Pi(P_i - P_j) \circ S_{ji} \\ &+ \frac{I}{2} \bullet (P_i - P_j) \circ \Pi(S_{ji}).\end{aligned}$$

From the other side, we have

$$\begin{aligned}\Pi(\frac{I}{2} \bullet (P_i - P_j) \circ S_{ii} + S_{ij} + S_{ji}) &= \Pi(\frac{I}{2}) \bullet (P_i - P_j) \circ (S_{ii} + S_{ij} + S_{ji}) \\ &+ \frac{I}{2} \bullet \Pi(P_i - P_j) \circ (S_{ii} + S_{ij} + S_{ji}) \\ &+ \frac{I}{2} \bullet (P_i - P_j) \circ \Pi(S_{ii} + S_{ij} + S_{ji}).\end{aligned}$$

From above two equations, we have $\frac{I}{2} \bullet (P_i - P_j) \circ M = 0$. That yields $P_iMP_i = 0$. Thus, $M = 0$ i.e.,

$$\Pi(S_{ii} + S_{ij} + S_{ji}) = \Pi(S_{ii}) + \Pi(S_{ij}) + \Pi(S_{ji}).$$

□

Lemma 2.5. *For any $S_{ij} \in \mathcal{A}_{ij}, 1 \leq i, j \leq 2$, we have*

$$\Pi(\sum_{i,j=1}^{2} S_{ij}) = \sum_{i,j=1}^{2} \Pi(S_{ij}).$$

Proof. Let $M = \Pi(S_{11} + S_{12} + S_{21} + S_{22}) - \Pi(S_{11}) - \Pi(S_{12}) - \Pi(S_{21}) - \Pi(S_{22})$.
On the one hand, it follows from $I \bullet P_2 \circ S_{11} = 0$ and using Lemma 2.4 that

$$\begin{aligned}\Pi(I \bullet P_2 \circ (S_{11} + S_{12} + S_{21} + S_{22})) &= \Pi(I \bullet P_2 \circ S_{11}) + \Pi(I \bullet P_2 \circ S_{12}) \\ &+ \Pi(I \bullet P_2 \circ S_{21}) + \Pi(I \bullet P_2 \circ S_{22}) \\ &= \Pi(I) \bullet P_2 \circ S_{11} + I \bullet \Pi(P_2) \circ S_{11} + I \bullet P_2 \circ \Pi(S_{11}) \\ &+ \Pi(I) \bullet P_2 \circ S_{12} + I \bullet \Pi(P_2) \circ S_{12} + I \bullet P_2 \circ \Pi(S_{12}) \\ &+ \Pi(I) \bullet P_2 \circ S_{21} + I \bullet \Pi(P_2) \circ S_{21} + I \bullet P_2 \circ \Pi(S_{21}) \\ &+ \Pi(I) \bullet P_2 \circ S_{22} + I \bullet \Pi(P_2) \circ S_{22} + I \bullet P_2 \circ \Pi(S_{22}).\end{aligned}$$

From the other side, we have

$$\begin{aligned}\Pi(I \bullet P_2 \circ (S_{11} + S_{12} + S_{21} + S_{22})) &= \Pi(I) \bullet P_2 \circ (S_{11} + S_{12} + S_{21} + S_{22}) \\ &+ I \bullet \Pi(P_2) \circ (S_{11} + S_{12} + S_{21} + S_{22}) \\ &+ I \bullet P_2 \circ \Pi(S_{11} + S_{12} + S_{21} + S_{22}).\end{aligned}$$

From the last two expressions, we get $I \bullet P_2 \circ M = 0$. This gives $P_1MP_2 = P_2MP_1 = P_2MP_2 = 0$. Similarly, $P_1MP_1 = 0$. Thus $M = 0$ i.e.,

$$\Pi(S_{11} + S_{12} + S_{21} + S_{22}) = \Pi(S_{11}) + \Pi(S_{12}) + \Pi(S_{21}) + \Pi(S_{22}).$$

□

Lemma 2.6. *For any $S_{ij}, T_{ij} \in \mathcal{A}_{ij}$ with $i \neq j$, $\Pi(S_{ij} + T_{ij}) = \Pi(S_{ij}) + \Pi(T_{ij})$.*

