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Existence and finite-time stability results for a class of nonlinear Hilfer fuzzy fractional differential equations with time-delays

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Abstract. In the current paper, we mainly investigate a novel class of nonlinear Hilfer fuzzy fractional differential equations (NHFFDEs) with time-delays. Firstly, using Laplace transform, we convert the system under consideration into an analogous integral system. Secondly, using Schauder's and Banach's fixed point theorems, the existence and uniqueness results of solutions for NHFFDEs are then established. Additionally, we explore the finite-time stability result of solution for the system under consideration.

1. Introduction

Nonlinear Hilfer fuzzy fractional differential equations with time-delays are a type of differential equation that combines several mathematical concepts, including fractional calculus, fuzzy logic [24] and time delays. Fractional calculus deals with derivatives and integrals of non-integer orders, while fuzzy logic is a type of logic that deals with uncertainty and imprecision. Time delays are also an important factor in many real-world systems, as they can cause oscillations and instability.
The Hilfer derivative [15] is a particular type of fractional derivative that has been shown to be a more accurate representation of many physical phenomena than the traditional Riemann-Liouville derivative. It has been widely used in modeling anomalous diffusion, viscoelasticity and other systems that exhibit power-law behavior. NHFFDEs with time-delays can be used to model a wide range of physical and engineering systems, including biological systems, chemical reactors, electrical circuits and many others. The solutions to these equations can provide insights into the behavior of these systems and can be used to make predictions and design control strategies. On the other hand, stability analysis is a fundamental aspect of mathematical analysis that is a crucial in avariety of engineering and science fields. Following that, many authors investigated various types of stability problems for different kinds of fractional differential equation and fuzzy differential equations using different methods (see [1]-[3],[14],[20],[17],[18],[21],[22], [6]-[12], [4]). Also, nany scholars have recently looked more into and explored the topic of existence and uniqueness for the solutions to linear and nonlinear fuzzy fractional differential equations in various aspects (see [5],[13],[16],[19]).

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Overall, the study of NHFFDEs is an important area of research that has the potential to impact many fields and applications. The novelties and main contributions of this manuscript are:

- We introduce a new kind of nonlinear Hilfer fuzzy fractional differential equations with time-delays.

- Existence result for the considered system is investigated by applying Schauder's fixed point theorem under the weaker non-Lipschitz condition.

- Uniqueness result is established by applying the Banach's contraction mapping.

- The finite-time stability is obtained which firstly promotes the further development of this type stability in fuzzy space, and secondly fills the gap of finite-time stability theory in the field of NHFFDEs.

The rest of the paper is organized as follows. In Section 2, We introduce some essential definitions and lemmas. The existence and uniqueness results for the NHFFDEs are given in Section 3. Afterwards, finite-time stability result for the considered system is established in Section 4. The last section is where you come to a conclusion.

2. Preliminaries

The definitions and lemmas that are utilized throughout this papeer will be introduced in this part.

Definition 2.1. *[22] The set of fuzzy subsets of* $\mathbb{R}^n$ *is denoted by* $\mathbf{E^n} := \{\Upsilon : \mathbb{R}^n \longrightarrow [0,1]\}$ *which satisfies:*

(i) Υ *is upper semicontinous on* $\mathbb{R}^n$,

(ii) Υ *is fuzzy convex, i.e, for* $0 \leq \lambda \leq 1$

$$\Upsilon\Big(\lambda z_1 + (1-\lambda)z_2\Big) \geq min\left\{\Upsilon(z_1), \Upsilon(z_2)\right\},\ \forall z_1, z_2 \in \mathbb{R}^n,$$

(iii) $[\Upsilon]^0 = \overline{\{z \in \mathbb{R}^n : \Upsilon(z) > 0\}}$ *is compact,*

(iv) Υ *is normal, i.e,* $\exists z_0 \in \mathbb{R}^n$ *such that* $\Upsilon(z_0) = 1$.

Remark 2.2. $\mathbf{E^n}$ *is called the space of fuzzy number.*

Definition 2.3. *[22] The p-level set of* $\Upsilon \in \mathbf{E^n}$ *is defined by:*
For $p \in (0,1]$, *we have* $[\Upsilon]^p = \{z \in \mathbb{R}^n | \Upsilon(z) \geq q\}$ *and for* $p = 0$ *we have* $[\Upsilon]^0 = \overline{\{z \in \mathbb{R}^n | \Upsilon(z) > 0\}}$.

Remark 2.4. *From Definition 2.1, it follows that the p-level set* $[\Upsilon]^p$ *of* Υ, *is a nonempty compact interval and* $[\Upsilon]^p = [\underline{\Upsilon}(p), \overline{\Upsilon}(p)]$. *Moreover,* $len([\Upsilon]^p) = \overline{\Upsilon}(p) - \underline{\Upsilon}(p)$.

Definition 2.5. *[22] For addition and scalar multiplication in fuzzy set space* $\mathbf{E^n}$, *we have*

$$[\Upsilon_1 + \Upsilon_2]^p = [\Upsilon_1]^p + [\Upsilon_2]^p = \{z_1 + z_2 \mid z_1 \in [\Upsilon_1]^p, z_2 \in [\Upsilon_2]^p\},$$

and

$$[\alpha\Upsilon]^p = \alpha[\Upsilon]^p = \{\alpha z \mid z \in [\Upsilon]^p\},$$

for all $p \in [0,1]$.

Definition 2.6. *[22] The Hausdorff distance is given by*

$$\mathbf{D}_\infty\Big(\Upsilon_1,\Upsilon_2\Big) = \sup_{0\le p\le 1}\Big\{|\underline{\Upsilon}_1(p)-\underline{\Upsilon}_2(p)|, |\overline{\Upsilon}_1(p)-\overline{\Upsilon}_2(p)|\Big\},$$
$$= \sup_{0\le p\le 1}\mathcal{D}_H\Big([\Upsilon_1]^p,[\Upsilon_2]^p\Big).$$

Remark 2.7. $\mathbf{E^n}$ *is complet metric space with the above definition (see [22]) and we have the following properties of* $\mathbf{D}_\infty$*:*

$$\mathbf{D}_\infty\Big(\Upsilon_1+\Upsilon_3,\Upsilon_2+\Upsilon_3\Big) = \mathbf{D}_\infty\Big(\Upsilon_1,\Upsilon_2\Big),$$
$$\mathbf{D}_\infty\Big(\lambda\Upsilon_1,\lambda\Upsilon_2\Big) = |\lambda|\mathbf{D}_\infty\Big(\Upsilon_1,\Upsilon_2\Big),$$
$$\mathbf{D}_\infty\Big(\Upsilon_1,\Upsilon_2\Big) \le \mathbf{D}_\infty\Big(\Upsilon_1,\Upsilon_3\Big)+\mathbf{D}_\infty\Big(\Upsilon_3,\Upsilon_2\Big),$$

for all $\Upsilon_1,\Upsilon_2,\Upsilon_3\in\mathbf{E^n}$ *and* $\lambda\in\mathbb{R}^n$.

