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The minimum number of chains in a noncrossing partition of a poset

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Abstract. The notion of noncrossing partitions of a partially ordered set (poset) is introduced here. When the poset in question is $[n] = \{1, 2, \ldots, n\}$ with the complete order of natural numbers, conventional noncrossing partitions arise. The minimum possible number of chains contained in a noncrossing partition of a poset clearly reflects the structural complexity of the poset. For the poset $[n]$, this number is just one. However, for a generic poset, it is a challenging task to determine the minimum number. Our main result in the paper is some characterization of this quantity.

1. Introduction

Partially ordered sets are well studied objects in discrete mathematics and we will basically follow the notation in Stanley [9]. A partially ordered set (poset) is a set P with a binary relation '$\leq$' among the elements in P, where the binary relation satisfies reflexivity, antisymmetry and transitivity. The poset will be denoted by $(P, \leq)$ or P for short. For simplicity, all posets discussed in this paper are assumed to be finite.

If two elements x and y in P satisfy $x \leq y$, we say x and y are comparable. We write $x < y$ if $x \leq y$ but $x \neq y$. A chain of P is a subset of elements such that any two elements there are comparable, while an antichain is a subset where any two elements are not comparable. A chain decomposition of P is a family of disjoint chains $\{C_1, C_2, \ldots, C_k\}$ such that $\bigcup_{i=1}^{k} C_i = P$. Let $Min(P)$ denote the minimum number of chains that are contained in a chain decomposition of P, and let $Anti(P)$ denote the maximum number of elements that can be contained in an antichain of P. These quantities reflect the structural complexity of the posets in question. For instance, if there is a complete order in P, then $Min(P) = 1$, and if there is no order at all, $Min(P) = |P|$ (i.e., the number of elements in P). The celebrated Dilworth's theorem [4] states that $Min(P) = Anti(P)$ for all finite P.

Chain decompositions with various constraints have been studied, e.g., symmetric chain decomposition [5], canonical symmetric chain decomposition [6], etc., which reflect the structural properties and complexity of posets from different angles. Here we introduce a new chain decomposition which can be viewed as a generalization of the ubiquitous object in combinatorics, i.e., noncrossing partitions (e.g., see Armstrong [1] and Simion [8]). As such, we call the new decompositions noncrossing partitions of posets. Specifically, a noncrossing partition of the set $[n] = \{1, 2, \ldots, n\}$ is merely a noncrossing partition of the poset $[n]$ with the natural order. Note that $[n]$ itself is a noncrossing partition of $[n]$. That is, the minimum

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number of chains contained in a noncrossing partition is simply one in this case. However, determining the minimum number of chains in a noncrossing partition for a general poset is a challenging task.

Our main goal of this note is to provide some characterization of the minimum possible number of chains contained in a noncrossing partition of a generic poset.

2. Noncrossing partitions of posets

Recall a noncrossing partition (see [1, 8]) of the set $[n]$ is a partition of $[n]$ into $k \geq 1$ blocks $B_1, B_2, \ldots, B_k$ such that there do not exist elements $a, b \in B_i$ and $c, d \in B_j$ ($i \neq j$) such that $a < c < b < d$. For example, for $n = 5$, $B_1 = \{1, 5\}$ and $B_2 = \{2, 3, 4\}$ give a noncrossing partition of $[5]$ into two blocks, while $B_1 = \{1, 3, 5\}$ and $B_2 = \{2, 4\}$ do not give a noncrossing partition. Evidently, the definition depends on the natural order on $[n]$. Regarding $[n]$ as a poset, B_i is just a chain and a partition is just a chain decomposition. What if we replace $[n]$ with an arbitrary poset?

Before we proceed, it will be convenient to represent a poset by introducing its Hasse diagram. For two elements x and y in a poset P, if $x < y$ and there does not exist z such that $x < z < y$, then we say y covers x. The Hasse diagram of P is the graph with the elements of P as the vertices, and with the covering relation giving the edges, and if y covers x then y is drawn "above" x (with an edge between x and y). Note the whole partial relation can be derived by applying the transitivity based on the Hasse diagram.

