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Convergence with respect to admissible sequences of weights for a holomorphic space of a sequence of Hilbert-valued rational functions

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Abstract. Let $D \subset \mathbb{C}^N$ be a domain, H be a locally convex space with the topology defined by a sequence of Hilbert semi-norms. Denote $\mathscr{H}(D, H)$ the space of H-valued holomorphic functions on D and $\mathscr{R}(\mathbb{C}^N, H)$ the space of H-valued rational functions on $\mathbb{C}^N$. In this paper, we set up terminology of *an admissible sequence* $\mathcal{W} = \{w_m\}_{m\geq 1}$ *of weights for* $\mathscr{H}(D, F)$ and study sufficient conditions on $\mathcal{W}$ to ensure convergence of $w_m(\|r_m - f\|_m^2)$ (pointwise/ in capacity/ uniformly on compact subsets) to 0 on the whole domain D except for a pluripolar subset provided the pointwise (rapid) $\mathcal{W}$-convergence of $\{r_m - f\}_{m\geq 1}$ to 0 occurs on a Borel non-pluripolar subset X for every $f \in \mathscr{H}(D, H)$ and $\{r_m\}_{m\geq 1} \subset \mathscr{R}(\mathbb{C}^N, H)$, in both of cases X lies in D, and X lies in the boundary ∂D of a bounded domain D.

1. Introduction

Let E, F be locally convex spaces over $\mathbb{C}$ and D be a domain in E. The problem of Tauberian convergence is to look for additional properties to ensure that every sequence of F-valued functions defined on D is convergent everywhere on D whenever it is convergent on a subset X of D. Vitali's theorem is an important result of this problem for sequences of scalar-valued holomorphic functions, where subsets X admitting a limit point in D are considered and local uniform boundedness is a possible additional property. A classical theorem of Vitali (see [9, Proposition 7]) says that a sequence $\{f_m\}_{m\geq 1}$ of holomorphic functions on a domain D in $\mathbb{C}^N$ converges uniformly on compact subsets of D to a holomorphic function f if it is uniformly bounded on compact subsets of D and converges pointwise to f on a subset X which is not contained in any proper analytic subset of D. Recently, some authors (see [1] and [10]) have proved that Vitali's theorem is still valid in several cases of vector-valued functions.

The subject of Tauberian convergence (in different modes) for scalar-valued rational functions of several complex variables has received considerable attention in the past many years. In 1974, Gonchar [4] proved that a sequence $\{r_m\}_{m\geq 1}$ of rational functions of $\deg r_m \leq m$ converging rapidly in measure on an open set X to a holomorphic function f defined on a domain $D \supset X$ in the sense that for every $\varepsilon > 0$

$$\lim_{m\to\infty} \lambda_{2n}(\{z \in X : |r_m(z) - f(z)|^{1/m} > \varepsilon\}) = 0,$$

must converge rapidly in measure to f on the whole domain D. Here, λ_{2n} is the Lebesgue measure in $\mathbb{R}^{2n}$. Later, Bloom proved an analogue result which says that if a sequence $\{r_m\}_{m\geq 1}$ of rational functions converges

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rapidly on a Borel non-pluripolar subset X of D, then the rapid convergence occurs on every Borel subset of D ([2, Theorem 2.1]). Furthermore, under the assumptions of Gonchar's theorem, Bloom showed that the convergence of $\{r_m\}_{m\geq 1}$ to f is rapid in capacity on D (see [2, Theorem 2.2]).

Motivating from the above results, we propose now to examine a more general framework for vector-valued case. We would like to find sequences $\mathcal{W} = \{w_m\}_{m\geq 1}$ of weights on $[0, \infty)$ such that from the convergence (pointwise/ in capacity) of $w_m(\|(r_m - f)(\cdot)\|_m^2)$ to 0 on a *Borel non-pluripolar subset* X of D (and of ∂D) in $\mathbb{C}^N$ it follows that $\{r_m\}_{m\geq 1}$ somehow converges to f on the *whole domain* D. Here, $r_m \in \mathscr{R}(\mathbb{C}^N, F)$, the space of F-valued rational functions on $\mathbb{C}^N$, $f \in \mathscr{H}(D, F)$, the space of F-valued holomorphic functions on D and $\mathcal{P} := \{\|\cdot\|_m\}_{m\geq 1}$ is a family of semi-norms defining the topology of the space F.

Note that, the studies in the scalar-case previously published are of the case $w_m(t) = t^{1/2m}$.

In our approach, the logarithmic plurisubharmonicity of functions $w(\|f(\cdot)\|^2)$, where $w \in \mathcal{W}$ and $\|\cdot\| \in \mathcal{P}$, for every $f \in \mathscr{H}(D, F)$ is very important. Let us orient weights w, which belong to the class $\mathscr{C}^2(0, +\infty)$, by requiring that they satisfy the following

$$w(t)[tw''(t) + w'(t)] \geq t[w'(t)]^2 \quad \forall t > 0$$

to guarantee the logarithmic plurisubharmonicity of $w(\|f(\cdot)\|^2)$. The idea for proposing this requirement arose from the fact that this inequality holds for the case when f is a scalar-valued holomorphic function of one complex variable. However, it is difficult to calculate derivatives of $w(\|f(\cdot)\|^2)$ in the general case. Fortunately, we can calculate explicitly the derivatives of higher orders of $\|\cdot\|^2$ in the case $\|\cdot\|$ is a Hilbert semi-norm. This is weighty reason why, in this paper, we consider the functions with values in a locally convex space with topology defined by a sequence of Hilbert semi-norms.

To implement this idea, in Section 3 we set up terminology of *an admissible sequence of weights for* $\mathscr{H}(D, F)$ which will be used throughout this paper. A sufficient condition for the admissibility of a sequence of weights in the $\mathscr{C}^2$-class will be proved in this section. Moreover, we also present several examples to show that the existence of this notion is not trivial.

In the foregoing section 2, beside the review of some standard notations in the theory of locally convex spaces and in pluripotential theory, we introduce a few concepts related to convergence with respect to a sequence $\mathcal{W}$ of weights and sequence $\mathcal{P}$ of continuous semi-norms in a locally convex space F, say $\mathcal{W}$-convergence, $\mathcal{WP}$-convergence (pointwise/in capacity) of a sequence $\{f_m\}_{m\geq 1}$ of F-valued functions on $D \subset \mathbb{C}^N$. A modification of the classical Bernstein-Walsh's inequality on estimating for a sequence of sup-norms of polynomials (Lemma 2.5) which help us in establishing some techniques results for the proofs of main theorems is also presented.

Section 4 is devoted to the stating the main theorems of the paper and some related comments. We have two cases to consider.

For the case where *the Borel non-pluripolar subset X lies in D*, in Theorem 4.2 we give conditions on an admissible sequence $\mathcal{W}$ of weights for $\mathscr{H}(D, H)$ to ensure the $\mathcal{W}$-convergence in capacity on the whole domain $D \subset \mathbb{C}^N$ of a sequence $\{r_m\}_{m\geq 1} \subset \mathscr{R}(\mathbb{C}^N, H)$, $\deg r_m \leq m$, to f provided that the pointwise $\mathcal{W}$-convergence occurs on X. Under this condition we also show that there is a pluripolar subset E of $\mathbb{C}^N$ such that on every affine complex subspace L through point $z_0 \in D \setminus E$ we can find a subsequence $\{r_{m_j}\}_{j\geq 1}$ of $\{r_m\}_{m\geq 1}$ such that it is $\mathcal{W}^{m_j}$-convergent in capacity (with respect to the connected component of $D \cap L$ containing z_0) to f, where $\mathcal{W}^{m_j} = \{w_{m_j}\}_{j\geq 1} \subset \mathcal{W}$. Moreover, a sufficient condition for $\mathcal{W}$-convergence uniformly on compact subsets of D of $\{r_m\}_{m\geq 1}$ is also given in this theorem.

Theorem 4.4 is another version Theorem 4.2, in which *the Borel non-pluripolar subset X is not inside D but lies in the boundary* ∂D of the bounded domain D and the function f is replaced by a sequence $\{f_m\}_{m\geq 1}$ of holomorphic functions that is convergent uniform on bounded, proper subsets of D.

It should be noted that in the particular case the admissible sequence $\mathcal{W}$ of weights $w_m(t) := t^{1/2m}$ (it is called the fundamental admissible sequence, in this paper) our results recover the above mentioned results of Bloom and Gonchar.

Section 4 ends up with analogous versions of Theorems 4.2, 4.4 for a faster $\mathcal{W}$-convergence in the sense of $\mathcal{WP}$-convergence where $\mathcal{P}$ is an increasing sequence of Hilbert semi-norms of a locally convex space (Theorem 4.6).

The proofs of the main theorems are presented in remain sections of the paper. Observe that, in contrast to the somewhat delicate methods used by Bloom, our approach is based only on basic properties of plurisubharmonic functions. Moreover, the method used in the original proof of Bloom does not seem tractable in the case where the set under consideration sits in the boundary of the domain, because it relies heavily on a comparison theorem of Alexander and Taylor on comparing the relative capacity of a Borel subset of a domain and its global (or Siciak) capacity.