Proof. From Lemma 2.3 and Lemma 2.4, we get

$$\begin{aligned}
\Pi((P_i + S_{ij}) \bullet (P_j + T_{ij}) \circ \frac{I}{2}) &= \Pi(P_i + S_{ij}) \bullet (P_j + T_{ij}) \circ \frac{I}{2} + (P_i + S_{ij}) \bullet \Pi(P_j + T_{ij}) \circ \frac{I}{2} \\
&\quad + (P_i + S_{ij}) \bullet (P_j + T_{ij}) \circ \Pi(\frac{I}{2}) \\
&= (\Pi(P_i) + \Pi(S_{ij})) \bullet (P_j + T_{ij}) \circ \frac{I}{2} \\
&\quad + (P_i + S_{ij}) \bullet (\Pi(P_j) + \Pi(T_{ij})) \circ \frac{I}{2} \\
&\quad + (P_i + S_{ij}) \bullet (P_j + T_{ij}) \circ \Pi(\frac{I}{2}) \\
&= \Pi(P_i \bullet P_j \circ \frac{I}{2}) + \Pi(P_i \bullet T_{ij} \circ \frac{I}{2}) \\
&\quad + \Pi(S_{ij} \bullet P_j \circ \frac{I}{2}) + \Pi(S_{ij} \bullet T_{ij} \circ \frac{I}{2}) \\
&= \Pi(T_{ij}) + \Pi(S_{ij} + S_{ij}^*) + \Pi(T_{ij} S_{ij}^*) \\
&= \Pi(T_{ij}) + \Pi(S_{ij}) + \Pi(S_{ij}^*) + \Pi(T_{ij} S_{ij}^*).
\end{aligned}$$

From the other side, using Lemma 2.4, we get

$$\begin{aligned}
\Pi((P_i + S_{ij}) \bullet (P_j + T_{ij}) \circ \frac{I}{2}) &= \Pi(S_{ij} + T_{ij} + S_{ij}^* + T_{ij} S_{ij}^*) \\
&= \Pi(S_{ij} + T_{ij}) + \Pi(S_{ij}^*) + \Pi(T_{ij} S_{ij}^*).
\end{aligned}$$

By comparing the above two equations, we get

$$\Pi(S_{ij} + T_{ij}) = \Pi(S_{ij}) + \Pi(T_{ij}).$$

□

Lemma 2.7. *For any $S_{ii}, T_{ii} \in \mathcal{A}_{ii}, 1 \leq i \leq 2$, we have*

1. $\Pi(S_{11} + T_{11}) = \Pi(S_{11}) + \Pi(T_{11})$.
2. $\Pi(S_{22} + T_{22}) = \Pi(S_{22}) + \Pi(T_{22})$.

Proof. (1) Let $M = \Pi(S_{11} + T_{11}) - \Pi(S_{11}) - \Pi(T_{11})$. Since $P_2 \bullet S_{11} \circ I = 0$ and using Lemma 2.2, we have

$$\begin{aligned}
\Pi(P_2 \bullet (S_{11} + T_{11}) \circ I) &= \Pi(P_2 \bullet S_{11} \circ I) + \Pi(P_2 \bullet T_{11} \circ I) \\
&= \Pi(P_2) \bullet S_{11} \circ I + P_2 \bullet \Pi(S_{11}) \circ I + P_2 \bullet S_{11} \circ \Pi(I) \\
&\quad + \Pi(P_2) \bullet T_{11} \circ I + P_2 \bullet \Pi(T_{11}) \circ I + P_2 \bullet T_{11} \circ \Pi(I).
\end{aligned}$$

Also, we have

$$\begin{aligned}
\Pi(P_2 \bullet (S_{11} + T_{11}) \circ I) &= \Pi(P_2) \bullet (S_{11} + T_{11}) \circ I + P_2 \bullet \Pi(S_{11} + T_{11}) \circ I \\
&\quad + P_2 \bullet (S_{11} + T_{11}) \circ \Pi(I).
\end{aligned}$$