Definition 2.1. *A chain decomposition of a poset P, $\{C_1, C_2, \ldots, C_k\}$, is called a noncrossing partition of P, if there do not exist elements $a, b \in C_i$ and $c, d \in C_j$ ($i \neq j$) such that $a < c < b < d$ in P.*

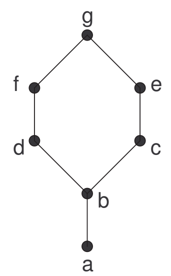

Figure 1: A poset of 7 elements represented by its Hasse diagram.

For example, for the poset in Figure 1, the partition $\{\{a, b, c, e\}, \{d, f, g\}\}$ is a noncrossing partition, while $\{\{a, c, e\}, \{b, d, f, g\}\}$ is not since $a < b < c < g$. We denote by $Min_{nc}(P)$ the minimum number of chains contained in a noncrossing partition of P. For P in Figure 1, $Min_{nc}(P) = 2$. Clearly, $Min_{nc}(P) = 1$ if and only if $P \sim [n]$. However, it is not easy to exactly determine this number for a generic poset. Nevertheless, by relating noncrossing partitions to other notion, we are able to prove some bounds.

Definition 2.2. *Let $(P, \leq)$ be a poset. A homogeneous chain decomposition (HCD) $\mathcal{C}$ of P is a collection of mutually disjoint chains $C_1, C_2, \ldots, C_n$ such that $\bigcup_i C_i = P$, and if $s_i \in C_i$ and $s_j \in C_j$ are comparable, then all elements in C_i and C_j are pairwise comparable.*

In the example of Figure 1, $\{\{a, b, g\}, \{c, e\}, \{d, f\}\}$ gives an HCD. When all elements in two chains C_i and C_j are pairwise comparable, i.e., $C_i \bigcup C_j$ is a chain, we say C_i and C_j are comparable for short. We also write $C = (\xi_1 < \xi_2 < \cdots < \xi_s)$ as a shorthand of that C is the chain $\{\xi_1, \xi_2, \ldots, \xi_s\}$ and $\xi_1 < \xi_2 < \cdots < \xi_s$.

Denote by $|\mathcal{C}|$ the number of chains contained in $\mathcal{C}$. Let $Min_h(P) = \min_{\mathcal{C}} |\mathcal{C}|$, where the minimization is over all HCDs of P. An HCD of P containing exactly $Min_h(P)$ chains is called a minimal homogeneous chain decomposition (MHCD) of P. It turns out there is only one such a decomposition. For instance, for P in Figure 1, $Min_h(P) = 3$ and $\{\{a, b, g\}, \{c, e\}, \{d, f\}\}$ is actually the only MHCD.

Proposition 2.3. *For any poset P, there exists a unique MHCD of P.*

Proof. Let $C = \{C_1, C_2, \ldots, C_m\}$ be a MHCD of P. If $|C_i| = 1$ for all $1 \le i \le m$, there is nothing to prove. Thus, we assume that there exists at least one i such that $|C_i| > 1$. Let $C' = \{C'_1, C'_2, \ldots, C'_m\}$ be a different MHCD of P. First, there exists k and s_{k1}, $s_{k2} \in C_k$ such that $s_{k1} \in C'_{j1}$, $s_{k2} \in C'_{j2}$ and $C'_{j1} \neq C'_{j2}$. Otherwise, it is not hard to argue $C = C'$. Next, since s_{k1} and s_{k2} are comparable, C'_{j1} and C'_{j2} are comparable. Thus, $C^* = C'_{j1} \bigcup C'_{j2}$ is a chain of P.

