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Multiple solutions in fractional Orlicz-Sobolev Spaces for a class of nonlocal Kirchhoff systems

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Abstract. The existence and multiplicity of solutions for a class of nonlocal elliptic systems are investigated in this paper. We show that there are precise intervals of positives parameters that the energy functional associated with the Kirchhoff system problem admit three critical points which are exactly the weak solutions to our problem, and another intervals of positives parameters that admit many solutions to the Kirchhoff system problem without the boundedness condition of the Kirchhoff function. Our strategy is focused on the use of variational approaches.

1. Introduction

In this paper, we focus on the existence of multiple solutions for the following Kirchhof-type systems involving two positives parameters in fractional Orlicz-Sobolev spaces

$$\begin{cases} -K_1\Big(\mathscr{S}_1(\omega_1)\Big)\Big((-\Delta)^s_{a_1(.)}\omega_1 + a_1(|\omega_1|)\omega_1\Big) = \lambda F_{\omega_1}(z,\omega_1,\omega_2,\dots,\omega_n) + \mu H_{\omega_1}(z,\omega_1,\omega_2,\dots,\omega_n) & in \quad \Omega, \\ \quad \vdots & \vdots \\ -K_n\Big(\mathscr{S}_n(\omega_n)\Big)\Big((-\Delta)^s_{a_n(.)}\omega_n + a_n(|\omega_n|)\omega_n\Big) = \lambda F_{\omega_n}(z,\omega_1,\omega_2,\dots,\omega_n) + \mu H_{\omega_n}(z,\omega_1,\omega_2,\dots,\omega_n) & in \quad \Omega, \\ \omega_1 = \omega_2 = \dots = \omega_n = 0 & on \ \mathbb{R}^n\backslash\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open domain in $\mathbb{R}^n$ ($n \geq 3$) with lipchitz boundary $\partial\Omega$, $K_\imath : \mathbb{R} \to \mathbb{R}$, are increasing continuous functions with $\imath = 1, 2, \dots, n$, and satisfying:

(M_1) there exists positives numbers $\alpha_\imath > 0$, such that : $\alpha_\imath \leq K_\imath(\gamma)$, $\ for\, all\, \gamma \geq 0$,

(M_2) $\exists\, \theta_\imath > 0$, such that : $\hat{K}_\imath(\gamma) \geq \theta_\imath K_\imath(\gamma)\gamma$, $\quad \hat{K}_\imath(\gamma) := \int_0^\gamma K_\imath(r)dr$, $\quad for\, all\, \gamma \geq 0$,

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and $\mathscr{S}_\imath(\omega_\imath) := \int_{\Omega\times\Omega} A_\imath(|D^s\omega_\imath|)d\mu + \int_\Omega A_\imath(|\omega_\imath|)dx$, for all $x \in \Omega$, $\omega_\imath \in W^s\mathbb{L}_{A_\imath}(\Omega)$, $d\mu = \frac{dxdy}{|x-y|^n}$. Also, F, H : $\Omega\times\mathbb{R}^n \to \mathbb{R}$ are functions such that $F(.,z), H(.,z)$ are measurable in respect to $x \in \Omega$ for all $z = (z_1,\dots,z_n) \in \mathbb{R}^n$ and is C^1 in respect to z for all $x \in \Omega$ and satisfies the following assumption
(G_0): for each $(z_1,\dots,z_n) \in \mathbb{R}^n$, $S > 0$ and $1 \le \imath \le n$,

$$\sup_{(z_1,\dots,z_n)\le S} |G_{z_\imath}(x, z_1,\dots,z_n)| \in \mathbb{L}^1(\Omega).$$

$F_{z_\imath}$ (resp. $H_{z_\imath}$) designates the partial derivative of F (resp. H) in respect to $z_\imath$. $s \in (0,1)$, λ, μ are two positives parameters, $(-\Delta)^s_{a_\imath(.)}$ is the nonlocal fractional $a_\imath(.)$-Laplacian operator which debuted in [9] as follows

$$(-\Delta)^s_{a_\imath(.)}\omega_\imath(x) = P.V\int_{\mathbb{R}^N} a_\imath\Big(|D^s\omega_\imath|\Big)D^s\omega_\imath \frac{dy}{|x-y|^n},$$

for all $x \in \mathbb{R}^n$, $D^s\omega_\imath = \dfrac{\omega_\imath(x)-\omega_\imath(y)}{|x-y|^s}$, is the s-Hölder quotient and the functions $a_\imath\colon \mathbb{R}^+ \to \mathbb{R}^+$ are non-decreasing and right continuous functions, with

$$a_\imath(0) = 0, \quad a_\imath(\gamma) > 0 \quad \text{for} \quad \gamma > 0 \quad \text{and} \quad \lim_{\gamma\to\infty} a_\imath(\gamma) = \infty, \tag{1.2}$$

which partnered with the function $\varphi_\imath\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi_\imath(\gamma) = \begin{cases} a_\imath(|\gamma|)\gamma & for\, \gamma \neq 0, \\ 0 & for\ \gamma = 0, \end{cases} \tag{1.3}$$

is such that is an odd, increasing homeomorphism from $\mathbb{R}$ onto itself.
Throughout this paper we assume that,

$$1 < \frac{n}{s'} < l_\imath := \inf_{\gamma>0}\frac{\gamma\varphi_\imath(\gamma)}{A_\imath(\gamma)} \le m_\imath := \sup_{\gamma>0}\frac{\gamma\varphi_\imath(\gamma)}{A_\imath(\gamma)} < \infty, \quad \gamma > 0,\ s' \in (0,s). \tag{1.4}$$

In this study, we use appropriate variational methods in the fractional Orlicz-Sobolev space $W^s\mathbb{L}_A(\Omega)$ to solve our problem. Bonder et al [9] was the first to introduce such space. Currently, $W^s\mathbb{L}_A(\Omega)$ is an extension of the traditional fractional Sobolev space $W^{s,p}(\Omega)$ [12]. As a result, Azroul [4] and El-houari [13] have extended a number of features of fractional Sobolev spaces to $W^s\mathbb{L}_A(\Omega)$. The applicability of these spaces in many branches of mathematics has piqued people's interest (see e.g. recent results contained in [10, 18, 25] and reference therein). It has been the topic of research in a variety of directions. It's impossible to cover every aspect of the subject, so we'll only present few instances for those who are interested. For instance, consider physics of plasmas and biophysics [21],

$$A(\omega) = \frac{1}{p}|\omega|^p + \frac{1}{q}|\omega|^q, \qquad 1 < p < n,\ q \in (p, p^*).$$

In nonlinear elasticity [15],

$$A(\omega) = (1+\omega^2)^p - 1, \qquad p \in \Big(1, \frac{n}{n-2}\Big), \quad n \ge 3.$$

Non-local elliptic problems with fractional $a(.)$-Laplacian operators and Dirichlet-type boundary conditions became more prevalent in recent years, see [6, 11, 13, 14, 19] and reference therein. For example. In [20], by using the mountain pass theorem, we proved the existence of non trivial weak solutions to the non-local Kirchhoff problem:

$$\begin{cases} K_1\Big(\int_{\Omega\times\Omega} A_1(|D^s\omega_1|)d\mu\Big)(-\Delta)^s_{a_1(.)}\omega_1 = F_{\omega_1}(x,\omega_1,\omega_2) & in\,\Omega,\\ K_2\Big(\int_{\Omega\times\Omega} A_2(|D^s\omega_2|)d\mu\Big)(-\Delta)^s_{a_1(.)}\omega_2 = F_{\omega_2}(x,\omega_1,\omega_2) & in\,\Omega,\\ \omega_1=\omega_2=0 & on\ \mathbb{R}^n\backslash\Omega. \end{cases} \tag{1.5}$$

Also, Heidarkhani et al. in [22] have studiend the following non-homogeneous Neumann problems involving two parameters

$$\begin{cases} -K\Big(\int_{\Omega}\Big[A(|\nabla\omega|)+A(|\omega|)\Big]dx\Big)\Big((-\Delta)_{\varphi}u+\varphi(|\omega|)\Big) = \lambda F_{\omega}(x,\omega)+\mu H_{\omega}(x,\omega)\ in\,\Omega,\\ \frac{\partial\omega}{\partial\nu}=0 & on\ \partial\Omega. \end{cases} \tag{1.6}$$

If $\varphi(\gamma)=p|\gamma|^{p-2}\gamma$, then the problem (1.6) becomes widely known p-Kirchhoff-type problem

$$\begin{cases} -K\Big(\int_{\Omega}\Big[|\nabla\omega|^p+|\omega|^p\Big]dx\Big)\Big((-\Delta)_{p}\omega+|\omega|^{p-2}\omega\Big) = \lambda F_{\omega}(x,\omega)+\mu H_{\omega}(x,\omega)\ in\ \Omega,\\ \frac{\partial\omega}{\partial\nu}=0 & on\ \partial\Omega, \end{cases} \tag{1.7}$$

which is associated with the stationary version of the Kirchhoff problem.

$$\varrho\frac{\partial^2\omega}{\partial\gamma^2}-\Big(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\Big|\frac{\partial_o m}{\partial x}\Big|^2\Big)\frac{\partial^2\omega}{\partial x^2}=0, \tag{1.8}$$

presented by Kirchhoff in 1883 [23]. This study is a natural extension of previous work on Kirchhoff type problems in (classical) Sobolev spaces and in Orlicz–Sobolev spaces based on Young functions in a broad family of functional spaces known as fractional Orlicz–Sobolev space. Although the strategy used in this study isn't new, the results are. We remember the following three critical points theorem, obtained by G. Bonanno and S.A. Marano in [8].

