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On strongly topological gyrogroups

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Abstract. A topological gyrogroup, as a generalization of the topological group, is defined as a gyrogroup endowed with a topology such that the binary operation is jointly continuous and the inverse mapping is also continuous. In this paper, we give an example such that a topological group G is a csf-countable k-space, but G is not sequential, which gives an answer to a question posed by Gabriyelyan, Kakol and Leiderman in [22]. Then it is proved that every feathered csf-countable strongly topological gyrogroup is metrizable. Finally, we extend some important results of topological groups to strongly topological gyrogroups.

1. Introduction

The concept of gyrogroups was introduced by A.A. Ungar [37] from the research of the c-ball of relativistically admissible velocities. A gyrogroup is a relaxation of a group such that the associativity condition has been replaced by a weaker one. In 2017, W. Atiponrat [4] defined the concept of topological gyrogroups as a generalization of topological groups and discussed some properties of them. A topological gyrogroup is a gyrogroup endowed with a topology such that the binary operation is jointly continuous and the inverse mapping is also continuous. It is obvious that every topological group is a topological gyrogroup. However, every topological gyrogroup whose gyrations are not identically equal to the identity is not a topological group, such as the classical Möbius gyrogroup and Einstein gyrogroup. Then, Cai, Lin and He in [17] proved that every topological gyrogroup is a rectifiable space, which implies that every first-countable Hausdorff topological gyrogroup is metrizable by [1]. By further study on the Möbius gyrogroups, Bao and Lin [10] introduced the concept of strongly topological gyrogroups. A topological gyrogroup G is called a strongly topological gyrogroup if there exists a neighborhood base $\mathscr{U}$ of 0 such that, for every $U \in \mathscr{U}$, $\text{gyr}[x, y](U) = U$ for any $x, y \in G$. Clearly, every topological group is a strongly topological gyrogroup. For more details about topological gyrogroups and strongly topological gyrogroups, see [5, 6, 14–16, 39]. In this paper, we would like to extend some well-known results of topological groups to topological gyrogroups and strongly topological gyrogroups.

In [22], Gabriyelyan, Kakol and Leiderman investigated topological groups with an ω^ω-base and showed that the k-property is equivalent with sequentiality for a topological group with an ω^ω-base, and they said

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that it would be interesting to know whether the k-property and sequentiality are equivalent for csf-countable topological groups, see the following question.

Question 1.1. *Let G be a csf-countable topological group. If G is a k-space, is it sequential?*

We give an example such that a topological group G is a csf-countable k-space, but G is not sequential, which gives a negative answer to Question 1.1. Furthermore, it was claimed in [22, Theorem 3.10] that every feathered topological group with an ω^{ω}-base is metrizable. Then we improve this result and show that every feathered csf-countable strongly topological gyrogroup is metrizable.

During the researches of quotient spaces of topological gyrogroups, Bao, Shen and Xu posed the following question.

Question 1.2. ([13, Question 4.6]) *Let G be a topological gyrogroup, K a compact subset and F a closed subset of G such that $K \cap F = \emptyset$. Then is there an open neighborhood V of e in G such that $(K \oplus V) \cap F = \emptyset$ and $(V \oplus K) \cap F = \emptyset$? What if G is a strongly topological gyrogroup?*

In Section 5, we give an affirmative answer to Question 1.2 on the case of strongly topological gyrogroups, see Theorem 5.2. According to the result, we extend an important result [3, Theorem 3.2.2] to strongly topological gyrogroups, that is, if G is a strongly topological gyrogroup and H is a locally compact strong subgyrogroup of G, then there exists an open neighborhood U of the identity element 0 such that $\pi(\overline{U})$ is closed in G/H and the restriction of π to $\overline{U}$ is a perfect mapping from $\overline{U}$ onto the subspace $\pi(\overline{U})$. Furthermore, we give some applications about it and show that if G is a strongly topological gyrogroup, H is a locally compact metrizable strong subgyrogroup of G and the quotient space G/H is sequential, then G is also sequential, which generalizes the result in [28, Theorem 4.3].

Our study also leads to some open problems, whose answers will deepen our understanding of the related spaces and structures.

2. Preliminary

Throughout this paper, all topological spaces are assumed to be Hausdorff, unless otherwise is explicitly stated. Let $\mathbb{N}$ be the set of all positive integers and ω the first infinite ordinal. The readers may consult [3, 19, 27, 38] for notation and terminology not explicitly given here. Next we recall some definitions and facts.

Definition 2.1. ([4]) Let G be a nonempty set, and let $\oplus : G \times G \to G$ be a binary operation on G. Then the pair $(G, \oplus)$ is called a *magma or groupoid*. A function f from a groupoid $(G_1, \oplus_1)$ to a groupoid $(G_2, \oplus_2)$ is called a *groupoid homomorphism* if $f(x \oplus_1 y) = f(x) \oplus_2 f(y)$ for any elements $x, y \in G_1$. Furthermore, a bijective groupoid homomorphism from a groupoid $(G, \oplus)$ to itself will be called a *groupoid automorphism*. We write $\mathrm{Aut}(G, \oplus)$ for the set of all automorphisms of a groupoid $(G, \oplus)$.

Definition 2.2. ([38]) Let $(G, \oplus)$ be a groupoid. The system $(G, \oplus)$ is called a *gyrogroup*, if its binary operation satisfies the following conditions:

$(G1)$ There exists a unique identity element $0 \in G$ such that $0 \oplus a = a = a \oplus 0$ for all $a \in G$.

$(G2)$ For each $x \in G$, there exists a unique inverse element $\ominus x \in G$ such that $\ominus x \oplus x = 0 = x \oplus (\ominus x)$.

$(G3)$ For all $x, y \in G$, there exists $\mathrm{gyr}[x, y] \in \mathrm{Aut}(G, \oplus)$ with the property that $x \oplus (y \oplus z) = (x \oplus y) \oplus \mathrm{gyr}[x, y](z)$ for all $z \in G$.

$(G4)$ For any $x, y \in G$, $\mathrm{gyr}[x \oplus y, y] = \mathrm{gyr}[x, y]$.