From the above two equations, we have $P_2 \bullet T \circ I = 0$, and hence $P_2 M P_1 = P_2 M P_2 = P_1 M P_2 = 0$. Now, for any $X_{12} \in \mathcal{A}_{12}$ and using Lemma 2.6, we have

$$\begin{aligned}
\Pi(P_1 \bullet (S_{11} + T_{11}) \circ X_{12}) &= \Pi(2S_{11}X_{12} + 2T_{11}X_{12}) \\
&= \Pi(2S_{11}X_{12}) + \Pi(2T_{11}X_{12}) \\
&= \Pi(P_1 \bullet S_{11} \circ X_{12}) + \Pi(P_1 \bullet T_{11} \circ X_{12}) \\
&= \Pi(P_1) \bullet S_{11} \circ X_{12} + P_1 \bullet \Pi(S_{11}) \circ X_{12} + P_1 \bullet S_{11} \circ \Pi(X_{12}) \\
&\quad + \Pi(P_1) \bullet T_{11} \circ X_{12} + P_1 \bullet \Pi(T_{11}) \circ X_{12} + P_1 \bullet T_{11} \circ \Pi(X_{12}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\Pi(P_1 \bullet (S_{11}+T_{11}) \circ X_{12}) &= \Pi(P_1) \bullet (S_{11}+T_{11}) \circ X_{12} + P_1 \bullet \Pi(S_{11}+T_{11}) \circ X_{12} \\ &\quad + P_1 \bullet (S_{11}+T_{11}) \circ \Pi(X_{12}).\end{aligned}$$

From the above two equations, we get $P_1 \bullet M \circ X_{12} = 0$. That means $P_1MX_{12} + MX_{12} + X_{12}MP_1 = 0$. Multiplying P_2 from the right and by P_1 from left, we get $P_1MX_{12} = 0$ i.e., $P_1MP_1XP_2 = 0$ for all $X \in \mathcal{A}$. It follows from (▲) and (▼) that $P_1MP_1 = 0$. Hence, $M = 0$.

(2) Using the same technique as used in the proof of (1), we can show that

$$\Pi(S_{22}+T_{22}) = \Pi(S_{22}) + \Pi(T_{22}).$$

□

Lemma 2.8. Π *is an additive map.*

Proof. For any $S, T \in \mathcal{A}$, we write $S = \sum_{i,j=1}^{2} S_{ij}$, $T = \sum_{i,j=1}^{2} T_{ij}$. By using Lemmas 2.5 - 2.7, we get

$$\begin{aligned}\Pi(S+T) &= \Pi\Big(\sum_{i,j=1}^{2} S_{ij} + \sum_{i,j=1}^{2} T_{ij}\Big) \\ &= \Pi\Big(\sum_{i,j=1}^{2} (S_{ij}+T_{ij})\Big) \\ &= \sum_{i,j=1}^{2} \Pi(S_{ij}+T_{ij}) \\ &= \sum_{i,j=1}^{2} \Pi(S_{ij}) + \Pi(T_{ij}) \\ &= \Pi\Big(\sum_{i,j=1}^{2} S_{ij}\Big) + \Pi\Big(\sum_{i,j=1}^{2} T_{ij}\Big) \\ &= \Pi(S) + \Pi(T).\end{aligned}$$

□

Lemma 2.9.
1. $P_1\Pi(P_1)P_2 = -P_1\Pi(P_2)P_2$.
2. $P_2\,\Pi(P_1)P_1 = -P_2\Pi(P_2)P_1$.
3. $P_1\Pi(P_2)P_1 = P_2\Pi(P_1)P_2 = 0$.
4. $P_1\Pi(P_2)^*P_1 = P_2\Pi(P_1)^*P_2 = 0$.

Proof. **(1)** It follows from $P_1 \bullet P_2 \circ P_1 = 0$ and using Lemma 2.2 that

$$\begin{aligned}0 &= \Pi(P_1 \bullet P_2 \circ P_1) \\ &= \Pi(P_1) \bullet P_2 \circ P_1 + P_1 \bullet \Pi(P_2) \circ P_1 \\ &= P_2\Pi(P_1)^*P_1 + P_1\Pi(P_1)P_2 + P_1\Pi(P_2)P_1 + \Pi(P_2)P_1 + P_1\Pi(P_2) + P_1\Pi(P_2)P_1.\end{aligned}$$

Multiplying both sides by P_2 from the right and by P_1 from the left, we get

$$P_1\Pi(P_1)P_2 = -P_1\Pi(P_2)P_2.$$

(2) Since $P_2 \bullet P_1 \circ P_2 = 0$ and using Lemma 2.2, we get

$$\begin{aligned} 0 &= \Pi(P_2 \bullet P_1 \circ P_2) \\ &= \Pi(P_2) \bullet P_1 \circ P_2 + P_2 \bullet \Pi(P_1) \circ P_2 \\ &= P_1\Pi(P_2)P_2 + P_2\Pi(P_2)P_1 + P_2\Pi(P_1)P_2 + \Pi(P_1)P_2 + P_2\Pi(P_1) + P_2\Pi(P_1)P_2. \end{aligned}$$

