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Refinements of generalized Euclidean operator radius inequalities of 2-tuple operators

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Abstract. We develop several upper and lower bounds for the A-Euclidean operator radius of 2-tuple operators admitting A-adjoint, and show that they refine the earlier related bounds. As an application of the bounds developed here, we obtain sharper A-numerical radius bounds.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and let $\|\cdot\|$ be the norm induced by the inner product. Let $\mathbb{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$. For $A \in \mathbb{B}(\mathcal{H})$, A^* denotes the adjoint of A, and $|A| = (A^*A)^{\frac{1}{2}}$. Also, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ denote the range and the kernel of A, respectively. Every positive operator A in $\mathbb{B}(\mathcal{H})$ defines the following positive semi-definite sesquilinear form:

$$\langle .,. \rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}, \quad (x, y) \to \langle x, y\rangle_A = \langle Ax, y\rangle.$$

Seminorm $\|\cdot\|_A$ induced by the semi-inner product $\langle .,.\rangle_A$, is given by $\|x\|_A = \langle Ax, x\rangle^{1/2} = \|A^{1/2}x\|$. This makes $\mathcal{H}$ into a semi-Hilbertian space. It is easy to verify that the seminorm induces a norm if and only if A is injective. Also, $(\mathcal{H}, \|\cdot\|_A)$ is complete if and only if $\mathcal{R}(A)$ is closed subspace of $\mathcal{H}$. Henceforth, we reserve the symbol A for a non-zero positive operator in $\mathbb{B}(\mathcal{H})$. We denote the A-unit sphere and A-unit ball of the semi-Hilbertian space $(\mathcal{H}, \|\cdot\|_A)$ by $\mathbb{S}_{\|\cdot\|_A}$ and $\mathbb{B}_{\|\cdot\|_A}$, respectively, i.e.,

$$\mathbb{S}_{\|\cdot\|_A} = \{x \in \mathcal{H} : \|x\|_A = 1\}, \ \mathbb{B}_{\|\cdot\|_A} = \{x \in \mathcal{H} : \|x\|_A \le 1\}.$$

For $T \in \mathbb{B}(\mathcal{H})$, let $c_A(T)$ and $w_A(T)$ denote the A-Crawford number and the A-numerical radius of T, respectively and are defined as

$$c_A(T) = \inf\left\{|\langle Tx, x\rangle_A| : x \in \mathbb{S}_{\|\cdot\|_A}\right\}, \ w_A(T) = \sup\left\{|\langle Tx, x\rangle_A| : x \in \mathbb{S}_{\|\cdot\|_A}\right\}.$$

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Note that $w_A(T)$ is not necessarily finite, see [8]. An operator $S \in \mathbb{B}(\mathcal{H})$ is called an A-adjoint of $T \in \mathbb{B}(\mathcal{H})$ if for every $x, y \in \mathcal{H}$, $\langle Tx, y\rangle_A = \langle x, Sy\rangle_A$ holds, i.e., S is a solution of the operator equation $AX = T^*A$. There are operators T for which A-adjoint may fail to exist, when it do exist then there may be more than one A-adjoint. The set of all operators in $\mathbb{B}(\mathcal{H})$ which possess A-adjoint is denoted by $\mathbb{B}_A(\mathcal{H})$. By Douglas theorem [12], we have

$$\begin{aligned}\mathbb{B}_A(\mathcal{H}) &= \{T \in \mathbb{B}(\mathcal{H}) : \mathcal{R}(T^*A) \subseteq \mathcal{R}(A)\}\\ &= \{T \in \mathbb{B}(\mathcal{H}) : \exists\, \lambda > 0 \text{ such that } \|ATx\| \leq \lambda\|Ax\|, \ \forall x \in \mathcal{H}\}.\end{aligned}$$

If $T \in \mathbb{B}_A(\mathcal{H})$, then there exists a unique solution of $AX = T^*A$, is denoted by $T^{\sharp_A}$, satisfying $\mathcal{R}(T^{\sharp_A}) \subseteq \overline{\mathcal{R}(A)}$, where $\overline{\mathcal{R}(A)}$ is the norm closure of $\mathcal{R}(A)$. For simplicity we will write $T^\sharp$ instead of $T^{\sharp_A}$. If $T \in \mathbb{B}_A(\mathcal{H})$, then $T^\sharp \in \mathbb{B}_A(\mathcal{H})$. Moreover, $\left[T^\sharp\right]^\sharp = P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}}$ and $\left[\left[T^\sharp\right]^\sharp\right]^\sharp = T^\sharp$, where $P_{\overline{\mathcal{R}(A)}}$ denotes the orthogonal projection onto $\overline{\mathcal{R}(A)}$. For more about $T^\sharp$, the reader can see [2, 3]. Again, clearly we have

$$\begin{aligned}\mathbb{B}_{A^{1/2}}(\mathcal{H}) &= \left\{T \in \mathbb{B}(\mathcal{H}) : \mathcal{R}(T^*A^{1/2}) \subseteq \mathcal{R}(A^{1/2})\right\}\\ &= \{T \in \mathbb{B}(\mathcal{H}) : \exists\, \lambda > 0 \text{ such that } \|Tx\|_A \leq \lambda\|x\|_A, \ \forall x \in \mathcal{H}\}.\end{aligned}$$

An operator in $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ is called A-bounded operator. The inclusion $\mathbb{B}_A(\mathcal{H}) \subseteq \mathbb{B}_{A^{1/2}}(\mathcal{H})$ always holds. Both of them are subalgebras of $\mathbb{B}(\mathcal{H})$ which are neither closed and nor dense in $\mathbb{B}(\mathcal{H})$. The semi-inner product $\langle ., .\rangle_A$ induces the A-operator seminorm on $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ defined as follows:

$$\|T\|_A = \sup_{\substack{x \in \overline{\mathcal{R}(A)}\\ x \neq 0}} \frac{\|Tx\|_A}{\|x\|_A} = \sup\left\{\|Tx\|_A : x \in \mathbb{S}_{\|\cdot\|_A}\right\} < \infty.$$

Also, it is easy to verify that

$$\|T\|_A = \sup\left\{|\langle Tx, y\rangle_A| : x, y \in \mathbb{S}_{\|\cdot\|_A}\right\}.$$