Theorem 1.1. *Let $\mathbb{X}$ be a reflexive real Banach space, $\mathscr{I}:\mathbb{X}\to\mathbb{R}$ be a function such that:*
i) $\mathscr{I}$ is continuous and $\mathscr{I}(0)=0$.
ii) $\mathscr{I}$ is Gâteaux differentiable and its Gâteaux derivative is compact.
Let $\mathscr{L}:\mathbb{X}\to\mathbb{R}$ be a function such that:
i) $\mathscr{L}$ is sequential weak lower semicontinuous and $\mathscr{L}(0)=0$.
ii) $\mathscr{L}$ is Gâteaux differentiable and its derivative is bounded on bounded subsets of $\mathbb{X}$.
iii) The Gâteaux derivative of $\mathscr{L}$ admits a continuous inverse on $\mathbb{X}^*$.
If there exist $\tau>0$ and $\tilde{v}\in\mathbb{X}$, with $\tau<\mathscr{L}(\tilde{v})$ such that:
$(\tau_1)\quad \dfrac{\sup_{w\in\mathscr{L}^{-1}(-\infty,\tau)}\mathscr{I}(\omega)}{\tau}<\dfrac{\mathscr{I}(\tilde{v})}{\mathscr{L}(\tilde{v})},$
(τ_2) *for each $\lambda\in\mathscr{U}_\tau:=\Big]\dfrac{\mathscr{I}(\tilde{v})}{\mathscr{L}(\tilde{v})},\dfrac{\tau}{\sup_{w\in\mathscr{L}^{-1}(-\infty,\tau)}\mathscr{I}(\omega)}\Big[$ the functional $\mathscr{L}-\lambda\mathscr{I}$ is coercive.*
Then, for every compact interval $[a,b]\subseteq\mathscr{U}_r$, there exists $\varrho>0$ with the property:
for every $\lambda\in[a,b]$, the equation $\mathscr{J}'(\omega)-\lambda\mathscr{I}'(\omega)=0$ has at least three solutions in $\mathbb{X}$ whose norms are less than ϱ.

We apply the following multiple critical point theorem, which is established in [7], to achieve the result of problem (1.1).

Theorem 1.2. *Let $\mathscr{L}$, $\mathscr{I}:\mathbb{X}\to\mathbb{R}$ be two Gâteaux differentiable functionals in reflexive (real Banach) space $\mathbb{X}$, such that $\mathscr{I}$ is sequentially (weakly) upper semicontinuous, $\mathscr{L}$ is continuous, coercive and sequential weak lower semicontinuous. For each $\tau>\inf_{\mathbb{X}}\mathscr{L}$, let*

$$\varpi(\tau):=\inf_{w\in\mathscr{L}^{-1}(-\infty,\tau)}\frac{\sup_{w\in\mathscr{L}^{-1}(-\infty,\tau)}\mathscr{I}(v)-\mathscr{I}(\omega)}{\tau-\mathscr{L}(w)},$$

and

$$\delta := \lim_{\tau \to (\inf_{\mathbb{X}} \mathscr{L})^+} \inf \varpi(\tau).$$

If $\delta < \infty$*, then for every* $\lambda \in (0, \frac{1}{\delta})$*, one of this holds:*
(i) *A global minimum of* $\mathscr{L}$ *exists, as well as, a local minimum of* $h_\lambda := \mathscr{L} - \lambda \mathscr{I}$*.*
(ii) *A sequence* $\{w_n\}$ *of pairwise distinct critical points (local minima) of* h_λ *exists, that converges (weakly) to a global minimum of* $\mathscr{L}$ *with* $\lim_{n\to+\infty} \mathscr{L}(w_n) = \inf_{\mathbb{X}} \mathscr{L}$*.*

This work is motived by [22] and paper above. The following is a breakdown of how this work is organized. We quickly review certain Orlicz and fractional Orlicz-Sobolev space features in Sect.2. Sect. 3 is concerned with defining the data assumptions and present the existing solutions to the problems (1.1) as well as their proofs. In Sect. 4, we give some examples of functions φ, A, K H and F for which the results of this paper can be applied.

2. Some preliminary results

The reader is refered to [1, 2, 5, 9, 24] to learn more about Orlicz and fractional Orlicz-Sobolev space.

We take notice of $\mathbf{N}$ the set of all N-functions. Let a be as in (1.2). We'll use the notation $A(z) = \int_0^z \varphi(r)\,\mathrm{d}r$, for every $z \in \mathbb{R}$, then, $A \in \mathbf{N}$ and its complementary $\overline{A}$ given by this relationship $\overline{A}(z) := \sup_{r\geq 0}\{zr - A(r)\}$, is also in $\mathbf{N}$. We point out that $A \in \Delta_2$. if for a certain constant $k > 0$,

$$A(2z) \leq k\,A(z), \quad \text{for every } z > 0. \tag{2.1}$$

We observe that A and $\overline{A}$ satisfies the following Young's inequality:

$$rz \leq A(r) + \overline{A}(z) \ \text{ for all } z,\ r \geq 0 \text{ and } x \in \Omega. \tag{2.2}$$

Recall that $A^* \in \mathbf{N}$ is defined by

$$(A^*)^{-1}(z) = \int_0^z \frac{A^{-1}(r)}{r^{\frac{n+s}{n}}} dr \quad for\, z \geq 0,$$

where we mention that

$$(H_0)\ \int_0^1 \frac{A^{-1}(z\gamma)}{z^{1+\frac{s}{n}}}\,\mathrm{d}z < \infty \ \text{ and } (H_\infty) \int_1^{+\infty} \frac{A^{-1}(z\gamma)}{z^{1+\frac{s}{n}}}\,\mathrm{d}z = +\infty, \text{for } s \in (0,1).$$

Let $(M, A) \in \mathbf{N}$. The notation $M << A$ means that, for each $\varepsilon > 0$,

$$\frac{M(\varepsilon z)}{A(z)} \to 0 \quad \text{as } z \to \infty. \tag{2.3}$$

The Orlicz space $\mathbb{L}_A(\omega)$ is defined as the mesurable functions $z : \Omega \to \mathbb{R}$ such that $\int_\Omega A\Big(d|z(x)|\Big)\,\mathrm{d}x < +\infty$ for some $d > 0$. The usual norm on $\mathbb{L}_A(\Omega)$ is $\|z\|_A = \inf\Big\{d > 0 \,/\, \int_\Omega A\Big(\frac{|z(x)|}{d}\Big)\,\mathrm{d}x \leq 1\Big\}$.
Recall that, the Hölder inequality holds

$$\int_\Omega |z(x)v(x)|\,\mathrm{d}x \leq \|z\|_A \|v\|_{\overline{A}} \text{ for all } z \in \mathbb{L}_A(\Omega) \text{ and } v \in \mathbb{L}_{\overline{A}}(\Omega).$$

One major inequality in $\mathbb{L}_A(\Omega)$ is:

$$\int_\Omega A\Big(\frac{|z(x)|}{\|z\|_A}\Big)\,\mathrm{d}x \leq 1, \quad \text{for all } z \in \mathbb{L}_A(\Omega) \setminus \{0\}. \tag{2.4}$$

After this, we list a few inequalities that will be used for our proofs. The proof is provided in [16].

Lemma 2.1. *Let $A \in \mathbf{N}$, then these assertions are equivalent:*
1)

$$1 < l := \inf_{z>0} \frac{\varphi(z)}{A(z)} \leq \sup_{z>0} \frac{\varphi(z)}{A(z)} := m < +\infty. \tag{2.5}$$

2)

$$\min\{z^l, z^m\}A(\varrho) \leq A(\varrho z) \leq \max\{z^l, z^m\}A(\varrho), \qquad \forall z, \varrho \geq 0. \tag{2.6}$$

3) $A \in \Delta_2$.

Lemma 2.2. *If $A \in \mathbf{N}$ satisfies (2.5) then we have*

$$\min\{\|z\|_A^l, \|z\|_A^m\} \leq \int_{\Omega} A(|z|)dx \leq \max\{\|z\|_A^l, \|z\|_A^m\}, \quad \forall z \in \mathbb{L}_A(\Omega). \tag{2.7}$$

Lemma 2.3. *we have $A << A^*$, i.e, $\lim_{\gamma\to\infty} \dfrac{A(kz)}{A^*(z)} = 0$, $\forall k > 0$.*

We now look at the definition of $W^s\mathbb{L}_A(\Omega)$, which defined as the mesurable functions $z \in \mathbb{L}_A(\Omega)$ such that

$$\int_{\Omega\times\Omega} A\Big(d|D^s z|\Big)d\mu < \infty \quad for\ some\ d > 0,$$

equipped with the norm,

$$\|z\|_{s,A} = \|z\|_A + [z]_{s,A}, \tag{2.8}$$

where $[.]_{s,A}$, is the Gagliardo semi-norm, given by

$$[z]_{s,A} = \inf\Big\{d > 0 : \int_{\Omega\times\Omega} A\Big(\frac{|D^s z|}{d}\Big)d\mu \leq 1\Big\}.$$

We set

$$W_0^s\mathbb{L}_A(\Omega) := \Big\{z \in W^s\mathbb{L}_A(\mathbb{R}^n) : z = 0\ a.e\ \mathbb{R}^N \setminus \Omega\Big\}.$$

In these spaces the generalized Poincaré inequality reads as follows (see [4])

$$\|z\|_A \leq C_A[z]_{s,A}, \quad \forall z \in W_0^s\mathbb{L}_A(\Omega), \tag{2.9}$$

where C_A is a positive constant. Then $(W_0^s\mathbb{L}_A(\Omega), [.]_{s,A})$ is a real Banach space (with $[.]_{s,A} \sim \|.\|_{s,A}$ when Ω is bounded). Also is a separable (resp. reflexive) space if and only if $A \in \Delta_2$ (resp. $A \in \Delta_2$ and $\overline{A} \in \Delta_2$). In addition, if $A \in \Delta_2$ and $A(\sqrt{.})$ is convex, then $W_0^s\mathbb{L}_A(\Omega)$ is uniformly convex, see [9].
The following embedding, will be used in this paper [4]:

$$W_0^s\mathbb{L}_A(\Omega) \overset{\text{cpt}}{\hookrightarrow} \mathbb{L}_B(\Omega), \qquad if\ B << A^*.$$

In particular, by Lemma 2.3, we have, $A << A^*$. Then

$$W_0^s\mathbb{L}_A(\Omega) \overset{\text{cpt}}{\hookrightarrow} \mathbb{L}_A(\Omega), \tag{2.10}$$

Moreover, if $s'l > N$. Then

$$W_0^s\mathbb{L}_A(\Omega) \overset{\text{cpt}}{\hookrightarrow} L^\infty(\Omega),$$

i.e, there exists $c > 0$, such that

$$|z|_\infty \leq c\|z\|_{s,A} \quad u \in W_0^s\mathbb{L}_A(\Omega), \tag{2.11}$$

where $|z|_\infty := \sup_{x\in\overline{\Omega}} |z(x)|$ and $0 < s' < s < 1$.