On multiplying P_1 from the right and by P_2 from the left to above equation, we get

$$P_2\Pi(P_2)P_1 = -P_2\Pi(P_1)P_1.$$

(3) In Lemma 2.9 (1), we have

$$0 = P_2\Pi(P_1)^*P_1 + P_1\Pi(P_1)P_2 + P_1\Pi(P_2)P_1 + \Pi(P_2)P_1 + P_1\Pi(P_2) + P_1\Pi(P_2)P_1.$$

By multiplying P_1 from the left and right sides to the above equation, we get $P_1\Pi(P_2)P_1 = 0$. Similarly, in Lemma 2.9 (2) we have

$$0 = P_1\Pi(P_2)P_2 + P_2\Pi(P_2)P_1 + P_2\Pi(P_1)P_2 + \Pi(P_1)P_2 + P_2\Pi(P_1) + P_2\Pi(P_1)P_2.$$

Multiplying both sides to above equation by P_2 from left and right, we get $P_2\Pi(P_1)P_2 = 0$.

(4) It follows from Lemma 2.2, that

$$\begin{aligned} 0 &= \Pi(P_1 \bullet P_2 \circ P_2) \\ &= \Pi(P_1) \bullet P_2 \circ P_2 + P_1 \bullet \Pi(P_2) \circ P_2 \\ &= \Pi(P_1)P_2 + P_2\Pi(P_1)^*P_2 + P_2\Pi(P_1)P_2 + P_2\Pi(P_1)^* + P_1\Pi(P_2)P_2 + P_2\Pi(P_2)P_1. \end{aligned}$$

Multiplying above equation by P_2 from left and right, and using $P_2\Pi(P_1)P_2 = 0$, we get $P_2\Pi(P_1)^*P_2 = 0$. Similarly, using the same technique as above on $\Pi(P_1 \bullet P_2 \circ P_2) = 0$, we can show that $P_1\Pi(P_2)^*P_1 = 0$. □

Lemma 2.10. $P_1\Pi(P_1)P_1 = P_2\Pi(P_2)P_2 = 0$.

Proof. For every $S_{12} \in \mathcal{A}_{12}$, in view of Lemma 2.8, we have

$$\Pi(P_1 \bullet S_{12} \circ P_1) = \Pi(S_{12}).$$

On the other hand, we have

$$\begin{aligned} \Pi(P_1 \bullet S_{12} \circ P_1) &= \Pi(P_1) \bullet S_{12} \circ P_1 + P_1 \bullet \Pi(S_{12}) \circ P_1 + P_1 \bullet S_{12} \circ \Pi(P_1) \\ &= S_{12}\Pi(P_1)^*P_1 + P_1\Pi(P_1)S_{12} + S_{12}\Pi(P_1)^* + P_1\Pi(S_{12})P_1 + \Pi(S_{12})P_1 \\ &\quad + P_1\Pi(S_{12}) + P_1\Pi(S_{12})P_1 + S_{12}\Pi(P_1) + \Pi(P_1)S_{12}. \end{aligned}$$

By comparing the above two equations, we get

$$\begin{aligned} 0 &= S_{12}\Pi(P_1)^*P_1 + P_1\Pi(P_1)S_{12} + S_{12}\Pi(P_1)^* + P_1\Pi(S_{12})P_1 + \Pi(S_{12})P_1 \\ &\quad + P_1\Pi(S_{12}) + P_1\Pi(S_{12})P_1 + S_{12}\Pi(P_1) + \Pi(P_1)S_{12} - \Pi(S_{12}). \end{aligned}$$

Multiplying the above equation by P_2 and P_1 from the right and left respectively, and using $P_2\Pi(P_1)^*P_2 = 0$, we get $P_1\Pi(P_1)S_{12} = 0$, i.e., $P_1\Pi(P_1)P_1SP_2 = 0$ for all $S \in \mathcal{A}$. It follows from (▲) and (▼) that $P_1\Pi(P_1)P_1 = 0$. Similarly, we can prove that $P_2\Pi(P_2)P_2 = 0$. □

Lemma 2.11.
1. $\Pi(P_1) = P_1\Pi(P_1)P_2 + P_2\Pi(P_1)P_1, \Pi(P_2) = P_1\Pi(P_2)P_2 + P_2\Pi(P_2)P_1$.
2. $\Pi(I) = 0$.