By Cauchy-Schwarz inequality, it follows that $|\langle Tx, x\rangle_A| \leq \|Tx\|_A\|x\|_A$ for all $x \in \mathcal{H}$, and so $w_A(T) \leq \|T\|_A$ for all $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. For A-selfadjoint operator T (i.e., $AT = T^*A$), we have $w_A(T) = \|T\|_A$, see in [26]. An operator $T \in \mathbb{B}_A(\mathcal{H})$ can be expressed as $T = \Re_A(T) + i\Im_A(T)$, where $\Re_A(T) = \frac{1}{2}(T + T^{\sharp_A})$ and $\Im_A(T) = \frac{1}{2i}(T - T^{\sharp_A})$. This decomposition is called A-Cartesian decomposition, using this we have $|\langle \Re_A(T)x, x\rangle_A|^2 + |\langle \Im_A(T)x, x\rangle_A|^2 = |\langle Tx, x\rangle_A|^2$ for all $x \in \mathcal{H}$. This implies $\|\Re_A(T)\|_A \leq w_A(T)$ and $\|\Im_A(T)\|_A \leq w_A(T)$, since $\Re_A(T)$ and $\Im_A(T)$ both are A-selfadjoint. Therefore, $\|T\|_A \leq \|\Re_A(T) + i\Im_A(T)\|_A \leq 2w_A(T)$. Thus, for every $T \in \mathbb{B}_A(\mathcal{H})$, we get $w_A(T) \leq \|T\|_A \leq 2w_A(T)$, (see also [26, Corollary 2.8]). One can also easily verify that the above inequality holds for every $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, and the A-power inequality $w_A(T^n) \leq [w_A(T)]^n$ holds for every positive integer n, see [4, 19].

Following [22], the A-Euclidean operator radius of d-tuple operators $\mathbf{T} = (T_1, T_2, ..., T_d) \in \mathbb{B}_{A^{1/2}}(\mathcal{H})^d$ is defined as

$$w_{A,e}(\mathbf{T}) = \sup\left\{\left(\sum_{k=1}^{d} |\langle T_k x, x\rangle_A|^2\right)^{1/2} : x \in \mathbb{S}_{\|\cdot\|_A}\right\}.$$

This is also known as A-joint numerical radius of $\mathbf{T}$. The A-Euclidean operator seminorm of d-tuple operators $\mathbf{T} = (T_1, T_2, ..., T_d) \in \mathbb{B}_{A^{1/2}}(\mathcal{H})^d$ is defined as

$$\|\mathbf{T}\|_A = \sup\left\{\left(\sum_{k=1}^{d} \|T_k x\|_A^2\right)^{1/2} : x \in \mathbb{S}_{\|\cdot\|_A}\right\}.$$

Clearly, the A-Euclidean operator radius and A-Euclidean operator seminorm of d-tuple operators are generalizations of A-numerical radius and A-operator seminorm of an operator in $\mathbb{B}_{A^{1/2}}(\mathcal{H})$. Observe that for $A = I$, $\|\cdot\|_A = \|\cdot\|$, $w_A(\cdot) = w(\cdot)$, $c_A(\cdot) = c(\cdot)$, $w_{A,e}(\cdot) = w_e(\cdot)$ and $\|\cdot\|_{A,e} = \|\cdot\|_e$ are the usual operator norm, numerical radius, Crawford number, Euclidean operator radius and Euclidean operator norm, respectively. For recent developments of A-numerical radius inequalities see [6, 7] and for Euclidean operator radius inequalities see [4, 11, 20, 23]. In this paper, we obtain several inequalities involving A-Euclidean operator radius and A-Euclidean operator seminorm of 2-tuple operators, and we show that these inequalities improve on the earlier related inequalities.

We end this introductory section with a brief description of the space $\mathbf{R}(A^{1/2})$ (see [1]) as follows: The semi-inner product $\langle .,.\rangle_A$ induces an inner product on the quotient space $\mathcal{H}/\mathcal{N}(A)$, defined by $[\overline{x},\overline{y}] = \langle Ax, y\rangle$, $\forall\ \overline{x},\overline{y} \in \mathcal{H}/\mathcal{N}(A)$. The space $(\mathcal{H}/\mathcal{N}(A), [.,.])$ is, in general, not a complete space. The completion of $(\mathcal{H}/\mathcal{N}(A), [.,.])$ is isometrically isomorphic to the Hilbert space $R(A^{1/2})$ via the canonical construction mentioned in [10], where $R(A^{1/2})$ is equipped with the inner product

$$(A^{1/2}x, A^{1/2}y) = \langle P_{\overline{\mathcal{R}(A)}}x, P_{\overline{\mathcal{R}(A)}}y\rangle,\ \ \forall x, y \in \mathcal{H}.$$

In the sequel, the Hilbert space $(\mathcal{R}(A^{1/2}), (.,.))$ will be denoted by $\mathbf{R}(A^{1/2})$ and we use the symbol $\|\cdot\|_{\mathbf{R}(A^{1/2})}$ to represent the norm induced by the inner product $(.,.)$. Note that, the fact $\mathcal{R}(A) \subseteq \mathcal{R}(A^{1/2})$ implies that $(Ax, Ay) = \langle x, y\rangle_A$, $\ \forall x, y \in \mathcal{H}$. This gives $\|Ax\|_{\mathcal{R}(A^{1/2})} = \|x\|_A$, $\ \forall x \in \mathcal{H}$. Now, we give a nice connection of an operator $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ with an operator $\widetilde{T} \in \mathbb{B}(\mathbf{R}(A^{1/2}))$, in the form of the following proposition, see [1].

Proposition 1.1. *Let $T \in \mathbb{B}(\mathcal{H})$. Then $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ if and only if there exist a unique $\widetilde{T} \in \mathbb{B}(\mathbf{R}(A^{1/2}))$ such that $Z_A T = \widetilde{T} Z_A$, where $Z_A : \mathcal{H} \to \mathbf{R}(A^{1/2})$ is defined by $Z_A x = Ax$.*

2. Main Results

We begin with the following sequence of known lemmas. First lemma is known as mixed Schwarz inequality.

Lemma 2.1. *[17] If $T \in \mathbb{B}(\mathcal{H})$ and $0 \le \alpha \le 1$, then*

$$|\langle Tx, y\rangle|^2 \le \langle |T|^{2\alpha}x, x\rangle\langle |T^*|^{2(1-\alpha)}y, y\rangle\ \forall\ x, y \in \mathcal{H}.$$

Second lemma is known as Holder-McCarthy inequality.