Throughout this paper, we will denote by $\mathcal{W}^{m_j}$ the subsequence $\{w_{m_j}\}_{j\geq 1}$ of $\mathcal{W}$.

2. Preliminaries and Auxiliary Results

2.1. Locally Convex Spaces

Standard notations of the theory of locally convex spaces as presented in the book of Meise and Vogt [7] will be used in the paper. A locally convex space always is a complex vector space with a locally convex Hausdorff topology.

A semi-norm p on a vector space E is called a Hilbert semi-norm if there exists a semi-scalar product $\langle \cdot, \cdot \rangle$ on E with $p(x) = \sqrt{\langle x, x\rangle}$ for all $x \in E$. Then the local Banach space E_p is obviously a Hilbert space.

2.2. Pluripotential Theory

For the reader convenience, we recall standard elements of pluripotential theory that will be need later on.

Let E and F be locally convex spaces and D be a domain in E. A function $f : D \to F$ is called *Gâteaux holomorphic* if for every $a \in D, b \in E$ and $\varphi \in F'$, the scalar-valued function of one complex variable

$$\lambda \mapsto (\varphi \circ f)(a + \lambda b)$$

is holomorphic on a neighborhood of $0 \in \mathbb{C}$.

The function f is said to be *holomorphic* if it is Gâteaux holomorphic and continuous.

By $\mathscr{H}(D, F)$ we denote the vector space of all holomorphic functions on D with values in F. The space $\mathscr{H}(D, F)$ equipped with the compact-open topology τ_0.

Let D be a domain in a locally convex space E. An upper-semicontinuous function $\varphi : D \to [-\infty, +\infty)$ is said to be *plurisubharmonic* (and write $\varphi \in psh(D)$) if φ is subharmonic on every one dimensional section of D.

A subset $B \subset E$ is said to be *pluripolar* in D if there exists $\varphi \in psh(D)$ such that $\varphi \not\equiv -\infty$ and $\varphi\big|_B = -\infty$. We say that B is *non-pluripolar* if B is not pluripolar.

Evidently, a proper complex subvariety of a domain $D \subset \mathbb{C}^n$ is pluripolar. The set $X := \prod_{j=1}^{\infty} X_j$ where $X_j \subset \mathbb{C}$, $j \in \mathbb{N}$, is non-pluripolar if and only if each X_j is non-pluripolar for all $j \in \mathbb{N}$ (see [11]).

For a Borel set X in a bounded domain $D \subset \mathbb{C}^n$, the relative capacity of X in D is defined as follows

$$\mathrm{cap}(X, D) := \sup\left\{ \int_X (dd^c u)^n : \ u \in psh(D), -1 < u < 0 \right\}$$

where $(dd^c)^n$ is the Monge-Ampère operator. The subadditivity and monotonicity under increasing sequences are some in many important properties of the relative capacity. Moreover, in Bedford-Taylor's theory, a deep result says that pluripolar sets in D are exactly those with vanishing relative capacity.

2.3. Notions of Weighted Convergences

Definition 2.1. *A nondecreasing continuous function* $w : [0, \infty) \to [0, \infty)$ *is called a weight if* $w \geq 0$, $w(0) = 0$, $w(t) > 0$ *on* $(0, \infty)$.

Let $\|\cdot\|$ be a continuous semi-norm in a locally convex space F.

Definition 2.2. *For a domain $D \subset \mathbb{C}^n$, we say that a function $f : D \to F$ is Borel $\|\cdot\|$-measurable on a Borel subset X of D if $\|f(\cdot)\|$ is Borel measurable on X.*

Definition 2.3. *Let $\{f_m\}_{m\geq 1}, f$ be F-valued Borel $\|\cdot\|$-measurable functions on a domain $D \subset \mathbb{C}^N$ and $\mathcal{W} = \{w_m\}_{m\geq 1}$ be a sequence of weights. For a Borel subset X of D, we say that the sequence $\{f_m\}_{m\geq 1}$ is*

(i) pointwise $\mathcal{W}$-convergent to f on X if

$$\lim_{m\to\infty} w_m(\|f_m(z) - f(z)\|^2) = 0 \quad \forall z \in X;$$

(ii) $\mathcal{W}$-convergent in capacity to f on X if for every $\varepsilon > 0$ we have

$$\lim_{m\to\infty} \operatorname{cap}(X_{m,\varepsilon}, D) = 0$$

where $X_{m,\varepsilon} := \{x \in X : \ w_m(\|f_m(z) - f(z)\|^2) > \varepsilon\}$;

(iii) $\mathcal{W}$-convergent in capacity to f on D if the property (ii) holds for every Borel subset X of D.

Remark 2.4. *(1) Obviously, if there exists $C > 0$ such that $w_n(t) \leq Ct$ for all $n \geq 1$ and for all $t > 0$, (normal) pointwise convergence implies $\mathcal{W}$-convergence.*

(2) If the condition (A_3(i')) (see Definition 3.5 below) holds for the sequence $\mathcal{W}$ then it follows easily that $\mathcal{W}$-convergence implies (normal) pointwise convergence.

(3) For the scalar-valued case, the notions of rapid convergence in normal sense and in capacity introduced in [2] are the pointwise $\mathcal{W}$-convergence and $\mathcal{W}$-convergence in capacity respectively with $w_m(t) = t^{1/2m}$, $t \in [0, \infty)$, for every $m \geq 1$ and $\|\cdot\| = |\cdot|$.

2.4. Vector-Valued Rational Functions

Let F be a locally convex space. We denote $\mathscr{P}_m(\mathbb{C}^N, F)$ the space of F-valued polynomials of degree $\leq m$ on $\mathbb{C}^N$. In the case $F = \mathbb{C}$ we write $\mathscr{P}_m(\mathbb{C}^N, \mathbb{C}) = \mathscr{P}_m(\mathbb{C}^N)$.

A function $r : \mathbb{C}^N \to F$ is said to be *rational* if and only if it can be written in the form

$$r(z) = \frac{p(z)}{q(z)}$$

where $p \in \mathscr{P}_m(\mathbb{C}^N, F)$ and $q \in \mathscr{P}_k(\mathbb{C}^N)$ for some m, k are called the numerator and the denominator of r respectively.

We denote $\mathscr{R}(\mathbb{C}^N, F)$ the space of F-valued rational functions on $\mathbb{C}^N$.

If p, q are non-constant and $1 \leq \deg p, \deg q \leq m$, we write $1 \leq \deg r \leq m$.

We say that the set

$$P(r) := \{z \in \mathbb{C}^n : \ p(z) \neq 0, q(z) = 0\}$$

is *the pole* of r.

2.5. Vector-Valued Version of Bernstein-Walsh's Inequality

The following result is a slight modification of the classical Bernstein-Walsh's inequality on estimating sup-norm of vector-valued polynomials.

Lemma 2.5. *Let X, K be compact sets in $\mathbb{C}^N$, with X is non-pluripolar. Let H be a locally convex space equipped with a system $\{\|\cdot\|_m\}_{m\geq 1}$ of Hilbert semi-norms. Then there exists $C_{X,K} > 0$ depending only on X and K such that for every $m \geq 1$ and for every continuous polynomial $p_m : \mathbb{C}^N \to H$ of degree m we have*

$$\frac{1}{m}\log\|p_m\|_{m,K} \leq \frac{1}{m}\log\|p_m\|_{m,X} + C_{X,K}$$

where $\|p_m\|_{m,A} := \sup_{z\in A}\|p_m(z)\|_m$.

Obviously, then for each $n \geq 1$ fixed the following inequality also holds for every $m \geq 1$

$$\frac{1}{m}\log\|p_m\|_{n,K} \leq \frac{1}{m}\log\|p_m\|_{n,X} + C_{X,K}.$$

Proof. Consider the Siciak extremal function

$$V_X(z) := \sup\{u(z) : u \in \mathscr{L}(\mathbb{C}^N) : u|_X \leq 0\},$$

where $\mathscr{L}(\mathbb{C}^N)$ is the Lelong class of plurisubharmonic functions φ with logarithmic growth on $\mathbb{C}^N$ i.e.,

$$\varphi(z) \leq \log(1 + |z|) + C \quad \forall z \in \mathbb{C}^N$$

where C is a constant independent of z. Since X is compact and non-pluripolar, by a well known result of Siciak (see Corollary 5.2.2 and Theorem 5.2.4 in [6]), the function V_X is bounded from above on compact subsets of $\mathbb{C}^N$.

On the other hand, for each $m \geq 1$, since $\deg p_m = m$,

$$\varphi_m(z) := \frac{1}{m}\log\|p_m(z)\|_m \quad \forall z \in \mathbb{C}^N$$

belongs to the class $\mathscr{L}(\mathbb{C}^N)$.

Indeed, put $M_m := \sup_{|z|\leq 1}\|p_m(z)\|_m$. Then by the Cauchy inequalities we have

$$\|p_m(z)\|_m \leq M_m(1 + |z| + \cdots + |z|^m) \leq M_m(1 + |z|)^m$$

whence $\varphi_m \in \mathscr{L}(\mathbb{C}^N)$ follows.