Proof. (1) From Peirce decomposition, we have

$$\Pi(P_1) = P_1\Pi(P_1)P_1 + P_1\Pi(P_1)P_2 + P_2\Pi(P_1)P_1 + P_2\Pi(P_1)P_2.$$

Now, by using Lemma 2.9 (3) and Lemma 2.10 , we get $\Pi(P_1) = P_1\Pi(P_1)P_2 + P_2\Pi(P_1)P_1$. Similarly, we can show that $\Pi(P_2) = P_1\Pi(P_2)P_2 + P_2\Pi(P_2)P_1$.

(2) From Lemma 2.8, Lemma 2.9 and Lemma 2.11 (1), we have

$$\Pi(I) = \Pi(P_1) + \Pi(P_2) = P_1\Pi(P_1)P_2 + P_2\Pi(P_1)P_1 + P_1\Pi(P_2)P_2 + P_2\Pi(P_2)P_1 = 0.$$

□

Lemma 2.12. $\Pi(S^*) = \Pi(S)^*$ *for all* $S \in \mathcal{A}$.

Proof. Using Lemma 2.8, we find that

$$\Pi(S \bullet I \circ I) = \Pi(2S + 2S^*) = 2\Pi(S) + 2\Pi(S^*).$$

From the other side, it follows from Lemma 2.11 (2) that

$$\Pi(S \bullet I \circ I) = \Pi(S) \bullet I \circ I = 2\Pi(S) + 2\Pi(S)^*.$$

From above two equations, we get $\Pi(S^*) = \Pi(S)^*$. □

Now, let $M = P_1\Pi(P_1)P_2 - P_2\Pi(P_1)P_1$. Then $M = -M^*$. Define a map $\Phi : \mathcal{A} \to \mathcal{A}$ by $\Phi(S) = \Pi(S) - (SM - MS)$ for all $S \in \mathcal{A}$. The following characteristics of Φ can be easily verified.

Remark 2.13. Φ *has the following properties.*

1. *For all* $S, T, U \in \mathcal{A}, \Phi(S \bullet T \circ U) = \Phi(S) \bullet T \circ U + S \bullet \Phi(T) \circ U + S \bullet T \circ \Phi(U)$.
2. $\Phi(S^*) = \Phi(S)^*$ *for all* $S \in \mathcal{A}$.
3. Φ *is additive.*
4. $\Phi(P_1) = \Phi(P_2) = 0$.
5. $\Phi(I) = 0$.
6. Φ *is a* $*$*-derivation if and only if* Π *is a* $*$*-derivation.*

Lemma 2.14. $\Phi(\mathcal{A}_{ij}) \subseteq \mathcal{A}_{ij}, i, j = 1, 2$.

Proof. First, we prove for $i = 1, j = 2$. For any $S_{12} \in \mathcal{A}_{12}$ and using Remark 2.13, we have

$$\Phi(I \bullet P_1 \circ S_{12}) = 2\Phi(S_{12}).$$

From the other side, we get

$$\Phi(I \bullet P_1 \circ S_{12}) = I \bullet P_1 \circ \Phi(S_{12}) = 2P_1\Phi(S_{12}) + 2\Phi(S_{12})P_1.$$

By comparing the above two equations, we have

$$2P_1\Phi(S_{12}) + 2\Phi(S_{12})P_1 - 2\Phi(S_{12}) = 0.$$

By multiplying P_1 from the left and right sides of the above equation, we get $P_1\Phi(S_{12})P_1 = 0$. Similarly, by multiplying P_2 on the left and right sides of the above equation, we obtain $P_2\Phi(S_{12})P_2 = 0$.
Again, by using Remark 2.13, we have

$$0 = \Phi(S_{12} \bullet P_1 \circ I) = \Phi(S_{12}) \bullet P_1 \circ I = 2\Phi(S_{12})P_1 + 2P_1\Phi(S_{12})^*.$$

By multiplying P_2 from left to the above equation, we get $P_2\Phi(S_{12})P_1 = 0$. Hence, $P_1\Phi(S_{12})P_1 = P_2\Phi(S_{12})P_2 = P_2\Phi(S_{12})P_1 = 0$. So, $\Phi(S_{12}) = P_1\Phi(S_{12})P_2 \in S_{12}$. Since, S_{12} is arbitarty, we have $\Phi(\mathcal{A}_{12}) \subseteq \mathcal{A}_{12}$. Similarly, we can show that $\Phi(\mathcal{A}_{21}) \subseteq \mathcal{A}_{21}$.