Definition 2.8. *[22] Let* Υ_1, $\Upsilon_2\in\mathbf{E^n}$, *if there exists* $\Upsilon_3\in\mathbf{E^n}$ *such that* $\Upsilon_1=\Upsilon_2+\Upsilon_3$, *then* Υ_3 *is called the Hukuhara difference of* Υ_1 *and* Υ_2 *noted by* $\Upsilon_1\ominus\Upsilon_2$.

Definition 2.9. *[5] The generalized Hukuhara difference (gH-difference) of* $\Upsilon_1,\Upsilon_2\in\mathbf{E^n}$ *is defined as follows:*

$$\Upsilon_1\ominus_{gH}\Upsilon_2=\Upsilon_3\Leftrightarrow\begin{cases}(i)\ \ \Upsilon_1=\Upsilon_2+\Upsilon_3,\ if\quad len([\Upsilon_1]^p)\ge len([\Upsilon_2]^p).\\ (ii)\ \ \Upsilon_2=\Upsilon_1+(-1)\Upsilon_3,\ if\quad len([\Upsilon_2]^p)\ge len([\Upsilon_1]^p).\end{cases}$$

Definition 2.10. *[22] Let a fuzzy function* $\Upsilon:[a,b]\longrightarrow\mathbf{E^n}$. *If for every* $p\in[0,1]$, *the function* $u\longmapsto len[\Upsilon(u)]^p$ *is increasing (decreasing) on* $[a,b]$, *then* Υ *is called increasing (decreasing) on* $[a,b]$.

Remark 2.11. *If* Υ *is increasing or decreasing, then we say that* Υ *is monotone on* $[a,b]$.

Notation:
- $C\Big([a,b],\mathbf{E^n}\Big)$ denote the set of all continuous fuzzy functions.
- $AC\Big([a,b],\mathbf{E^n}\Big)$ denote the set of all absolutely continuous fuzzy functions on $[a,b]$ with value in $\mathbf{E^n}$.
- $AC^1\Big([a,b],\mathbf{E^n}\Big)$ the set of all absolutely continuously differentiable fuzzy functions on $[a,b]$ with value in $\mathbf{E^n}$.
- For $\zeta\in(0,1)$, let $C_\zeta\Big([a,b],\mathbf{E^n}\Big)$ denote the space of continuous functions defined by

$$C_\delta\Big([a,b],\mathbf{E^n}\Big):=\Big\{\mathbf{z}\in(a,b]\longrightarrow\mathbf{E^n}:\ \ (u-a)^{1-\delta}\mathbf{z}(u)\in C[a,b]\Big\}.$$

- Denote by $L\Big([a,b],\mathbf{E^n}\Big)$ the set of all fuzzy functions $\mathbf{z}:[a,b]\longrightarrow\mathbf{E^n}$ such that $u\mapsto\mathbf{D}_\infty[\mathbf{z}(u),\hat{0}]$ belong to $L^1[a,b]$.

Definition 2.12. *[5] The Riemann–Liouville fractional integral of order* $\gamma>0$ *of a continuous function is defined by*

$$\mathcal{I}^\gamma_{a+}\mathbf{z}(u)=\frac{1}{\Gamma(\gamma)}\int_a^u(u-s)^{\gamma-1}\mathbf{z}(s)ds.$$

Definition 2.13. *[5] The Riemann–Liouville fractional derivative of order* $\gamma>0$ *of a continuous function* $\mathbf{z}$ *is given by*

$${}^{RL}\mathcal{D}^\gamma_{a^+}\mathbf{z}(u):=D^n\mathcal{I}^{n-\gamma}_{a^+}\mathbf{z}(u),$$
$$=\frac{1}{\Gamma(n-\gamma)}\Big(\frac{d}{du}\Big)^n\int_a^u(u-s)^{n-\gamma-1}\mathbf{z}(s)ds,$$

where $n=[\gamma]+1$.

For $\mathbf{z} \in L\Big([a,b], \mathbf{E^n}\Big)$, we define the Riemann–Liouville fractional integral of order γ of the fuzzy function $\mathbf{z}$:

$$\mathbf{z}_\gamma(u) := \mathcal{I}^{\gamma}_{a^+}\mathbf{z}(u) = \frac{1}{\Gamma(\gamma)}\int_a^u (u-s)^{\gamma-1}\mathbf{z}(s)ds, \quad u \geq a.$$

Since $[\mathbf{z}(u)]^p = \Big[\underline{\mathbf{z}}(u,p), \overline{\mathbf{z}}(u,p)\Big]$, we can define the fuzzy Riemann–Liouville fractional integral of fuzzy function $\mathbf{z}$ based on lower and upper functions

$$[\mathcal{I}^{\gamma}_{a^+}\mathbf{z}(u)]^p = \Big[\mathcal{I}^{\gamma}_{a^+}\underline{\mathbf{z}}(u,p), \mathcal{I}^{\gamma}_{a^+}\overline{\mathbf{z}}(u,p)\Big], \qquad u \geq a.$$

Where

$$\mathcal{I}^{\gamma}_{a^+}\underline{\mathbf{z}}(u,p) = \frac{1}{\Gamma(\gamma)}\int_a^u (u-s)^{\gamma-1}\underline{\mathbf{z}}(s,p)ds,$$

and

$$\mathcal{I}^{\gamma}_{a^+}\overline{\mathbf{z}}(u,p) = \frac{1}{\Gamma(\gamma)}\int_a^u (u-s)^{\gamma-1}\overline{\mathbf{z}}(s,p)ds.$$

It follows that the operator $\mathbf{z}_\gamma(u)$ is linear and bounded from $C\Big([a,b], \mathbf{E^n}\Big)$ to $C\Big([a,b], \mathbf{E^n}\Big)$.

Definition 2.14. *[17] The fuzzy Hilfer fractional derivative of order γ and parameter β of a function $\mathbf{z} \in C_{1-\delta}[a,b]$ is defined by*

$${}^{H}\mathcal{D}^{\gamma,\beta}_{a^+}\mathbf{z}(u) = \mathcal{I}^{\beta(n-\gamma)}_{a^+}\left(\frac{d}{du}\right)^n \mathcal{I}^{(1-\beta)(n-\gamma)}_{a^+}\mathbf{z}(u),$$

if the gH-derivative $\mathbf{z}'_{1-\delta}(u)$ exists, where $n-1 < \gamma < n$, and $0 \leq \beta \leq 1$.

Definition 2.15. *[17] Mittag-Leffler function with two parameter is defined as*

$$\mathfrak{M}_{\gamma,\beta}(u) = \sum_{j=0}^{\infty} \frac{u^j}{\Gamma(\gamma j+\beta)}, \quad \gamma,\beta > 0.$$

Particularly, when $\beta = 1$, two parameter will degenerate into one parameter function, i.e $\mathfrak{M}_{\gamma,1}(u) = \mathfrak{M}_{\gamma}(u)$.