The fractional $a(.)$-Laplacian operator specified in (**??**) is defined between $W_0^s\mathbb{L}_A(\Omega)$ and its dual space $(W_0^s\mathbb{L}_A(\Omega))^*$ and the following expression is found ([9], Theorem 6.12)

$$\langle \mathcal{G}'(z), v\rangle = \int_{\Omega\times\Omega} a\Big(|D^s z|\Big)D^s z D^s v d\mu = \langle (-\Delta)^s_{a(.)} z, v\rangle, \tag{2.12}$$

for all $z, v \in W_0^s\mathbb{L}_A(\Omega)$, where $\mathcal{G}(z) := \int_{\Omega\times\Omega} A(|D^s z|)d\mu$.

Proposition 2.4. *[6] Suppose that $A(\sqrt{.})$ is convex, $z_k \rightharpoonup z$ in $W_0^s\mathbb{L}_A(\Omega)$ and $\limsup\langle \mathcal{G}'(z_k), z_k - z\rangle \leq 0$. Then $z_k \to z \in W_0^s\mathbb{L}_A(\Omega)$.*

Lastly, the next Lemmas, will be useful in what follows.

Lemma 2.5. *[5] The following properties are true:*
1)

$$\mathcal{G}\Big(\frac{z}{[z]_{s,A}}\Big) \leq 1, \quad for\, all\, z \in W_0^s\mathbb{L}_A(\Omega) \setminus \{0\}.$$

2)

$$\min\{[z]^l_{s,A}, [z]^m_{s,A}\} \leq \mathcal{G}(z) \leq \max\{[z]^l_{s,A}, [z]^m_{s,A}\}, \quad for \quad all \quad z \in W_0^s\mathbb{L}_A(\Omega).$$

Lemma 2.6. *Let $z \in W_0^s\mathbb{L}_A(\Omega)$. Then*

$$\int_{\Omega\times\Omega} A(|D^s z|)d\mu + \int_\Omega A(|z|)dx \geq \|z\|^l_{s,A}, \quad if \quad \|z\|_{s,A} < 1.$$

$$\int_{\Omega\times\Omega} A(|D^s z|)d\mu + \int_\Omega A(|z|)dx \geq \|z\|^m_{s,A}, \quad if \quad \|z\|_{s,A} > 1.$$

Proof. By similar argument in [22], we prove this Lemma. Let $\beta \in (1, \|z\|_{s,A})$ with $\|z\|_{s,A} > 1$. By (2.6) we have

$$\int_{\Omega\times\Omega} A(|D^s z|)d\mu + \int_\Omega A(|z|)dx \geq \beta^l \int_{\Omega\times\Omega} A\Big(\frac{|D^s z|}{\beta}\Big)d\mu + \int_\Omega A\Big(\frac{|z|}{\beta}\Big)dx$$

Since $\beta < \|z\|_{s,A}$ we find

$$\int_{\Omega\times\Omega} A\Big(\frac{|D^s z|}{\beta}\Big)d\mu + \int_\Omega A\Big(\frac{|z|}{\beta}\Big)dx > 1.$$

Thus,

$$\int_{\Omega\times\Omega} A(|D^s z|)d\mu + \int_\Omega A(|z|)dx \geq \beta^l.$$

Letting $\beta \nearrow \|z\|_{s,A}$ in the inequality above, we get

$$\int_{\Omega\times\Omega} A(|D^s z|)d\mu + \int_\Omega A(|z|)dx \geq \|z\|^l_{s,A}. \tag{2.13}$$

Next, Assume that $\|z\|_{s,A} < 1$. Let $\zeta \in (0, \|z\|_{s,A})$. By (2.6) we have

$$\int_{\Omega\times\Omega} A(|D^s z|)d\mu + \int_{\Omega} A(|z|)dx \geq \zeta^m \int_{\Omega\times\Omega} A\Big(\frac{|D^s z|}{\zeta}\Big)d\mu + \int_{\Omega} A\Big(\frac{|z|}{\zeta}\Big)dx.$$

Set $v(x) = \frac{z(x)}{\zeta}$. Then we have $\|v\|_{s,A} = \frac{\|z\|_{s,A}}{\zeta} > 1$. By (2.13), we infer that

$$\int_{\Omega\times\Omega} A(|D^s v|)d\mu + \int_{\Omega} A(|v|)dx \geq \|v\|^l_{s,A} > 1. \tag{2.14}$$

Observe that,

$$A(\gamma) \geq \tau^m A(\frac{\gamma}{\tau}), \quad for\ all\ \gamma > 0,\ \tau \in (0,1). \tag{2.15}$$

From (2.14) and (2.15), we infer that

$$\int_{\Omega\times\Omega} A(|D^s z|)d\mu + \int_{\Omega} A(|z|)dx \geq \zeta^m.$$

Letting $\zeta \nearrow \|z\|_{s,A}$ in the inequality above, we obtain

$$\int_{\Omega\times\Omega} A(|D^s z|)d\mu + \int_{\Omega} A(|z|)dx \geq \|z\|^m_{s,A}.$$

□

Proposition 2.7. *Let $z \in W^s_0\mathbb{L}_A(\Omega)$ and assume that $\mathscr{S}(z) \leq \tau$, for $0 < \tau < 1$. Then, $\|z\|_{s,A} < 1$.*

Proof. Let $z \in W^s_0\mathbb{L}_A(\Omega)$. By (2.16), if $\mathscr{S}(z) \leq \tau$ holds, then $\|z\|_{s,A} \leq 1$. Now, claim that $\|z\|_{s,A} \neq 1$. Arguing by contradiction, assume that there exists $z \in W^s_0\mathbb{L}_A(\Omega)$ with $\|z\|_{s,A} = 1$ and $\mathscr{S}(z) \leq \tau$ holds. Let us take $\beta \in (0,1)$, for all $x \in \Omega$. By similar argument in Lemma 2.6 we obtain that,

$$\int_{\Omega\times\Omega} A\Big(|D^s z|\Big)d\mu + \int_{\Omega} A\Big(|z|\Big)dx \geq \beta^m.$$

Letting $\beta \nearrow 1$ in the above inequality we obtain

$$\int_{\Omega\times\Omega} A\Big(|D^s z|\Big)d\mu + \int_{\Omega} A\Big(|z|\Big)dx \geq 1.$$

that contradicts condition $\mathscr{S}(z) \leq \tau$. The proof is complete. □

Remark 2.8. *Using Lemma 2.1 and Lemma 2.5, we can see that*

$$\|z\|_{s,A} \sim \|z\| = \inf\{\lambda > 0 : \mathscr{S}\Big(\frac{z}{\lambda}\Big) \leq 1\}, \tag{2.16}$$

for all z in $W^s_0\mathbb{L}_A(\Omega)$.

We define the space $\mathbb{X} := \prod_{\imath=1}^n W^s_0\mathbb{L}_{A_\imath}(\Omega)$ for problem (1.1), which is a reflexive Banach space, with respect to the norm

$$\|z\| := \sum_{\imath=1}^{n} \|z_\imath\|_{s,A_\imath}, \quad z = (z_1, z_2, \ldots, z_n) \in \mathbb{X}.$$

Hence $\mathbb{X} \hookrightarrow L^{\infty}(\Omega) \times ... \times L^{\infty}(\Omega)$ is compact. We set $C > 0$, such that

$$C := \max\Big\{ \sup_{z_\iota \in W_0^s \mathbb{L}_{A_\iota} \setminus \{0\}} \frac{\max_{x\in\overline{\Omega}} |z_\iota(x)|}{\|z_\iota\|_{s,A_\iota}} \ : \ for\ 1 \le \iota \le n \Big\} < +\infty. \tag{2.17}$$

Put

$$\mathscr{F}_\iota(z) := \int_{\Omega} F(x,z)dx,$$

Lemma 2.9. *([3] Lemma 3, [17] Lemma 3.4) The functions* $,\mathscr{S}_\iota, \mathscr{F}_\iota : W^s\mathbb{L}_{A_\iota}(\Omega) \to \mathbb{R}$ *are well defined and its the* $C^1(W^s\mathbb{L}_{A_\iota}(\Omega), \mathbb{R})$ *and we have*

$$\langle \mathscr{S}_\iota'(z_\iota), \overline{z}_\iota \rangle = \int_{\Omega\times\Omega} a_\iota\Big(|D^s z_\iota|\Big) D^s z_\iota D^s \overline{z}_\iota d\mu + \int_{\Omega} a_\iota\Big(|z_\iota|\Big) z_\iota \overline{z}_\iota dx, \tag{2.18}$$

$$\langle \mathscr{F}_\iota'(z_\iota), \overline{z}_\iota \rangle = \int_{\Omega} F_{z_\iota}(x, z_\iota)\overline{z}_\iota dx,$$

for all $\overline{z}_\iota \in W^s\mathbb{L}_{A_\iota}(\Omega)$.