Notice that a group is a gyrogroup $(G, \oplus)$ such that $\mathrm{gyr}[x, y]$ is the identity function for all $x, y \in G$. The definition of a subgyrogroup is given as follows.

Definition 2.3. ([33]) Let $(G, \oplus)$ be a gyrogroup. A nonempty subset H of G is called a *subgyrogroup*, denoted by $H \leq G$, if H forms a gyrogroup under the operation inherited from G and the restriction of $\text{gyr}[a, b]$ to H is an automorphism of H for all $a, b \in H$.

Furthermore, a subgyrogroup H of G is said to be an *L-subgyrogroup*, denoted by $H \leq_L G$, if $\text{gyr}[a, h](H) = H$ for all $a \in G$ and $h \in H$.

Lemma 2.4. ([38]). *Let $(G, \oplus)$ be a gyrogroup. Then for any $x, y, z \in G$, we obtain the following:*

1. $(\ominus x) \oplus (x \oplus y) = y$.
2. $(x \oplus (\ominus y)) \oplus gyr[x, \ominus y](y) = x$.
3. $(x \oplus gyr[x, y](\ominus y)) \oplus y = x$.
4. $gyr[x, y](z) = \ominus(x \oplus y) \oplus (x \oplus (y \oplus z))$.

Definition 2.5. ([4]) A triple $(G, \tau, \oplus)$ is called a *topological gyrogroup* if the following statements hold:

(1) (G, τ) is a topological space.

(2) $(G, \oplus)$ is a gyrogroup.

(3) The binary operation $\oplus : G \times G \to G$ is jointly continuous while $G \times G$ is endowed with the product topology, and the operation of taking the inverse $\ominus(\cdot) : G \to G$, i.e. $x \to \ominus x$, is also continuous.

Obviously, every topological group is a topological gyrogroup. However, every topological gyrogroup whose gyrations are not identically equal to the identity is not a topological group.

Example 2.6. ([4, Example 3]) The Einstein gyrogroup with the standard topology is a topological gyrogroup but not a topological group.

Let $\mathbb{R}^3_{\mathbf{c}} = \{\mathbf{v} \in \mathbb{R}^3 : \|\mathbf{v}\| < \mathbf{c}\}$, where $\mathbf{c}$ is the vacuum speed of light, and $\|\mathbf{v}\|$ is the Euclidean norm of a vector $\mathbf{v} \in \mathbb{R}^3$. The Einstein velocity addition $\oplus_E : \mathbb{R}^3_{\mathbf{c}} \times \mathbb{R}^3_{\mathbf{c}} \to \mathbb{R}^3_{\mathbf{c}}$ is given as follows:

$$\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1 + \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{c}^2}} \left(\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}} \mathbf{v} + \frac{1}{\mathbf{c}^2} \frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}} (\mathbf{u} \cdot \mathbf{v}) \mathbf{u}\right),$$

for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c$, $\mathbf{u} \cdot \mathbf{v}$ is the usual dot product of vectors in $\mathbb{R}^3$, and $\gamma_{\mathbf{u}}$ is the gamma factor which is given by

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \frac{\mathbf{u} \cdot \mathbf{u}}{\mathbf{c}^2}}}.$$

It was proved in [38] that $(\mathbb{R}^3_c, \oplus_E)$ is a gyrogroup but not a group. Moreover, with the standard topology inherited from $\mathbb{R}^3$, it is clear that $\oplus_E$ is continuous. Finally, $-\mathbf{u}$ is the inverse of $\mathbf{u} \in \mathbb{R}^3$ and the operation of taking the inverse is also continuous. Therefore, the Einstein gyrogroup $(\mathbb{R}^3_c, \oplus_E)$ with the standard topology inherited from $\mathbb{R}^3$ is a topological gyrogroup but not a topological group.

Definition 2.7. Let X be a topological space.

(1) X is called a *weakly first-countable space* or *gf-countable space* [2] if for each point $x \in X$ it is possible to assign a sequence $\{B(n, x) : n \in \mathbb{N}\}$ of subsets of X containing x in such a way that $B(n + 1, x) \subseteq B(n, x)$ and so that a set U is open if, and only if, for each $x \in U$ there exists $n \in \mathbb{N}$ such that $B(n, x) \subseteq U$.

(2) X is called a *sequential space* [20] if for each non-closed subset $A \subseteq X$, there are a point $x \in X \setminus A$ and a sequence in A converging to x in X.

(3) X is called a *Fréchet-Urysohn space* [20] if for any subset $A \subseteq X$ and $x \in \overline{A}$, there is a sequence in A converging to x in X.

(4) X is called a *strongly Fréchet-Urysohn space* [32] if the following condition is satisfied:

(SFU) For each $x \in X$ and every sequence $\xi = \{A_n : n \in \mathbb{N}\}$ of subsets of X such that $x \in \bigcap_{n \in \mathbb{N}} \overline{A_n}$, there exists a sequence $\eta = \{b_n : n \in \mathbb{N}\}$ in X converging to x and intersecting infinitely many members of ξ.

Definition 2.8. Let $\mathscr{P}$ be a family of subsets of a space X with $x \in \bigcap \mathscr{P}$.

(1) The family $\mathscr{P}$ is called a *network at* x [19] if for each neighborhood U of x there exists $P \in \mathscr{P}$ such that $P \subseteq U$.

(2) The family $\mathscr{P}$ is called a *cs-network at* x [26] if for any sequence L converging to x and a neighborhood U of x, there exists $P \in \mathscr{P}$ such that L is eventually in P and $P \subseteq U$.

(3) The family $\mathscr{P}$ is called an *sn-network at* x [25] if $\mathscr{P}$ is a network at x and each element of $\mathscr{P}$ is a sequential neighborhood of x.

(4) The family $\mathscr{P}$ is called an *so-network at* x [25] if $\mathscr{P}$ is a network at x and each element of $\mathscr{P}$ is a sequential open subset of X.

(5) A space X is called *csf-countable* (resp., *snf-countable*, *sof-countable*) [25] if for each $x \in X$, there is a countable *cs*-network (resp., *sn*-network, *so*-network) at x.