We claim $(C' \setminus \{C'_{j1}, C'_{j2}\}) \bigcup \{C^*\}$ is an HCD of P. For any $j \notin \{j1, j2\}$, C'_j is either comparable to C'_{j1} or not comparable to C'_{j1}. For the former case, there exists $s_j \in C'_j$ comparable to s_{k1}. Since s_{k1} and s_{k2} come from the same chain C_k, regardless of whether $s_j \in C_k$, s_{k2} must be comparable to s_j as well. Hence, C'_j is also comparable to C'_{j2} so that C'_j is comparable to C^*. For the latter case, we can analogously show C'_j is not comparable to C^*. Thus, the claim holds. However, this contradicts the assumption that C' is minimum. Hence, C is the unique MHCD of P. □

HCDs were first introduced in Chen and Reidys [3] in the context of studying the interaction between incidence algebras of posets and linear sequential dynamical systems, where in particular, it was shown that the Möbius function of any poset can be efficiently computed via a sequential dynamical system and a cut theorem concerning HCDs of posets holds.

Another notion that we need is a generalization of 132-avoiding permutations, another ubiquitous object in combinatorics and computer science.

Definition 2.4. *A permutation $\pi = \pi_1\pi_2\cdots\pi_n$ of the elements of P is called* 132*-avoiding if no three-element subsequence $\pi_{i_1}\pi_{i_2}\pi_{i_3}$ in π satisfies $i_1 < i_2 < i_3$ while $\pi_{i_1} < \pi_{i_3} < \pi_{i_2}$ in P.*

In the case of $P = [n]$, conventional 132-avoiding permutations arise. For example, when $P = [5]$, 53241 is a 132-avoiding permutation, while 21453 is not. Because in the latter, we realized that the subsequences 243, 253, 143, 153 all violate the definition. A linear extension of P is a permutation $e = e_1e_2\cdots e_n$ of P-elements such that $e_i < e_j$ implies $i < j$. For example, for the poset P in Figure 1, $abcdefg$ and $abcedfg$ are linear extensions. There are more than one linear extension unless $P \sim [n]$.

Definition 2.5. *Let $e = e_1e_2\cdots e_n$ be a linear extension of P. A permutation $\pi = \pi_1\pi_2\cdots\pi_n$ of the elements of P is called* 132^e*-avoiding (i.e.,* 132*-avoiding with respect to e) if there does not exist a subsequence $\pi_{i_1}\pi_{i_2}\pi_{i_3} = e_{j_1}e_{j_2}e_{j_3}$ such that $i_1 < i_2 < i_3$ and $j_1 < j_3 < j_2$.*

For example, $gedfbac$ is 132-avoiding w.r.t. the linear extension $abcdefg$ of P in Figure 1, while $abcedfg$ is not due to the appearance of the subsequence ced. It is easily seen that a 132^e-avoiding permutation is a 132-avoiding permutation of P. Given a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ of P, i is called a p-descent of π if $\pi_i > \pi_{i+1}$ or π_i is not comparable with π_{i+1} in P or $i = n$. The number of p-descents in π is denoted by $d_P(\pi)$. For P in Figure 1 and $\pi = gedfbac$, it can be checked that $d_P(\pi) = 5$, i.e., $i = 1, 2, 4, 5, 7$. Let

$$Min_d(P) = \min\{d_P(\pi) : \pi \text{ is a 132-avoiding permutation of } P\},$$
$$Min_d^e(P) = \min\{d_P(\pi) : \pi \text{ is a } 132^e\text{-avoiding permutation of } P\}.$$

Now we are in a position to present our main result.

Theorem 2.6 (Main theorem). *For any poset P, there exists a linear extension e of P such that*

$$Min_{nc}(P) \le Min_d(P) \le Min_d^e(P) \le Min_h(P). \tag{1}$$

Moreover, all inequalities are sharp.