Let $C\Big(\mathbf{I}, \mathbf{E^n}\Big)$ be the Banach space of all continuous process from $\mathbf{I} := [0,T]$ into $\mathbf{E^n}$ such that $\mathbf{D}_\infty(\zeta_1, \zeta_2) < \infty$.

Lemma 2.16. *Let $\gamma \in (0,1)$ and $0 \leq s < u$, we get*

$$u^\gamma - s^\gamma \leq (u-s)^\gamma.$$

Proof. We have

$$u^\gamma - s^\gamma = \gamma\int_s^u z^{\gamma-1}dz,$$

and

$$(u-s)^\gamma = \gamma\int_s^u (z-s)^{\gamma-1}dz.$$

Since $s \leq z \leq u$ and $-1 < \gamma - 1 < 0$, we have $z^{\gamma-1} \leq (z-s)^{\gamma-1}$. Then, $u^\gamma - s^\gamma \leq (u-s)^\gamma$. □

3. Existence and uniqueness results

The following nonlinear Hilfer fuzzy fractional differential equations with time-delays are discussed in this section.

$$\begin{cases} {}^{H}\mathcal{D}_{0^+}^{\gamma,\beta}\mathbf{z}(u) = A\mathbf{z}(u) + \mathfrak{H}\Big(u, \mathbf{z}(u), \mathbf{z}(u-\tau)\Big), & u \in \mathbf{I}, \\ \mathbf{z}(u) = \phi(u), \qquad u \in [-\tau, 0], & \\ \mathbf{I}_{0^+}^{1-\mu}\mathbf{z}(0) = \phi(0), \qquad \mu = \gamma + \beta - \gamma\beta. & \end{cases} \tag{1}$$

where ${}^{H}\mathcal{D}_{0^+}^{\gamma,\beta}$ is the Hilfer fractional derivative with $0 \le \gamma \le 1$, $\frac{1}{2} < \beta < 1$, A is an n-dimensional matrix and $\mathfrak{H} : \mathbf{I} \times \mathbf{E^n} \times \mathbf{E^n} \longrightarrow \mathbf{E^n}$ is continuous function, $\mathbf{z}(t)$ is fuzzy variable, $\tau \in \mathbb{R}^+$ represents the delay and $\phi : [-\tau, 0] \longrightarrow \mathbf{E^n}$ is a continuous function satisfying $\mathbf{D}_\infty[\phi(0), \hat{0}] < \infty$.

Lemma 3.1. *System* (1) *is equivalent to the following integral equation:*

$$\mathbf{z}(u) = \begin{cases} \phi(u), \quad -\tau \le u \le 0, \\ \mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)\phi(0) + \displaystyle\int_0^u \frac{\mathfrak{M}_{\beta,\beta}\Big(A(u-s)^\beta\Big)}{(u-s)^{1-\beta}} \mathfrak{H}\Big(s, \mathbf{z}(s), \mathbf{z}(s-\tau)\Big)ds, \quad u \in \mathbf{I}. \end{cases} \tag{2}$$

Proof. Following the Laplace transform on (1), we obtain

$$\mathcal{L}\Big[{}^{H}\mathcal{D}_{0^+}^{\gamma,\beta}\mathbf{z}(u)\Big]^p = A\mathcal{L}\Big[\mathbf{z}(u)\Big]^p + \mathcal{L}\Big[\mathfrak{H}\Big(u, \mathbf{z}(u), \mathbf{z}(u-\tau)\Big)\Big]^p,$$

then

$$\Big[s^\gamma \hat{\mathbf{z}}(s) - s^{\beta(\gamma-1)}\phi(0)\Big]^p = A\Big[\hat{\mathbf{z}}(u)\Big]^p + \Big[\hat{\mathfrak{H}}\Big(u, \mathbf{z}(u), \mathbf{z}(u-\tau)\Big)\Big]^p,$$

where $\hat{\mathbf{z}}(s)$ and $\hat{\mathfrak{H}}\Big(u, \mathbf{z}(u), \mathbf{z}(u-\tau)\Big)$ denote the Laplace transformation of $\mathbf{z}(s)$ and $\mathfrak{H}\Big(u, \mathbf{z}(u), \mathbf{z}(u-\tau)\Big)$ respectively. Therefore

$$\Big[\hat{\mathbf{z}}(s)\Big]^p = \frac{s^{\beta(\gamma-1)}\Big[\phi(0)\Big]^p}{s^\gamma I - A} + \frac{\Big[\hat{\mathfrak{H}}\Big(u, \mathbf{z}(u), \mathbf{z}(u-\tau)\Big)\Big]^p}{s^\gamma I - A},$$

where I denotes the identity matrix. Next, using the inverse Laplace transform, we get

$$\mathcal{L}^{-1}\Big[\hat{\mathbf{z}}(s)\Big]^p = \mathcal{L}^{-1}\Big\{\frac{s^{\beta(\gamma-1)}}{s^\gamma I - A}\Big\}\Big[\phi(0)\Big]^p + \mathcal{L}^{-1}\Big\{\frac{1}{s^\gamma I - A}\Big\} * \mathcal{L}^{-1}\Big[\hat{\mathfrak{H}}\Big(u, \mathbf{z}(u), \mathbf{z}(u-\tau)\Big)\Big]^p.$$

We derive the fuzzy Laplace transformation replacement in terms of the Mittag Leffler function, we get

$$\Big[\mathbf{z}(s)\Big]^p = \mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)\Big[\phi(0)\Big]^p + \int_0^u \frac{\mathfrak{M}_{\beta,\beta}\Big(A(u-s)^\beta\Big)}{(u-s)^{1-\beta}}\Big[\mathfrak{H}\Big(s, \mathbf{z}(s), \mathbf{z}(s-\tau)\Big)\Big]^p ds,$$

then, we get $\mathbf{z}(s) = \Big[\mathbf{z}(s)\Big]^p$, $\phi(0) = \Big[\phi(0)\Big]^p$ and $\mathfrak{H}\Big(s, \mathbf{z}(s), \mathbf{z}(s-\tau)\Big) = \Big[\mathfrak{H}\Big(s, \mathbf{z}(s), \mathbf{z}(s-\tau)\Big)\Big]^p$. Therefore, we obtain (2). □

We make the following hypotheses concerning the coefficients of the system under consideration:

($\mathcal{H}1$) For all $\varphi_1, \varphi_2, \psi_1, \psi_2 \in \mathbf{E^n}$ and for all $u \in \mathbf{I}$, we have

$$\mathbf{D}_\infty\left[\mathfrak{H}(u, \varphi_1, \psi_1), \mathfrak{H}(u, \varphi_2, \psi_2)\right] \le H\Big(u, \mathbf{D}_\infty[\varphi_1, \varphi_2], \mathbf{D}_\infty[\psi_1, \psi_2]\Big),$$

where $H : \mathbf{I} \times \mathbf{E^n} \times \mathbf{E^n} \longrightarrow \mathbf{E^n}$ is a monotone increasing, continuous and concave function with $H(u, 0, 0) = 0$ and $H(u, \mathbf{z}(u), \mathbf{z}(u)) = kH(u, \mathbf{z}(u))$, k is a constant.