Then, since the function

$$\psi_m : z \mapsto \frac{1}{m}\log\|p_m(z)\|_m - \frac{1}{m}\log\|p_m\|_{m,X}$$

belongs to $\mathscr{L}(\mathbb{C}^N)$ and $\psi\big|_X \leq 0$ we have $\psi(z) \leq V_K(z)$ for all $z \in \mathbb{C}^N$. Hence, letting $C_{X,K} := \sup_{z\in K} V_X(z)$, we see that

$$\begin{aligned}\sup_{z\in K}\frac{1}{m}\log\|p_m(z)\|_m = \frac{1}{m}\log\|p\|_{m,K} &\leq \frac{1}{m}\log\|p\|_{m,X} + \sup_{z\in K}V_X(z)\\ &\leq \frac{1}{m}\log\|p_m\|_{m,X} + C_{X,K}.\end{aligned}$$

□

3. Admissible Sequences of Weights for $\mathscr{H}(D, F)$

The aim of this section is set up terminology of *an admissible sequence of weights for* $\mathscr{H}(D, F)$ which will be used throughout this paper.

Definition 3.1. *For $\alpha, \beta > 0$, a weight w is called (α, β)-subadditive if*

$$w(t + s) \leq \alpha[w(\beta t) + w(\beta s)] \quad \forall\, t, s \in [0, \infty).$$

When $\alpha = \beta = 1$, w is subadditive.

Example 3.2. *Let $w : [0, \infty) \to [0, \infty)$ be a weight.*

(i) *If w is convex in $[0, \infty)$, i.e.*

$$w(\alpha t + (1 - \alpha)s) \leq \alpha w(t) + (1 - \alpha)w(s) \quad \forall\, t, s \geq 0, \alpha \in [0, 1],$$

then w is $(2^{-1}, 2)$-subadditive.

(ii) *If w is concave in* $[0,\infty)$, *i.e.*

$$w(\alpha t+(1-\alpha)s)\geq \alpha w(t)+(1-\alpha)w(s)\quad \forall\, t,s\geq 0,\alpha\in[0,1],$$

then w is subadditive.

Indeed, the case (i) is trivial. For the case (ii), by taking $s=0$ we have $\alpha w(t)\leq w(\alpha t)$. Now for $\alpha=\frac{t}{t+s}\in[0,1]$ we get

$$\begin{aligned} w(t) &= w(\alpha(t+s))\geq \alpha w(t+s),\\ w(s) &= w((1-\alpha)(t+s))\geq (1-\alpha)w(t+s). \end{aligned}$$

It implies that w is subadditive.

Definition 3.3. *Let F be a locally convex space. A weight w is called logarithmic plurisubharmonic for $\mathscr{H}(D,F)$ and write $w\in lpsh(\mathscr{H}(D,F))$, if for every continuous semi-norm $\|\cdot\|$ on F, and for every $f\in\mathscr{H}(D,F)$, the functions $w(\|f(\cdot)\|^2)$ are logarithmic plurisubharmonic on D, i.e.* $\log[w(\|f(\cdot)\|^2)]\in psh(D)$.

In order to give a sufficient condition for $w\in \mathrm{lpsh}(\mathscr{H}(D,F))$ we recall some notations.

Let D be a domain in $\mathbb{C}$ and F,G be locally convex spaces. For $g\in\mathscr{C}^1(D,F)$, we denote $Dg(a)$ is the real differential of g at a and define $D'g(x)$ and $D''g(x)$ as follows

$$\begin{aligned} D'g(a)(t) &= \frac{1}{2}[Dg(a)(t)-iDg(a)(it)],\\ D''g(a)(t) &= \frac{1}{2}[Dg(a)(t)+iDg(a)(it)]. \end{aligned}$$

For $g\in\mathscr{C}^1(D,F)$ and $h\in\mathscr{C}^1(V,G)$ with $g(D)\subset V$ we have

$$\begin{aligned} D'(h\circ g)(x) &= D'h(g(x))\circ D'g(x)+D''h(g(x))\circ D''g(x),\\ D''(h\circ g)(x) &= D'h(g(x))\circ D''g(x)+D''h(g(x))\circ D'g(x). \end{aligned}$$

Note that, if $g\in\mathscr{C}^2(D,F)$ then

$$D'D''g(a)(s,t)=\frac{\partial^2 g}{\partial z\partial\bar z}(a)s.\bar t$$

and if $f\in\mathscr{H}(D,F)$ then $D''f=0$, hence $Df=D'f=f'$.

By a direct computation, it is easy to check that if $\|\cdot\|$ is a semi-norm on a locally convex space F such that the function $h^2:=\|f\|^2\in\mathscr{C}^2(D)$ for every $f\in\mathscr{H}(D,F)$ then a weight $w\in\mathscr{C}^2[0,\infty)$ will belong to $\mathrm{lpsh}(\mathscr{H}(D,F))$ if

$$w(h^2)\Big\{2w''(h^2)h^2\frac{\partial h}{\partial\bar z}\frac{\partial h}{\partial z}+w'(h^2)\Big[\frac{\partial h}{\partial\bar z}\frac{\partial h}{\partial z}+h\frac{\partial^2 h}{\partial\bar z\partial z}\Big]\Big\}\geq 2[w'(h^2)]^2h^2\frac{\partial h}{\partial\bar z}\frac{\partial h}{\partial z}.$$

We do not know, in general, that $h\frac{\partial^2 h}{\partial\bar z\partial z}\geq\frac{\partial h}{\partial\bar z}\frac{\partial h}{\partial z}$. However, this inequality is true in the case $\|\cdot\|$ is a Hilbert semi-norm. We have the following

Proposition 3.4. *Let H be a locally convex space equipped with a system of Hilbert semi-norms. If a weight $w\in\mathscr{C}^2[0,+\infty)$ satisfies the following*

$$w(t)[tw''(t)+w'(t)]\geq t[w'(t)]^2\quad \forall t>0 \tag{1}$$

then $w\in lpsh(\mathscr{H}(D,H))$ with D is a domain in $\mathbb{C}^N$.

Proof. Without loss of generality we may assume that $D \subset \mathbb{C}$ and H is a Hilbert space with the norm $\|\cdot\|$ is induced by the scalar product $\langle \cdot, \cdot \rangle$.

First, it is known that the function $x \mapsto \|x\|^2$ is of $\mathscr{C}^\infty$-class on H (see [8, Exercices 13.E and 14.C]) and

$$D(\|x\|^2)(y) = 2\mathrm{Re}\langle y, x\rangle, \quad D^2(\|x\|^2)(y, y) = 2\langle y, y\rangle \quad \forall x, y \in H.$$

Now, let $f \in \mathscr{H}(D, H)$. Then, for each $a \in \mathbb{C}$ we have

$$\frac{\partial \log w(\|f\|^2)}{\partial \bar{z}}(a) = \frac{w'(\|f(a)\|^2) D''(\|f(a)\|^2)}{w(\|f(a)\|^2)}.$$

Note that $D'f(a) \in L(\mathbb{C}, H) \cong H$, with $M := w^2(\|f(a)\|^2)$ we have the following estimate

$$\begin{aligned}
&\frac{\partial^2 \log w(\|f(a)\|^2)}{\partial z \partial \bar{z}} \\
&= \frac{1}{M}\Big\{[w''(\|f(a)\|^2) D'(\|f(a)\|^2) D''(\|f(a)\|^2) + w'(\|f(a)\|^2) D' D''(\|f(a)\|^2)] w(\|f(a)\|^2) - \\
&\qquad - [w'(\|f(a)\|^2)]^2 D'(\|f(a)\|^2) D''(\|f(a)\|^2)\Big\} \\
&= \frac{1}{M}\Big\{[4w''(\|f(a)\|^2)(\mathrm{Re}\langle f(a), f'(a)\rangle)^2 + 2w'(\|f(a)\|^2) D'(\mathrm{Re}\langle f(z), f'(a)\rangle)] w(\|f(a)\|^2) - \\
&\qquad - 4[w'(\|f(a)\|^2)]^2 (\mathrm{Re}\langle f(a), f'(a)\rangle)^2\Big\} \\
&= \frac{1}{M}\Big\{\Big[4w''(\|f(a)\|^2)\|f(a)\|^2 \Big(\mathrm{Re}\Big\langle \frac{f(a)}{\|f(a)\|}, f'(a)\Big\rangle\Big)^2 + \\
&\qquad + 2w'(\|f(a)\|^2) D'\Big(\|f(a)\| \mathrm{Re}\Big\langle \frac{f(a)}{\|f(a)\|}, f'(a)\Big\rangle\Big)\Big] w(\|f(a)\|^2) - \\
&\qquad - 4[w'(\|f(a)\|^2)]^2 (\mathrm{Re}\langle f(a), f'(a)\rangle)^2\Big\} \\
&= \frac{1}{M}\Big\{\Big[4w''(\|f(a)\|^2)\|f(a)\|^2 \Big(\mathrm{Re}\Big\langle \frac{f(a)}{\|f(a)\|}, f'(a)\Big\rangle\Big)^2 + \\
&\qquad + 2w'(\|f(a)\|^2)\Big(\Big(\mathrm{Re}\Big\langle \frac{f(a)}{\|f(a)\|}, f'(a)\Big\rangle\Big)^2 + \langle f'(a), f'(a)\rangle\Big)\Big] w(\|f(a)\|^2) - \\
&\qquad - 4[w'(\|f(a)\|^2)]^2 (\mathrm{Re}\langle f(a), f'(a)\rangle)^2\Big\} \\
&= \frac{1}{M}\Big\{\Big[4w''(\|f(a)\|^2)\|f(a)\|^2 \Big(\mathrm{Re}\Big\langle \frac{f(a)}{\|f(a)\|}, f'(a)\Big\rangle\Big)^2 + \\
&\qquad + 2w'(\|f(a)\|^2)\Big(\mathrm{Re}\Big\langle \frac{f(a)}{\|f(a)\|}, f'(a)\Big\rangle\Big)^2 + \Big\|\frac{f(a)}{\|f(a)\|}\Big\|^2 \|f'(a)\|^2\Big)\Big] w(\|f(a)\|^2) - \\
&\qquad - 4[w'(\|f(a)\|^2)]^2 \|f(a)\|^2 \Big(\mathrm{Re}\Big\langle \frac{f(a)}{\|f(a)\|}, f'(a)\Big\rangle\Big)^2\Big\}
\end{aligned}$$