Lemma 2.2. *[18] If $T \in \mathcal{B}(\mathcal{H})$ is positive, then the following inequalities hold: For any $x \in \mathcal{H}$,*

$$\langle T^r x, x\rangle \ge \|x\|^{2(1-r)}\langle Tx, x\rangle^r, \quad for\ \ r \ge 1$$

and

$$\langle T^r x, x\rangle \le \|x\|^{2(1-r)}\langle Tx, x\rangle^r, \quad for\ \ 0 \le r \le 1.$$

Third lemma is related to A-selfadjoint operators.

Lemma 2.3. *[15] Let $T \in \mathcal{B}(\mathcal{H})$ be A-selfadjoint. Then $T^\sharp$ is also A-selfadjoint and $[T^\sharp]^\sharp = T^\sharp$.*

Fourth lemma is related to semi-Hilbertian space operator T and Hilbert space operator $\widetilde{T}$.

Lemma 2.4. *[1, 13] Let $T \in \mathcal{B}_A(\mathcal{H})$. Then*
(i) $\widetilde{T^\sharp} = \left(\widetilde{T}\right)^*$ and $\widetilde{(T^{\sharp_A})^{\sharp_A}} = \widetilde{T}$.
(ii) $\|T\|_A = \|\widetilde{T}\|_{\mathcal{B}(\mathbf{R}(A^{1/2}))}$, $w_A(T) = w(\widetilde{T})$ and $c_A(T) = c(\widetilde{T})$.
(Here $\|\widetilde{T}\|_{\mathcal{B}(\mathbf{R}(A^{1/2}))}$ denotes the usual operator norm of $\widetilde{T}$).

Now, we prove the following result related to A-Euclidean operator radius and Euclidean operator radius by using a similar technique as used in [25, Proposition 2.5].

Theorem 2.5. *Let* $\mathbf{T} = (T_1, T_2, \dots, T_d) \in \mathbb{B}_{A^{1/2}}(\mathcal{H})^d$. *Then*

$$w_{A,e}(\mathbf{T}) = w_{A,e}(T_1, T_2, ..., T_d) = w_e(\widetilde{T_1}, \widetilde{T_2}, ..., \widetilde{T_d}) = w_e(\widetilde{\mathbf{T}})$$

where $\widetilde{\mathbf{T}} = (\widetilde{T_1}, \widetilde{T_2}, ..., \widetilde{T_d}) \in \mathbb{B}(\mathbf{R}(A^{1/2}))^d$.

Proof. First we prove $w_{A,e}(\mathbf{T}) \leq w_e(\widetilde{\mathbf{T}})$. We recall that

$$\begin{aligned} w_{A,e}(\mathbf{T}) &= \sup\left\{\left(\sum_{i=1}^{d} |\langle T_i x, x\rangle|^2\right)^{\frac{1}{2}} : x \in \mathcal{H}, \|x\|_A = 1\right\} \\ &= \sup\left\{\left(\sum_{i=1}^{d} |(AT_i x, Ax)|^2\right)^{\frac{1}{2}} : x \in \mathcal{H}, \|Ax\|_{\mathbf{R}(A^{1/2})} = 1\right\} \\ &= \sup\left\{\left(\sum_{i=1}^{d} |(\widetilde{T_i} Ax, Ax)|^2\right)^{\frac{1}{2}} : x \in \mathcal{H}, \|Ax\|_{\mathbf{R}(A^{1/2})} = 1\right\} \\ &\qquad (\textit{using Proposition 1.1}). \end{aligned}$$

From the decomposition $\mathcal{H} = \mathcal{N}(A^{1/2}) \oplus \overline{\mathcal{R}(A^{1/2})}$, we obtain that

$$w_{A,e}(\mathbf{T}) = \sup\left\{\left(\sum_{i=1}^{d} |(\widetilde{T_i} Ax, Ax)|^2\right)^{\frac{1}{2}} : x \in \overline{\mathcal{R}(A^{1/2})}, \|Ax\|_{\mathbf{R}(A^{1/2})} = 1\right\}. \tag{1}$$

Now,

$$\begin{aligned} & w_e(\widetilde{\mathbf{T}}) \\ =\ & \sup\left\{\left(\sum_{i=1}^{d} |(\widetilde{T_i} y, y)|^2\right)^{\frac{1}{2}} : y \in \mathcal{R}(A^{1/2}), \|y\|_{\mathbf{R}(A^{1/2})} = 1\right\} \\ =\ & \sup\left\{\left(\sum_{i=1}^{d} |(\widetilde{T_i} A^{1/2}x, A^{1/2}x)|^2\right)^{\frac{1}{2}} : x \in \mathcal{H}, \|A^{1/2}x\|_{\mathbf{R}(A^{1/2})} = 1\right\} \\ =\ & \sup\left\{\left(\sum_{i=1}^{d} |(\widetilde{T_i} A^{1/2}x, A^{1/2}x)|^2\right)^{\frac{1}{2}} : x \in \overline{\mathcal{R}(A^{1/2})}, \|A^{1/2}x\|_{\mathbf{R}(A^{1/2})} = 1\right\}. \end{aligned} \tag{2}$$

Since $\mathcal{R}(A) \subseteq \mathcal{R}(A^{1/2})$, (1) together with (2) implies $w_{A,e}(\mathbf{T}) \leq w_e(\widetilde{\mathbf{T}})$.

Next we show the reverse inequality, i.e, $w_A(\widetilde{\mathbf{T}}) \leq w_{A,e}(\mathbf{T})$. Suppose that

$$\beta \in \left\{\left(\sum_{i=1}^{d} |(\widetilde{T_i} A^{1/2}x, A^{1/2}x)|^2\right)^{\frac{1}{2}} : x \in \overline{\mathcal{R}(A^{1/2})}, \|A^{1/2}x\|_{\mathbf{R}(A^{1/2})} = 1\right\} = W_e(\widetilde{\mathbf{T}}), (\textit{say}).$$

So, there exists $x \in \overline{\mathcal{R}(A^{1/2})}$ with $\|A^{1/2}x\|_{\mathbf{R}(A^{1/2})} = 1$ such that

$$\beta = \left(\sum_{i=1}^{d} |(\widetilde{T_i} A^{1/2}x, A^{1/2}x)|^2\right)^{\frac{1}{2}}.$$