Note that in [7], csf-countable spaces and snf-countable spaces are called *spaces with countable cs*-character* and *spaces with countable sb-character*, respectively.

3. On *csf*-countable topological gyrogroups

In this section, an example is posed such that a topological group G is a csf-countable k-space, but G is not sequential, which gives an answer to a question in [22].

Definition 3.1. ([9, 21, 23]) A point x of a topological space X is said to have a *neighborhood ω^{ω}-base* or a *local $\mathfrak{G}$-base* if there exists a base of neighborhoods at x of the form $\{U_{\alpha}(x) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ such that $U_{\beta}(x) \subseteq U_{\alpha}(x)$ for all elements $\alpha \leq \beta$ in $\mathbb{N}^{\mathbb{N}}$, where $\mathbb{N}^{\mathbb{N}}$ consisting of all functions from $\mathbb{N}$ to $\mathbb{N}$ is endowed with the natural partial order, ie., $f \leq g$ if and only if $f(n) \leq g(n)$ for all $n \in \mathbb{N}$. The space X is said to have an *ω^{ω}-base* or a *$\mathfrak{G}$-base* if it has a neighborhood ω^{ω}-base or a local $\mathfrak{G}$-base at every point $x \in X$.

In [22, Theorem 3.9], it was proved that every precompact subset of a topological group with an ω^{ω}-base is metrizable. It is easy to obtain the following result by the similar proof.

Proposition 3.2. *Let G be a topological gyrogroup with an ω^{ω}-base, K an arbitrary compact subset of G. Then K is metrizable.*

Banakh in [8, Theorem 1.1] showed that each non-metrizable sequential csf-countable rectifiable space X contains a clopen rectifiable submetrizable k_{ω}-subspace. Indeed, by the proof of [8, Theorem 1.1], it is not difficult to see that if X is a non-metrizable sequential csf-countable topological gyrogroup, then it contains an open and closed subgyrogroup which is a submetrizable k_{ω}-space. For a topological space X, Chasco, Martín and Tarieladze in [18, Lemma 1.5] showed that if X is sequential, then it is a k-space and if X is a Hausdorff k-space and its compact subsets are sequential (in particular first countable or metrizable), then X is sequential. Furthermore, it was proved in [16, Theorem 3.8] that if a topological gyrogroup G has an ω^{ω}-base, then it is csf-countable. Therefore, by Proposition 3.2 and these results, we obtain:

Theorem 3.3. *If a topological gyrogroup G has an ω^{ω}-base, then the following conditions are equivalent.*

1. *G is a k-space;*
2. *G is sequential;*
3. *G is metrizable or contains an open submetrizable k_{ω}-subgyrogroup.*

In [40, Theorem 3.5], the authors showed that if G is a sequential topological gyrogroup with an ω^{ω}-base, then G has the strong Pytkeev property. Therefore, Theorem 3.3 poses the following result directly.

Theorem 3.4. *Let G be a topological gyrogroup with an ω^{ω}-base. If G is a k-space, then G has the strong Pytkeev property.*

In 2015, Gabriyelyan, Kakol and Leiderman said that it would be interesting to know whether the k-property and sequentiality are equivalent for csf-countable topological groups, see Question 1.1. Combining [30, Example 4.5], we pose the following example which shows that there is a non-metrizable snf-countable topological group which is a k-space, but it is not sequential, which gives a negative answer to Question 1.1.

Example 3.5. There exists a csf-countable topological group G such that G is a k-space, but G is not sequential.

Let G be a free (Abelian) topological group over the compact space $\beta\mathbb{N}$. It follows from [3, Corollary 7.4.2] that G is a k-space. Clearly, G is not metrizable. According to [34, Proposition 2.4], G does not contain non-trivial convergent sequences. Therefore, $\{x\}$ is a sequential neighborhood of x for each $x \in G$. Then, G is snf-countable and hence G is csf-countable.

Then it is easy to see that G is not sequential. Suppose on the contrary, we assume that G is sequential. Since every sequential snf-countable topological space is weakly first-countable, we know that G is a weakly first-countable topological group. Then it is well-known that every weakly first-countable topological group is metrizable, it follows that G is metrizable, which is a contradiction.

Since every topological group is a topological gyrogroup, it also shows that it is not equivalent between the k-property and sequentiality for csf-countable topological gyrogroups.

4. On feathered strongly topological gyrogroups

In this section, we show that every feathered csf-countable strongly topological gyrogroup is metrizable. First, we introduce the concept of strongly topological gyrogroups.

Definition 4.1. ([10]) Let G be a topological gyrogroup. We say that G is a *strongly topological gyrogroup* if there exists a neighborhood base $\mathscr{U}$ of 0 such that, for every $U \in \mathscr{U}$, $\mathrm{gyr}[x, y](U) = U$ for any $x, y \in G$. For convenience, we say that G is a strongly topological gyrogroup with neighborhood base $\mathscr{U}$ of 0.

For each $U \in \mathscr{U}$, we can set $V = U \cup (\ominus U)$. Then,

$$\mathrm{gyr}[x, y](V) = \mathrm{gyr}[x, y](U \cup (\ominus U)) = \mathrm{gyr}[x, y](U) \cup (\ominus \mathrm{gyr}[x, y](U)) = U \cup (\ominus U) = V,$$

for all $x, y \in G$. Obviously, the family $\{U \cup (\ominus U) : U \in \mathscr{U}\}$ is also a neighborhood base of 0. Therefore, we may assume that U is symmetric for each $U \in \mathscr{U}$ in Definition 4.1.

A topological gyrogroup G is *feathered* if it contains a non-empty compact set K of countable character in G. It was proved in [19, 3.1 E(b) and 3.3 H(a)] that every locally compact topological gyrogroup is feathered and every metrizable topological gyrogroup is feathered. A strongly topological gyrogroup G is feathered if and only if it contains a compact L-subgyrogroup H such that the quotient space G/H is metrizable by [10, Theorem 3.14].

In [22, Theorem 3.10], it was proved that every feathered topological group with an ω^ω-base is metrizable. By the similar method, it is not difficult to obtain the following result.