($\mathcal{H}2$) For all $u \geq 0$, $\exists \lambda > 0$ such that

$$\mathbf{D}_\infty[\mathfrak{H}(u, \hat{0}, \hat{0}), \hat{0}] \leq \lambda.$$

($\mathcal{H}3$) For any $\varphi, \psi \in \mathbf{E^n}$, we suppose that there exists a function $h \in C(\mathbf{I}, \mathbf{E^n})$ such that

$$H\Big(u, \mathbf{D}_\infty[\varphi, \psi]\Big) \leq h(u)\mathbf{D}_\infty[\varphi, \psi].$$

We will now use Schauder's fixed point theorem to demonstrate our result.

Theorem 3.2. *Suppose that* $\mathfrak{H} : \mathbf{I} \times \mathbf{E^n} \times \mathbf{E^n} \longrightarrow \mathbf{E^n}$ *is continuous and satisfying the hypotheses* ($\mathcal{H}1$)-($\mathcal{H}3$). *Then, there exist at least a solution to the system* (1).

Proof. Consider the operator $\mathfrak{L}$ on $\mathcal{B}_h$ defined as follows

$$\mathfrak{L}(\mathbf{z}(u)) = \begin{cases} \phi(u), \quad u \in [-\tau, 0], \\ \mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)\phi(0) + \displaystyle\int_0^u \frac{\mathfrak{M}_{\beta,\beta}\Big(A(u-s)^\beta\Big)}{(u-s)^{1-\beta}} \mathfrak{H}(s, \mathbf{z}(s), \mathbf{z}(s-\tau))ds, \quad u \in \mathbf{I}. \end{cases} \tag{3}$$

To prove this result, we divide the subsequent proof into two steps.
Step 1: $\mathfrak{L}$ is completely continuous. For this, let us prove that:
ⓐ- $\mathfrak{L}$ is continuous. Indeed, for any integer $n \geq 1$, define $\mathbf{z}_n(u) = \phi(0)$ for all $u \in [-\tau, 0]$. For all $u \in \mathbf{I}$

$$\mathbf{z}_n(u) = \mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)\phi(0) + \int_0^u (u-s)^{\beta-1}\mathfrak{M}_{\beta,\beta}\Big(A(u-s)^\beta\Big)\mathfrak{H}(s, \mathbf{z}(s), \mathbf{z}(s-\tau))ds. \tag{4}$$

Let $\overline{\mathfrak{M}}_1 = \sup\limits_{0 \leq u \leq T} \mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)$, with the aid of the hypotheses ($\mathcal{H}1$)-($\mathcal{H}3$), we get

$$\begin{aligned} \mathbf{D}_\infty\Big[\mathfrak{L}(\mathbf{z}_n(u)), \mathfrak{L}(\mathbf{z}(u))\Big] &= \mathbf{D}_\infty\Big[\mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)\phi(0) + \int_0^u (u-s)^{\beta-1}\mathfrak{M}_{\beta,\beta}\Big(A(u-s)^\beta\Big)\mathfrak{H}(s, \mathbf{z}_n(s), \mathbf{z}_n(s-\tau))ds, \\ &\qquad \mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)\phi(0) + \int_0^u (u-s)^{\beta-1}\mathfrak{M}_{\beta,\beta}\Big(A(u-s)^\beta\Big)\mathfrak{H}(s, \mathbf{z}(s), \mathbf{z}(s-\tau))ds\Big], \\ &= \mathbf{D}_\infty\Big[\int_0^u \frac{\mathfrak{M}_{\beta,\beta}\Big(A(u-s)^\beta\Big)}{(u-s)^{1-\beta}}\mathfrak{H}(s, \mathbf{z}_n(s), \mathbf{z}_n(s-\tau))ds, \int_0^u \frac{\mathfrak{M}_{\beta,\beta}\Big(A(u-s)^\beta\Big)}{(u-s)^{1-\beta}}\mathfrak{H}(s, \mathbf{z}(s), \mathbf{z}(s-\tau))ds\Big], \\ &\leq \int_0^u \frac{\mathfrak{M}_{\beta,\beta}\Big(A(u-s)^\beta\Big)}{(u-s)^{1-\beta}}\mathbf{D}_\infty\Big[\mathfrak{H}(s, \mathbf{z}_n(s), \mathbf{z}_n(s-\tau)), \mathfrak{H}(s, \mathbf{z}(s), \mathbf{z}(s-\tau))\Big]ds, \\ &\leq \frac{T^\beta \overline{\mathfrak{M}}_1}{\beta} D_\infty\Big[\mathfrak{H}(s, \mathbf{z}_n(s), \mathbf{z}_n(s-\tau)), \mathfrak{H}(s, \mathbf{z}(s), \mathbf{z}(s-\tau))\Big]ds, \\ &\leq \frac{T^\beta \overline{\mathfrak{M}}_1}{\beta} H\Big(s, \mathbf{D}_\infty\Big[\mathbf{z}_n(s), \mathbf{z}(s)\Big], \mathbf{D}_\infty\Big[\mathbf{z}_n(s-\tau), \mathbf{z}(s-\tau)\Big]\Big), \end{aligned}$$

or, by using the definition of $\mathbf{D}_\infty$, we have

$$\begin{aligned}\mathbf{D}_\infty[\mathbf{z}_n(s-\tau),\mathbf{z}(s-\tau)] &= \sup_{0\le r\le 1}\max_{0\le s\le u}\{|\underline{\mathbf{z}_n}(s-\tau,r)-\underline{\mathbf{z}}(s-\tau,r)|,|\overline{\mathbf{z}_n}(s-\tau,r)-\overline{\mathbf{z}}(s-\tau,r)|\},\\ &= \sup_{0\le r\le 1}\max_{-\tau\le\mu\le u-\tau}\{|\underline{\mathbf{z}_n}(\mu,r)-\underline{\mathbf{z}}(\mu,r)|,|\overline{\mathbf{z}_n}(\mu,r)-\overline{\mathbf{z}}(\mu,r)|\},\\ &\le \sup_{0\le r\le 1}\max_{-\tau\le\mu\le 0}\{|\underline{\mathbf{z}_n}(\mu,r)-\underline{\mathbf{z}}(\mu,r)|,|\overline{\mathbf{z}_n}(\mu,r)-\overline{\mathbf{z}}(\mu,r)|\}\\ &\qquad+\sup_{0\le r\le 1}\max_{0\le\mu\le u-\tau}\{|\underline{\mathbf{z}_n}(\mu,r)-\underline{\mathbf{z}}(\mu,r)|,|\overline{\mathbf{z}_n}(\mu,r)-\overline{\mathbf{z}}(\mu,r)|\},\\ &\le \sup_{0\le r\le 1}\max_{0\le s\le\tau}\{|\underline{\mathbf{z}_n}(s,r)-\underline{\mathbf{z}}(s,r)|,|\overline{\mathbf{z}_n}(s,r)-\overline{\mathbf{z}}(s,r)|\}\\ &\qquad+\sup_{0\le r\le 1}\max_{\tau\le s\le u}\{|\underline{\mathbf{z}_n}(s,r)-\underline{\mathbf{z}}(s,r)|,|\overline{\mathbf{z}_n}(s,r)-\overline{\mathbf{z}}(s,r)|\},\\ &\le \sup_{0\le r\le 1}\max_{0\le s\le u}\{|\underline{\mathbf{z}_n}(s,r)-\underline{\mathbf{z}}(s,r)|,|\overline{\mathbf{z}_n}(s,r)-\overline{\mathbf{z}}(s,r)|\}=\mathbf{D}_\infty[\mathbf{z}_n(s),\mathbf{z}(s)].\end{aligned}$$