At this point, we set the definition of our weak solution, we say that $z = (z_1, \ldots, z_n) \in \mathbb{X}$ is a weak solution for problem (1.1) if

$$\sum_{\iota=1}^{n} K_\iota\Big(\mathscr{S}_\iota(z_\iota)\Big)\langle \mathscr{S}_\iota'(z_\iota), \overline{z}_\iota \rangle - \lambda \int_{\Omega} \sum_{\iota=1}^{n} F_{z_\iota}(x, z_1, \ldots, z_n)\overline{z}_\iota dx = 0,$$

for all $\overline{z} = (\overline{z}_1, \ldots, \overline{z}_n) \in \mathbb{X}$.

Proposition 2.10. *Let* $\mathfrak{T} : \mathbb{X} \to \mathbb{X}^*$ *be the operator defined by*

$$\mathfrak{T}(\omega)(w) \quad = \sum_{\iota=1}^{n} K_\iota\Big(\mathscr{S}_\iota(\omega_\iota)\Big)\langle \mathscr{S}_\iota'(\omega_\iota), w_\iota \rangle$$

for each $\omega = (\omega_1, \ldots, \omega_n),\ w = (w_1, \ldots, w_n) \in \mathbb{X}$. *Then* $\mathfrak{T}$ *has a continuous inverse on the dual space* $\mathbb{X}^*$ *of* $\mathbb{X}$.

Proof. Due to Minty–Browder theorem [26]. It is enough to check that $\mathfrak{T}$ is hemicontinuous, coercive, and uniformly monotone. For every $\omega \in \mathbb{X}$, with $\|\omega_\iota\|_{s,A_\iota} > 1$, we have

$$\begin{aligned}\mathfrak{T}(\omega_1, \ldots, \omega_n)(\omega_1, \ldots, \omega_n) \quad &= \sum_{\iota=1}^{n} K_\iota\Big(\mathscr{S}_\iota(\omega_\iota)\Big) \\ &\times\Big(\int_{\Omega\times\Omega} a_\iota\Big(|D^s\omega_\iota|\Big)|D^s\omega_\iota|^2 d\mu + \int_{\Omega} a_\iota\Big(|\omega_\iota|\Big)|\omega_\iota|^2 dx\Big).\end{aligned}$$

Appliying (1.4),

$$\mathfrak{T}(\omega_1, \ldots, \omega_2)(\omega_1, \ldots, \omega_n) \quad \ge l_\iota \sum_{\iota=1}^{n} K_\iota\Big(\mathscr{S}_\iota(\omega_\iota)\Big)\mathscr{S}_\iota(\omega_\iota).$$

Using Lemma 2.6 and (M_1), then

$$\mathfrak{T}(\omega_1, \ldots, \omega_2)(\omega_1, \ldots, \omega_n) \quad \ge l_\iota \alpha_0 \sum_{\iota=1}^{n} \|\omega_\iota\|_{s,A_\iota}^{2l_\iota}.$$

where $\alpha_0 = \min \alpha_\imath$. So $\mathfrak{T}$ is coercive. From the continuity of the function $K_\imath$ and Lemma 2.9 we can verified that $\mathfrak{T}$ is hemicontinuous. Now let $u, w \in \mathbb{X}$ such that $u \neq w$. Since $a_\imath(|\gamma|)\gamma$, $K_\imath(\gamma)$ are increasing and from (M_1) we have

$$\begin{aligned}
\langle \mathfrak{T}(\omega) - \mathfrak{T}(w), \omega - w\rangle &= \langle \mathfrak{T}(\omega), \omega - w\rangle - \langle \mathfrak{T}(w), \omega - w\rangle \\
&= \sum_{\imath=1}^{n} K_\imath\Big(\mathscr{S}_\imath(\omega_\imath)\Big)\Big[\int_{\Omega\times\Omega}\Big[\Big(a_\imath(|D^s\omega_\imath|)D^s\omega_\imath\Big(D^s\omega_\imath - D^s w_\imath\Big)d\mu \\
&+ \int_{\Omega}\Big(a_\imath(|\omega_\imath|)\omega_\imath\Big(\omega_\imath - w_\imath\Big)dx\Big)\Big] \\
&- \sum_{\imath=1}^{n} K_\imath\Big(\mathscr{S}_\imath(w_\imath)\Big)\Big[\int_{\Omega\times\Omega}\Big[\Big(a_\imath(|D^s w_\imath|)D^s w_\imath\Big(D^s\omega_\imath - D^s w_\imath\Big)d\mu \\
&+ \int_{\Omega}\Big(a_\imath(|w_\imath|)w_\imath\Big(\omega_\imath - w_\imath\Big)dx\Big)\Big] \\
&\geq \alpha_0\Big[\int_{\Omega\times\Omega}\Big(a_\imath(|D^s\omega_\imath|)D^s\omega_\imath\Big(D^s\omega_\imath - D^s w_\imath\Big)d\mu + \int_{\Omega}\Big(a_\imath(|\omega_\imath|)\omega_\imath\Big(\omega_\imath - w_\imath\Big)dx\Big)\Big] \\
&- \alpha_0\Big[\int_{\Omega\times\Omega}\Big(a_\imath(|D^s w_\imath|)D^s w_\imath\Big(D^s\omega_\imath - D^s w_\imath\Big)d\mu + \int_{\Omega}\Big(a_\imath(|w_\imath|)w_\imath\Big(\omega_\imath - w_\imath\Big)dx\Big)\Big] \\
&= \alpha_0\Big[\int_{\Omega\times\Omega}\Big(a_\imath(|D^s\omega_\imath|)D^s\omega_\imath - a_\imath(|D^s w_\imath|)D^s w_\imath\Big)\Big(D^s\omega_\imath - D^s w_\imath\Big)d\mu \\
&+ \int_{\Omega}\Big(a_\imath(|\omega_\imath|)\omega_\imath - a_\imath(|w_\imath|)w_\imath\Big)\Big(\omega_\imath - w_\imath\Big)dx\Big] > 0.
\end{aligned}$$

So, $\mathfrak{T} : \mathbb{X} \to \mathbb{X}^*$ is strictly monotone. Thus, from Minty-Browder theorem, $\mathfrak{T}^{-1} : \mathbb{X}^* \to \mathbb{X}$ exists and it is bounded. By demonstrating that $\mathfrak{T}^{-1}$ is sequentially continuous, we can prove that it is continuous. Let $\omega_n \to \omega$ in $\mathbb{X}^*$, let $w_n = \mathfrak{T}^{-1}(\omega_n)$ and $w = \mathfrak{T}^{-1}(\omega)$. Then w_n is bounded in $\mathbb{X}$, so, $w_n \rightharpoonup w_0$ in $\mathbb{X}$. Since $\omega_n \to \omega$, we have

$$\lim_{n\to\infty}\langle \mathfrak{T}(w_n), w_n - w_0\rangle = \lim_{n\to\infty}\langle \omega_n, w_n - w_0\rangle = 0$$

that is,

$$\begin{aligned}
\sum_{\imath=1}^{n} K_\imath\Big(\mathscr{S}_\imath(w_n)\Big) \quad &\Big(\int_{\Omega\times\Omega} a_\imath(|D^s w_n|)D^s w_n(D^s w_n - D^s w_0)d\mu \\
&+ \int_{\Omega} a_\imath(|w_n|)w_n\Big(w_n - w_0\Big)dx\Big) = 0.
\end{aligned} \tag{2.19}$$

From the continuity of the functions $K_\imath$, Proposition 2.4 and the last equation, the fact that $w_n \rightharpoonup w_0$ in $\mathbb{X}$, we conclude that $w_n \to w_0$ in $\mathbb{X}$. □

Now we define the problem's energy functional $g_\lambda : \mathbb{X} \to \mathbb{R}$ by:

$$g_\lambda(\omega) = \Psi(\omega) - \lambda\mathscr{F}(\omega),$$

for all $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{X}$, where

$$\Psi(\omega) = \sum_{\imath=1}^{n} \hat{K}_\imath\Big(\mathscr{S}_\imath(\omega_\imath)\Big).$$

Note that, the weak solutions of (1.1) are exactly the critical points of g_λ. It is well known that Ψ and $\mathscr{F}$ are two continuous, Gâteaux differentiable functions and whose Gâteaux differentials at the point

$\omega = (\omega_1, \dots, \omega_n) \in \mathbb{X}$ are the functionals Ψ' and $\mathscr{F}'$ given by

$$\Psi'(\omega)(v) = \sum_{\imath=1}^{n} K_\imath\Big(\mathscr{S}_\imath(\omega)\Big)\Big(\int_{\Omega\times\Omega} a_\imath(|D^s\omega_\imath|)D^s\omega_\imath D^s v_\imath d\mu + \int_\Omega a_\imath(|\omega_\imath|)\omega_\imath v_\imath dx\Big), \tag{2.20}$$

and

$$\mathscr{F}'(\omega)(v) = \int_\Omega \sum_{\imath=1}^{n} F_{\omega_\imath}(x, \omega_1, \dots, \omega_n)v_\imath(x)dx,$$

also $\mathscr{F}' : \mathbb{X} \to (\mathbb{X})^*$ is a compact operator.