Now, we prove for $i = j = 1$. For any $S_{11} \in \mathcal{A}_{11}$, it follows from Remark 2.13 and using $I \bullet S_{11} \circ P_2 = 0$, we have

$$0 = \Phi(I \bullet P_2 \circ S_{11}) = I \bullet P_2 \circ \Phi(S_{11}) = 2\Phi(S_{11})P_2 + 2P_2\Phi(S_{11}).$$

This implies that $P_2\, \Phi(S_{11})P_2 = P_2\, \Phi(S_{11})P_1 = P_1\, \Phi(S_{11})P_2 = 0$. Hence $\Phi(\mathcal{A}_{11}) \subseteq \mathcal{A}_{11}$. Similarly, $\Phi(\mathcal{A}_{22}) \subseteq \mathcal{A}_{22}$. □

Lemma 2.15. *For any $S_{ij}, T_{ij} \in \mathcal{A}_{ij}, 1 \leq i, j \leq 2$, we have*

1. $\Phi(S_{11}T_{12}) = \Phi(S_{11})T_{12} + S_{11}\Phi(T_{12})$ *and* $\Phi(S_{22}T_{21}) = \Phi(S_{22})T_{21} + S_{22}\Phi(T_{21})$.
2. $\Phi(S_{12}T_{21}) = \Phi(S_{12})T_{21} + S_{12}\Phi(T_{21})$ *and* $\Phi(S_{21}T_{12}) = \Phi(S_{21})T_{12} + S_{21}\Phi(T_{12})$.
3. $\Phi(S_{11}T_{11}) = \Phi(S_{11})T_{11} + S_{11}\Phi(T_{11})$ *and* $\Phi(S_{22}T_{22}) = \Phi(S_{22})T_{22} + S_{22}\Phi(T_{22})$.
4. $\Phi(S_{12}T_{22}) = \Phi(S_{12})T_{22} + S_{12}\Phi(T_{22})$ *and* $\Phi(S_{21}T_{11}) = \Phi(S_{21})T_{11} + S_{21}\Phi(T_{11})$.

Proof. (**1**) By using Remark 2.13, we have

$$\Phi(P_1 \bullet S_{11} \circ T_{12}) = 2\, \Phi(S_{11}T_{12}).$$

From the other side, using Lemma 2.14 and $\Phi(P_1) = 0$, we get

$$\Phi(P_1 \bullet S_{11} \circ T_{12}) = P_1 \bullet \Phi(S_{11}) \circ T_{12} + P_1 \bullet S_{11} \circ \Phi(T_{12}) = 2\Phi(S_{11})T_{12} + 2S_{11}\Phi(T_{12}).$$

By comparing the above two equations, we get $\Phi(S_{11}T_{12}) = \Phi(S_{11})T_{12} + S_{11}\Phi(T_{12})$.
Similarly, we can show that $\Phi(S_{22}T_{21}) = \Phi(S_{22})T_{21} + S_{22}\Phi(T_{21})$.

(**2**) By using Remark 2.13 that Φ is additive, we have

$$\Phi(P_1 \bullet S_{12} \circ T_{21}) = 2\, \Phi(S_{12}T_{21}).$$

From the other side, using Lemma 2.14 and $\Phi(P_1) = 0$, we have

$$\Phi(P_1 \bullet S_{12} \circ T_{21}) = P_1 \bullet \Phi(S_{12}) \circ T_{21} + P_1 \bullet S_{12} \circ \Phi(T_{21}) = 2\, \Phi(S_{12})T_{21} + 2\, S_{12}\, \Phi(T_{21}).$$

By comparing the above two equations, we get $\Phi(S_{12}T_{21}) = \Phi(S_{12})T_{21} + S_{12}\Phi(T_{21})$.
Similarly, we can prove that $\Phi(S_{21}T_{12}) = \Phi(S_{21})T_{12} + S_{21}\Phi(T_{12})$.