Then, using the hypothesis $(\mathcal{H}1)$, we have

$$\begin{aligned}\mathbf{D}_\infty\Big[\mathfrak{L}(\mathbf{z}_n(u)),\mathfrak{L}(\mathbf{z}(u))\Big] &\le \frac{T^\beta\overline{\mathfrak{M}}_1}{\beta}H\Big(s,\mathbf{D}_\infty\Big[\mathbf{z}_n(s),\mathbf{z}(s)\Big],\mathbf{D}_\infty\Big[\mathbf{z}_n(s),\mathbf{z}(s)\Big]\Big),\\ &\le \frac{T^\beta\overline{\mathfrak{M}}_1 k}{\beta}H\Big(s,\mathbf{D}_\infty\Big[\mathbf{z}_n(s),\mathbf{z}(s)\Big]\Big),\\ &\le \frac{T^\beta\overline{\mathfrak{M}}_1 k}{\beta}H\Big(u,\mathbf{D}_\infty\Big[\mathbf{z}_n(u),\mathbf{z}(u)\Big]\Big).\end{aligned}$$

Since H is continuous, we can conclude that $\mathbf{D}_\infty\Big[\mathfrak{L}(\mathbf{z}_n(u)),\mathfrak{L}(\mathbf{z}(u))\Big]\longrightarrow 0$ as $n\longrightarrow\infty$. Hence, $\mathfrak{L}$ is continuous.
ⓑ- We prove that there exists a positive constant ξ_1 and for all $\varsigma_1>0$ satisfying for all $\mathbf{z}(u)\in\mathcal{B}_{\varsigma_1}:=\Big\{\mathbf{z}(u)\in C\Big([-\tau,T],\mathbf{E^n}\Big)|\mathbf{D}_\infty[\mathbf{z}(u),\hat{0}]\le\varsigma_1\Big\}$ one has $\mathbf{D}_\infty\Big[\mathfrak{L}\Big(\mathbf{z}(u)\Big),\hat{0}\Big]\le\xi_1$. So, let $\overline{\mathfrak{M}}_2=\sup_{0\le u\le T}\mathfrak{M}_{\beta,\beta}(A(u-s)^\beta)$ and for all $u\in\mathbf{I}$ and $\mathbf{z}(u)\in\mathcal{B}_{\varsigma_1}$, we have

$$\begin{aligned}\mathbf{D}_\infty\Big[\mathfrak{L}\Big(\mathbf{z}(u)\Big),\hat{0}\Big] &= \mathbf{D}_\infty\Big[\mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)\phi(0)+\int_0^u(u-s)^{\beta-1}\mathfrak{M}_{\beta,\beta}\Big(A(u-s)^\beta\Big)\mathfrak{H}(s,\mathbf{z}(s),\mathbf{z}(s-\tau))ds,\hat{0}\Big],\\ &\le \mathbf{D}_\infty\Big[\mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)\phi(0),\hat{0}\Big]+\mathbf{D}_\infty\Big[\int_0^u\frac{\mathfrak{M}_{\beta,\beta}\Big(A(u-s)^\beta\Big)}{(u-s)^{1-\beta}}\mathfrak{H}(s,\mathbf{z}(s),\mathbf{z}(s-\tau))ds,\hat{0}\Big],\\ &\le \overline{\mathfrak{M}}_1\mathbf{D}_\infty\Big[\phi(0),\hat{0}\Big]+\int_0^u\frac{\mathfrak{M}_{\beta,\beta}\Big(A(u-s)^\beta\Big)}{(u-s)^{1-\beta}}\mathbf{D}_\infty\Big[\mathfrak{H}(s,\mathbf{z}(s),\mathbf{z}(s-\tau)),\hat{0}\Big]ds,\\ &\le \overline{\mathfrak{M}}_1\mathbf{D}_\infty\Big[\phi(0),\hat{0}\Big]+\frac{T^\beta\overline{\mathfrak{M}}_2}{\beta}\mathbf{D}_\infty\Big[\mathfrak{H}(s,\mathbf{z}(s),\mathbf{z}(s-\tau)),\hat{0}\Big].\end{aligned}$$

Since the function $\mathfrak{H}$ is continuous, there is a constant $N_{\mathfrak{H}}>0$ such that $\mathbf{D}_\infty\Big[\mathfrak{H}(u,\varphi,\psi),\hat{0}\Big]\le N_{\mathfrak{H}}$. Then,

$$\mathbf{D}_\infty\Big[\mathfrak{L}\Big(\mathbf{z}(u)\Big),\hat{0}\Big]\le\overline{\mathfrak{M}}_1\mathbf{D}_\infty\Big[\phi(0),\hat{0}\Big]+\frac{T^\beta\overline{\mathfrak{M}}_2}{\beta}:=\xi_1.$$

Therefore, for every $\mathbf{z}(u)\in\mathcal{B}_{\varsigma_1}$, we have $\mathbf{D}_\infty\Big[\mathfrak{L}\Big(\mathbf{z}(u)\Big),\hat{0}\Big]\le\xi_1$, this implies that $\mathfrak{L}(\mathcal{B}_{\varsigma_1})\subseteq\mathcal{B}_{\xi_1}$.
ⓒ- $\mathfrak{L}$ maps bounded set into equi-continuous set. Indeed, for each $\mathbf{z}(u)\in\mathcal{B}_{\varsigma_2}$ and $u_1,u_2\in\mathbf{I}$ such that