We remark that the rightmost inequality is not necessarily true for an arbitrary linear extension. For example, for the poset P in Figure 2, it is easy to see $Min_h(P) = 2$. However, for its linear extension $e = abxy$, there are 14 132^e-avoiding permutations all of which have at least three p-descents. In Figure 2, the number of descents of a 132^e-avoiding permutation is written right after the 132^e-avoiding permutation.

For instance, "*baxy* : 3" means that *baxy* has three p-descents. Moreover, the reason that we are interested in $Min_d^e(P)$ is as follows: while it may be hard to generate all 132-avoiding permutations of P to compute $Min_d(P)$, it is easy to generate all 132^e-avoiding permutations for any linear extension e of P as we shall see it is essentially generating all plane trees. A proof of the above theorem follows from a series of properties that we are about to present.

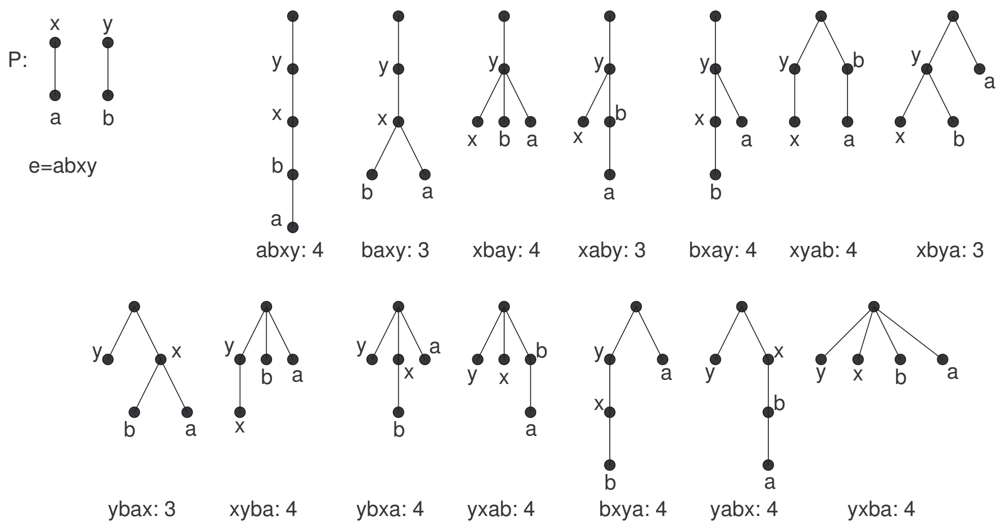

Figure 2: A poset P with a linear extension e such that $Min_d^e(P) > Min_h(P)$.

Assume $\pi = \pi_1\pi_2\cdots\pi_n$ is a permutation of a poset P. Read π from left to right and collect these elements between two consecutive p-descents, excluding the first one and including the second. By definition of p-descents, these elements comprise a chain. In this way, all p-descents of π induce a chain decomposition of P.

Proposition 2.7. *Let π be a* 132*-avoiding permutation of a poset P. Then, the induced chain decomposition by the p-descents of π is a noncrossing partition of P.*

Proof. If not, without loss of generality, suppose π_1, π_2 from the first induced chain and π_3, π_4 from the second induced chain cross, i.e., $\pi_1 < \pi_3 < \pi_2 < \pi_4$ or $\pi_3 < \pi_1 < \pi_4 < \pi_2$. Obviously, either case implies a 132 pattern in π, a contradiction whence the proposition. □

As a result, we immediately have $Min_{nc}(P) \le Min_d(P) \le Min_d^e(P)$ for any linear extension e of P. If otherwise explicitly stated, we assume the following notation in the rest of the section. Let $C = \{C_1, C_2, \ldots, C_k\}$ be the MHCD of P, where

$$C_i = (s_{i1} < s_{i2} < \cdots < s_{im_i}), \quad \sum_{i=1}^{k} m_i = n.$$

Lemma 2.8. *If C_i and C_j are comparable, then there exists $0 \le l \le m_i$ such that*

$$s_{i1} < s_{i2} < \cdots < s_{il} < s_{j1} < s_{j2} < \cdots < s_{jm_j} < s_{i(l+1)} < s_{i(l+2)} < \cdots < s_{im_i}.$$