Theorem 4.2. *Let G be a strongly topological gyrogroup. Then G is feathered and has an ω^ω-base if and only if G is metrizable.*

Proof. Suppose that G is a feathered strongly topological gyrogroup and has an ω^ω-base. Then G contains a compact L-subgyrogroup H such that the quotient space G/H is metrizable. It follows from Proposition 3.2 that the subgyrogroup H is metrizable. Since each compact subset of a Hausdorff space is closed, it is clear that H is a closed L-subgyrogroup of G. Then, by [15, Corollary 4.3], if G is a topological gyrogroup and H is a closed L-subgyrogroup of G and if the spaces H and G/H are metrizable, then the space G is also metrizable. Therefore, we obtain that G is a metrizable space. □

Corollary 4.3. *Let G be a locally compact strongly topological gyrogroup. Then G has an* ω^ω*-base if and only if G is metrizable.*

Since every topological gyrogroup with an ω^ω-base is csf-countable, it is natural to pose the following question.

Question 4.4. *Let G be a feathered* csf*-countable strongly topological gyrogroup. Is G metrizable?*

Then, we give an affirmative answer to Question 4.4. We note that Uspenskiĭ [35, 36] proved that all compact rectifiable spaces are dyadic. Since every topological gyrogroup is a rectifiable space, it is trivial that each compact topological gyrogroup is dyadic. Moreover, Banakh and Zdomskyy in [7, Proposition 7] claimed that a dyadic compactum is metrizable if and only if it is csf-countable.

Proposition 4.5. *Every compact* csf*-countable topological gyrogroup is metrizable.*

The following result improves [22, Theorem 3.10].

Theorem 4.6. *Every feathered* csf*-countable strongly topological gyrogroup is metrizable.*

Proof. Since G is a feathered strongly topological gyrogroup, there exists a compact L-subgyrogroup H of G such that the quotient space G/H is metrizable. By the hypothesis, G is csf-countable, then H is also csf-countable, which deduces that the compact csf-countable subgyrogroup H is metrizable, and it follows from [15, Corollary 4.3] that G is metrizable. □

Corollary 4.7. *Every locally compact* csf*-countable strongly topological gyrogroup is metrizable.*

In [14], Bao and Xu showed that every csf-countable Fréchet-Urysohn rectifiable space is metrizable, then the following is achieved by Theorems 4.2 and 4.6.

Theorem 4.8. *Let G be a strongly topological gyrogroup. Then the following conditions are equivalent:*

1. *G is metrizable;*
2. *G is Fréchet-Urysohn and* csf*-countable;*
3. *G is Fréchet-Urysohn and has an* ω^ω*-base;*
4. *G is feathered and* csf*-countable;*
5. *G is feathered and has an* ω^ω*-base.*

Recall that *a topological magma* [8] is a topological space X endowed with a continuous binary operation $\cdot : X \times X \to X$. A topological magma X is called *a topological left-loop* [8] if X has a right unit e and the map $X \times X \to X \times X$, $(x, y) \mapsto (x, xy)$, is a homeomorphism. A topological magma X is called *a topological lop* [8] if X has a unit e and the map $X \times X \to X \times X$, $(x, y) \mapsto (x, xy)$, is a homeomorphism. It was showed in [8] that each rectifiable space is homeomorphic to a topological left-loop and even to a topological lop. Moreover, it is well-known that every topological gyrogroup is a rectifiable space by [17].

Banakh and Repovš [8, Lemma 5.1] showed that suppose that G is a topological lop and $F \subseteq G$ is a subset containing the unit e of G, then put $F_1 = F$ and $F_{n+1} = F_n^{-1}F_n$ for $n \in \mathbb{N}$. If F is a sequential space containing no closed topological copy of the Fréchet-Urysohn fan S_ω and each space F_n, $n \in \mathbb{N}$, is csf-countable at e, then F is snf-countable at e; if F is sequential and each space $F_n, n \in \mathbb{N}$, is snf-countable at e, then F is first-countable at e. Then, Shen [31, Proposition 2.6 and Theorem 2.7] showed that every snf-countable paratopological left-loop is sof-countable and a sequential, regular csf-countable paratopological left-loop G is first-countable if and only if G contains no closed copy of S_ω. These results in both of two articles can obtain the following results immediately.

Proposition 4.9. *If G is a sequential* csf*-countable topological gyrogroup containing no closed copy of* S_ω*, then G is* snf*-countable.*

Proposition 4.10. *Every snf-countable topological gyrogroup is sof-countable.*

Therefore, the following result is obtained since a space X is first-countable if and only if X is sequential and sof-countable.

Corollary 4.11. *If G is a sequential csf-countable topological gyrogroup containing no closed copy of S_ω, then G is metrizable.*

Theorem 4.12. *A strongly topological gyrogroup G is metrizable if and only if G is an snf-countable k-space of countable pseudocharacter.*

Proof. The necessity is trivial, it suffices to claim the sufficiency.

Let a strongly topological gyrogroup G be an snf-countable k-space of countable pseudocharacter. It follows from [11, Theorem 4.3] that every strongly topological gyrogroup with countable pseudocharacter is submetrizable. We obtain that every compact subset of G is metrizable. Since G is a k-space, it is easy to see that G is sequential. From Proposition 4.10, it follows that G is sof-countable and hence metrizable, since a space is first-countable if and only if it is sequential and sof-countable. □

As every strongly topological gyrogroup is a topological gyrogroup, it is natural to pose the following questions.

Question 4.13. *Is each feathered csf-countable topological gyrogroup metrizable?*

Question 4.14. *Let G be a feathered topological gyrogroup with an ω^ω-base. Is G metrizable?*

5. Quotient with respect to strong subgyrogroups

In this section, we give an affirmative answer to Question 1.2 on the case of strongly topological gyrogroups, see Theorem 5.2.