and by Schwarz's inequatily

$$\geq \frac{1}{M}\Big\{\Big[4w''(\|f(a)\|^2)\|f(a)\|^2\Big(\mathrm{Re}\Big\langle\frac{f(a)}{\|f(a)\|}, f'(a)\Big\rangle\Big)^2 +$$

$$+ 2w'(\|f(a)\|^2)\Big(\Big(\mathrm{Re}\Big\langle\frac{f(a)}{\|f(a)\|}, f'(a)\Big\rangle\Big)^2 + \Big(\mathrm{Re}\Big\langle\frac{f(a)}{\|f(a)\|}, f'(a)\Big\rangle\Big)^2\Big)\Big]w(\|f(a)\|^2) -$$

$$- 4[w'(\|f(a)\|^2)]^2\|f(z)\|^2\Big(\mathrm{Re}\Big\langle\frac{f(a)}{\|f(a)\|}, f'(a)\Big\rangle\Big)^2\Big\}$$

$$= \frac{4}{M}\Big\{\Big[w''(\|f(a)\|^2)\|f(a)\|^2\Big(\mathrm{Re}\Big\langle\frac{f(a)}{\|f(a)\|}, f'(a)\Big\rangle\Big)^2 +$$

$$+ w'(\|f(a)\|^2)\Big(\mathrm{Re}\Big\langle\frac{f(a)}{\|f(a)\|}, f'(a)\Big\rangle\Big)^2\Big]w(\|f(a)\|^2) -$$

$$- [w'(\|f(a)\|^2)]^2\|f(a)\|^2\Big(\mathrm{Re}\Big\langle\frac{f(a)}{\|f(a)\|}, f'(a)\Big\rangle\Big)^2\Big\}$$

$$= \frac{4}{M}b^2\{w(t)(tw''(t) + w'(t)) - t[w'(t)]^2\}$$

where $t = \|f(a)\|^2$ and $b = \mathrm{Re}\Big\langle\frac{f(a)}{\|f(a)\|}, f'(a)\Big\rangle$.

Hence, if w satisfies (1) then $w \in \mathrm{lpsh}(\mathscr{H}(D, F))$. □

Throughout the forthcoming, unless otherwise specified, we shall denote by H a locally convex space equipped with a sequence of Hilbert semi-norms and D is a domain in $\mathbb{C}^N$.

Definition 3.5. *A sequence $\{w_m\}_{m\geq1}$ of weights is called admissible for $\mathscr{H}(D, H)$ if:*

(A_1) $w_m \in lpsh(\mathscr{H}(D, H))$ for every $m \geq 1$;

(A_2) There exist $\alpha, \beta > 0$ such that w_m is (α, β)-subadditive for every $m \geq 1$;

(A_3) There exists a sequence $\{\widetilde{w}_m\}_{m\geq1} \subset lpsh(\mathscr{H}(D, H))$ of weights such that

- *(i)* $\inf_{m\geq1} \widetilde{w}_m(1) > 0$;
- *(ii)* $\sup\limits_{m\geq1}\ \sup\limits_{0<s\leq a^{2m}} \widetilde{w}_m(s) < \infty$ *for every $a > 0$;*
- *(iii)* $\sup\limits_{m\geq1}\ \sup\limits_{\substack{0<t\leq a^{2m}\\ 0<s\leq b^{2m}}} w_m\Big(\frac{t}{s}\Big)\widetilde{w}_m(s) < \infty$ *for every $a, b > 0$.*

It should be noted that the condition (A_3(i)) is a special case of the following

$$\inf_{m\geq1} \widetilde{w}_m(t_m) = 0 \quad \Rightarrow \quad \inf_{m\geq1} t_m = 0. \tag{A_3(i')}$$

Example 3.6. *For each $m \geq 1$, it is easy to check that the function w_m given by $w_m(t) = t^{1/2m}$ satisfies (1) and is concave, therefore, $w_m \in lpsh(\mathscr{H}(D, H))$ and is subadditive.*

On the other hand, by choosing the sequence $\{\widetilde{w}_m\}_{m\geq1} = \{w_m\}_{m\geq1}$ we can check that the sequence $\{w_m\}_{m\geq1}$ satisfies the condition (A_3). Hence the sequence $\{w_m\}_{m\geq1}$ is admissible admissible for $\mathscr{H}(D, H)$.

Note that *the equality* in (1) holds for w_m. This is the reason we will call this sequence is *the fundamental admissible sequence of weights admissible for $\mathscr{H}(D, H)$.*

Example 3.7. *For each $m \geq 1$, let φ_m be a real valued increasing function, $\varphi_m \in \mathscr{C}^1(0, \infty)$, such that $0 < \varphi_m(x) < \frac{1}{2m}$ on $(0, \infty)$. Put*

$$\psi_m(t) := \int_1^t \frac{\varphi_m(x)}{x}dx.$$

Consequently,

$$0 < \psi_m(t) \leq -\frac{1}{2m}\log t \quad on\ (0,1)$$
$$0 \leq \psi_m(t) \leq \frac{1}{2m}\log t \quad on\ [1,\infty).$$

This implies that

$$\begin{aligned} t^{1/2m} &\leq w_m(t) < 1 \quad on\ (0,1) \\ 1 &\leq w_m(t) \leq t^{1/2m} \quad on\ [1,\infty). \end{aligned} \tag{2}$$

Then we can define the weight w given by

$$w_m(t) = \begin{cases} e^{\psi_m(t)} & if\ t \in (0,\infty) \\ \lim_{s\to 0} w_m(s) = e^{-\int_0^1 \frac{\varphi_m(x)}{x}dx} & if\ t = 0. \end{cases}$$

Under the assumptions of φ_m it is easy to check that $w_m \in \mathscr{C}^2[0,\infty)$.

By a direct computation and $\varphi'_m(t) \geq 0$ we get

$$w_m(t)[tw''_m(t) + w'_m(t)] = w_m^2(t)\Big[\varphi'_m(t) + \frac{\varphi_m^2(t)}{t}\Big] \geq tw_m^2(t)\frac{\varphi_m^2(t)}{t^2} = t[w'_m(t)]^2.$$

Thus, by Proposition 3.4, $w_m \in lpsh(\mathscr{H}(D,H))$.

Obviously, the function ψ_m is increasing because $\psi'_m(t) = \frac{\varphi_m(t)}{t} > 0$. Then w_m is convex, hence, by Example 3.2, w_m is $(2^{-1},2)$-subadditive.

Now we consider $\{\widetilde{w}_m\}_{m\geq 1}$ as the fundamental admissible sequence of weights for $\mathscr{H}(D,H)$. Then, the sequence $\{w_m\}_{m\geq 1}$ satisfies (A_3) because for every $a > 0$ and for every $m \geq 1$, from (2) we obtain

$$w_m\Big(\frac{t}{s}\Big)\widetilde{w}_m(s) < s^{1/2m} \leq a < \infty \quad if\ \ 0 \leq t < s \leq a^{2m}$$

and

$$w_m\Big(\frac{t}{s}\Big)\widetilde{w}_m(s) \leq \Big(\frac{t}{s}\Big)^{1/2m} s^{1/2m} = t^{1/2m} \leq a < \infty \quad if\ \ 0 < s \leq t \leq a^{2m}.$$

Hence, $\{w_m\}_{m\geq 1}$ is admissible for $\mathscr{H}(D,H)$.

Note that, it follows from (2) that, in a neighbourhood 0 the fact $w_m \to 0$ as $m \to \infty$ implies the convergence to 0 of the fundamental admissible sequence for $\mathscr{H}(D,H)$. The next example gives an admissible sequence for which this phenomenon does not happen.