Proof. In order to prove the lemma, it suffices to show that there does not exist $0 < l_1 < m_i$ and $0 < l_2 < m_j$ such that

$$s_{il_2} < s_{jl_1} < s_{i(l_2+1)} < s_{j(l_1+1)}.$$

Assume by contradiction that such l_1 and l_2 exist. For any other chain C_k, if C_k is comparable to C_i and $x \in C_k$, then either $x < s_{i(l_2+1)}$ or $x > s_{i(l_2+1)}$. In any case, we conclude that an element in C_j is comparable to x

whence C_j and C_k are comparable. By similar analysis, we can conclude that if C_k is not comparable to C_i, then C_k is not comparable to C_j either. Therefore, $(C \setminus \{C_i, C_j\}) \bigcup \{C_i \cup C_j\}$ is an HCD of P. This contradicts the assumption that C is the minimum and the lemma follows. □

Consider the relation $\leq_b$ on C that $C_i \leq_b C_j$ if there exist elements $x, z \in C_j$ and $y \in C_i$ such that $x < y < z$ or $\min(C_i) > \max(C_j)$. As for the first case, we say C_j wrap around C_i or C_i can be wrapped around by C_j. In view of Lemma 2.8, we leave it to the reader to verify that $(C, \leq_b)$ is a well-defined poset.

Proposition 2.9. *Suppose $C_1 C_2 \cdots C_k$ is a linear extension of $(C, \leq_b)$. Then the following permutation π is 132-avoiding and has k p-descents:*

$$\pi = s_{11} s_{12} \cdots s_{1m_1} s_{21} \cdots s_{2m_2} \cdots s_{k1} \cdots s_{km_k}.$$

Proof. By definition, it is easy to see there are exactly k p-descents in π. We prove the rest by contradiction. Suppose $\pi_{l_1}\pi_{l_2}\pi_{l_3}$ is a 132 pattern in π. Since each C_i appears as an increasing chain in π, we have only two possible cases:

- $\pi_{l_1}, \pi_{l_2} \in C_i$, $\pi_{l_3} \in C_j$, and $i < j$;
- $\pi_{l_1} \in C_i, \pi_{l_2} \in C_j$, $\pi_{l_3} \in C_k$, and $i < j < k$.

The first case cannot happen because the condition implies that $C_j <_b C_i$ in the light of Lemma 2.8, contradicting the assumption of the proposition. Next suppose the second case occurs. First, $\pi_{l_1} < \pi_{l_2}$ and $C_i <_b C_j$ imply that $\pi_{l_1} > x_j$ for some $x_j \in C_j$, i.e., C_j wrap around C_i. Analogously, C_k wrap around C_i. Secondly, $\pi_{l_2} > \pi_{l_3}$ and $C_j <_b C_k$ imply that either $\min(C_j) > \max(C_k)$ or C_k wrap around C_j. Since both C_j and C_k can wrap around C_i, the former is absurd. On the other hand, that C_k wrap around C_j while C_j wrap around C_i makes it impossible to have a 132 pattern $\pi_{l_1}\pi_{l_2}\pi_{l_3}$ such that $\pi_{l_1} \in C_i, \pi_{l_2} \in C_j$, $\pi_{l_3} \in C_k$. Hence, no 132 patterns exist in π, completing the proof. □

From Lemma 2.8 and Proposition 2.9, we conclude

$$Min_{nc}(P) \leq Min_d(P) \leq Min_h(P).$$

But we cannot conclude $Min_d^e(P) \leq Min_h(P)$ for an arbitrary linear extension e.