Lemma 5.1. *Let G be a topological gyrogroup. For each open neighborhood U of 0 and $x \in G$, there exists an open neighborhood V of 0 such that $V \oplus x \subseteq x \oplus U$.*

Proof. For each open neighborhood U of 0 and $x \in G$, define $L_{\ominus x}(y) : G \to G$ by $L_{\ominus x}(y) = \ominus x \oplus y$ and $R_x : G \to G$ by $R_x(y) = y \oplus x$. For each open neighborhood U of 0, we can find an open neighborhood W of x such that $L_{\ominus x}(W) = (\ominus x) \oplus W \subseteq U$, as $L_{\ominus x}$ is continuous. By the same method, we can find an open neighborhood V of 0 such that $R_x(V) = V \oplus x \subseteq W$. Then $(\ominus x) \oplus (V \oplus x) \subseteq U$, which means that $V \oplus x \subseteq x \oplus U$. □

Theorem 5.2. *Let G be a strongly topological gyrogroup with neighborhood base $\mathscr{U}$ of 0, F a compact subset of G, and P a closed subset of G such that $F \cap P = \emptyset$. Then there exists an open neighborhood V of the identity element 0 such that $(F \oplus V) \cap P = \emptyset$ and $(V \oplus F) \cap P = \emptyset$.*

Proof. For each $x \in F$, we can find an open neighborhood W_x of 0 in G such that $(x \oplus W_x) \cap P = \emptyset$. Then choose an open neighborhood $O_x \in \mathscr{U}$ with $O_x \oplus O_x \subseteq W_x$. Since F is compact and $F \subseteq \bigcup_{x\in F}\{x \oplus O_x\}$, we can find a finite subset $C \subseteq F$ such that $F \subseteq \bigcup_{x\in C}\{x \oplus O_x\}$. Put $Q = \bigcap_{x\in C} O_x$. For each $y \in F$, there exists $x \in C$ with $y \in x \oplus O_x$. Then

$$
\begin{aligned}
y \oplus Q &\subseteq (x \oplus O_x) \oplus Q \\
&\subseteq (x \oplus O_x) \oplus O_x \\
&= \bigcup_{s,t\in O_x} \{(x \oplus s) \oplus t\} \\
&= \bigcup_{s,t\in O_x} \{x \oplus (s \oplus \mathrm{gyr}[s, x](t))\} \\
&= x \oplus (O_x \oplus O_x) \\
&\subseteq x \oplus W_x \\
&\subseteq G \setminus P.
\end{aligned}
$$

Therefore, $(F \oplus Q) \cap P = \emptyset$.

On the other hand, choose $Q_1 \in \mathscr{U}$ such that $Q_1 \oplus Q_1 \subseteq Q$. By Lemma 5.1, for each $y \in F$, we can find an open neighborhood U_y of 0 such that $U_y \oplus y \subseteq y \oplus Q_1$. Then $F \subseteq \bigcup_{y \in F}\{U_y \oplus y\}$. Since F is compact, there is a finite set $E \subseteq F$ such that $F \subseteq \bigcup_{y \in E}\{U_y \oplus y\}$. Put $U = \bigcap_{y \in E} U_y$. For each $t \in F$, we can find $y \in E$ with $t \in U_y \oplus y$. Then

$$\begin{aligned}
U \oplus t &\subseteq U \oplus (U_y \oplus y) \\
&\subseteq U \oplus (y \oplus Q_1) \\
&= \bigcup\{u \oplus (y \oplus q) : u \in U, q \in Q_1\} \\
&= \bigcup\{(u \oplus y) \oplus \text{gyr}[u, y](q) : u \in U, q \in Q_1\} \\
&= (U \oplus y) \oplus Q_1 \\
&\subseteq (U_y \oplus y) \oplus Q_1 \\
&\subseteq (y \oplus Q_1) \oplus Q_1 \\
&= \bigcup\{(y \oplus p) \oplus q : p, q \in Q_1\} \\
&= \bigcup\{y \oplus (p \oplus \text{gyr}[p, y](q)) : p, q \in Q_1\} \\
&= y \oplus (Q_1 \oplus Q_1) \\
&\subseteq y \oplus Q \\
&\subseteq G \setminus P.
\end{aligned}$$

Therefore, $(U \oplus F) \cap P = \emptyset$. Finally, put $V = Q \cap U$, and we obtain that $(F \oplus V) \cap P = \emptyset$ and $(V \oplus F) \cap P = \emptyset$. □

The following result is well-known on topological groups, it is very important for the researches of quotient spaces of topological groups.

Theorem 5.3. ([3, Theorem 3.2.2]) *Let G be a topological group, H a locally compact subgyrogroup of G. Then there exists an open neighborhood U of e such that $\pi(\overline{U})$ is closed in G/H and $\pi\restriction_{\overline{U}}$ is a perfect mapping, where $\pi : G \to G/H$ is the natural quotient mapping.*

Then, according to Theorem 5.2, we would like to extend this result to strongly topological gyrogroups with respect to closed strong subgyrogroups.

First, we recall the following concept of the coset space of a topological gyrogroup.

Let G be a topological gyrogroup and H an L-subgyrogroup of G. It follows from [33, Theorem 20] that $G/H = \{a \oplus H : a \in G\}$ is a coset space which defines a partition of G. We denote by π the mapping $a \mapsto a \oplus H$ from G onto G/H. Clearly, for each $a \in G$, we have $\pi^{-1}(\pi(a)) = a \oplus H$. Furthermore, denote by $\tau(G)$ the topology of G, the quotient topology on G/H is as follows:

$$\tau(G/H) = \{O \subseteq G/H : \pi^{-1}(O) \in \tau(G)\}.$$

In this section, denote by π the natural homomorphism from a topological gyrogroup G to its quotient topology on G/H.

Definition 5.4. ([14, Definition 3.9]) A subgyrogroup H of a topological gyrogroup G is called *strong subgyrogroup* if for any $x, y \in G$, we have $gyr[x, y](H) = H$.

Obviously, every strong subgyrogroup is an L-subgyrogroup. Moreover, it was claimed that every strongly topological gyrogroup G contains some strong subgyrogroups which are union-generated from open neighborhoods of the identity element by construction, see [14, Proposition 3.11].