Example 3.8. *For each $m \geq 1$, let φ_m, ψ_m be real valued increasing functions, $\varphi_m, \psi_m \in C^2(0,\infty)$, such that*

1. $\varphi_m(x) \leq x^{1/2m}$, $\varphi_m(x) \leq 1$ *for every* $x \geq 0$,
2. $\psi_m(x) \leq x^{1/2m}$, $\inf_{m\geq 1}\psi_m(1) \geq -1$,
3. $\psi_m(x) \leq \varphi_m\big(\frac{y}{x}\big)[\varphi_m(x) - 1]$ *for every* $x, y > 0$,
4. $\lim_{x\to 0}\varphi_m(x) = \lim_{x\to 0}\psi_m(x) = -\infty$,
5. $x\varphi''_m(x) + \varphi'_m(x) \geq 0, \quad x\psi''_m(x) + \psi'_m(x) \geq 0.$

It is easy to check that the functions φ_m and ψ_m given by

$$\varphi_m(x) = -\frac{1}{mx} \quad and \quad \psi_m(x) = -\frac{1}{m^2x} \quad \forall x > 0,$$

satisfy the conditions (1)–(5).

Now put

$$w_m(x) := \begin{cases} e^{\varphi_m(x)} & \text{for } x > 0 \\ e^{\lim_{y\to 0}\varphi_m(y)} & \text{for } x = 0 \end{cases} \quad \text{and} \quad \widetilde{w}_m(x) := \begin{cases} e^{\psi_m(x)} & \text{for } x > 0 \\ e^{\lim_{y\to 0}\psi_m(y)} & \text{for } x = 0. \end{cases}$$

Obviously, $w_m, \widetilde{w}_m \in \mathscr{C}^2[0,\infty)$. By a direct computation, from (5) we get $w_m, \widetilde{w}_m$ satisfy Proposition 3.4, hence, $w_m, \widetilde{w}_m \in lpsh(\mathscr{H}(D,H))$ for every $m \geq 1$. Moreover, since φ_m and ψ_m are increasing, the functions $w_m, \widetilde{w}_m$ are convex, hence, they are $(2^{-1}, 2)$-subadditive.

Next, it follows from (2) that

$$\inf_{m\geq 1} \widetilde{w}_m(1) = e^{-1} > 0 \quad \text{and} \quad \sup_{m\geq 1}\sup_{0<s\leq a^{2m}} \widetilde{w}_m(s) \leq \sup_{m\geq 1}\sup_{0<s\leq a^{2m}} e^{s^{1/2m}} \leq e^a < \infty.$$

Finally, from the conditions (1) and (3) we obtain

$$\begin{aligned} w_m\Big(\frac{t}{s}\Big)\widetilde{w}_m(s) &= e^{\varphi_m(\frac{t}{s})}e^{\psi(s)} \\ &\leq e^{\varphi_m(\frac{t}{s})+\varphi_m(\frac{t}{s})[\varphi_m(s)-1]} = e^{\varphi_m(\frac{t}{s})\varphi_m(s)} \\ &\leq e^{(\frac{t}{s})^{1/2m}s^{1/2m}} = e^{t^{1/2m}} \leq e^a < \infty \quad \text{for } t \in [0, a^{2m}]. \end{aligned}$$

Hence, $\{w_m\}_{m\geq 1}$ is admissible for $\mathscr{H}(D,H)$.

4. The main results

We begin this section by a discussion about a standard relation between $\mathcal{W}$-convergence in capacity and pointwise $\mathcal{W}$-convergence. As in classical measure theory, we have the following relation of which the proof will be omited because it is a combination of the subadditivity and monotonicity of relative capacities with the method used in the proof of standard theorem of measure theory.

Proposition 4.1. *Let $\|\cdot\|$ be a continuous semi-norm in a locally convex space F, and f, f_m, $m \geq 1$, be F-valued Borel $\|\cdot\|$-measurable functions on a domain $D \subset \mathbb{C}^N$ and X a Borel subset of D. Let $\mathcal{W} = \{w_m\}_{m\geq 1}$ be a sequence of weights. Then, if $\{f_m\}_{m\geq 1}$ is $\mathcal{W}$-convergent in capacity to f on X then there exists a subsequence $\{f_{m_j}\}_{j\geq 1}$ of $\{f_m\}_{m\geq 1}$ and a pluripolar subset E of X such that $\{f_{m_j}\}_{j\geq 1}$ is pointwise $\mathcal{W}^{m_j}$-convergent to f on $X \setminus E$.*

Now we state the main theorems of the paper which are concerned with the convergence of Tauberian type of sequences in $\mathscr{R}(\mathbb{C}^N, H)$ from a Borel non-pluripolar subset of D. They are the refinement of theorems of Gonchar [4, Theorem 2] and of Bloom [2, Theorem 2.1] in the Hilbert-valued case in which rapidly convergence in measure and in capacity is replaced by $\mathcal{W}$-convergence. The proofs will be presented in the next sections.

Let D be a domain in $\mathbb{C}^N$ and $\|\cdot\|$ be a Hilbert semi-norm of a locally convex space H and $\mathcal{W} = \{w_m\}_{m\geq 1}$ be an admissible sequence of weights for $\mathscr{H}(D,H)$.

The first result is for the case where the Borel non-pluripolar subset X of D lies in D.

Theorem 4.2. *Let X be a Borel non-pluripolar subset of D and $f \in \mathscr{H}(D,H)$. Assume that there exists a sequence $\{r_m\}_{m\geq 1} \subset \mathscr{R}(\mathbb{C}^N, H)$, $\deg r_m \leq m$, which is pointwise $\mathcal{W}$-convergent to f on X. Then the following assertions hold:*

- *(a) $\{r_m\}_{m\geq 1}$ is $\mathcal{W}$-convergent in capacity to f on D.*
- *(b) There exists a pluripolar subset (possibly empty) E of $\mathbb{C}^N$ satisfying the following property: For every $z_0 \in D \setminus E$ and every affine complex subspace L of $\mathbb{C}^N$ passing through z_0 there exists a subsequence $\{r_{m_j}\}_{j\geq 1} \subset \{r_m\}_{m\geq 1}$ (dependent only on z_0) such that $\{r_{m_j}\big|_{D_{z_0}}\}$ is $\mathcal{W}^{m_j}$-convergent in capacity (with respect to D_{z_0}) to $f\big|_{D_{z_0}}$ where D_{z_0} is the connected component of $D \cap L$ that contains z_0.*

(c) *If $P(r_m) \cap D = \emptyset$ for every $m \geq 1$ and $\inf_{m\geq 1} \widetilde{w}_m(a^{2m}) > 0$ for every $a > 0$ then $\{r_m\}_{m\geq 1}$ is $\mathcal{W}$-convergent to f uniformly on any compact set $K \subset D$, i.e.*

$$\lim_{m\to\infty} \sup_{z\in K} w_m(\|f(z) - r_m(z)\|^2) = 0.$$

Remark 4.3. *(1) The assertion (a) of Theorem 4.2 might be considered as a converse of Proposition 4.1.*

(2) The additional assumption on $\{\widetilde{w}_m\}_{m\geq 1}$ in the assertion (c) implies the condition (A_3(i)). The admissible sequences of weights in the examples 3.6, 3.7 and 3.8 satisfy this assumption.

(3) In the special case where $\{r_m\}_{m\geq 1}$ is a sequence of polynomials the assertion (b) was proved by Quang and his colleagues for the functions between locally convex spaces (see [11]).

The next one deals with the case where the Borel non-pluripolar X is located in the boundary ∂D.

Theorem 4.4. *Assume that D is a bounded domain. Let $\{f_m\}_{m\geq 1} \subset \mathcal{H}(D, H)$ which is uniformly bounded on bounded, proper subsets of D, and $\{r_m\}_{m\geq 1} \subset \mathcal{R}(\mathbb{C}^N, H)$ with $\deg r_m \leq m$. Assume that there exists a Borel non-pluripolar subset X of ∂D satisfying*

(i) *There exists an open set $U \subset \mathbb{C}^N$ containing X such that the pluriharmonic measure of X relative to $\Omega := U \cap D$*

$$\omega(z, X, \Omega) := \sup\left\{u(z) : \ u \in psh(\Omega), u < 0, \limsup_{\xi\to x, \xi\in\Omega} u(\xi) \leq -1 \ \forall x \in X\right\}$$

is negative on Ω,

(ii) *For every $z \in X$ and every sequence $\{z_m\}_{m\geq 1} \subset D$ with $z_m \to z$ we have*

$$\lim_{m\to\infty} w_m(\|(f_m - r_m)(z_m)\|^2) = 0.$$

Then the following assertions hold:

(a) *The sequence $\{f_m - r_m\}_{m\geq 1}$ is $\mathcal{W}$-convergent in capacity to 0 on D.*