We proceed with further analysis below, where on the way we need to make use of plane trees. A plane tree T can be recursively defined as an unlabeled tree with one distinguished vertex called the root of T, where the unlabeled trees obtained by deleting the root as well as its adjacent edges from T are linearly ordered, and they are plane trees with the vertices adjacent to the root of T in T as their respective roots. These subtrees are pictured as locating below the root and appearing left to right. A non-root vertex without any child is called a leaf, and an internal vertex otherwise. The root is always treated as internal. A labelled plane tree is a plane tree where vertices carry mutually distinct labels from a certain set of labels. The preorder of the vertices in a labelled plane tree T is the sequence obtained by travelling T in a left-to-right depth-first manner and recording the label of a vertex when it is first visited. See Figure 3 for an example.

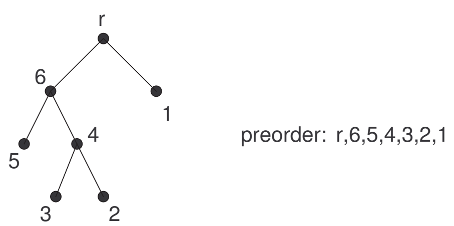

Figure 3: A labelled plane tree and the preorder of its vertices.

There is a bijection between plane trees and conventional 132-avoiding permutations given by Jani and Rieper [7]. The following is how it works. Let T be a plane tree of n edges. We use a preorder traversal of T to label the non-root vertices with the integers $n, n-1, \ldots, 1$. As such, the first vertex visited gets the label n and the last receives 1. A permutation written as a word is next obtained by reading the labelled tree in postorder, that is, traverse the tree from left to right and record the label of a vertex when it is last visited. It was shown [7] that the obtained permutation is 132-avoiding on $[n]$. As an example, for the tree in Figure 3, the obtained 132-avoiding permutation is 532461.

The reverse from a 132-avoiding permutation to a plane tree was not explicitly presented in Jani and Rieper [7]. A reverse procedure was proposed in [2] and is presented here for later use. Let π be a 132-avoiding permutation on $[n]$. Suppose the (increasing) chains induced by the p-descents of π from left to right are $\tau_1, \tau_2, \ldots, \tau_k$. Start with τ_k and make it into a path with the maximum (i.e., rightmost) element in τ_k attaching to the root of the expected plane tree T. For example, suppose $\pi = 532461$. Then, $k = 4$ and $\tau_k = 1$, and the path will be the path from vertex 1 to the root of the tree in Figure 3. After τ_i is "integrated" into the (partial) tree, we find the minimum element u in the path from the leftmost leaf to the root in the current partial tree that is larger than the maximum element x in τ_{i-1}, and attach the path induced by τ_{i-1} to the tree such that u and x are adjacent; if no such a u exists, we attach the path induced by τ_{i-1} to the root of the current tree. Iteration of the procedure eventually yields a labelled plane tree. (The vertex labels are uniquely determined by the underlying plane tree.)

In the forthcoming result, a straightforward generalization of 132^e-avoiding permutations of P from a linear extention e to an arbitrary permutation e of P will be used.

Proposition 2.10. *Suppose $C_1C_2\cdots C_k$ is a linear extension of the poset $(\mathcal{C}, \leq_b)$. Then, there exists a labelled plane tree T with non-root vertex labels from P such that π in Proposition 2.9 is 132^e-avoiding, where e is the reverse of the preorder of the vertices other than the root of T.*

Proof. First, we use the chains C_i to build a labelled plane tree following the same procedure from the induced chains of 132-avoiding permutations to plane trees described above. We then claim the obtained tree is the desired T. To see this, one has to realize that the Jani-Rieper bijection essentially encodes the relation among the non-root vertex sequences from the preorder, postorder and the reverse of the preorder. In a word, the postorder is 132-avoiding with respect to the reverse of the preorder. Actually, this is how we constructed all 132^e-avoiding permutations in Figure 2. In particular, when the preorder is $n, n-1, \ldots, 1$, the postorder gives a conventional 132-avoiding permutation on $[n]$. The rest is clear, completing the proof. □

It remains to prove that there exists a linear extension of $(\mathcal{C}, \leq_b)$ of which the corresponding e is in fact a linear extension of P. We need one more important lemma to that end.