Lemma 2.11. *Ψ is sequential weak lower semicontinuous and coercive.*

Proof. For $\gamma \geq 0$, using (M_1), we have

$$\Psi(\omega) = \sum_{\imath=1}^{n} \hat{K}_\imath\Big(\mathscr{S}_\imath(\omega_\imath)\Big) \geq \sum_{\imath=1}^{n} \theta_\imath\alpha_\imath\mathscr{S}_\imath(\omega_\imath) \geq \theta_0\alpha_0 \sum_{\imath=1}^{n} \mathscr{S}_\imath(\omega_\imath),$$

where $\theta_0 = \min\theta_\imath$ and by Lemma 2.6, for all $u \in \mathbb{X}$ with $\|\omega_\imath\|_{s,A_\imath} > 1$, we have

$$\Psi(\omega) \geq \theta_0\alpha_0 \sum_{\imath=1}^{n} \|\omega_\imath\|^{l_\imath}_{s,A_\imath},$$

from which it follows that Ψ is coercive. Moreover, because of $A_\imath$ are convex, then Ψ is also convex function, thus, it is sequentially weakly lower semicontinuous. □

3. Main results.

Our main results are stated below

Theorem 3.1. *Assume that conditions (M_1), (M_2) hold, (1.4), $A_\imath(\sqrt{\cdot})$ are convex and*
(F_0): $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfying condition (G_0) and $F(x, 0, \dots, 0) = 0$ for each $x \in \Omega$,
(F_1): there exist $h \in \mathbb{L}^1(\Omega)$ and positive constants $d_\imath$, with $d_\imath < l_\imath$ for $1 \leq \imath \leq n$, such that

$$0 \leq F(x, \gamma_1, \dots, \gamma_n) \leq h(x)\sum_{\imath=1}^{n} |\gamma_\imath|^{d_\imath},$$

for each $x \in \Omega$ and every $(\gamma_1, \dots, \gamma_n) \in \mathbb{R}^n$,
(F_2) : there exist $0 < c_\imath < \psi := \frac{C}{\theta_0\alpha_0}$ and $\xi_\imath \in \mathbb{R}$ for $1 \leq \imath \leq n$, with

$$\theta_0\alpha_0|\omega|\sum_{\imath=1}^{n} A_\imath(|z_\imath|) > \min\Big\{\Big(\frac{c_\imath}{\psi}\Big)^m : 1 \leq \imath \leq n\Big\},$$

such that

$$\int_\Omega \sup_{|\gamma_1|\leq c_1,\dots,|\gamma_n|\leq c_n} F(x, \gamma_1, \dots, \gamma_n)dx < \frac{\min\Big\{\Big(\frac{c_\imath}{\psi}\Big)^m : 1 \leq \imath \leq n\Big\}}{\hat{K}(1)|\Omega|^{\frac{1}{\theta_0}} \sum_{\imath=1}^{n}(A_\imath(\xi_\imath))^{\frac{1}{\theta_\imath}}} \int_\Omega F(x, \xi_1, \dots, \xi_n)dx, \tag{3.1}$$

where $\hat{K}(1) = \max\hat{K}_\imath(1)$ and $\theta_0 = \min\theta_\imath$. Then, setting

$$\mathscr{U}_r := \left(\frac{\hat{K}(1)|\Omega|^{\frac{1}{\theta_0}} \sum_{\imath=1}^{n}(A_\imath(\xi_\imath))^{\frac{1}{\theta_0}}}{\displaystyle\int_\Omega F(x, \xi_1, \dots, \xi_n)dx}, \frac{\min\Big\{\Big(\frac{c_\imath}{\psi}\Big)^m : 1 \leq \imath \leq n\Big\}}{\displaystyle\int_\Omega \sup_{|\gamma_1|\leq c_1,\dots,|\gamma_n|\leq c_n} F(x, \gamma_1, \dots, \gamma_n)dx}\right)$$

for each compact interval $[a,b] \subset \mathscr{U}_r$*, there exists* $\varrho > 0$ *with the property: For each function* $H : \Omega \times \mathbb{R}^n \to \mathbb{R}$ *satisfying condition* (G_0)*, for all* $\lambda \in [a,b]$*, there exists* $\delta > 0$ *such that, for each* $\mu \in [0,\delta]$*, the system* (1.1) *has at least three weak solutions in* $\mathbb{X}$ *whose norms are less than* ϱ*.*

We set

$$B := \liminf_{\zeta\to 0^+} \frac{\int_{\Omega} \sup_{(\gamma_1,\ldots,\gamma_n)\in R(\zeta)} F(x,\gamma_1,\ldots,\gamma_n)dx}{\zeta^{\bar{m}}}$$

and

$$D := \limsup_{(\zeta_1,\ldots,\zeta_n)\to(0^+,\ldots,0^+)} \frac{\int_{\Omega} F(x,\zeta_1,\ldots,\zeta_n)dx}{\sum_{\imath=1}^n \hat{K}_\imath(|\Omega|\varrho\zeta_\imath^{m_\imath})},$$

where $R(\zeta) := \left\{(\gamma_1,\ldots,\gamma_n)\in\mathbb{R}^n : \sum_{\imath=1}^n |\gamma_\imath| < \zeta\right\}$ for all $\zeta > 0$.

Theorem 3.2. *Assume that conditions* (M_1)*,* (M_2) *and* (1.4) *hold and consider the following:*
(F_3)*:* $H(x,\gamma_1,\ldots,\gamma_n) \geq 0$ *for all* $(x,\gamma_1,\ldots,\gamma_n) \in \Omega\times\mathbb{R}^n_+$
let $\varrho > 0$ *such that*

$$\lim_{\gamma\to 0^+} \frac{A_\imath(\gamma)}{\gamma^{m_\imath}} < \varrho. \tag{3.2}$$

Further, assume that

$$\liminf_{\zeta\to 0^+} \frac{\int_{\Omega} \sup_{(\gamma_1,\ldots,\gamma_n)\in R(\zeta)} F(x,\gamma_1,\ldots,\gamma_n)dx}{\zeta^{\bar{m}}} < L \limsup_{(\zeta_1,\ldots,\zeta_n)\to(0^+,\ldots,0^+)} \frac{\int_{\Omega} F(x,\zeta_1,\ldots,\zeta_n)dx}{\sum_{\imath=1}^n \hat{K}_\imath(|\Omega|\varrho\zeta_\imath^{m_\imath})}, \tag{3.3}$$

where $\bar{m} = \max\, m_\imath$ *and*

$$L = \min\left\{L_{m_\imath} = \frac{1}{\left(\sum_{\imath=1}^n \left(\frac{C}{\theta_0\alpha_0}\right)^{\frac{1}{m_\imath}}\right)^{\bar{m}}} \ : \ for\ 0 \leq \imath \leq n\right\}. \tag{3.4}$$

If,

$$H_0 := \Big(\sum_{\imath=1}^n \Big(\frac{C}{\theta_0\alpha_0}\Big)^{\frac{1}{m_\imath}}\Big)^{\bar{m}} \liminf_{\zeta\to 0^+} \frac{\int_{\Omega} \sup_{(\gamma_1,\ldots,\gamma_n)\in R(\zeta)} H(x,\gamma_1,\ldots,\gamma_n)dx}{\zeta^{\bar{m}}} < \infty. \tag{3.5}$$

Then, for every

$$\lambda \in \mathscr{D} := \frac{1}{\left(\sum_{\imath=1}^n \left(\frac{C}{\theta_0\alpha_0}\right)^{\frac{1}{m_\imath}}\right)^{\bar{m}}} \Big]\frac{1}{LD}, \frac{1}{B}\Big[,$$

and for every $\mu \geq 0$ *with,*

$$\mu < \mu_\lambda := \frac{1}{H_0}\Big(1 - \lambda D\Big(\sum_{\imath=1}^n \Big(\frac{C}{\theta_0\alpha_0}\Big)^{\frac{1}{m_\imath}}\Big)^{\bar{m}}\Big),$$

problem (1.1) *has a sequence of pairwise distinct weak solutions, which converges (strongly) to zero in* $\mathbb{X}$*.*

Proof of Theorem 3.1

Proof. From the definitions of Ψ, from condition (F_0), we have

$$\Psi(0) = \mathscr{F}(0) = 0.$$

Notice that

$$|w(x) - w(y)| \leq |x - y| \|\nabla w\|_{L^\infty(\mathbb{R}^n)}. \tag{3.6}$$

Set $w(x) := (\xi_1, \dots, \xi_n)$ for any $x \in \mathbb{R}^n$ and $\xi_\imath \in \mathbb{R}$, by (3.6), we deduce $w(y) = w(x) = (\xi_1, \dots, \xi_n)$ for any $x, y \in \mathbb{R}^{2n}$. Clearly, $w \in \mathbb{X}$, and from (M_2) we have

$$\begin{aligned}\Psi(w) &= \sum_{\imath=1}^{n} \hat{K}_\imath\Big(\mathscr{S}_\imath(w_\imath)\Big) \geq \sum_{\imath=1}^{n} \theta_\imath \alpha_\imath \mathscr{S}_\imath(w_\imath) \geq \theta_0\alpha_0 \sum_{\imath=1}^{n} \mathscr{S}_\imath(w_\imath)\\ &= \theta_0\alpha_0|\Omega| \sum_{\imath=1}^{n} A_\imath(\xi_\imath) > \min\Big\{\Big(\frac{c_\imath}{\psi}\Big)^m : 1 \leq \imath \leq n\Big\} := \tau.\end{aligned}$$

Moreover, (M_2) implies that, $\hat{K}_\imath(\gamma) \leq \hat{K}_\imath(1)\gamma^{\frac{1}{\theta_\imath}}$, then we have,

$$\Psi(w) = \sum_{\imath=1}^{n} \hat{K}_\imath\Big(\mathscr{S}_\imath(w_\imath)\Big) \leq \sum_{\imath=1}^{n} \hat{K}_\imath(1)(\mathscr{S}_\imath(w_\imath))^{\frac{1}{\theta_\imath}} \leq \hat{K}(1)|\Omega|^{\frac{1}{\theta_0}} \sum_{\imath=1}^{n} (A_\imath(\xi_\imath))^{\frac{1}{\theta_\imath}}.$$