$0 \leq u_1 < u_2 \leq T$ and using Lemma 2.16 we have

$$\begin{aligned}
\mathbf{D}_\infty\Big[\mathfrak{L}\Big(\mathbf{z}(u_1)\Big), \mathfrak{L}\Big(\mathbf{z}(u_2)\Big)\Big] &= \mathbf{D}_\infty\Big[\mathfrak{M}_{\beta,\gamma(1-\beta)}(Au_1)\phi(0) + \int_0^{u_1} \frac{\mathfrak{M}_{\beta,\beta}\Big(A(u_1-s)^\beta\Big)}{(u_1-s)^{1-\beta}}\mathfrak{H}(s,\mathbf{z}(s),\mathbf{z}(s-\tau))ds \\
&\quad , \mathfrak{M}_{\beta,\gamma(1-\beta)}(Au_2)\phi(0) + \int_0^{u_2} \frac{\mathfrak{M}_{\beta,\beta}\Big(A(u_2-s)^\beta\Big)}{(u_2-s)^{1-\beta}}\mathfrak{H}(s,\mathbf{z}(s),\mathbf{z}(s-\tau))ds\Big], \\
&\leq \mathbf{D}_\infty\Big[\int_0^{u_1} \frac{\mathfrak{M}_{\beta,\beta}\Big(A(u_1-s)^\beta\Big)}{(u_1-s)^{1-\beta}}\mathfrak{H}(s,\mathbf{z}(s),\mathbf{z}(s-\tau))ds, \int_0^{u_2} \frac{\mathfrak{M}_{\beta,\beta}\Big(A(u_2-s)^\beta\Big)}{(u_2-s)^{1-\beta}}\mathfrak{H}(s,\mathbf{z}(s),\mathbf{z}(s-\tau))ds\Big], \\
&\leq \mathbf{D}_\infty\Big[\int_0^{u_1} \frac{\mathfrak{M}_{\beta,\beta}\Big(A(u_1-s)^\beta\Big)}{(u_1-s)^{1-\beta}}\mathfrak{H}(s,\mathbf{z}(s),\mathbf{z}(s-\tau))ds, \int_0^{u_1} \frac{\mathfrak{M}_{\beta,\beta}\Big(A(u_2-s)^\beta\Big)}{(u_2-s)^{1-\beta}}\mathfrak{H}(s,\mathbf{z}(s),\mathbf{z}(s-\tau))ds \\
&\quad + \int_{u_1}^{u_2} \frac{\mathfrak{M}_{\beta,\beta}\Big(A(u_2-s)^\beta\Big)}{(u_2-s)^{1-\beta}}\mathfrak{H}(s,\mathbf{z}(s),\mathbf{z}(s-\tau))ds\Big], \\
&\leq \mathbf{D}_\infty\Big[\int_0^{u_1} \frac{\mathfrak{M}_{\beta,\beta}\Big(A(u_1-s)^\beta\Big)}{(u_1-s)^{1-\beta}}\mathfrak{H}(s,\mathbf{z}(s),\mathbf{z}(s-\tau))ds, \int_0^{u_1} \frac{\mathfrak{M}_{\beta,\beta}\Big(A(u_2-s)^\beta\Big)}{(u_2-s)^{1-\beta}}\mathfrak{H}(s,\mathbf{z}(s),\mathbf{z}(s-\tau))ds\Big] \\
&\quad + \mathbf{D}_\infty\Big[\int_{u_1}^{u_2} \frac{\mathfrak{M}_{\beta,\beta}\Big(A(u_2-s)^\beta\Big)}{(u_2-s)^{1-\beta}}\mathfrak{H}(s,\mathbf{z}(s),\mathbf{z}(s-\tau))ds, \hat{0}\Big], \\
&\leq \int_0^{u_1} \mathbf{D}_\infty\Big[\frac{\mathfrak{M}_{\beta,\beta}\Big(A(u_1-s)^\beta\Big)}{(u_1-s)^{1-\beta}}\mathfrak{H}(s,\mathbf{z}(s),\mathbf{z}(s-\tau)), \frac{\mathfrak{M}_{\beta,\beta}\Big(A(u_2-s)^\beta\Big)}{(u_2-s)^{1-\beta}}\mathfrak{H}(s,\mathbf{z}(s),\mathbf{z}(s-\tau))\Big]ds \\
&\quad + \int_{u_1}^{u_2} \frac{\mathfrak{M}_{\beta,\beta}\Big(A(u_2-s)^\beta\Big)}{(u_2-s)^{1-\beta}}\mathbf{D}_\infty\Big[\mathfrak{H}(s,\mathbf{z}(s),\mathbf{z}(s-\tau)), \hat{0}\Big]ds, \\
&\leq \overline{\mathfrak{M}}_2 \int_0^{u_1} \mid (u_1-s)^{\beta-1} - (u_2-s)^{\beta-1} \mid \mathbf{D}_\infty\Big[\mathfrak{H}(s,\mathbf{z}(s),\mathbf{z}(s-\tau)), \hat{0}\Big]ds \\
&\quad + \frac{\overline{\mathfrak{M}}_2(u_2-u_1)^\beta}{\beta}\mathbf{D}_\infty\Big[\mathfrak{H}(s,\mathbf{z}(s),\mathbf{z}(s-\tau)), \hat{0}\Big], \\
&\leq \overline{\mathfrak{M}}_2\Big(\frac{(u_2-u_1)^\beta}{\beta} + \frac{u_2^\beta - u_1^\beta}{\beta}\Big)\mathbf{D}_\infty\Big[\mathfrak{H}(s,\mathbf{z}(s),\mathbf{z}(s-\tau)), \hat{0}\Big] + \frac{\overline{\mathfrak{M}}_2(u_2-u_1)^\beta}{\beta}\mathbf{D}_\infty\Big[\mathfrak{H}(s,\mathbf{z}(s),\mathbf{z}(s-\tau)), \hat{0}\Big].
\end{aligned}$$

Therefore, we get

$$\mathbf{D}_\infty\Big[\mathfrak{L}\Big(\mathbf{z}(u_1)\Big), \mathfrak{L}\Big(\mathbf{z}(u_2)\Big)\Big] \leq \frac{3\overline{\mathfrak{M}}_2(u_2-u_1)^\beta}{\beta}\mathbf{D}_\infty\Big[\mathfrak{H}(s,\mathbf{z}(s),\mathbf{z}(s-\tau)), \hat{0}\Big].$$

We have $\frac{3\overline{\mathfrak{M}}_2(u_2-u_1)^\beta}{\beta}$ is independent of $\mathbf{z}(u)$ and $\frac{3\overline{\mathfrak{M}}_2(u_2-u_1)^\beta}{\beta} \longrightarrow 0$ as $u_2 \longrightarrow u_1$. Then, we obtain

$$\mathbf{D}_\infty\Big[\mathfrak{L}\Big(\mathbf{z}(u_1)\Big), \mathfrak{L}\Big(\mathbf{z}(u_2)\Big)\Big] \longrightarrow 0.$$

It means that $\mathfrak{L}(\mathcal{B}_{\varsigma_2})$ is equi-continuous. Then, according to Arzela-Ascoli Theorem, $\mathfrak{L}$ is completely continuous.