Lemma 5.5. ([10, Theorem 3.7]) *Let G be a topological gyrogroup and H an L-subgyrogroup of G. Then the natural homomorphism π from a topological gyrogroup G to its quotient topology on G/H is an open and continuous mapping.*

Lemma 5.6. ([14, Theorem 3.13]) *Let G be a strongly topological gyrogroup and H a closed strong subgyrogroup of G. Then the family $\{\pi(x \oplus V) : V \in \tau, 0 \in U\}$ is a local base of the space G/H at the point $x \oplus H \in G/H$, and G/H is a homogeneous T_1-space.*

Lemma 5.7. *Let G be a strongly topological gyrogroup and H a closed strong subgyrogroup of G. If U and V are open neighborhoods of 0 in G with $(\ominus V) \oplus V \subseteq U$. Then $\overline{\pi(V)} \subseteq \pi(U)$.*

Proof. Take any $x \in G$ such that $\pi(x) \in \overline{\pi(V)}$. Since $V \oplus x$ is an open neighborhood of x, $\pi(V \oplus x)$ is an open neighborhood of $\pi(x)$ by Lemma 5.5. Then $\pi(V \oplus x) \cap \pi(V) \neq \emptyset$. We can find $a \in V$ and $b \in V$ such that $\pi(a \oplus x) = \pi(b)$, that is, $a \oplus x = b \oplus h$, for some $h \in H$. Then

$$\begin{aligned} x &= (\ominus a) \oplus (b \oplus h) \\ &= ((\ominus a) \oplus b) \oplus \mathrm{gyr}[\ominus a, b](h) \\ &\in ((\ominus a) \oplus b) \oplus H \\ &\subseteq ((\ominus V) \oplus V) \oplus H \\ &\subseteq U \oplus H. \end{aligned}$$

Thus, $\pi(x) \in \pi(U \oplus H) = \pi(U)$, which means that $\overline{\pi(V)} \subseteq \pi(U)$. □

Theorem 5.8. *Let G be a strongly topological gyrogroup and H a closed strong subgyrogroup of G. Then the quotient space G/H is regular.*

Proof. Let W be an arbitrary open neighborhood of $\pi(0)$ in G/H. By Lemma 5.5, the natural quotient mapping π is open and continuous, so there exists an open neighborhood U of 0 in G such that $\pi(U) \subseteq W$. It follows from [4, Proposition 8] that we can find an open neighborhood V of 0 such that $(\ominus V) \oplus V \subseteq U$. By Lemma 5.7, $\overline{\pi(V)} \subseteq \pi(U) \subseteq W$. It is clear that $\pi(V)$ is an open neighborhood of $\pi(0)$ in G/H, so we obtain that G/H is regular at the point $\pi(0)$. Moreover, it is verified that G/H is homogeneous by Lemma 5.6, hence the quotient space G/H is regular. □

Proposition 5.9. *If H is a locally compact subgyrogroup of a topological gyrogroup G, then H is closed in G.*

Proof. We show that $H = \overline{H}$ in G. By [4, Proposition 7], $\overline{H}$ is a subgyrogroup of G. Since H is a dense locally compact subspace of $\overline{H}$, we know that H is open in $\overline{H}$. Therefore, H is closed, which means that $H = \overline{H}$. Therefore, H is a closed subgyrogroup in G. □

The proof of the following Lemma depends on Theorem 5.2.

Lemma 5.10. *Let G be a strongly topological gyrogroup and H a locally compact strong subgyrogroup of G. Suppose that P is a closed symmetric subset of G such that P contains an open neighborhood of 0 in G, and $\overline{P \oplus (P \oplus P)} \cap H$ is compact. Then the restriction f of π to P is a perfect mapping from P onto the subspace $\pi(P)$ of G/H.*

Proof. It is obvious that f is continuous. Moreover, since H is a locally compact subgyrogroup of G, we know that H is closed in G by Proposition 5.9.

Claim 1. $f^{-1}(f(a))$ is compact for each $a \in P$.

Indeed, from the definition of f, we have $f^{-1}(f(a)) = (a \oplus H) \cap P$. By [4, Proposition 3], the left gyrotranslation is a homeomorphism, so the subspace $(a \oplus H) \cap P$ and $H \cap ((\ominus a) \oplus P)$ are homeomorphic, thus both of them are closed in G. From $\ominus a \in \ominus P = P$, it follows that

$$H \cap ((\ominus a) \oplus P) \subseteq H \cap (P \oplus P) \subseteq \overline{P \oplus (P \oplus P)} \cap H.$$

Hence, $H \cap ((\ominus a) \oplus P)$ is compact and so is the set $f^{-1} f(a)$.

Claim 2. f is a closed mapping.

Let us fix any closed subset M of P and let a be an any point of P such that $f(a) \in \overline{f(M)}$. It suffices to show that $f(a) \in f(M)$.

Suppose on the contrary. Then $(a \oplus H) \cap (M \oplus H) \cap P = \emptyset$. Since H is a strong subgyrogroup in G, $(a\oplus H)\oplus H = \bigcup_{x,y\in H}\{(a\oplus x)\oplus y\} = \bigcup_{x,y\in H}\{a\oplus(x\oplus \mathrm{gyr}[x,a](y))\} = a\oplus(H\oplus H) = a\oplus H$. Then $(a\oplus H)\cap M\cap P = \emptyset$, and $(a \oplus H) \cap \overline{(P \oplus P)} \cap M = \emptyset$ since $M \subseteq P$. Obviously, $(a \oplus H) \cap \overline{(P \oplus P)}$ is compact. Since M is a closed and disjoint from the compact subset $(a \oplus H) \cap \overline{(P \oplus P)}$, by Theorem 5.2, there exists an open neighborhood $W \in \mathscr{U}$ such that $W \subseteq P$ and $(W \oplus ((a \oplus H) \cap \overline{(P \oplus P)})) \cap M = \emptyset$.