Since $A^{1/2}x \in \mathbf{R}(A^{1/2})$ and $\mathcal{R}(A)$ is dense in $\mathbf{R}(A^{1/2})$, there exist a sequence $\{x_n\}$ in $\mathcal{H}$ such that $\lim_{n\to\infty} \|Ax_n - A^{1/2}x\|_{\mathbf{R}(A^{1/2})} = 0$. Hence $\beta = \lim_{n\to\infty} \left(\sum_{i=1}^{d} |(\widetilde{T_i}Ax_n, Ax_n)|^2\right)^{\frac{1}{2}}$ and $\lim_{n\to\infty} \|Ax_n\|_{\mathbf{R}(A^{1/2})} = 1$. Now, let $y_n = \frac{x_n}{\|Ax_n\|_{\mathbf{R}(A^{1/2})}}$. Then clearly we have, $\beta = \lim_{n\to\infty} \left(\sum_{i=1}^{d} |(\widetilde{T_i}Ay_n, Ay_n)|^2\right)^{\frac{1}{2}}$ and $\|Ay_n\|_{\mathbf{R}(A^{1/2})} = 1$. Therefore,

$$\beta \in \overline{\left\{\left(\sum_{i=1}^{n} |(\widetilde{T_i}Ax, Ax)|^2\right)^{\frac{1}{2}} : x \in \overline{\mathcal{R}(A^{1/2})},\ \|Ax\|_{\mathbf{R}(A^{1/2})} = 1\right\}} = \overline{W_{A,e}(\mathbf{T})},\ (say).$$

Hence, $W_e(\widetilde{\mathbf{T}}) \subseteq \overline{W_{A,e}(\mathbf{T})}$. This implies $w_e(\widetilde{\mathbf{T}}) \leq w_{A,e}(\mathbf{T})$, and this completes the proof. □

Now, we are in a position to prove the bounds of A-Euclidean operator radius. In the following theorem we obtain upper and lower bound for the A-Euclidean operator radius of 2-tuple operators in $\mathbb{B}_A(\mathcal{H})$ involving A-numerical radius.

Theorem 2.6. *Let $B, C \in \mathbb{B}_A(\mathcal{H})$, then*

$$\begin{aligned} &\frac{1}{2}w_A(B^2 + C^2) + \frac{1}{2}\max\{w_A(B), w_A(C)\}\left|w_A(B + C) - w_A(B - C)\right| \\ &\leq w_{A,e}^2(B, C) \\ &\leq \frac{1}{\sqrt{2}}w_A((B^\sharp B + C^\sharp C) + i(BB^\sharp + CC^\sharp)). \end{aligned}$$

Proof. Let $x \in \mathcal{H}$ with $\|x\|_A = 1$. Then we have,

$$\begin{aligned} |\langle Bx, x\rangle_A|^2 + |\langle Cx, x\rangle_A|^2 &\geq \frac{1}{2}\left(|\langle Bx, x\rangle_A| + |\langle Cx, x\rangle_A|\right)^2 \\ &\geq \frac{1}{2}\left(|\langle Bx, x\rangle_A \pm \langle Cx, x\rangle_A|\right)^2 \\ &= \frac{1}{2}|\langle (B \pm C)x, x\rangle_A|^2. \end{aligned}$$

Taking supremum over all $x \in \mathcal{H}$, $\|x\|_A = 1$, we get

$$w_{A,e}^2(B, C) \geq \frac{1}{2}w_A^2(B \pm C). \tag{3}$$

Therefore, it follows from the inequalities in (3) that

$$\begin{aligned} w_{A,e}^2(B, C) &\geq \frac{1}{2}\max\{w_A^2(B + C), w_A^2(B - C)\} \\ &= \frac{w_A^2(B + C) + w_A^2(B - C)}{4} + \frac{\left|w_A^2(B + C) - w_A^2(B - C)\right|}{4} \\ &\geq \frac{w_A((B + C)^2) + w_A((B - C)^2)}{4} \\ &\quad + (w_A(B + C) + w_A(B - C))\frac{\left|w_A(B + C) - w_A(B - C)\right|}{4} \\ &\geq \frac{w_A((B + C)^2 + (B - C)^2)}{4} \\ &\quad + w_A((B + C) + (B - C))\frac{\left|w_A(B + C) - w_A(B - C)\right|}{4}. \end{aligned}$$

Therefore,

$$w_{A,e}^2(B,C) \geq \frac{w_A(B^2+C^2)}{2} + \frac{w_A(B)}{2}\Big|w_A(B+C) - w_A(B-C)\Big|. \tag{4}$$

Interchanging B and C in (4), we arrive

$$w_{A,e}^2(B,C) \geq \frac{w_A(B^2+C^2)}{2} + \frac{w_A(C)}{2}\Big|w_A(B+C) - w_A(B-C)\Big|. \tag{5}$$

The inequality (4) together with (5), gives the first inequality.

Next, we prove the second inequality. Let $x \in \mathcal{H}$ with $\|x\| = 1$. Then we have,

$$\begin{aligned}
&(|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2)^2 \\
&\leq (\langle |B|x, x\rangle \langle |B^*|x, x\rangle + \langle |C|x, x\rangle \langle |C^*|x, x\rangle)^2 \ \ (\textit{using Lemma 2.1}) \\
&\leq (\langle |B|x, x\rangle^2 + \langle |C|x, x\rangle^2)(\langle |B^*|x, x\rangle^2 + \langle |C^*|x, x\rangle^2) \\
&\qquad\qquad (\textit{since } (ab+cd)^2 \leq (a^2+c^2)(b^2+d^2) \textit{ for all } a,b,c,d \in \mathbb{R}) \\
&\leq (\left\langle |B|^2x, x\right\rangle + \left\langle |C|^2x, x\right\rangle)(\left\langle |B^*|^2x, x\right\rangle + \left\langle |C^*|^2x, x\right\rangle) \ \ (\textit{using Lemma 2.2}) \\
&= \langle (B^*B + C^*C)x, x\rangle \langle (BB^* + CC^*)x, x\rangle \\
&\leq \frac{1}{2}\left\{\langle (B^*B + C^*C)x, x\rangle^2 + \langle (BB^* + CC^*)x, x\rangle^2\right\} \\
&= \frac{1}{2}|\langle (B^*B + C^*C)x, x\rangle + i\langle (BB^* + CC^*)x, x\rangle|^2 \\
&= \frac{1}{2}|\langle ((B^*B + C^*C) + i(BB^* + CC^*))x, x\rangle|^2 \\
&\leq \frac{1}{2}w^2((B^*B + C^*C) + i(BB^* + CC^*)).
\end{aligned}$$