Step 2: In this step, we will demonstrate that there is a closed, convex and bounded subset $\mathcal{B}_\xi = \Big\{\mathbf{z}(u) \in C\Big([-\tau, T], \mathbf{E^n}\Big) | \mathbf{D}_\infty[\mathbf{z}(u), \hat{0}] \leq \xi\Big\}$ such that $\mathfrak{L}(\mathcal{B}_\xi) \subseteq \mathcal{B}_\xi$. We know that $\mathcal{B}_\xi$ is a closed, convex and bounded subset of $C\Big([-\tau, T], \mathbf{E^n}\Big)$ for all $\xi > 0$. Suppose that for all $\xi > 0$, $\exists \mathbf{z}_\xi(u) \in \mathcal{B}_\xi$ such that $\mathfrak{L}(\mathbf{z}_\xi(u)) \notin \mathcal{B}_\xi$, that is

$\mathbf{D}_\infty\Big[\mathfrak{L}(\mathbf{z}_\xi(u)),\hat{0}\Big] > \xi$. Then

$$\begin{aligned}\xi < \mathbf{D}_\infty\Big[\mathfrak{L}(\mathbf{z}_\xi(u)),\hat{0}\Big] &= \mathbf{D}_\infty\Big[\mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)\phi(0) + \int_0^u \frac{\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)}{(u-s)^{1-\beta}}\mathfrak{H}(s,\mathbf{z}_\xi(s),\mathbf{z}_\xi(s-\tau))ds,\hat{0}\Big],\\ &\le \overline{\mathfrak{M}}_1\mathbf{D}_\infty\Big[\phi(t),\hat{0}\Big] + \int_0^u \frac{\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)}{(u-s)^{1-\beta}}\mathbf{D}_\infty\Big[\mathfrak{H}(s,\mathbf{z}_\xi(s),\mathbf{z}_\xi(s-\tau)),\hat{0}\Big]ds,\\ &\le \overline{\mathfrak{M}}_1\mathbf{D}_\infty\Big[\phi(t),\hat{0}\Big] + \frac{T^\beta\overline{\mathfrak{M}}_2}{\beta}N_{\mathfrak{H}}.\end{aligned}$$

Taking limit as $\xi \longrightarrow +\infty$, we obtain that $\overline{\mathfrak{M}}_1\mathbf{D}_\infty\Big[\phi(t),\hat{0}\Big] + \frac{T^\beta\overline{\mathfrak{M}}_2}{\beta}N_{\mathfrak{H}} \longrightarrow +\infty$ which is in contradiction with $\overline{\mathfrak{M}}_1\mathbf{D}_\infty\Big[\phi(t),\hat{0}\Big] + \frac{T^\beta\overline{\mathfrak{M}}_2}{\beta}N_{\mathfrak{H}}$ is bounded. Therefore, for every positive constant ξ, we obtain $\mathfrak{L}(\mathcal{B}_\xi) \subseteq \mathcal{B}_\xi$. By means of Schauder's fixed point Theorem implying that there is at least one solution to the system (1). □

For the uniqueness result, we have the following theorem:

Theorem 3.3. *Assume that the hypotheses* ($\mathcal{H}$1)-($\mathcal{H}$3) *holds. Then, if*

$$\sup_{0\le u\le T} h(u) \le \frac{\beta}{T^\beta\overline{\mathfrak{M}}_2 k},$$

then the solution of system (1) *is unique.*

Proof. We know that $\mathbf{z}(u)$ is a solution of system (1) if

$$\mathbf{z}(u) = \mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)\phi(0) + \int_0^u \frac{\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)}{(u-s)^{1-\beta}}\mathfrak{H}(s,\mathbf{z}(s),\mathbf{z}(s-\tau))ds,$$

hold. If $\mathbf{z}(u) \in C\Big([-\tau,T],\mathbf{E^n}\Big)$ is a fixed point of $\mathfrak{L}$ which define as in Theorem 3.2, therefore $\mathbf{z}(u)$ is the solution of system (1). Let $\mathbf{z}_1(u),\mathbf{z}_2(u) \in C\Big([-\tau,T],\mathbf{E^n}\Big)$ and for $u \in [-\tau,0]$, $\mathbf{z}_1(u) = \mathbf{z}_2(u) = \phi(u)$. For all $u \in \mathbf{I}$, we have

$$\begin{aligned}\mathbf{D}_\infty\Big[\mathfrak{L}(\mathbf{z}_1(u)),\mathfrak{L}(\mathbf{z}_2(u))\Big] &= \mathbf{D}_\infty\Big[\mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)\phi(0) + \int_0^u \frac{\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)}{(u-s)^{1-\beta}}\mathfrak{H}(s,\mathbf{z}_1(s),\mathbf{z}_1(s-\tau))ds,\\ &\qquad \mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)\phi(0) + \int_0^u \frac{\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)}{(u-s)^{1-\beta}}\mathfrak{H}(s,\mathbf{z}_2(s),\mathbf{z}_2(s-\tau))ds\Big],\\ &\le \int_0^u \frac{\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)}{(u-s)^{1-\beta}}\mathbf{D}_\infty\Big[\mathfrak{H}(s,\mathbf{z}_1(s),\mathbf{z}_1(s-\tau)),\mathfrak{H}(s,\mathbf{z}_2(s),\mathbf{z}_2(s-\tau))\Big]ds,\\ &\le \frac{T^\beta\overline{\mathfrak{M}}_2}{\beta}H\Big(u,\mathbf{D}_\infty\Big[\mathbf{z}_1(u),\mathbf{z}_2(u)\Big],\mathbf{D}_\infty\Big[\mathbf{z}_1(u-\tau),\mathbf{z}_2(u-\tau)\Big]\Big),\\ &\le \frac{T^\beta\overline{\mathfrak{M}}_2 k}{\beta}\sup_{0\le u\le T} H\Big(u,\mathbf{D}_\infty\Big[\mathbf{z}_1(u),\mathbf{z}_2(u)\Big]\Big),\\ &\le \frac{T^\beta\overline{\mathfrak{M}}_2 k}{\beta}\sup_{0\le u\le T} h(u)\mathbf{D}_\infty\Big[\mathbf{z}_1(u),\mathbf{z}_2(u)\Big],\end{aligned}$$

since $\sup_{0\le u\le T} h(u) \le \frac{\beta}{T^\beta\overline{\mathfrak{M}}_2 k}$, we have

$$\mathbf{D}_\infty\Big[\mathfrak{L}(\mathbf{z}_1(u)),\mathfrak{L}(\mathbf{z}_2(u))\Big] \le \mathbf{D}_\infty\Big[\mathbf{z}_1(u),\mathbf{z}_2(u)\Big].$$

Based on the Banach contraction principle, $\mathfrak{L}$ has an unique fixed point $\mathbf{z}(u)$. □

4. Finite-time stability result

In this section, finite-time stability result for the system (1) is provided. First, we recall the definition of this type of stability.