Since the quotient mapping π is open by Lemma 5.5 and $W \oplus a$ is an open neighborhood of a, the set $\pi(W\oplus a)$ is an open neighborhood of $\pi(a)$ in G/H. Therefore, the set $\pi(W\oplus a)\cap\pi(M) \neq \emptyset$ and we can fix $m \in M$ and $y \in W$ such that $\pi(m) = \pi(y\oplus a)$, that is, $m \in (y\oplus a)\oplus H$. Then, $(y\oplus a)\oplus H = y\oplus(a\oplus gyr[a, y](H)) = y\oplus(a\oplus H)$. Hence, there exists an $h \in H$ such that $a \oplus h = \ominus y \oplus m$. Since $\ominus y \in \ominus W = W \subseteq P$ and $m \in M \subseteq P$, we have that $a\oplus h = \ominus y\oplus m \in \overline{(P \oplus P)}$. In addition, $a\oplus h \in a\oplus H$. Hence, $a\oplus h \in ((a\oplus H)\cap\overline{(P \oplus P)})$ and $m \in W\oplus((a\oplus H)\cap\overline{(P \oplus P)})$. Thus, $M \cap (W \oplus ((a \oplus H) \cap \overline{(P \oplus P)})) \neq \emptyset$, which is a contradiction.

Therefore, $f(a) \in f(M)$ and $f(M)$ is closed in $f(P)$. Then, since $a \in P$ is arbitrarily taken, we conclude that the mapping f is perfect. □

The following result generalizes Theorem 5.3.

Theorem 5.11. *Let G be a strongly topological gyrogroup and H a locally compact strong subgyrogroup of G. Then there exists an open neighborhood U of the identity element 0 such that $\pi(\overline{U})$ is closed in G/H and the restriction of π to $\overline{U}$ is a perfect mapping from $\overline{U}$ onto the subspace $\pi(\overline{U})$.*

Proof. Since H is locally compact, we know that H is closed in G by Proposition 5.9 and we can find an open neighborhood V of 0 in G such that $\overline{V \cap H}$ is compact. Since G is regular, we can choose an open neighborhood W of 0 such that $\overline{W} \subseteq V$. As a closed subspace of the compact set $\overline{V \cap H}$, $\overline{W} \cap H$ is compact. Let U_0 be an arbitrary symmetric open neighborhood of 0 such that $U_0 \oplus (U_0 \oplus U_0) \subseteq W$. By the joint continuity, we have $\overline{U_0} \oplus (\overline{U_0} \oplus \overline{U_0}) \subseteq \overline{U_0 \oplus (U_0 \oplus U_0)}$. Then the set $P = \overline{U_0}$ satisfies all restrictions on P in Lemma 5.10. It follows from Lemma 5.10 that the restriction of π to P is a perfect mapping from P onto the subspace $\pi(P)$.

It follows from Lemma 5.5 that π is an open mapping, the set $\pi(U_0)$ is open in G/H. It follows from Theorem 5.8 that the space G/H is regular, then we can find an open neighborhood V_0 of $\pi(0)$ in G/H such that $\overline{V_0} \subseteq \pi(U_0)$. Hence $U = \pi^{-1}(V_0) \cap U_0$ is an open neighborhood of 0 contained in P such that the restriction f of π to $\overline{U}$ is a perfect mapping from $\overline{U}$ onto the subspace $\pi(\overline{U})$. Furthermore, $\pi(\overline{U})$ is closed in $\pi(P)$, and $\pi(\overline{U}) \subseteq \overline{V_0} \subseteq \pi(U_0) \subseteq \pi(P)$. Then $\pi(\overline{U})$ is closed in $\overline{V_0}$, so that $\pi(\overline{U})$ is closed in G/H. □

Remark 5.12. Indeed, Theorem 5.2 plays a leading role in the proofs of Lemma 5.10 and Theorem 5.11. It is not difficult to see that if Theorem 5.2 holds on topological gyrogroups, then Lemma 5.10 and Theorem 5.11 also hold on topological gyrogroups by the same proof.

Since paracompactness is inherited by regular closed sets and preserved by perfect preimages and it was proved in [12, Theorem 4.6] that every locally paracompact strongly topological gyrogroup is paracompact, the following result is trivial.

Corollary 5.13. *Let G be a strongly topological gyrogroup and H a locally compact strong subgyrogroup of G. If the quotient space G/H is locally paracompact, then G is a paracompact space.*

Corollary 5.14. *Let G be a strongly topological gyrogroup and H a locally compact strong subgyrogroup of G. If the quotient space G/H is a k-space, then G is also a k-space.*

Proof. Since the property of being a k-space is invariant under taking perfect preimages and a locally k-space is a k-space, see [19, Section 3.3], it follows that G is also a k-space. □

Then we give some applications of Theorem 5.11.

Theorem 5.15. *Let G be a strongly topological gyrogroup and H a locally compact metrizable strong subgyrogroup of G. If the quotient space G/H is strongly Fréchet-Urysohn, then the space G is also strongly Fréchet-Urysohn.*

Proof. By Theorem 5.11, there exists an open neighborhood U of the identity element 0 in G such that $\pi\restriction_{\overline{U}} : \overline{U} \to \pi(\overline{U})$ is a perfect mapping and $\pi(\overline{U})$ is closed in G/H.

Put $f = \pi\restriction_{\overline{U}} : \overline{U} \to \pi(\overline{U})$. Then $f(\overline{U}) = \pi(\overline{U})$ is strongly Fréchet-Urysohn. For each $b \in \overline{U}$, $f^{-1}(f(b)) = \pi^{-1}(\pi(b)) \cap \overline{U} = (b \oplus H) \cap \overline{U}$ is metrizable. Therefore, the singleton $\{b\}$ is a G_δ-set in the space $f^{-1}(f(b))$. Moreover, since the quotient space G/H is strongly Fréchet-Urysohn, the space G is locally Fréchet-Urysohn by [3, Proposition 4.7.18]. Hence, G is Fréchet-Urysohn. Furthermore, every Fréchet-Urysohn topological gyrogroup is strongly Fréchet-Urysohn by [24, Corollary 5.2]. So G is strongly Fréchet-Urysohn. □

According to Theorem 5.11, the following theorem generalizes an important result of topological groups in [28, Theorem 4.3].