Taking supremum over all $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$w_e^2(B,C) \leq \frac{1}{\sqrt{2}}w((B^*B + C^*C) + i(BB^* + CC^*)). \tag{6}$$

As $B, C \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, following Proposition 1.1, there exist unique $\widetilde{B}$ and $\widetilde{C}$ in $\mathbb{B}(\mathbf{R}(A^{1/2}))$ such that $Z_AB = \widetilde{B}Z_A$ and $Z_AC = \widetilde{C}Z_A$. The inequality (6) implies that

$$w_e^2(\widetilde{B},\widetilde{C}) \leq \frac{1}{\sqrt{2}}w((\widetilde{B}^*\widetilde{B} + \widetilde{C}^*\widetilde{C}) + i(\widetilde{B}\widetilde{B}^* + \widetilde{C}\widetilde{C}^*)). \tag{7}$$

Since $(\widetilde{B})^* = \widetilde{B^\sharp}$, the inequality (7) becomes

$$w_e^2(\widetilde{B},\widetilde{C}) \leq \frac{1}{\sqrt{2}}w((\widetilde{B^\sharp}\widetilde{B} + \widetilde{C^\sharp}\widetilde{C}) + i(\widetilde{B}\widetilde{B^\sharp} + \widetilde{C}\widetilde{C^\sharp})). \tag{8}$$

For any $S, T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, it is easy to see that $\widetilde{S}\widetilde{T} = \widetilde{ST}$ and $\widetilde{S + \lambda T} = \widetilde{S} + \lambda\widetilde{T}$ for all $\lambda \in \mathbb{C}$. So, the inequality (8) is of the following form

$$w_e^2(\widetilde{B},\widetilde{C}) \leq \frac{1}{\sqrt{2}}w(\widetilde{(B^\sharp B + C^\sharp C) + i(BB^\sharp + CC^\sharp)}). \tag{9}$$

Now, by applying Theorem 2.5 and Lemma 2.4, we have

$$w_{A,e}^2(B,C) \leq \frac{1}{\sqrt{2}}w_A((B^\sharp B + C^\sharp C) + i(BB^\sharp + CC^\sharp)).$$

This completes the proof. □

Remark 2.7. *(i) The lower bound of $w_e(B,C)$ in Theorem 2.6 is stronger than the lower bound in [14, Theorem 2.8], namely,* $\frac{1}{2}w_A(B^2+C^2) \le w^2_{A,e}(B,C)$. *Also, it is not difficult to verify that*

$$\frac{1}{\sqrt{2}}w_A((B^\sharp B + C^\sharp C) + i(BB^\sharp + CC^\sharp)) \le \frac{1}{\sqrt{2}}\left\{\|B^\sharp B + C^\sharp C\|_A^2 + \|BB^\sharp + CC^\sharp\|_A^2\right\}^{\frac{1}{2}}.$$

Therefore, the upper bound of $w_{A,e}(B,C)$ in Theorem 2.6 is better than the upper bound in [14, Theorem 2.8], namely, $w^2_{A,e}(B,C) \le \|BB^\sharp + CC^\sharp\|_A$ *if* $\|BB^\sharp + CC^\sharp\|_A \le \|B^\sharp B + C^\sharp C\|_A$.

(ii) Following Theorem 2.6, $w^2_{A,e}(B,C) = \frac{1}{2}w_A(B^2+C^2)$ *implies* $w_A(B+C) = w_A(B-C)$. *However, the converse is not true, in general.*

The following corollary is an immediate consequence of Theorem 2.6.

Corollary 2.8. *If $B, C \in \mathbb{B}_A(\mathcal{H})$ are A-selfadjoint, then*

$$\frac{1}{2}\|B^2+C^2\|_A + \frac{1}{2}\max\{\|B\|_A, \|C\|_A\}\Big|\|B+C\|_A - \|B-C\|_A\Big| \le w^2_{A,e}(B,C).$$

In particular, considering $B = [\Re_A(T)]^\sharp$ and $C = [\Im_A(T)]^\sharp$ in Theorem 2.6, and the using the Lemma 2.3. we obtain the following new upper and lower bounds for the A-numerical radius of a bounded linear operator $T \in \mathbb{B}_A(\mathcal{H})$.

Corollary 2.9. *If $T \in \mathbb{B}_A(\mathcal{H})$, then*

$$\frac{1}{4}\|T^\sharp T + TT^\sharp\|_A + \frac{m}{2}\max\{\|\Re_A(T)\|_A, \|\Im_A(T)\|_A\} \le w_A^2(T) \le \frac{1}{2}\|TT^\sharp + T^\sharp T\|_A,$$

where $m = \Big|\|\Re_A(T) + \Im_A(T)\|_A - \|\Re_A(T) - \Im_A(T)\|_A\Big|$.

Again, considering $B = T$ and $C = T^\sharp$ in Theorem 2.6, we get the following new lower bound for the A-numerical radius of $T \in \mathbb{B}_A(\mathcal{H})$.

Corollary 2.10. *Let $T \in \mathbb{B}_A(\mathcal{H})$, then*

$$\frac{1}{2}\|\Re_A(T^2)\|_A + \frac{1}{2}w_A(T)\Big|\|\Re_A(T)\|_A - \|\Im_A(T)\|_A\Big| \le w_A^2(T).$$

To prove our next theorem, we need the following lemma, known as Bohr's inequality.