(**3**) For every $X_{12} \in \mathcal{A}_{12}$, we have

$$\Phi(S_{11}T_{11}X_{12}) = \Phi(S_{11}T_{11})X_{12} + S_{11}T_{11}\Phi(X_{12})$$

Again, for every $X_{12} \in \mathcal{A}_{12}$, we have from Lemma 2.15(1) that

$$\begin{aligned}\Phi(S_{11}T_{11}X_{12}) &= \Phi(S_{11})T_{11}X_{12} + S_{11}\Phi(T_{11}X_{12}) \\ &= \Phi(S_{11})T_{11}X_{12} + S_{11}\Phi(T_{11})X_{12} + S_{11}T_{11}\Phi(X_{12}).\end{aligned}$$

By comparing the above two equations, we get

$$(\, \Phi(S_{11}T_{11}) - \Phi(S_{11})T_{11} - S_{11}\Phi(T_{11})\,)X_{12} = 0, \quad \forall\, X_{12} \in \mathcal{A}_{12}.$$

From (▲) and (▼), we get $\Phi(S_{11}T_{11}) = \Phi(S_{11})T_{11} + S_{11}\Phi(T_{11})$.
Similarly, $\Phi(S_{22}T_{22}) = \Phi(S_{22})T_{22} + S_{22}\Phi(T_{22})$.

(**4**) It follows from Remark 2.13, Lemma 2.14 and Lemma 2.15(1), we get

$$\begin{aligned}\Phi(S_{12} \bullet T_{22} \circ P_2) &= \Phi(S_{12}T_{22} + T_{22}S_{12}^*) \\ &= \Phi(S_{12}T_{22}) + \Phi(T_{22}S_{12}^*) \\ &= \Phi(S_{12}T_{22}) + \Phi(T_{22})S_{12}^* + T_{22}\Phi(S_{12})^*.\end{aligned}$$

On the other hand, using Lemma 2.14 , we get

$$\begin{aligned}\Phi(S_{12} \bullet T_{22} \circ P_2) &= \Phi(S_{12}) \bullet T_{22} \circ P_2 + S_{12} \bullet \Phi(T_{22}) \circ P_2 \\ &= \Phi(S_{12})T_{22} + T_{22}\Phi(S_{12})^* + S_{12}\Phi(T_{22}) + \Phi(T_{22})S_{12}^*\end{aligned}$$

By comparing the above two equations, we get $\Phi(S_{12}T_{22}) = \Phi(S_{12})T_{22} + S_{12}\Phi(T_{22})$. Similarly, $\Phi(S_{21}T_{11}) = \Phi(S_{21})T_{11} + S_{21}\Phi(T_{11})$. □

Lemma 2.16. *$\Phi(ST) = \Phi(S)T + S\Phi(T)$ for all $S, T \in \mathcal{A}$.*

Proof. For every $S, T \in \mathcal{A}$, we can write $S = S_{11} + S_{12} + S_{21} + S_{22}$ and $T = T_{11} + T_{12} + T_{21} + T_{22}$. Since, Φ is additive and using Lemma 2.15, we get

$$\begin{aligned}\Phi(ST) &= \Phi(S_{11}T_{11} + S_{11}T_{12} + S_{12}T_{21} + S_{12}T_{22} \\ &\quad + S_{21}T_{11} + S_{21}T_{12} + S_{22}T_{21} + S_{22}T_{22}) \\ &= \Phi(S_{11}T_{11}) + \Phi(S_{11}T_{12}) + \Phi(S_{12}T_{21}) + \Phi(S_{12}T_{22}) \\ &\quad + \Phi(S_{21}T_{11}) + \Phi(S_{21}T_{12}) + \Phi(S_{22}T_{21}) + \Phi(S_{22}T_{22}) \\ &= \Phi(S_{11} + S_{12} + S_{21} + S_{22})(T_{11} + T_{12} + T_{21} + T_{22}) \\ &\quad + (S_{11} + S_{12} + S_{21} + S_{22})\,\Phi(T_{11} + T_{12} + T_{21} + T_{22}) \\ &= \Phi(S)T + S\Phi(T).\end{aligned}$$

Hence, Φ is a derivation. □

Now, from Remark 2.13 and Lemma 2.16, we get Φ is an additive $*$-derivation. Hence, Π is an additive $*$-derivation. This completes the proof of Theorem 2.1.

Let H be a complex Hilbert space, and $B(H)$ be the algebra of all bounded linear operators on H. An algebra of operators on a Hilbert space H is termed a von Neumann algebra if it is both weakly closed and self-adjoint, and it contains the identity operator I.