Definition 4.1. *[25] We say that the system* (1) *is finite-time stable with respect to* $\{\varrho, \rho, T\}$, $0 < \varrho < \rho$, *if* $\mathbf{D}_\infty\Big[\phi(0), \hat{0}\Big] \leq \varrho$ *implies* $\sup\limits_{0\leq u\leq T} \mathbf{D}_\infty\Big[\mathbf{z}(u), \hat{0}\Big] \leq \rho$, *for* $u \in \mathbf{I}$.

Theorem 4.2. *Assume that the hypotheses* ($\mathcal{H}1$)-($\mathcal{H}3$) *holds and there exist two positive constants* ϱ, ρ *such that* $\varrho < \rho$ *and* $\mathbf{D}_\infty\Big[\phi(0), \hat{0}\Big] \leq \varrho$, *then system* (1) *is finite-time stable on* $[-\tau, T]$ *provided that*

$$\overline{\mathfrak{M}}_1 \beta \varrho \leq \rho\Big(\beta - T^\beta \overline{\mathfrak{M}}_2 k\delta\Big) - T^\beta \overline{\mathfrak{M}}_2 \lambda.$$

Proof. Using the Definition 4.1, we have for all $u \in \mathbf{I}$

$$\begin{aligned}
\mathbf{D}_\infty\Big[\mathbf{z}(u), \hat{0}\Big] &= \mathbf{D}_\infty\Big[\mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)\phi(0) + \int_0^u \frac{\mathfrak{M}_{\beta,\beta}\Big(A(u-s)^\beta\Big)}{(u-s)^{1-\beta}} \mathfrak{H}(s, \mathbf{z}(s), \mathbf{z}(s-\tau))ds, \hat{0}\Big],\\
&\leq \overline{\mathfrak{M}}_1 \mathbf{D}_\infty\Big[\phi(0), \hat{0}\Big] + \mathbf{D}_\infty\Big[\int_0^u (u-s)^{\beta-1}\mathfrak{M}_{\beta,\beta}\Big(A(u-s)^\beta\Big)\mathfrak{H}(s, \mathbf{z}(s), \mathbf{z}(s-\tau))ds, \hat{0}\Big],\\
&\leq \overline{\mathfrak{M}}_1 \mathbf{D}_\infty\Big[\phi(0), \hat{0}\Big] + \int_0^u (u-s)^{\beta-1}\mathfrak{M}_{\beta,\beta}\Big(A(u-s)^\beta\Big)\mathbf{D}_\infty\Big[\mathfrak{H}(s, \mathbf{z}(s), \mathbf{z}(s-\tau)), \hat{0}\Big]ds,\\
&\leq \overline{\mathfrak{M}}_1 \mathbf{D}_\infty\Big[\phi(0), \hat{0}\Big] + \int_0^u (u-s)^{\beta-1}\mathfrak{M}_{\beta,\beta}\Big(A(u-s)^\beta\Big)\mathbf{D}_\infty\Big[\mathfrak{H}(s, \mathbf{z}(s), \mathbf{z}(s-\tau)), \mathfrak{H}(s, \hat{0}, \hat{0})\Big]ds\\
&\qquad + \int_0^u (u-s)^{\beta-1}\mathfrak{M}_{\beta,\beta}\Big(A(u-s)^\beta\Big)\mathbf{D}_\infty\Big[\mathfrak{H}(s, \hat{0}, \hat{0}), \hat{0}\Big]ds,\\
&\leq \overline{\mathfrak{M}}_1 \mathbf{D}_\infty\Big[\phi(t), \hat{0}\Big] + \frac{T^\beta\overline{\mathfrak{M}}_2}{\beta}\mathbf{D}_\infty\Big[\mathfrak{H}(s, \mathbf{z}(s), \mathbf{z}(s-\tau)), \mathfrak{H}(s, \hat{0}, \hat{0})\Big] + \frac{T^\beta\overline{\mathfrak{M}}_2}{\beta}\mathbf{D}_\infty\Big[\mathfrak{H}(s, \hat{0}, \hat{0}), \hat{0}\Big],\\
&\leq \overline{\mathfrak{M}}_1 \mathbf{D}_\infty\Big[\phi(t), \hat{0}\Big] + \frac{T^\beta\overline{\mathfrak{M}}_2}{\beta}H\Big(s, \mathbf{D}_\infty\Big[\mathbf{z}(s), \hat{0}\Big], \mathbf{D}_\infty\Big[\mathbf{z}(s-\tau), \hat{0}\Big]\Big) + \frac{T^\beta\overline{\mathfrak{M}}_2\lambda}{\beta},\\
&\leq \overline{\mathfrak{M}}_1 \mathbf{D}_\infty\Big[\phi(t), \hat{0}\Big] + \frac{T^\beta\overline{\mathfrak{M}}_2 k}{\beta}\sup_{0\leq u\leq T} h(u)\mathbf{D}_\infty\Big[\mathbf{z}(u), \hat{0}\Big] + \frac{T^\beta\overline{\mathfrak{M}}_2\lambda}{\beta},
\end{aligned}$$

then,

$$\sup_{0\leq u\leq T}\mathbf{D}_\infty\Big[\mathbf{z}(u), \hat{0}\Big] \leq \overline{\mathfrak{M}}_1 \mathbf{D}_\infty\Big[\phi(t), \hat{0}\Big] + \frac{T^\beta\overline{\mathfrak{M}}_2 k\delta}{\beta}\sup_{0\leq u\leq T}\mathbf{D}_\infty\Big[\mathbf{z}(u), \hat{0}\Big] + \frac{T^\beta\overline{\mathfrak{M}}_2\lambda}{\beta},$$

where $\delta = \sup\limits_{0\leq u\leq T} h(u)$. Hence,

$$\sup_{0\leq u\leq T}\mathbf{D}_\infty\Big[\mathbf{z}(u), \hat{0}\Big] \leq \frac{\overline{\mathfrak{M}}_1\beta\mathbf{D}_\infty\Big[\phi(t), \hat{0}\Big] + T^\beta\overline{\mathfrak{M}}_2\lambda}{\beta - T^\beta\overline{\mathfrak{M}}_2 k\delta}.$$

Therefore, if $\mathbf{D}_\infty\Big[\phi(0), \hat{0}\Big] \leq \varrho$, we have $\sup\limits_{0\leq u\leq T}\mathbf{D}_\infty\Big[\mathbf{z}(u), \hat{0}\Big] \leq \rho$. Which means that system (1) is finite-time stable on $[-\tau, T]$. □

5. Conclusion

This research has examined a class of nonlinear Hilfer fuzzy fractional differential equations with time-delays. Schauder and Banach fixed point theorems are employed under non-Lipschitz conditions to demonstrate the existence and uniqueness of solution results. In addition, finite-time stability result for the main system is provided.

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