Lemma 2.11. *[24]. Suppose $a_i \ge 0$ for $i = 1, 2, \ldots\ldots, n$. Then*

$$\left(\sum_{i=1}^{k} a_i\right)^r = k^{r-1}\sum_{i=1}^{k} a_i^r \ \ for \ \ r \ge 1.$$

Theorem 2.12. *If $B, C \in \mathbb{B}_A(\mathcal{H})$, then*

$$\frac{1}{8}\|B+C\|_A^4 \le w_{A,e}(B^\sharp B, C^\sharp C)w_{A,e}(BB^\sharp, CC^\sharp).$$

Proof. Let $x, y \in \mathcal{H}$ with $\|x\| = \|y\| = 1$. Then we have,

$$\begin{aligned}
& |\langle (B+C)x, y\rangle|^4 \\
= \ & |\langle Bx, y\rangle + \langle Cx, y\rangle|^4 \\
\le \ & (|\langle Bx, y\rangle| + |\langle Cx, y\rangle|)^4 \\
\le \ & 8(|\langle Bx, y\rangle|^4 + |\langle Cx, y\rangle|^4) \ (\textit{using Lemma 2.11}) \\
\le \ & 8(\langle |B|x, x\rangle^2\langle |B^*|y, y\rangle^2 + \langle |C|)x, x\rangle^2\langle |C^*|y, y\rangle^2) \ (\textit{using Lemma 2.1}) \\
\le \ & 8(\langle B^*Bx, x\rangle\langle BB^*y, y\rangle + \langle C^*Cx, x\rangle\langle CC^*y, y\rangle) \ (\textit{using Lemma 2.2}) \\
\le \ & 8(\langle B^*Bx, x\rangle^2 + \langle C^*Cx, x\rangle^2)^{\frac{1}{2}}(\langle BB^*y, y\rangle^2 + \langle CC^*y, y\rangle^2)^{\frac{1}{2}} \\
\le \ & 8w_e(B^*B, C^*C)w_e(BB^*, CC^*).
\end{aligned}$$

Taking supremum over $\|x\| = \|y\| = 1$, we get

$$\frac{1}{8}\|B + C\|^4 \leq w_e(B^*B, C^*C)w_e(BB^*, CC^*). \tag{10}$$

As $B, C \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, following Proposition 1.1, there exist unique $\widetilde{B}$ and $\widetilde{C}$ in $\mathbb{B}(\mathbf{R}(A^{1/2}))$ such that $Z_A B = \widetilde{B} Z_A$ and $Z_A C = \widetilde{C} Z_A$. The inequality (10) implies that

$$\frac{1}{8}\|\widetilde{B} + \widetilde{C}\|^4_{\mathbb{B}(\mathbf{R}(\mathbf{A}^{1/2}))} \leq w_e(\widetilde{B}^*\widetilde{B}, \widetilde{C}^*\widetilde{C})w_e(\widetilde{B}\widetilde{B}^*, \widetilde{C}\widetilde{C}^*). \tag{11}$$

Since $(\widetilde{B})^* = \widetilde{B^\sharp}$, the inequality (11) becomes

$$\frac{1}{8}\|\widetilde{B} + \widetilde{C}\|^4_{\mathbb{B}(\mathbf{R}(\mathbf{A}^{1/2}))} \leq w_e(\widetilde{B^\sharp}\widetilde{B}, \widetilde{C^\sharp}\widetilde{C})w_e(\widetilde{B}\widetilde{B^\sharp}, \widetilde{C}\widetilde{C^\sharp}), \tag{12}$$

that is,

$$\frac{1}{8}\|\widetilde{B + C}\|^4_{\mathbb{B}(\mathbf{R}(\mathbf{A}^{1/2}))} \leq w_e(\widetilde{B^\sharp B}, \widetilde{C^\sharp C})w_e(\widetilde{BB^\sharp}, \widetilde{CC^\sharp}). \tag{13}$$

By using Lemma 2.4 and Theorem 2.5 in the above inequality (13), we obtain

$$\frac{1}{8}\|B + C\|^4_A \leq w_{A,e}(B^\sharp B, C^\sharp C)w_{A,e}(BB^\sharp, CC^\sharp),$$

as desired. □

Next we obtain an upper bound for the A-Euclidean operator radius of 2-tuple operators admitting A-adjoint. For this we need the following lemma.

Lemma 2.13. *[16] If $x, y, e \in \mathcal{H}$ with $\|e\|_A = 1$, then*

$$|\langle x, e\rangle_A \langle e, y\rangle_A| \leq \frac{|\langle x, y\rangle_A| + \max\{1, |\alpha - 1|\}\|x\|_A\|y\|_A}{|\alpha|},$$

for all non-zero scalar α.

Theorem 2.14. *If $B, C \in \mathbb{B}_A(\mathcal{H})$, then*

$$w^2_{A,e}(B, C) \leq \frac{\max\{1, |1 - \alpha|\}\|(B, C)\|_{A,e}\|(B^\sharp, C^\sharp)\|_{A,e} + w_A(B^2) + w_A(C^2)}{|\alpha|},$$

for any non-zero scalar α.

Proof. Let $x \in \mathcal{H}$ with $\|x\|_A = 1$. Then we have,

$$
\begin{aligned}
& |\langle Bx, x\rangle_A|^2 + |\langle Cx, x\rangle_A|^2 \\
= \;& |\langle Bx, x\rangle_A \langle x, B^\sharp x\rangle_A| + |\langle Cx, x\rangle_A \langle x, C^\sharp x\rangle_A| \\
\leq \;& \frac{\max\{1, |\alpha - 1|\}\|Bx\|_A\|B^\sharp x\|_A + |\langle Bx, B^\sharp x\rangle_A|}{|\alpha|} \\
& + \frac{\max\{1, |\alpha - 1|\}\|Cx\|_A\|C^\sharp x\|_A + |\langle Cx, C^\sharp x\rangle_A|}{|\alpha|} \;(\textit{using Lemma 2.13}) \\
= \;& \frac{\max\{1, |\alpha - 1|\}(\|Bx\|_A\|B^\sharp x\|_A + \|Cx\|_A\|C^\sharp x\|_A)}{|\alpha|} \\
& + \frac{|\langle Bx, B^\sharp x\rangle_A| + |\langle Cx, C^\sharp x\rangle_A|}{|\alpha|} \\
\leq \;& \frac{\max\{1, |\alpha - 1|\}(\|Bx\|_A^2 + \|Cx\|_A^2)^{\frac{1}{2}}(\|B^\sharp x\|_A^2 + \|C^\sharp x\|_A^2)^{\frac{1}{2}}}{|\alpha|} \\
& + \frac{|\langle B^2x, x\rangle_A| + |\langle C^2x, x\rangle_A|}{|\alpha|} \\
\leq \;& \frac{\max\{1, |\alpha - 1|\}\|(B, C)\|_{A,e}\|(B^\sharp, C^\sharp)\|_{A,e}}{|\alpha|} + \frac{w_A(B^2) + w_A(C^2)}{|\alpha|}.
\end{aligned}
$$

Taking supremum over all $x \in \mathcal{H}$ with $\|x\|_A = 1$, we get the desired inequality. □

In particular, considering $B = C = T$ in Theorem 2.14, we obtain the following corollary.