Moreover, when $\Psi(\omega) \leq \tau$ for $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{X}$, by (M_2) and Lemma 2.6, we have

$$\theta_0\alpha_0\|\omega_\imath\|_{s,A_\imath} \leq \max\{\tau^{\frac{1}{l_\imath}}, \tau^{\frac{1}{m_\imath}}\}.$$

Then, by (2.17) we obtain

$$|\omega_\imath|_\infty \leq \frac{Cr^{\frac{1}{m_\imath}}}{\alpha_0\theta_0} = c_\imath.$$

Therfore, for every $\omega \in \mathbb{X}$,

$$\begin{aligned}\sup_{\omega\in\Psi^{-1}((-\infty,\tau))} \mathscr{F}(\omega) &= \sup_{\omega\in\Psi^{-1}((-\infty,\tau))} \int_\Omega F(x, \omega_1(x), \dots, \omega_n(x))dx\\ &\leq \int_\Omega \sup_{|\gamma_1|\leq c_1,\dots,|\gamma_n|\leq c_n} F(x, \gamma_1, \dots, \gamma_n)dx.\end{aligned}$$

Condition (F_2) implies

$$\begin{aligned}\frac{\sup_{\omega\in\Psi^{-1}((-\infty,\tau))} \mathscr{F}(\omega)}{\tau} &\leq \frac{\int_\Omega \sup_{|\gamma_1|\leq c_1,\dots,|\gamma_n|\leq c_n} F(x, \gamma_1, \dots, \gamma_n)dx}{\min\Big\{\Big(\frac{c_\imath}{\psi}\Big)^m : 1 \leq \imath \leq n\Big\}}\\ &\leq \frac{\int_\Omega F(x, \xi_1, \dots, \xi_n)dx}{\hat{K}(1)|\Omega|^{\frac{1}{\theta_0}} \sum_{\imath=1}^{n} (A_\imath(\xi_\imath))^{\frac{1}{\theta_\imath}}}\\ &\leq \frac{\mathscr{F}(w)}{\Psi(w)}.\end{aligned}$$

Thus, assumption (τ_1) of Theorem 1.1 is satisfied. From (τ_2), as a result, for every positive parameter λ, the function $\Psi - \lambda\mathscr{F}$ is coercive, in particular for every

$$\lambda \in \mathscr{U}_\tau \subseteq \left(\frac{\Psi(w)}{\mathscr{F}(w)}, \frac{\tau}{\sup_{\Psi(\omega)\le\tau}\mathscr{F}(\omega)}\right),$$

Indeed, from (F_1) we have

$$\int_\Omega F(x,\omega_1,\dots,\omega_n)dx \le \int_\Omega h(x)\sum_{\imath=1}^n |\omega_\imath|^{d_\imath} \le |h|_\infty \sum_{\imath=1}^n \|\omega_\imath\|^{d_\imath}_{L^{d_\imath}(\Omega)} \le C|h|_\infty \sum_{\imath=1}^n \|\omega_\imath\|^{d_\imath}_{s,A_\imath},$$

then, from Lemma 2.11 we deduce that

$$\begin{aligned}\Psi(\omega) - \lambda\mathscr{F}(\omega) &\ge \theta_0\alpha_0\sum_{\imath=1}^n \|\omega_\imath\|^{l_\imath}_{s,A_\imath} - \lambda C|h|_\infty\sum_{\imath=1}^n \|\omega_\imath\|^{d_\imath}_{s,A_\imath}\\ &= \sum_{\imath=1}^n \|\omega_\imath\|^{l_\imath}_{s,A_\imath}\Big(\theta_0\alpha_0 - \lambda C|h|_\infty\sum_{\imath=1}^n \|\omega_\imath\|^{d_\imath-l_\imath}_{s,A_\imath}\Big).\end{aligned}$$

Since $d_\imath < l_\imath$, hence, condition (τ_2) of Theorem 1.1 holds. Then all the assumptions of Theorem 1.1 are then satisfied. In addition, for every function $H : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfiying (G_0) the function:

$$\Gamma(\omega) := \int_\Omega H(x,\omega_1,\dots,\omega_n)dx,$$

is well defined, continuously Gateaux differentiable on $\mathbb{X}$, with a compact derivative provided by

$$\Gamma'(\omega)(v) = \int_\Omega \sum_{\imath=1}^n H_{\omega_\imath}(x,\omega_1,\dots,\omega_n)v_\imath(x)dx.$$

Thus, all the conditions of Theorem 1.1 are satisfied. Also, the solution of the following equation

$$\Psi'(\omega) - \lambda\mathscr{F}'(\omega) - \mu\Gamma'(\omega) = 0 \tag{3.7}$$

are exactly the weak solutions of (1.1). As a result, Theorem1.1 leads to the conclusion. □

Proof of Theorem 3.2

First, let fix $\bar{\lambda} \in \mathscr{D}$ and for all $\mu \ge 0$, assume that $\mu < \mu_\lambda$. Since $\bar{\lambda} < \frac{1}{\left(\sum_{\imath=1}^n \left(\frac{C}{\theta_0\alpha_0}\right)^{\frac{1}{m_\imath}}\right)^{\bar{m}} B}$, one has $\mu_\lambda > 0$. Fix $\bar{\mu} \in [0, \mu_\lambda[$. Therefore, from (3.4) we infer that

$$\frac{1}{L} \ge \left(\sum_{\imath=1}^n \Big(\frac{C}{\theta_0\alpha_0}\Big)^{\frac{1}{m_\imath}}\right)^{\bar{m}}, \quad \text{for all } 1 \le \imath \le n. \tag{3.8}$$

Put $a := \frac{1}{D}$ and $b := \frac{1}{\frac{\bar{\mu}}{\bar{\lambda}}H_0 + B\left(\sum_{\imath=1}^n \left(\frac{C}{\theta_0\alpha_0}\right)^{\frac{1}{m_\imath}}\right)^{\bar{m}}}$. If $H_0 = 0$, clearly, $a = \frac{1}{D}$, $b = \frac{1}{\left(\sum_{\imath=1}^n \left(\frac{C}{\theta_0\alpha_0}\right)^{\frac{1}{m_\imath}}\right)^{\bar{m}} B}$.
Since, the fact that $\bar{\lambda} \in \mathscr{D}$ and (3.8) we obtain,

$$\bar{\lambda} > \frac{1}{LD\Big(\sum_{\imath=1}^n \Big(\frac{C}{\theta_0\alpha_0}\Big)^{\frac{1}{m_\imath}}\Big)^{\bar{m}}} > \frac{1}{D}.$$

Then, $\bar{\lambda} \in]a, b[$. If $H_0 \neq 0$, since $\bar{\mu} < \mu_\lambda$, which implies that

$$\bar{\mu} H_0 + \bar{\lambda} B\Big(\sum_{\imath=1}^{n} \Big(\frac{C}{\theta_0 \alpha_0}\Big)^{\frac{1}{m_\imath}}\Big)^{\bar{m}} < 1,$$

then

$$\bar{\lambda} < \frac{1}{\frac{\bar{\mu}}{\bar{\lambda}} H_0 + B\Big(\sum_{\imath=1}^{n} \Big(\frac{C}{\theta_0 \alpha_0}\Big)^{\frac{1}{m_\imath}}\Big)^{\bar{m}}}.$$

Then $\bar{\lambda} < b$. Since $\bar{\lambda} > \frac{1}{D}$, then one has $\bar{\lambda} \in]a, b[$.

On the other hand, put $Q(w) = F(x, w) + \dfrac{\bar{\mu}}{\bar{\lambda}} H(x, w)$, for all $x \in \Omega$, $w \in \mathbb{R}^n$ and we introduce the functional $\mathscr{Q} : \mathbb{X} \to \mathbb{R}$ for each $u \in \mathbb{X}$, as follows

$$\mathscr{Q}(\omega) = \int_\Omega F(x, \omega) dx + \frac{\bar{\mu}}{\bar{\lambda}} \int_\Omega H(x, \omega) dx.$$

It is generally known that $\mathscr{Q}$ is a Gâteaux differentiable functional and sequentially weakly upper semicontinuous, with a Gâteaux derivative at the point $u \in \mathbb{X}$, is the functional $\mathscr{Q}'(\omega) \in \mathbb{X}^*$ given by

$$\mathscr{Q}'(\omega)(v) = \int_\Omega \sum_{\imath=1}^{n} F_{\omega_\imath}(x, \omega_1, \ldots, \omega_n) v_\imath(x) dx + \frac{\bar{\mu}}{\bar{\lambda}} \int_\Omega \sum_{\imath=1}^{n} H_{\omega_\imath}(x, \omega_1, \ldots, \omega_n) v_\imath(x) dx.$$

Put $h_{\bar{\lambda}}(\omega) := \Psi(\omega) - \bar{\lambda}\mathscr{Q}(\omega)$.In view of (3.7), we can easily see that the weak solutions of the problem (1.1) are also weak solutions to the equation $h'_{\bar{\lambda}}(\omega) = 0$. Now, we want to show that $\delta < +\infty$. We can seek for weak solutions of problem (1.1) by applying Theorem 1.2. For that let $\{\zeta_k\}$ be a real sequence such that $\lim_{n\to\infty} \zeta_k = 0$. Then,

$$\begin{aligned} \lim_{k\to+\infty} \; & \frac{\displaystyle\int_\Omega \sup_{(\gamma_1,\ldots,\gamma_n)\in R(\zeta_k)} F(x, \gamma_1, \ldots, \gamma_n) dx}{\zeta_k^{\bar{m}}} \\ &= \lim_{\zeta\to 0^+} \frac{\displaystyle\int_\Omega \sup_{(\gamma_1,\ldots,\gamma_n)\in R(\zeta)} F(x, \gamma_1, \ldots, \gamma_n) dx}{\zeta^{\bar{m}}} \\ &= B < +\infty. \end{aligned} \tag{3.9}$$