(b) *There exists a pluripolar subset (possibly empty) E of $\mathbb{C}^N$ satisfying the following property: For every $z_0 \in D \setminus E$ and every affine complex subspace L of $\mathbb{C}^N$ passing through z_0 there exists a subsequence $\{r_{m_j}\}_{j\geq 1} \subset \{r_m\}_{m\geq 1}$ (dependent only on z_0) such that $\{(f_{m_j} - r_{m_j})\big|_{D_{z_0}}\}$ is $\mathcal{W}^{m_j}$-convergent in capacity (with respect to D_{z_0}) to 0 where D_{z_0} is the connected component of $D \cap L$ that contains z_0.*

(c) *If $P(r_m) \cap D = \emptyset$ for every $m \geq 1$ and $\inf_{m\geq 1} \widetilde{w}_m(a^{2m}) > 0$ for every $a > 0$ then $\{r_m - f_m\}_{m\geq 1}$ is $\mathcal{W}$-convergent to 0 uniformly on any compact set $K \subset D$, i.e.*

$$\lim_{m\to\infty} \sup_{z\in K} w_m(\|f_m(z) - r_m(z)\|^2) = 0.$$

Remark 4.5. *(1) In general, the non-pluripolarity of X is not sufficient to guarantee the assumption (i). Indeed, let $D \subset \mathbb{C}$ be the unit disk, X be the circular Cantor middle-third set. Then X is non-pluripolar, but of harmonic measure zero (See [12, Exercise 5.3.7]). On the other hand, if D is the unit ball in $\mathbb{C}^N$ and X is an open subset of ∂D then by the maximum principle we can see that X satisfies the assumption (i).*

(2) The sequence $\{f_m\}_{m\geq 1}$ is not assumed to be bounded uniformly on the whole bounded domain D. It is enough to suppose that the uniform boundedness of $\{f_m\}_{m\geq 1}$ occurs on bounded, "proper" subsets of D. However, in fact, it suffices to assume that $\{f_m\}_{m\geq 1}$ is bounded uniformly on compact subsets of D and on $U_X \cap D$ where U_X is some sufficiently small open set containing X. This assumption is essential for the proof. We do not know if the theorem is still valid without this hypothesis.

The remainder of this section discusses only a few aspects of the above results for the case of *rapid $\mathcal{W}$-convergence* in the following sense:

Let $\mathcal{P} = \{\|\cdot\|_n\}_{n\geq 1}$ be a family of continuous semi-norms in a locally convex space F. For a domain $D \subset \mathbb{C}^n$, we say that a function $f : D \to F$ is Borel $\mathcal{P}$-measurable on a Borel subset X of D if f is Borel $\|\cdot\|_n$-measurable on X for every $n \geq 1$.

Let $\{f_m\}_{m\geq 1}, f$ be F-valued Borel $\mathcal{P}$-measurable functions on a domain $D \subset \mathbb{C}^N$ and $\mathcal{W} = \{w_m\}_{m\geq 1}$ be a sequence of weights. For a Borel subset X of D, we say that the sequence $\{f_m\}_{m\geq 1}$ is

(i) pointwise $\mathcal{WP}$-convergent to f on X if

$$\lim_{m\to\infty} w_m(\|f_m(z) - f(z)\|_m^2) = 0 \quad \forall z \in X;$$

(ii) $\mathcal{WP}$-convergent in capacity to f on X if for every $\varepsilon > 0$ we have

$$\lim_{m\to\infty} \operatorname{cap}(X_{m,\varepsilon}, D) = 0$$

where $X_{m,\varepsilon} := \{x \in X : \ w_m(\|f_m(z) - f(z)\|_m^2) > \varepsilon\}$;

(iii) $\mathcal{WP}$-convergent in capacity to f on D if the property (ii) holds for every Borel subset X of D.

We get the analogous results for $\mathcal{WP}$-convergence with additional conditions for the function f and the sequence $\{f_m\}_{m\geq 1}$ respectively as follows: For each compact (resp. bounded, proper) subset K of D then

$$\sup_{m\geq 1} \|f\|_{m,K} := \sup_{m\geq 1} \sup_{z\in K} \|f(z)\|_m < \infty, \tag{3}$$

respectively,

$$\sup_{m\geq 1} \|f_m\|_{m,K} := \sup_{m\geq 1} \sup_{z\in K} \|f_m(z)\|_m < \infty. \tag{4}$$

Theorem 4.6. *Let $\mathcal{P} := \{\|\cdot\|_m\}_{m\geq 1}$ be an increasing sequence of Hilbert semi-norms of a locally convex space H. Then,*

(a) *Under the same hypotheses as in Theorem 4.2 on the subset X, the functions f, r_m and the additional condition (3), we obtain the assertions as in Theorem 4.2 for the $\mathcal{WP}$-convergences;*

(b) *Under the same hypotheses as in Theorem 4.4 on the subset X, the functions f_m, r_m but the property "uniform boundedness of $\{f_m\}_{m\geq 1}$ on bounded, proper subsets" is replaced by (4) and the conditions (ii) by the following one:*

(ii') For every $z \in X$ and every sequence $\{z_m\}_{m\geq 1} \subset D$ with $z_m \to z$ we have

$$\lim_{m\to\infty} w_m(\|(f_m - r_m)(z_m)\|_m^2) = 0,$$

we obtain the assertions as in Theorem 4.4 for the $\mathcal{WP}$-convergences.

5. Proof of Theorem 4.2

We need the following auxiliary result to prove Theorems 4.2.

Proposition 5.1. *Under the hypotheses of Theorem 4.2 the functions ψ_m given by*

$$\psi_m(z) = \log\left[w_m(\|(f - r_m)(z)\|^2)\widetilde{w}_m(|q_m(z)|^2)\right] \quad \forall z \in D \setminus q_m^{-1}(0),$$

where $q_m \in \mathscr{P}_m(\mathbb{C}^n)$ is the denominator of r_m for every $m \geq 1$, can be extended to plurisubharmonic functions (still denoted by ψ_m) on D and the sequence $\{\psi_m\}_{m\geq 1}$ is uniformly bounded from above on compact subsets of D and converges to $-\infty$ uniformly on compact subsets of D.

Proof. By shrinking X we may achieve that $P(r_m) \cap X = \emptyset$ for every $m \geq 1$. By the condition (A_2), suppose that w_m is (α, β)-subadditive for every $m \geq 1$. We have

$$\begin{aligned} w_m\Big(\frac{\|r_m(z)\|^2}{2\beta}\Big) &= w_m\Big(\frac{\|r_m(z) - f(z) + f(z)\|^2}{2\beta}\Big) \\ &\leq w_m\Big(\frac{\|r_m(z) - f(z)\|^2}{\beta} + \frac{\|f(z)\|^2}{\beta}\Big) \\ &\leq \alpha[w_m(\|f(z) - r_m(z)\|^2) + w_m(\|f(z)\|^2)] \\ &\leq \alpha\Big[w_m(\|f(z) - r_m(z)\|^2) + w_m(\|f(z)\|^2)\frac{\widetilde{w}_m(1)}{\inf_{k\geq 1}\widetilde{w}_k(1)}\Big]. \end{aligned}$$

Then, since $\{r_m\}_{m\geq 1}$ is pointwise $\mathcal{W}$-convergent to f on X and by the condition (A_3) it implies that

$$\sup_{m\geq 1} w_m\Big(\frac{\|r_m(z)\|^2}{2\beta}\Big) < \infty \quad \text{for every } z \in X.$$

For $k \geq 1$, we set

$$X_k := \Big\{x \in X : w_m\Big(\frac{\|r_m(z)\|^2}{2\beta}\Big) \leq k,\ \forall m \geq 1\Big\}.$$

By the assumptions we infer that

$$\bigcup_{k\geq 1} X_k = X.$$

Since X is non-pluripolar in $\mathbb{C}^N$, there exists $k_0 \geq 1$ such that $X' := X_{k_0}$ is non-pluripolar. This means that

$$w_m\Big(\frac{\|r_m(z)\|^2}{2\beta}\Big)) \leq k_0 \quad \forall x \in X', \forall m \geq 1.$$

Now we write $r_m = \frac{p_m}{q_m}$, where $p_m \in \mathscr{P}_m(\mathbb{C}^n, H)$ and $q_m \in \mathscr{P}_m(\mathbb{C}^n)$. Since X' is bounded we may normalize q_m such that

$$\|q_m\|_{X'} := \sup_{x\in X'} |q_m(x)| = 1 \quad \forall m \geq 1. \tag{5}$$

Note that, for every compact subset K of D, using Lemma 2.5 we can find a constant $C_K > 0$ such that

$$\max(\|p_m\|_K, \|q_m\|_K) \leq e^{mC_K} \quad \forall m \geq 1, \tag{6}$$

where $\|p_m\|_K = \sup_{z\in K} \|p_m(z)\|$ and $\|q_m\|_K = \sup_{z\in K} |q_m(z)|$.