Lemma 2.11. *Suppose $\{C_{i_1}, C_{i_2}, \ldots, C_{i_{k'}}\}$ is a subposet of $(\mathcal{C}, \leq_b)$, and $C_{i_1}, C_{i_2}, \ldots, C_{i_m}$ are the maximal elements of the subposet. Then, any C_{i_j} for $m+1 \leq j \leq k'$ satisfies either one of the cases:*

(1) *for at least one t ($1 \leq t \leq m$), $\min(C_{i_j}) > \max(C_{i_t})$;*

(2) *for a unique t ($1 \leq t \leq m$), C_{i_t} wrap around C_{i_j}.*

In addition, two case (2) *elements wrapped around by distinct maximal elements are not comparable, while a case* (1) *element is smaller than a case* (2) *element if comparable and the minimal (P-element) of the former is greater than the maximal of the latter.*

Proof. For any $m+1 \leq j \leq k'$, by assumption, C_{i_j} is smaller than at least one maximal element. We first show that C_{i_j} cannot satisfy both cases. Suppose $\min(C_{i_j}) > \max(C_{i_t})$ for some $1 \leq t \leq m$. If C_{i_j} can be wrapped around by another maximal element $C_{i_{t'}}$, then it is easy to see that C_{i_t} and $C_{i_{t'}}$ are comparable, a contradiction. Analogously, an element satisfying (2) cannot satisfy (1) at the same time. Moreover, an element cannot be wrapped around by more than one maximal element.

If two case (2) elements wrapped around by distinct maximal elements are comparable, either the minimal P-element (i.e., element in P) of one is greater than the maximal P-element of the other or one wrap around the other. Either case implies the two involved distinct maximal elements are comparable, contradicting the maximality. The remaining statement can be similarly verified, and the proof follows. □

Proposition 2.12. *There exists a linear extension of* $(C, \leq_b)$*, still denoted by* $C_1C_2\cdots C_k$*, of which the corresponding* e *referred to in Proposition 2.10 is a linear extension of* P*.*

Proof. Our strategy here is to construct a linear extension of $(C, \leq_b)$ first and then argue the corresponding e is a linear extension of P.

Construct a linear extension of $(C, \leq_b)$**.** First, we apply the following procedure.

(i) Initialize $j = 0$, $W_0 = (C, \leq_b)$ and $F = W_0$;

(ii) Arrange the maximal elements of F on a line in an arbitrary way;

(iii) Put each of those case (2) elements w.r.t. F right before the maximal element in F that wrap around it (and after the preceeding maximal element) and order those right before the same maximal element later. Next, denote the set of case (1) elements w.r.t. F by W_{j+1} and put them in front of the current "partially linearized" sequence in an arbitrary way. Update F to W_{j+1} and j to $j + 1$;

(iv) Iterate (ii) and (iii) until F is an empty set.

At this point, all elements of $(C, \leq_b)$ are in a sense grouped into disjoint ordered groups. The involved maximal C-elements (w.r.t. a certain iteration) serve as a kind of group markers. (The group marker of a group is on the right-hand side.) See Figure 4 for an illustration, where C_m and the case (2) elements wrapped around by C_m give an example of a group. The groups obtained so far will be referred to as type I groups. By construction, the maximum P-element contained in a group marker is larger than (in terms of $(P, \leq)$) all other P-elements contained in the chains in the same group. Moreover, in view of Lemma 2.11, any C-element in a left group is smaller than any C-element in a right group if comparable, not violating the current sequence to possibly become a linear extension of $(C, \leq_b)$.