Corollary 2.15. *If $T \in \mathbb{B}_A(\mathcal{H})$, then*

$$w_A^2(T) \leq \frac{\max\{1, |1 - \alpha|\}\|T\|_A^2 + w_A(T^2)}{|\alpha|},$$

for any non-zero scalar α.

The above inequality also studied in [16, Corollary 2.5]. For $\alpha = 2$,

$$w_A^2(T) \leq \frac{1}{2}\left(\|T\|_A^2 + w_A(T^2)\right),$$

which was also obtained in [14, Corollary 2.5] and [16, Remark 2.6].

Next bound reads as follows.

Theorem 2.16. *If $B, C \in \mathbb{B}_A(\mathcal{H})$, then*

$$
\begin{aligned}
w_{A,e}^2(B, C) \;\leq\; & \min\{w_A^2(B - C), w_A^2(B + C)\} \\
& + \frac{\max\{1, |1 - \alpha|\}\|C^\sharp C + BB^\sharp\|_A + 2w_A(BC)}{|\alpha|},
\end{aligned}
$$

for any non-zero scalar α.

Proof. Let $x \in \mathcal{H}$ with $\|x\|_A = 1$. Then we have,

$$
\begin{aligned}
|\langle Cx, x\rangle_A|^2 - 2Re[\langle Cx, x\rangle_A \overline{\langle Bx, x\rangle_A}] + |\langle Bx, x\rangle_A|^2 \;&=\; |\langle Cx, x\rangle_A - \langle Bx, x\rangle_A|^2 \\
&=\; |\langle (C - B)x, x\rangle_A|^2 \\
&\leq\; w_A^2(C - B).
\end{aligned}
$$

Thus,

$$\begin{aligned}
& |\langle Cx,x\rangle_A|^2+|\langle Bx,x\rangle_A|^2 \\
\leq\ & w_A^2(C-B)+2Re[\langle Cx,x\rangle_A\overline{\langle Bx,x\rangle_A}] \\
\leq\ & w_A^2(C-B)+2|\langle Cx,x\rangle_A\langle Bx,x\rangle_A| \\
\leq\ & w_A^2(C-B)+\frac{2\max\{1,|\alpha-1|\}\|Cx\|_A\|B^\sharp x\|_A+2|\langle Cx,B^\sharp x\rangle_A|}{|\alpha|}\ (\textit{by Lemma 2.13}) \\
\leq\ & w_A^2(C-B)+\frac{\max\{1,|1-\alpha|\}(\|Cx\|_A^2+\|B^\sharp x\|_A^2)+2w_A(BC)}{|\alpha|} \\
\leq\ & w_A^2(C-B)+\frac{\max\{1,|1-\alpha|\}\|C^\sharp C+BB^\sharp\|_A+2w_A(BC)}{|\alpha|}.
\end{aligned}$$

Taking supremum over all $x\in\mathcal{H}$ with $\|x\|_A=1$, we get

$$w_{A,e}^2(B,C)\leq w_A^2(B-C)+\frac{\max\{1,|1-\alpha|\}\|C^\sharp C+BB^\sharp\|_A+2w_A(BC)}{|\alpha|}. \tag{14}$$

Replacing C by $-C$, we obtain that

$$w_{A,e}^2(B,C)\leq w_A^2(B+C)+\frac{\max\{1,|1-\alpha|\}\|C^\sharp C+BB^\sharp\|_A+2w_A(BC)}{|\alpha|}. \tag{15}$$

Following the inequality (15) together with (14), we get the desired inequality. □

In particular, considering $\alpha=2$ in Theorem 2.16, we get

$$w_{A,e}^2(B,C)\leq \min\{w_A^2(B-C),w_A^2(B+C)\}+\frac{\|C^\sharp C+BB^\sharp\|_A+2w_A(BC)}{2}. \tag{16}$$

Again, considering $B=C=T$ in Theorem 2.16, we get the following upper bound for the A-numerical radius of $T\in\mathbb{B}_A(\mathcal{H})$:

$$w_A^2(T)\leq\frac{\frac{1}{2}\max\{1,|1-\alpha|\}\|T^\sharp T+TT^\sharp\|_A+w_A(T^2)}{|\alpha|}. \tag{17}$$

Putting $\alpha=2$ in (17), we get

$$w_A^2(T)\leq\frac{1}{4}\|T^\sharp T+TT^\sharp\|_A+\frac{1}{2}w_A(T^2),$$

which was also obtained in [26, Theorem 2.11].

Next, in the following theorem we obtain a lower bound for $w_{A,e}(B,C)$.

Theorem 2.17. *If $B,C\in\mathbb{B}_A(\mathcal{H})$, then*

$$\frac{1}{2}\max\left\{w_A^2(B+C)+c_A^2(B-C),w_A^2(B-C)+c_A^2(B+C)\right\}\leq w_{A,e}^2(B,C).$$

Proof. Let $x\in\mathcal{H}$ with $\|x\|_A=1$. Then we have,

$$|\langle Bx,x\rangle_A+\langle Cx,x\rangle_A|^2+|\langle Bx,x\rangle_A-\langle Cx,x\rangle_A|^2=2(|\langle Bx,x\rangle_A|^2+|\langle Cx,x\rangle_A|^2).$$

This implies that

$$\begin{aligned}
|\langle (B+C)x,x\rangle_A|^2+|\langle (B-C)x,x\rangle_A|^2 &= 2(|\langle Bx,x\rangle_A|^2+|\langle Cx,x\rangle_A|^2) \\
&\leq 2w_{A,e}^2(B,C).
\end{aligned}$$

Thus,

$$\begin{aligned} |\langle (B+C)x,x\rangle_A|^2 &\le 2w_{A,e}^2(B,C) - |\langle (B-C)x,x\rangle_A|^2 \\ &\le 2w_{A,e}^2(B,C) - c_A^2(B-C). \end{aligned}$$