Next, put $u_m(z) := \log w_m(\|(r_m - f)(z)\|^2)$, $v_m(z) := \log \widetilde{w}_m(|q_m(z)|^2)$ and

$$\psi_m(z) := u_m(z) + v_m(z)$$

for every $z \in D_m := D \setminus q_m^{-1}(0)$. Since $w_m, \widetilde{w}_m \in \mathrm{lpsh}(\mathscr{H}(D, H))$, we have ψ_m is plurisubharmonic on D_m.

First we show that for every compact subset K of D the sequence $\{\psi_m\}_{m\geq 1}$ is uniformly bounded from above on K. Indeed, it follows from (6) that

$$\|q_m f - p_m\|_K \leq (1 + \|f\|_K)e^{C_K m} \leq e^{C'_K m} \quad \forall m \geq 1. \tag{7}$$

By the condition (A_3) and (7) we get

$$\psi_m(z) = \log\bigg(w_m\Big(\frac{\|q_m(z)f(z) - p_m(z)\|^2}{|q_m(z)|^2}\Big)\widetilde{w}_m(|q_m(z)|^2)\bigg) \leq C''_K$$

where C''_K does not depend on z and m.

Hence, since $q_m^{-1}(0)$ is pluripolar, by [6, Theorem 2.9.22] ψ_m can be extended to a plurisubharmonic function (still denoted by ψ_m) on D. Moreover, by the above estimate, $\{\psi_m\}_{m\geq 1}$ is uniformly bounded from above on K.

It follows from Theorem 3.2.12 in [5] that either ψ_m goes to $-\infty$ uniformly on compact subsets on D or there exists a subsequence $\{\psi_{m_j}\}_{j\geq 1}$ converges to $\psi \in \mathrm{psh}(D)$, $\psi \not\equiv -\infty$, in $L^1_{loc}(D)$. We will show that the second case does not occur. Assume the contrary, we can assume that the sequence ψ_m itself converges almost everywhere on D to ψ. Applying Lemma 3.8 in [3] we have $\limsup_{m\to\infty} \psi_m = \psi$ outside a pluripolar set. Note that, it follows from (6) and the condition $(A_3(\mathrm{ii}))$ that

$$\sup_{m\geq 1} v_m(z) < \infty \quad \forall z \in \mathbb{C}^N.$$

Then, since $\limsup_{m\to\infty} \psi_m = -\infty$ for every $z \in X$ we infer

$$\lim_{m\to\infty} \psi_m(z) = -\infty, \quad \forall z \in X.$$

This implies $\psi = -\infty$ on a non-pluripolar subset of D, a contradiction. It follows that ψ_m converges to $-\infty$ uniformly on compact subsets of D. This proves our assertion. □

With proven facts in hand, we now can prove the theorem 4.2 as follows.

Proof. [Proof of Theorem 4.2] (a) By (5) and $(A_3(\mathrm{i}))$, we have

$$-\infty < \inf_{m\geq 1} \log \widetilde{w}_m(1) \leq \sup_{m\geq 1} \sup_{x\in X'} \log \widetilde{w}_m(|q_m(x)|^2) = \sup_{m\geq 1} \sup_{x\in X'} v_m(x).$$

Thus $\{v_m\}_{m\geq 1}$ does not converge to $-\infty$ uniformly on X'. Finally, by Proposition 5.1, applying Lemma 2.9 in [3] to the sequences $\{u_m\}_{m\geq 1}$ and $\{v_m\}_{m\geq 1}$ we get $\{w_m(\|(r_m - f)(\cdot)\|^2\}_{m\geq 1}$ converges to 0 in capacity on D. This means that (a) is proved.

(b) By Proposition 5.1 the sequence $\{\psi_m\}_{m\geq 1}$ convergences to $-\infty$ uniformly on compact subsets on D. We also note that by (6) and the condition $(A_3(\mathrm{ii}))$ again, the sequence $\{v_m\}_{m\geq 1}$ is uniformly bounded from above on compact sets of $\mathbb{C}^N$. Since $\{v_m\}_{m\geq 1}$ does not converge to $-\infty$ uniformly on X' (see the proof of (a)), using again Lemma 3.8 in [3] we deduce that $\limsup_{m\to\infty} v_m > -\infty$ on $D \setminus E$ with

$$E := \Big\{z \in \mathbb{C}^N : \lim_{m\to\infty} v_m(z) = -\infty\}$$

is pluripolar.

We will show that E has the desired property. Fix a point $z_0 \in D \setminus E$ and an affine complex subspace L containing z_0. Then, we can choose a subsequence $\{v_{m_j}\}_{j\geq 1}$ such that

$$\inf_{j\geq 1} v_{m_j}(z) > -\infty.$$

Denote $u_j^0 := u_{m_j}\big|_{D\cap L}$ and $v_j^0 := v_{m_j}\big|_{D\cap L}$. As shown in (a), $\psi_j^0 := u_j^0 + v_j^0$ converges to $-\infty$ uniformly on compact subsets of D_{z_0}. Now, applying Lemma 2.9 in [3] again to the sequences $\{u_j^0\}_{j\geq 1}$ and $\{v_j^0\}_{j\geq 1}$ we get $\{w_{m_j}(\|(r_{m_j} - f)(\cdot)\|^2\}_{j\geq 1}$ converges to 0 in capacity on D_{z_0}. This is what needs to be shown.

(c) Suppose that $\{r_m\}_{m\geq 1}$ is not $\mathcal{W}$-convergent to f on some compact subset K of D. Then there exist $\varepsilon > 0$ and a subsequence $\{r_{m_k}\}_{k\geq 1}$ of $\{r_m\}_{m\geq 1}$ such that

$$\sup_{z\in K} w_{m_k}(\|(r_{m_k} - f)(z)\|^2) > \varepsilon,$$

or equivalently,

$$\inf_{k\geq 1} \sup_{z\in K} w_{m_k}(\|(r_{m_k} - f)(z)\|^2) \geq \varepsilon > 0. \tag{8}$$

Since $P(r_m) \cap D = \emptyset$ for every $m \geq 1$, the functions $\lambda_{m_k}(\cdot) := \frac{1}{m_k} \log |q_{m_k}(\cdot)|$ are pluriharmonic on D. By (6) the sequence $\{\lambda_{m_k}\}_{k\geq 1}$ is uniformly bounded from above on compact subsets of D. On the other hand, since λ_{m_k} is pluriharmonic on D and $\sup_{x\in X'} \lambda_{m_k}(x) = 0$ for every $k \geq 1$ we infer that every accumulation point of the sequence $\{\lambda_{m_k}\}_{k\geq 1}$ taken in the topology of locally uniform convergence is not identically $-\infty$. Thus, each accumulation point of this sequence must be a (real valued) pluriharmonic function on D. It follows that $\{\lambda_{m_k}\}_{k\geq 1}$ is uniformly bounded from below on compact sets of D. Thus for each compact subset K of D we get a constant $a > 0$ (depending on K) such that

$$a^{m_k} \leq |q_{m_k}(z)| \quad \forall z \in K. \tag{9}$$

Then, by the same reasoning as in (a) we obtain

$$\psi_{m_k} = u_{m_k} + v_{m_k} \to -\infty \quad \text{uniformly on } K. \tag{10}$$

Thus, by the assumption $\inf_{m\geq 1} \widetilde{w}_m(a^m) > 0$ for every $a > 0$ and (9) we get

$$\inf_{k\geq 1} \inf_{z\in K} \log \widetilde{w}_{m_k}(|q_{m_k}(z)|^2) \geq \inf_{k\geq 1} \inf_{z\in K} \log \widetilde{w}_{m_k}(a^{2m_k}) > -\infty. \tag{11}$$

We can now combine (8), (10) and (11) together to obtain a contradiction.

The theorem is proved. □

6. Proof of Theorem 4.4

The proposition below is a version of Proposition 5.1 for the case where X is a Borel non-pluripolar subset of the boundary ∂D of a bounded domain D of $\mathbb{C}^N$.

Proposition 6.1. *Under the hypotheses of Theorem 4.4 the functions φ_m given by*

$$\varphi_m(z) = \log \Big[w_m(\|(f_m - r_m)(z)\|^2) \widetilde{w}_m(|q_m(z)|^2) \Big] \quad \forall z \in D \setminus q_m^{-1}(0),$$

where $q_m \in \mathscr{P}_m(\mathbb{C}^n)$ is the denominator of r_m for every $m \geq 1$, can be extended to plurisubharmonic functions (still denoted by φ_m) on D and the sequence $\{\varphi_m\}_{m\geq 1}$ is uniformly bounded from above on bounded, proper subsets of D and converges to $-\infty$ uniformly on compact subsets of D.