Put $\tau_k = \dfrac{\zeta_k^{\bar{m}}}{\Big(\sum_{\imath=1}^{n} \Big(\frac{C}{\theta_0 \alpha_0}\Big)^{\frac{1}{m_\imath}}\Big)^{\bar{m}}}$ for all $k \in \mathbb{N}$. Then, by Lemmas 2.6 and Proposition 2.7, we can deduce that

$$\Psi^{-1}(]-\infty, \tau_k[) := \Big\{u = (\omega_1, \ldots, \omega_k) \in \mathbb{X} : \Psi(\omega) < \tau_k\Big\} \subseteq \Big\{u \in \mathbb{X} : \sum_{\imath=1}^{n} \hat{K}_\imath\Big(\mathscr{S}_\imath(\omega_\imath)\Big) < \tau_k\Big\}.$$

By (M_2) and Lemma 2.6, we have, for k large enough $(0 < \tau_k < 1)$,

$$\theta_0 \alpha_0 \|\omega_\imath\|_{s, A_\imath}^{m_\imath} < \tau_k,$$

and from (2.17) we have $\max_{x\in\overline{\omega}} |\omega_\imath(x)| \leq C\|\omega_\imath\|_{s;A_\imath}$. Then for all $x \in \Omega$

$$|\omega_\imath(x)| \leq \Big(\frac{C\tau_k}{\theta_0 \alpha_0}\Big)^{\frac{1}{m_\imath}}.$$

Thus,

$$\sum_{\imath=1}^{n} |\omega_\imath(x)| \leq \sum_{\imath=1}^{n} \Big(\frac{C\tau_k}{\theta_0 \alpha_0}\Big)^{\frac{1}{m_\imath}} \leq \tau_k^{\frac{1}{\bar{m}}} \sum_{\imath=1}^{n} \Big(\frac{C}{\theta_0 \alpha_0}\Big)^{\frac{1}{m_\imath}} = \zeta_k.$$

Then we have

$$\Psi^{-1}(]-\infty,\tau_k[) := \Big\{u \in \mathbb{X} : \sum_{\imath=1}^{n} |\omega_\imath(x)| < \zeta_k\Big\},$$

Consequently,

$$\sup_{u\in\Psi^{-1}(]-\infty,\tau_k[)} \mathscr{Q}(\omega) \leq \int_{\Omega} \sup_{(\gamma_1,\ldots,\gamma_n)\in R(\zeta_k)} Q(x,\gamma_1,\ldots,\gamma_n)dx.$$

Considering that $\Psi(0) = 0$ and $\mathscr{Q}(0) = 0$, for all $x \in \Omega$, for all $n \in \mathbb{N}$ one has

$$\begin{aligned}
\varpi(\tau_k) = &\inf_{u\in\Psi^{-1}(]-\infty,\tau_k[)} \frac{\sup_{v\in\Psi^{-1}(]-\infty,\tau_k[)} \mathscr{Q}(v) - \mathscr{Q}(\omega)}{\tau_k - \Psi(\omega)} \leq \frac{\sup_{v\in\Psi^{-1}(]-\infty,\tau_k[)} \mathscr{Q}(v)}{\tau_k} \\
\leq &\Big(\sum_{\imath=1}^{n}\Big(\frac{C}{\theta_0\alpha_0}\Big)^{\frac{1}{m_\imath}}\Big)^{\bar m} \frac{\int_{\Omega} \sup_{(\gamma_1,\ldots,\gamma_n)\in R(\zeta_k)} Q(x,\gamma_1,\ldots,\gamma_n)dx}{\zeta_k^{\bar m}} \\
= &\Big(\sum_{\imath=1}^{n}\Big(\frac{C}{\theta_0\alpha_0}\Big)^{\frac{1}{m_\imath}}\Big)^{\bar m} \frac{\int_{\Omega} \sup_{(\gamma_1,\ldots,\gamma_n)\in R(\zeta_k)} F(x,\gamma_1,\ldots,\gamma_n)dx}{\zeta_k^{\bar m}} \\
+ &\frac{\bar\mu}{\bar\lambda}\Big(\sum_{\imath=1}^{n}\Big(\frac{C}{\theta_0\alpha_0}\Big)^{\frac{1}{m_\imath}}\Big)^{\bar m} \frac{\int_{\Omega} \sup_{(\gamma_1,\ldots,\gamma_n)\in R(\zeta_k)} H(x,\gamma_1,\ldots,\gamma_n)dx}{\zeta_k^{\bar m}}.
\end{aligned}$$

hence, by (3.3), (3.5), (3.8) and (3.9) , we infer that

$$\begin{aligned}
\delta := \ & \liminf_{\tau\to 0^+} \varpi(\tau) \leq \liminf_{k\to\infty} \varpi(\tau_k) \\
\leq \ & \Big(\sum_{\imath=1}^{n}\Big(\frac{C}{\theta_0\alpha_0}\Big)^{\frac{1}{m_\imath}}\Big)^{\bar m} \lim_{k\to\infty} \frac{\int_{\Omega} \sup_{(\gamma_1,\ldots,\gamma_n)\in R(\zeta_k)} F(x,\gamma_1,\ldots,\gamma_n)dx}{\zeta_k^{\bar m}} \\
+ \ & \frac{\bar\mu}{\bar\lambda}\Big(\sum_{\imath=1}^{n}\Big(\frac{C}{\theta_0\alpha_0}\Big)^{\frac{1}{m_\imath}}\Big)^{\bar m} \lim_{k\to\infty} \frac{\int_{\Omega} \sup_{(\gamma_1,\ldots,\gamma_n)\in R(\zeta_k)} H(x,\gamma_1,\ldots,\gamma_n)dx}{\zeta_k^{\bar m}} \\
\leq \ & \frac{1}{L} \lim_{k\to\infty} \frac{\int_{\Omega} \sup_{(\gamma_1,\ldots,\gamma_n)\in R(\zeta_k)} F(x,\gamma_1,\ldots,\gamma_n)dx}{\zeta_k^{\bar m}} + \frac{\bar\mu}{\bar\lambda} H_0 \\
< \ & \limsup_{(\zeta_1,\ldots,\zeta_n)\to(0^+,\ldots,0^+)} \frac{\int_{\Omega} F(x,\zeta_1,\ldots,\zeta_n)dx}{\sum_{\imath=1}^{n} \hat{K}_\imath(|\Omega|\varrho\zeta_\imath^{m_\imath})} + \frac{\bar\mu}{\bar\lambda} H_0 = D + \frac{\bar\mu}{\bar\lambda} H_0 < +\infty.
\end{aligned} \tag{3.10}$$

Moreover, since H is nonnegative we have

$$\limsup_{(\zeta_1,\ldots,\zeta_n)\to(0^+,\ldots,0^+)} \frac{\int_{\Omega} Q(x,\zeta_1,\ldots,\zeta_n)dx}{\sum_{\imath=1}^{n} \hat{K}_\imath(|\Omega|\varrho\zeta_\imath^{m_\imath})} \geq \limsup_{(\zeta_1,\ldots,\zeta_n)\to(0^+,\ldots,0^+)} \frac{\int_{\Omega} F(x,\zeta_1,\ldots,\zeta_n)dx}{\sum_{\imath=1}^{n} \hat{K}_\imath(|\Omega|\varrho\zeta_\imath^{m_\imath})}, \tag{3.11}$$

Therefore, from assumption (3.3), (3.10) and (3.11), we observe that

$$\bar\lambda \subseteq \Big]a, b\Big[\subseteq \Big]\lambda_1, \lambda_2\Big[\subseteq \Big]0, \frac{1}{\delta}\Big[,$$

where

$$\lambda_1 = \frac{1}{\limsup_{(\zeta_1,\ldots,\zeta_n)\to(0^+,\ldots,0^+)} \frac{\int_{\Omega} Q(x,\zeta_1,\ldots,\zeta_n)dx}{\sum_{\imath=1}^n \hat{K}_\imath(|\Omega|\varrho\zeta_\imath^{m_\imath})}}$$

and

$$\lambda_2 = \frac{1}{\left(\sum_{\imath=1}^n \left(\frac{C}{\theta_0\alpha_0}\right)^{\frac{1}{m_\imath}}\right)^{\bar{m}} \liminf_{\zeta\to 0^+} \frac{\int_{(\gamma_1,\ldots,\gamma_n)\in R(\zeta)} Q(x,\gamma_1,\ldots,\gamma_n)dx}{\zeta^{\bar{m}}}}.$$

So $\mathscr{D} \subseteq]0, \frac{1}{\delta}[$. For a fixed $\bar{\lambda} \in \mathscr{D}$, the functional $h_{\bar{\lambda}}$ is unbounded from below. Indeed, since

$$\frac{1}{\bar{\lambda}} < D,$$

consider n positive real sequences $\{\zeta_{\imath,k}\}$ and $\eta > 0$ such that $\zeta_{\imath,k} \to 0$ as $k \to +\infty$ and

$$\frac{1}{\bar{\lambda}} < \eta < \frac{\int_{\Omega} F(x,\zeta_{1,k},\ldots,\zeta_{n,k})dx}{\sum_{\imath=1}^n \hat{K}_\imath(|\Omega|\varrho\zeta_{\imath,k}^{m_\imath})}. \tag{3.12}$$

Let $\{\omega_k(x) = (\zeta_{1,k},\ldots,\zeta_{n,k})\} \subseteq \mathbb{X}$ be a sequence for all $x \in \Omega$. From (3.6), we have

$$\Psi(\omega_k(x)) = \sum_{\imath=1}^n \hat{K}_\imath\Big(\mathscr{S}_\imath(\zeta_{\imath,k})\Big) = \sum_{\imath=1}^n \hat{K}_\imath\Big(|\Omega|A_\imath(\zeta_{\imath,k})\Big).$$