Lemma 2.17. *[7, Lemma 2.2] Let $\mathcal{A}$ be a von Neumann algebra on a Hilbert space H. Let $\mathcal{A} \in B(H)$ and $P \in \mathcal{A}$ be a projection with $\overline{P} = I$. If $\mathcal{A}BP = 0$ for all $B \in \mathcal{A}$, then $\mathcal{A} = 0$.*

As a direct result of Theorem 2.1, we have an following corollaries:

Corollary 2.18. *Let $\mathcal{A}$ be a standard operator algebra on an infinite dimensional complex Hilbert space $\mathcal{H}$ containing identity operator I. Suppose that $\mathcal{A}$ is closed under adjoint operation. Define $\Pi : \mathcal{A} \to \mathcal{A}$ such that*

$$\Pi(S \bullet T \circ U) = \Pi(S) \bullet T \circ U + S \bullet \Pi(T) \circ U + S \bullet T \circ \Pi(U)$$

for all $S, T, U \in \mathcal{A}$. Then Π is an additive $*$-derivation.

Proof. It is a fact that every standard operator algebra $\mathcal{A}$ is prime algebra. Which is a consequences of Hahn-Banach theorem. Then by the definition of primeness, $\mathcal{A}$ also satisfies (▲) and (▼), hence by Theorem 2.1, Π is an additive $*$-derivation.
□

Corollary 2.19. *Let $\mathcal{M}$ ba a factor von Neumann algebra with $\dim\mathcal{M} \geq 2$. Define $\Pi : \mathcal{M} \to \mathcal{M}$ such that*

$$\Pi(S \bullet T \circ U) = \Pi(S) \bullet T \circ U + S \bullet \Pi(T) \circ U + S \bullet T \circ \Pi(U)$$

for all $S, T, U \in \mathcal{A}$. Then Π is an additive $*$-derivation.

Proof. It follows from Lemma 2.17 that every factor von Neumann algebra $\mathcal{M}$ satifies (▲) and (▼). Hence by using Theorem 2.1, Π is an additive $*$-derivation. □

Corollary 2.20. *Let $\mathcal{A}$ be a prime $*$-algebra with unit I containing non-trivial projection P. A map $\Pi : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\Pi(S \bullet T \circ U) = \Pi(S) \bullet T \circ U + S \bullet \Pi(T) \circ U + S \bullet T \circ \Pi(U)$$

for all $S, T, U \in \mathcal{A}$. Then Π is an additive $*$-derivation.

Proof. By the definition of primeness of $\mathcal{A}$, it is easy to see that $\mathcal{A}$ also satisfies (▲) and (▼), hence by Theorem 2.1, Π is an additive $*$-derivation. □

Now, we construct an example to show that the conditions (▲) and (▼) in Theorem 2.1 are necessary.

Example 2.21. *Consider $\mathcal{A} = \mathbb{C}[X] \times M_2(\mathbb{C})$, where $\mathbb{C}[X]$ and $M_2(\mathbb{C})$ are a polynomial ring and a matrix ring over complex numbers $\mathbb{C}$, respectively. Let $I = I_1 \times I_2$, where I_1 and $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are units of $\mathbb{C}[X]$ and $M_2(\mathbb{C})$, respectively. For any $Y = (q(x), B) \in \mathcal{A}$, where $\overline{q(x)}$ is a conjugate of $q(x)$ and B^T is a transpose of B. Then $* : \mathcal{A} \to \mathcal{A}$ such that $Y^* = (\overline{q(x)}, B^T)$ is an involution. Thus, $\mathcal{A}$ is a unital $*$-algebra with unity I. Define a map $\Pi : \mathcal{A} \to \mathcal{A}$ such that $\Pi(q(x), B) = (iq(x)', 0)$, where $i = \sqrt{-1} \in \mathbb{C}$ and $q(x)'$ is a derivative of $q(x)$. Then*

$$\Pi(S \bullet T \circ U) = \Pi(S) \bullet T \circ U + S \bullet \Pi(T) \circ U + S \bullet T \circ \Pi(U) \tag{1}$$

for all $S, T, U \in \mathcal{A}$. Note that $d(x) = iq(x)'$ is a derivation on $\mathbb{C}[X]$ and so the above condition (1) satisfies. Let $P = \{0\} \times \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ be a non-trivial projection so $P^2 = P$ and $P^* = P$. For $W = (r(x), 0) \neq (0, 0)$ where $0 \neq r \in \mathbb{C}$, we have $W\mathcal{A}P = (0)$ but $0 \neq W \in \mathcal{A}$. However, Π is not an additive $*$-derivation because $\Pi(S^*) \neq (\Pi(S))^*$ for some $S \in \mathcal{A}$.

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