Proof. As the above, we may achieve that $P(r_m) \cap X = \emptyset$ for every $m \geq 1$. We shall continue using the notation of the previous proof with following minor modifications:

$$u_m(z) := \log w_m(\|(r_m - f_m)(z)\|^2) \quad \text{and} \quad \varphi_m(z) := u_m(z) + v_m(z).$$

As in the previous theorem, we first prove that

$$\sup_{m\geq 1} w_m\Big(\frac{\|r_m(z)\|^2}{4\beta^2}\Big) < \infty \quad \text{for every } z \in X. \tag{12}$$

Fix $z \in X$. Since $P(r_m) \cap X = \emptyset$, the function $\log w_m(\|r_m(\cdot)\|^2/2\beta)$ is continuous at z for every $m \geq 1$. Then we can find a sequence $\{z_m\}_{m\geq 1} \subset D$ with $|z_m - z| < m^{-1}$ and

$$\log w_m\Big(\frac{\|r_m(z) - r_m(z_m)\|^2}{2\beta}\Big) < 1 \tag{13}$$

for every $m \geq 1$. Since $z_m \to z$, by the assumption (ii) we have

$$\sup_{m\geq 1} w_m(\|f_m(z_m) - r_m(z_m)\|^2) < \infty. \tag{14}$$

Moreover, it follows from the boundedness of the set $\{z_m, m \geq 1\}$ and the assumption on $\{f_m\}_{m\geq 1}$ that $\|f_m(z_m)\|^2 \leq \max(1, [\sup_{k\geq 1} \|f_k(z_k)\|]^{2m})$ for every $m \geq 1$. Therefore, by the condition (A_3) we obtain

$$\sup_{m\geq 1} w_m(\|f_m(z_m)\|^2) \leq \sup_{m\geq 1} w_m(\|f_m(z_m)\|^2)\frac{\widetilde{w}_m(1)}{\inf_{k\geq 1}\widetilde{w}_k(1)} < \infty. \tag{15}$$

Then, from (13), (14) and (15) we get

$$\begin{aligned} w_m\Big(\frac{\|r_m(z)\|^2}{4\beta^2}\Big) &\leq w_m\Big(\frac{\|r_m(z) - r_m(z_m)\|^2}{2\beta^2} + \frac{\|r_m(z_m)\|^2}{2\beta^2}\Big) \\ &\leq \alpha\Big[w_m\Big(\frac{\|r_m(z) - r_m(z_m)\|^2}{2\beta}\Big) + w_m\Big(\frac{\|r_m(z_m)\|^2}{2\beta}\Big)\Big] \\ &\leq \alpha w_m\Big(\frac{\|r_m(z) - r_m(z_m)\|^2}{2\beta}\Big) + \alpha w_m\Big(\frac{\|r_m(z_m) - f_m(z_m)\|^2}{\beta} + \frac{\|f_m(z_m)\|^2}{\beta}\Big) \\ &\leq \alpha w_m\Big(\frac{\|r_m(z) - r_m(z_m)\|^2}{2\beta}\Big) + \alpha^2[w_m(\|r_m(z) - r_m(z_m)\|^2) + w_m(\|f_m(z_m)\|^2)] \\ &< \infty \quad \text{for every } m \geq 1. \end{aligned}$$

Thus, (12) is proved, hence as in the previous theorem, there exists a non-pluripolar subset X' such that $\sup_{m\geq 1} w_m\Big(\frac{\|r_m(z)\|^2}{4\beta^2}\Big) < \infty$ for every $z \in X'$.

On the other hand, by the assumption on $\{f_m\}_{m\geq 1}$ again we also have the estimate that similar to (7) for each bounded proper subset K of D :

$$\|q_m f_m - p_m\|_K \leq (1 + \|f_m\|_K)e^{C_K m} \leq e^{C'_K m} \quad \forall m \geq 1.$$

From now, by an argument analogous to the previous one we can show that φ_m can be extended to a plurisubharmonic function (still denoted by φ_m) on D, moreover, $\{\varphi_m\}_{m\geq 1}$ is uniformly bounded from above on bounded proper subsets of D.

Next, we prove that sequence $\{\varphi_m\}_{m\geq 1}$ converges to $-\infty$ uniformly on compact subsets of D.

Suppose otherwise, by Lemma 3.8 in [3] we can find a subsequence $\{\varphi_{m_j}\}_{j\geq 1} \subset \{\varphi_m\}_{m\geq 1}$ and a function $\varphi \in \mathrm{psh}(D)$, $\varphi \not\equiv -\infty$ such that the set $F := \{z \in D : \limsup_{j\to\infty} \varphi_{m_j}(z) \neq \varphi(z)\}$ is pluripolar. Now we fix $z \in X$ and a sequence $\{z_j\}_{j\geq 1} \subset D$ which converges to z, and prove that

$$\lim_{j\to\infty} \varphi(z_j) = -\infty. \tag{16}$$

Suppose (16) is not true. Then φ is bounded from below on a subsequence of $\{z_j\}_{j\geq 1}$. For brevity of presentation we shall assume that φ is bounded from below on $\{z_j\}_{j\geq 1}$. Then there is $M > 0$ such that $\varphi(z_j) > M$ for every $j \geq 1$. Since F is pluripolar, for each $j \geq 1$ there exists a complex line L_j through z_j such that $L_j \cap F$ is polar (in L_j). Then by [12, Theorem 5.4.2], the set $L_j \setminus (F \cup \{z_j\})$ is non-thin at the point z_j. Thus we can choose $x_j \in D$ such that

$$|z_j - x_j| < j^{-1}, \quad \varphi(x_j) > M, \quad x_j \notin F.$$

From the definition of F, we can find a sequence $\{L(j)\}_{j\geq 1} \uparrow \infty$ such that

$$\varphi_{m_{L(j)}}(x_j) > M \quad \forall j \geq 1.$$

On the other hand, by what we have proved in (b) of the previous theorem, $\{v_m\}_{m\geq 1}$ is uniformly bounded from above on compact sets of $\mathbb{C}^N$ we have

$$\sup_{m\geq 1}\sup_{j\geq 1} v_m(x_j) < \infty.$$

Consequently, there exists $\varepsilon > 0$ such that

$$w_{m_{L(j)}}(\|(f_{m_{L(j)}} - r_{m_{L(j)}})(x_j)\|^2) > \varepsilon \quad \forall j \geq 1.$$

This contradicts the assumption (iii). Thus (16) follows.

Finally, consider the open set Ω that exists according to the assumption (i). It implies from the uniform boundedness from above on Ω of $\{\varphi_m\}_{m\geq 1}$ that $\sup_{z\in\Omega} \varphi(z) = N < \infty$. Then, for every $n > 0$ we have

$$\varphi(z) \leq N + n\omega(z, X, \Omega) \quad \forall z \in \Omega. \tag{17}$$

Letting $n \to \infty$ in (17) and using the assumption (i) we obtain $\varphi \equiv -\infty$ on Ω. This is impossible. Thus we have proved that $\{\varphi_m\}_{m\geq 1}$ converges to $-\infty$ uniformly on compact subsets of D. □

Proof. [Proof of Theorem 4.4] The proof parallels that of Theorem 4.2 with using Proposition 6.1 in place of Proposition 5.1, so it will be omitted. □

7. Some Remarks on Proof of Theorem 4.6

We omit the proof of Theorem 4.6 because it is similar to the previous theorems with the following remarks:

Remark 7.1. *(1) The assumptions (3) and (4) are essential for our proof because then we will receive the following estimates which like the estimate (7):*

$$\|q_m f - p_m\|_{m,K} \leq (1 + \|f\|_{m,K})e^{C_K m} \leq e^{C'_K m} \quad \forall m \geq 1,$$

$$\|q_m f_m - p_m\|_{m,K} \leq (1 + \|f_m\|_{m,K})e^{C_K m} \leq e^{C'_K m} \quad \forall m \geq 1.$$

On the other hand, we obtain from (3) that $\|f(z)\|_m^2 \leq \max(1, [\sup_{k\geq 1} \|f(z)\|_k^2]^m)$ *for every* $m \geq 1$ *and every* $z \in X$*, and, hence, by* (A_3)

$$\sup_{m\geq 1} w_m(\|f(z)\|_m^2) \leq \sup_{m\geq 1} w_m(\|f(z)\|_m^2)\frac{\widetilde{w}_m(1)}{\inf_{k\geq 1}\widetilde{w}_k(1)} < \infty$$

which is useful for the proof of the assertion (a).

Similarly, it implies from (4) that

$$\|f_m(z)\|_m^2 \leq \max(1, [\sup_{k\geq 1} \|f_k(z)\|_k^2]^m),$$
$$\|f_m(z_m)\|_m^2 \leq \max(1, [\sup_{k\geq 1} \|f_k(z_k)\|_k^2]^m)$$

for every $m \geq 1$*, every* $z \in X$*, and every sequence* $\{z_m\}$ *converging to* z*. Therefore, by* (A_3)

$$\sup_{m\geq 1} w_m(\|f_m(z)\|_m^2) = \sup_{m\geq 1} w_m(\|f_m(z)\|_m^2)\frac{\widetilde{w}_m(1)}{\inf_{k\geq 1}\widetilde{w}_k(1)} < \infty,$$

$$\sup_{m\geq 1} w_m(\|f_m(z_m)\|_m^2) = \sup_{m\geq 1} w_m(\|f_m(z_m)\|_m^2)\frac{\widetilde{w}_m(1)}{\inf_{k\geq 1}\widetilde{w}_k(1)} < \infty$$

which are useful for the proof of the assertion (b).

We do not know if the theorem is still valid without the hypothesis (3) or (4).

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