Iteratively apply the above procedure to each type I group with the group marker excluded and each of those newly generated groups (excluding the group markers) in the process until each non-empty group contains a single element. It is a kind of successive "linearization". Eventually, we obtain a linear extension of $(C, \leq_b)$.

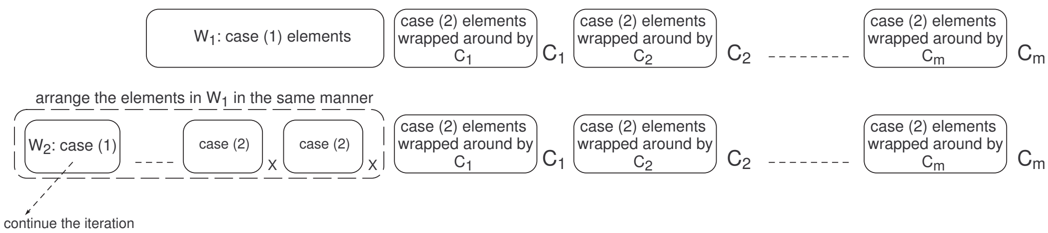

Figure 4: Construct a linear extension of $(C, \leq_b)$.

Assume the resulting linear extension is $C_1C_2\cdots C_k$, and its corresponding tree from the reverse procedure of the Jani-Rieper bijection is T. Note that in terms of Figure 4, C_k here is actually C_m, i.e., the rightmost chain. We next show that **the reverse** e **of the preorder of the vertices other than the root of** T **is a linear extension of** P**,** which is equivalent to showing that for any two entries in the preorder, the left one is greater than the right one if comparable in P. To this end, for any vertex u in T, consider the subtree T_u with u as the root. It suffices to show: (i) u is greater than any of its descendants (in terms of the labels in P) where the root of T is assumed to be an artificial maximum element added into P; (ii) any vertex in a left subtree of T_u is greater than any vertex in a right subtree of T_u if comparable.

Suppose u is the root of T. Then, $T_u = T$. According to the construction of the linear extension $C_1C_2\cdots C_k$ and the tree T, P-elements contained in chains belonging to distinct type I groups (including respective group markers) are contained in distinct subtrees of T. Thus, a P-element in a left subtree of T_u is greater than a P-element in a right subtree of T_u if comparable. So, the above two requirements (i) and (ii) hold for this case.

We next examine the cases where u is the root of a subtree of T. Recall that the maximum P-element contained in a group marker is the maximum of the whole group. Then, the maximum P-element must be the root of the corresponding subtree of T formed by the P-elements in the group in view of the building process of T. Without loss of generality, we take the rightmost type I group, i.e., the one with C_k as the group marker, to continue the analysis. In this case, $u = s_{km_k}$. The requirement (i) is clear since s_{km_k} is the maximum P-element. As for the requirement (ii), suppose in the linear extension $C_1C_2 \cdots C_k$, the chains contained in the subsequence $C_lC_{l+1} \cdots C_k$ constitute the type I group with C_k as the group marker. Noticing that when restricted to this subsequence, the constructed plane tree is exactly the subtree T_u with s_{km_k} as the root. Then the requirement (ii) concerning the vertices in the subtrees of T_u follows by the same token as that for the subtrees of T verified above.

Iterating the above reasoning for u being a vertex in T in a kind of "top-down" manner, we conclude that the requirements (i) and (ii) hold for all vertices. Therefore, e, the reverse of the preorder of T is a linear extension of P, completing the proof. □

Now it is not hard to piece all properties above together to arrive at Theorem 2.6. Obviously, when P is itself a chain, all inequalities become equalities whence the sharpness claim. We end this paper with some **future study problems**: (1) in which more general cases can some of the equalities be achieved in Theorem 2.6, e.g., when $Min_{nc}(P) = Min_h(P)$? and (2) how many noncrossing partitions are there for a given poset P? Note that the answer is given by the famous Catalan numbers when $P = [n]$.

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