Moreover, from (3.2) and since $\lim_{k\to+\infty} \zeta_{\imath,k} = 0$, there exist $\xi > 0$ and $n_\imath \in \mathbb{N}$ such that $\zeta_{\imath,k} \in (0,\xi)$ and

$$A_\imath(\zeta_{\imath,k}) < \varrho\zeta_{\imath,k}^{m_\imath} \quad for\, all\ n \geq n_\imath.$$

Since $\hat{K}$ is increasing, it follows that, for all $n \geq \max\{n_1,\ldots,n_2\}$, we have

$$\Psi(\omega_k(x)) = \sum_{\imath=1}^n \hat{K}_\imath\Big(|\Omega|A_\imath(\zeta_{\imath,k})\Big) \leq \sum_{\imath=1}^n \hat{K}_\imath\Big(\varrho|\Omega|\zeta_{\imath,k}^{m_\imath}\Big). \tag{3.13}$$

Moreover, since H is nonnegative we have

$$\mathscr{Q}(\omega_k(x)) \geq \int_{\Omega} F(x,\zeta_{1,k},\ldots,\zeta_{n,k})dx \tag{3.14}$$

By (3.12), (3.13) and (3.14), we have

$$\begin{aligned} h_{\bar{\lambda}}(\omega_k(x)) &= \Psi(\omega_k(x)) - \bar{\lambda}\mathscr{Q}(\omega_k(x)) \\ &\leq \sum_{\imath=1}^n \hat{K}_\imath\Big(\varrho|\Omega|\zeta_{\imath,k}^{m_\imath}\Big) - \bar{\lambda}\int_{\Omega} F(x,\zeta_{1,k},\ldots,\zeta_{n,k})dx \\ &< (1-\lambda\eta)\sum_{\imath=1}^n \hat{K}_\imath\Big(\varrho|\Omega|\zeta_{\imath,k}^{m_\imath}\Big). \end{aligned}$$

Since $\frac{1}{\bar{\lambda}} < \eta$, then $1 - \lambda\eta < 0$ which infer that $h_{\bar{\lambda}}(\omega_k(x)) < 0 = h_{\bar{\lambda}}(0,\ldots,0)$. Then $(0,\ldots,0)$ isn't a local minimum of $h_{\bar{\lambda}}$. Thus, owing to the fact that $(0,\ldots,0)$ is the unique global minimum of Ψ, there exists a sequence $\{\omega_k\}$ of pairwise distinct critical points of $h_{\bar{\lambda}}$ such that $\lim_{k\to\infty} \|\omega_k\||_{s,A_\imath} = 0$, and this completes the proof.

4. Example

In this section, we point out certain examples of functions φ, $A(\gamma)$, K, H and F for which the results of this paper can be applied.

Let $\Omega \in \mathbb{R}^3$ be a bounded domain with $|\Omega| = 1$ and $\imath = 1, 2$. We can take K, due to Kirchhoff, as

$$K_\imath(\gamma) = a_\imath + b_\imath \gamma^{\beta_\imath - 1}, \quad a_\imath, b_\imath \geq 0,\ a_\imath + b_\imath > 0,\ \gamma \geq 0,\ \beta_\imath \geq 1. \tag{4.1}$$

and

$$\begin{cases} \beta_\imath \in (1, +\infty), & if\ b_\imath > 0, \\ \beta_\imath = 1, & if\ b_\imath = 0 \end{cases} \tag{4.2}$$

So we can see that,

$$\hat{K}_\imath(\gamma) = a_\imath \gamma + \frac{b_\imath}{\beta_\imath} \gamma^{\beta_\imath}$$

Further, it is clear that

$$K_\imath(\gamma) = a_\imath + b_\imath \gamma^{\beta_\imath - 1} \geq a_\imath > 0 \quad \forall \gamma \geq 0$$

and

$$\hat{K}_\imath(\gamma) = \int_0^\gamma K_\imath(r) dr \geq \frac{1}{\beta_\imath} K_\imath(\gamma)\gamma \quad \forall \gamma \geq 0.$$

Hence, (M_1) and (M_2) holds true (choose $\theta_\imath = \frac{1}{\beta_\imath}$). At this point, we take $a_1 = b_1 = 1$, $\beta_1 = \beta_2 = 1$ and $b_1 = b_2 = 0$. Now take

$$\varphi_1(\gamma) = \begin{cases} \frac{|\gamma|^4 \gamma}{\log(1+|\gamma|)} & if\ \gamma \neq 0, \\ 0 & if\ \gamma = 0, \end{cases} \quad and \quad \varphi_2(\gamma) = \log(1 + |\gamma|^2)|\gamma|^2\gamma, \quad \gamma \in \mathbb{R},$$

Similar to Remark 3.6 in [22], we have $l_1 = 5 < m_1 = 6$ and $l_2 = 4 < m_2 = 6$. Thus, the condition (1.4) is satisfied. Also we deduce that $\bar{m} = \max m_\imath = 6$. Moreover, owing to

$$\lim_{\gamma \to 0^+} \frac{1}{\gamma^5} \int_0^\gamma \frac{|r|^4 r}{\log(1 + |r|)} dr = \frac{1}{5}, \quad and \quad \lim_{\gamma \to 0^+} \frac{1}{\gamma^4} \int_0^t \log(1 + |r|^2)|r|^2 r = 0,$$

the condition (3.2) is also fulfilled (choose $\varrho = \frac{1}{n} = \frac{1}{3}$). Also we deduce that $L = \dfrac{\alpha_0}{2^8 C}$.
Let $F : \mathbb{R}^2 \to [0, \infty)$ be a continuous function defined by

$$F(r, \gamma) = \begin{cases} r^6(1 + \sin(\ln(1 + |\gamma|))) & if\ (r, \gamma) \neq (0, 0), \\ 0 & if (r, \gamma) = (0, 0), \end{cases}$$

and

$$H(r, \gamma) = \begin{cases} (1 + \cos(|r|)))\gamma^6 e^{-t} & if\ (r, \gamma) \neq (0, 0), \\ 0 & if (r, \gamma) = (0, 0). \end{cases}$$

Then,

$$B = \liminf_{\zeta \to 0^+} \frac{\int_\Omega \max_{|r|+|\gamma| \leq \zeta} F(r, \gamma) dx}{\zeta^6} = |\Omega| \liminf_{\zeta \to 0^+} \frac{\max_{|r|+|\gamma| \leq \zeta} F(r, \gamma)}{\zeta^6} = 2,$$

$$D = \limsup_{(r,\gamma) \to (0^+, 0^+)} \frac{\int_\Omega F(x, r, \gamma) dx}{\hat{K}_1(\frac{1}{3} r^6) + \hat{K}_2(\frac{1}{3}\gamma^6)} = |\Omega| \limsup_{(r,\gamma) \to (0^+, 0^+)} \frac{F(r, \gamma)}{\frac{1}{3} r^6 + \frac{1}{3}\gamma^6} = 3.$$

$$H_0 = \frac{1}{L} \liminf_{\zeta \to 0^+} \frac{\int_{\Omega} \max_{|r|+|\gamma| \le \zeta} H(r,\gamma) dx}{\zeta^6} = \frac{|\Omega| 2^8 C}{\alpha_0} \liminf_{\zeta \to 0^+} \frac{\max_{|r|+|\gamma|\le\zeta} H(r,\gamma)}{\zeta^6} = \frac{2^9 C}{\alpha_0},$$

$$\mu_\lambda = \frac{1}{H_0}\Big(1 - \lambda D\Big(\sum_{\imath=1}^{n}\Big(\frac{C}{\theta_0 \alpha_0}\Big)^{\frac{1}{m_\imath}}\Big)^{\bar{m}}\Big) = \frac{\alpha_0}{2^9 C}\Big(1 - 3\lambda \frac{2^8 C}{\alpha_0}\Big).$$

Thus, for all $\bar{\mu} \in [0, \mu_\lambda[$ and $\bar{\lambda} \in \mathscr{D} =]\frac{1}{3}, \frac{\alpha_0}{2^9 C}[$, with this condition $\frac{2^8}{3} < \frac{\alpha_0}{C}$. Then for every $\lambda \in]a = \frac{1}{D} = \frac{1}{3}, b = \frac{\bar{\lambda}\alpha_0}{2^9 C(\bar{\mu}+\bar{\lambda})}[$ and $\mu \in [0, \mu_{h,\lambda}[$ the following system:

$$\begin{cases} -K_1\Big(\mathscr{S}_1(\omega_1)\Big)\Big((-\Delta)^s_{\frac{|.|^4}{\log(1+|.|)}} \omega_1 + \dfrac{|\omega_1|^4 \omega_1}{\log(1+|\omega_1|)}\Big) = 6\lambda \omega_1^5 (1 + \sin(\ln(1+|\omega_2|))) \\ \qquad\qquad -\mu \omega_2^6 e^{-\omega_2} \sin(|\omega_1|) \ in\, \Omega, \\ -K_2\Big(\mathscr{S}_2(\omega_2)\Big)\Big((-\Delta)^s_{\log(1+|.|^2)|.|^2} \omega_2 + \log(1+|\omega_2|^2)|\omega_2|^2\omega_2\Big) = \lambda \dfrac{\omega_1^6}{1+\omega_2} \cos(\ln(1+|\omega_2|)) \\ \qquad\qquad +\mu \omega_2^5 e^{-\omega_2}(6-\omega_2)(1+\cos(|\omega_1|)) \ in\, \Omega, \\ \omega_1 = \omega_2 = 0 \qquad\qquad on\ \mathbb{R}^3 \backslash \Omega, \end{cases}$$

admits a sequence of pairwise distinct weak solutions which strongly converges to zero in $W_0^s \mathbb{L}_{A_1}(\Omega) \times W_0^s \mathbb{L}_{A_2}(\Omega)$.

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