Theorem 5.16. *Let G be a strongly topological gyrogroup and H a locally compact metrizable strong subgyrogroup of G. If the quotient space G/H is sequential, then G is also sequential.*

Proof. By Theorem 5.11, there exists an open neighborhood U of the identity element 0 in G such that $\pi\restriction_{\overline{U}} : \overline{U} \to \pi(\overline{U})$ is a perfect mapping and $\pi(\overline{U})$ is closed in G/H.

First, we show that if $\{x_n\}_n$ is a sequence in $\overline{U}$ such that $\{\pi(x_n)\}_n$ is a convergent sequence in $\pi(\overline{U})$ and if x is an accumulation point of the sequence $\{x_n\}_n$, then there is a subsequence of $\{x_n\}_n$ which converges to x.

Since $\pi\restriction_{\overline{U}}$ is perfect, every subsequence of $\{x_n\}_n$ has an accumulation point in $\overline{U}$. Put $F = \pi^{-1}(\pi(x)) \cap \overline{U}$. Since H is metrizable, $\pi^{-1}(\pi(x)) = x \oplus H$ is also metrizable. Since every topological gyrogroup is regular, there exists a sequence $\{U_k\}_k$ of open subsets in G such that $\overline{U_{k+1}} \subseteq U_k$ for each $k \in \mathbb{N}$ and $\{x\} = F \cap \bigcap_{k\in\mathbb{N}} U_k$. Choose a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}_n$ such that $x_{n_k} \in U_k$ for each $k \in \mathbb{N}$. For an arbitrary accumulation point p of a subsequence of the sequence $\{x_{n_k}\}_k$, we have $\pi(p) = \pi(x)$ and $p \in \bigcap_{k\in\mathbb{N}} \overline{U_k}$. Thus $p = x$. Therefore, $\{x_{n_k}\}_k$ converges to x.

Then choose an open neighborhood V of 0 such that $\overline{V} \subseteq U$ and we show that $\overline{V}$ is a sequential subspace.

Suppose that $\overline{V}$ is not a sequential subspace, so we can find a non-closed and sequentially closed subset A of $\overline{V}$. Then there exists a point x such that $x \in cl_{\overline{V}}(A) \setminus A$. It is clear that $cl_{\overline{V}}(A) = \overline{A}$. Let $f = \pi\restriction_{\overline{V}} : \overline{V} \to \pi(\overline{V})$ and $B = A \cap f^{-1}(f(x))$. Since B is a closed subset of A, B is sequentially closed. Moreover, the fiber $f^{-1}(f(x)) = (\pi^{-1}(\pi(x))) \cap \overline{V}$ is sequential, so B is closed in $\overline{V}$. Since $x \notin B$, there exists an open neighborhood W of x in $\overline{V}$ such that $\overline{W} \cap B = \emptyset$. Let $C = \overline{W} \cap A$, then C is also sequentially closed as a closed subset of A and $x \in \overline{C} \setminus C$. Therefore, $C \cap f^{-1}(f(x)) = \overline{W} \cap B = \emptyset$, then $f(x) \in \overline{f(C)} \setminus f(C)$. So $f(C) = \pi(C)$ is not closed in $\pi(\overline{V})$. However, this is impossible, as it is easy to verify that the image of each sequentially closed subset of $\overline{V}$ is closed in $\pi(\overline{V})$.

Indeed, let C be sequentially closed in $\overline{V}$ and $\{y_n\}_n$ a sequence in $\pi(C)$ such that $y_n \to y$ in $\pi(\overline{V})$. Choose $x_n \in C$ with $\pi(x_n) = y_n$ for each $n \in \mathbb{N}$. Since every subsequence of the sequence $\{x_n\}_n$ has an accumulation point, there exist a point $x \in \pi^{-1}(y)$ and a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}_n$ such that $x_{n_k} \to x$. Since C is sequentially closed, we obtain $x \in C$ and $y \in \pi(C)$. Therefore, $\pi(C)$ is sequentially closed in $\pi(\overline{V})$. Since $\pi\restriction_{\overline{U}} : \overline{U} \to \pi(\overline{U})$ is a closed mapping and $\pi(\overline{U})$ is closed in G/H, $\pi(\overline{V})$ is closed in G/H. Since G/H is sequential, $\pi(\overline{V})$ is also sequential and then $\pi(C)$ is closed in $\pi(\overline{V})$.

Since G is homogeneous and $\overline{V}$ is a sequential subspace, we conclude that G is a locally sequential space. Thus, G is a sequential space. □

6. Open problems

The (strongly) topological gyrogroup, as a generalization of a topological group is an interesting and meaningful topic. In particular, since the classical Möbius gyrogroups, Einstein gyrogroups, and Proper Velocity gyrogroups with standard topology are all strongly topological gyrogroups, it is necessary to find a topological gyrogroup G such that G is not a strongly topological gyrogroup. Indeed, during the proofs of Lemma 5.7, Theorem 5.8 and [14, Theorem 3.13], it is easy to see that G just needs to be a topological gyrogroup. However, we do not find any topological gyrogroup G such that G has some non-trivial strong subgyrogroups and G is not a strongly topological gyrogroup. Therefore, we pose the following questions.

Question 6.1. *Is there a topological gyrogroup G but not a strongly topological gyrogroup? Especially with* $|G| \geq \omega$.

Question 6.2. *Is there a topological gyrogroup G such that G is not a strongly topological gyrogroup and G contains some non-trivial strong subgyrogroups?*

In Theorems 4.2 and 4.6, it is clear that the characterization of feathered strongly topological gyrogroup playes an important role in the proof. However, we do not know whether the characterization of feathered holds in topological gyrogroups. If it holds in topological gyrogroups, both of Theorems 4.2, and 4.6 can be extended to topological gyrogroups immediately.

Question 6.3. *If G is a feathered topological gyrogroup, is there a compact L-subgyrogroup of G such that the quotient space G/H metrizable?*

The following question is also open.

Question 6.4. ([4, Question 1]) *Is every Hausdorff topological gyrogroup completely regular?*

Finally, we pose the following question, which is important for the researches of quotient spaces of strongly topological gyrogroups with respect to closed strong subgyrogroups.

Question 6.5. *Let G be a strongly topological gyrogroup and H a closed strong subgyrogroup of G. Is the quotient space G/H completely regular?*

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