Taking supremum over all $x \in \mathcal{H}$ with $\|x\|_A = 1$, we get

$$w_A^2(B+C) \le 2w_{A,e}^2(B,C) - c_A^2(B-C),$$

that is,

$$w_A^2(B+C) + c_A^2(B-C) \le 2w_{A,e}^2(B,C). \tag{18}$$

Similarly,

$$w_A^2(B-C) + c_A^2(B+C) \le 2w_{A,e}^2(B,C). \tag{19}$$

Combining the inequalities (18) and (19) we obtain

$$\frac{1}{2}\max\left\{w_A^2(B+C) + c_A^2(B-C), w_A^2(B-C) + c_A^2(B+C)\right\} \le w_{A,e}^2(B,C),$$

as desired. □

Note that, for A-selfadjoint operators B and C, the bound in Theorem 2.17 is of the form

$$\frac{1}{2}\max\left\{\|B+C\|_A^2 + c_A^2(B-C), \|B-C\|_A^2 + c_A^2(B+C)\right\} \le w_{A,e}^2(B,C). \tag{20}$$

Also observe that the bound obtained in Theorem 2.17 is stronger then the first bound in [14, Theorem 2.7]. Next inequality reads as follows:

Theorem 2.18. *If $B, C \in \mathbb{B}_A(\mathcal{H})$, then*

$$\max\left\{w_A^2(B) + c_A^2(C), w_A^2(C) + c_A^2(B)\right\} \le w_{A,e}^2(B,C).$$

Proof. Let $x \in \mathcal{H}$ with $\|x\|_A = 1$. Then we have,

$$|\langle Bx,x\rangle_A + \langle Cx,x\rangle_A|^2 + |\langle Bx,x\rangle_A - \langle Cx,x\rangle_A|^2 = 2(|\langle Bx,x\rangle_A|^2 + |\langle Cx,x\rangle_A|^2),$$

that is,

$$|\langle (B+C)x,x\rangle_A|^2 + |\langle (B-C)x,x\rangle_A|^2 = 2(|\langle Bx,x\rangle_A|^2 + |\langle Cx,x\rangle_A|^2).$$

This implies that

$$w_{A,e}^2(B+C, B-C) = 2w_{A,e}^2(B,C). \tag{21}$$

Now, replacing B by $B+C$ and C by $B-C$ in Theorem 2.17, we obtain

$$2\max\left\{w_A^2(B) + c_A^2(C), w_A^2(C) + c_A^2(B)\right\} \le w_{A,e}^2(B+C, B-C). \tag{22}$$

The desired inequality follows from (22) together with the equality (21).
□

Finally, we obtain the following upper and lower bounds for A-Euclidean operator radius involving A-numerical radius.

Theorem 2.19. *Let $B, C \in \mathbb{B}(\mathcal{H})$, then*

$$w_A^2(\sqrt{\alpha}B \pm \sqrt{1-\alpha}C) \le w_{A,e}^2(B,C) \le w_A^2(\sqrt{\alpha}B + \sqrt{1-\alpha}C) + w_A^2(\sqrt{1-\alpha}B + \sqrt{\alpha}C),$$

for all $\alpha \in [0,1]$.

Proof. Let $x \in \mathcal{H}$ with $\|x\|_A = 1$. Then we have,

$$\begin{aligned}
&\sqrt{\alpha}|\langle Bx, x\rangle_A| + \sqrt{1-\alpha}|\langle Cx, x\rangle_A| \\
&\leq (|\langle Bx, x\rangle_A|^2 + |\langle Cx, x\rangle_A|^2)^{\frac{1}{2}}((\sqrt{\alpha})^2 + (\sqrt{1-\alpha})^2)^{\frac{1}{2}} \\
&= (|\langle Bx, x\rangle_A|^2 + |\langle Cx, x\rangle_A|^2)^{\frac{1}{2}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(|\langle Bx, x\rangle_A|^2 + |\langle Cx, x\rangle_A|^2)^{\frac{1}{2}} &\geq |\langle \sqrt{\alpha}Bx, x\rangle_A| + |\langle \sqrt{1-\alpha}Cx, x\rangle_A| \\
&\geq |\langle \sqrt{\alpha}Bx, x\rangle_A \pm \langle \sqrt{1-\alpha}Cx, x\rangle_A| \\
&= |\langle \left(\sqrt{\alpha}B \pm \sqrt{1-\alpha}C\right)x, x\rangle_A|.
\end{aligned}$$

Taking supremum over all x in $\mathcal{H}$ with $\|x\|_A = 1$, we get the first inequality, i.e.,

$$w_{A,e}(B, C) \geq w_A(\sqrt{\alpha}B \pm \sqrt{1-\alpha}C).$$

Next, we prove the second inequality. By simple calculation, we get

$$\begin{aligned}
&|\langle Bx, x\rangle_A|^2 + |\langle Cx, x\rangle_A|^2 \\
= \;& |\langle \sqrt{\alpha}Bx, x\rangle_A + \langle \sqrt{1-\alpha}Cx, x\rangle_A|^2 + |\langle \sqrt{1-\alpha}Bx, x\rangle_A - \langle \sqrt{\alpha}Cx, x\rangle_A|^2 \\
= \;& |\langle(\sqrt{\alpha}B + \sqrt{1-\alpha}C)x, x\rangle_A|^2 + |\langle(\sqrt{1-\alpha}B - \sqrt{\alpha}C)x, x\rangle_A|^2 \\
\leq \;& w_A^2(\sqrt{\alpha}B + \sqrt{1-\alpha}C) + w_A^2(\sqrt{1-\alpha}B - \sqrt{\alpha}C).
\end{aligned}$$

Taking supremum over all x in $\mathcal{H}$ with $\|x\|_A = 1$, we get

$$w_{A,e}^2(B, C) \leq w_A^2(\sqrt{\alpha}B + \sqrt{1-\alpha}C) + w_A^2(\sqrt{1-\alpha}B - \sqrt{\alpha}C),$$

as desired. □

Remark 2.20. *(i) It is easy to verify that*

$$\begin{aligned}
w_{A,e}^2(B, C) &\geq \max_{0\leq\alpha\leq 1} w_A^2(\sqrt{\alpha}B \pm \sqrt{1-\alpha}C) \\
&\geq \frac{1}{2}\max w_A^2(B \pm C) \\
&\geq \frac{1}{2}w_A(B^2 + C^2).
\end{aligned}$$

(ii) Putting $B = \Re_A(T)$ and $C = \Im_A(T)$ in (i) we obtain that

$$\begin{aligned}
w_A^2(T) &\geq \frac{1}{2}\max\left\|\Re_A(T) \pm \Im_A(T)\right\|_A^2 \\
&\geq \frac{1}{4}\|T^\sharp T + TT^\sharp\|_A.
\end{aligned}$$

See also [16].

Declarations.
The authors have no competing interests to declare that are relevant to the content of this article.

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