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Enrichment and internalization in tricategories, the case of tensor categories and alternative notion to intercategories

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Abstract. This paper emerged as a result of tackling the following three issues. Firstly, we would like the well known embedding of bicategories into pseudo double categories to be monoidal, which it is not if one uses the usual notion of a monoidal pseudo double category. Secondly, in [3] the question was raised: which would be an alternative notion to intercategories of Grandis and Paré, so that monoids in Böhm's monoidal category $(Dbl, \otimes)$ of strict double categories and strict double functors with a Gray type monoidal product be an example of it? We obtain and prove that precisely the monoidal structure of $(Dbl, \otimes)$ resolves the first question. On the other hand, resolving the second question, we upgrade the category *Dbl* to a tricategory DblPs and propose to consider internal categories in this tricategory. For this purpose we define categories internal to tricategories (of the type of DblPs), which simultaneously serves our third motive. Apart from monoids in $(Dbl, \otimes)$ - more importantly, weak pseudomonoids in a tricategory containing $(Dbl, \otimes)$ as a sub 1-category - most of the examples of intercategories are also examples of this new gadget. The ones that escape are duoidal categories and Gray categories, as their monoidal product induces a lax double functor on the Cartesian product. What our third motive concerns, inspired by the tricategory and (1×2)-category of tensor categories, we prove under mild conditions that categories enriched over certain type of tricategories may be made into categories internal in them. We illustrate this occurrence for tensor categories with respect to the ambient tricategory 2-Cat_{wk} of 2-categories, pseudofunctors, pseudonatural transformations and modifications.

Contents

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1. Introduction

It is well-known that 2-categories embed in strict double categories and that bicategories embed in pseudo double categories. However, it is not clear which of the definitions of a monoidal pseudo double category existent in the literature would be suitable to have a monoidal version of the result, that monoidal bicategories embed into monoidal pseudo double categories. This question we resolve in Subsection 2.2. Namely, seeing a monoidal bicategory as a one-object tricategory, we consider the equivalent one-object Gray 3-category (by the coherence of tricategories of [17]), which is nothing but a monoid in the monoidal category *Gray*, *i.e.* a Gray monoid (see [8], [1, Lemma 4]). We then prove that *Gray* embeds as a monoidal category in the monoidal category *Dbl* from [3] of strict double categories and strict double functors. For the above-mentioned embedding we give an explicit description of the monoidal structure of *Dbl* (in [3] only an explicit description of the structure of a monoid is given). Analogously as in [23], we introduce a *cubical double functor* along the way. We also show why the other notions of monoidal double categories (monoids in the category of strict double categories and strict double functors from [6], and pseudomonoids in the 2-category *PsDbl* of pseudo double categories, pseudo double functors and vertical transformations from [30]) do not obey the embedding in question.

Then we turn to the following question of Böhm: if a monoid in her monoidal category *Dbl* could fit some framework similar to intercategories of Grandis and Paré. Observe that neither of the two notions would be more general than the other. While intercategories are categories internal in the 2-category *LxDbl* of pseudo double categories, lax double functors and horizontal transformations, in the structure of Böhm's monoid in *Dbl* the relevant objects are *strict* double categories and morphisms *double pseudo functors* in the sense of [31] (they are given by isomorphisms in both directions). By Strictification Theorem of [22, Section 7.5] every pseudo double category is equivalent to a strict double category, thus on the level of objects in the ambient category basically nothing is changed. Though, going from lax double functors to double pseudo functors, one tightens in one direction and weakens in the other.

In the search for a desired framework, we assume that the ambient category for internalization has strict double categories for 0-cells and double pseudo functors for 1-cells. Then we define 2-cells among double pseudo functors and we get that instead of an ambient 2-category, we indeed have a tricategory structure, including modifications as 3-cells. This led us to propose an alternative notion for intercategories, as categories internal in this tricategory, which we denote by DblPs. Given the extensiveness of the proof that DblPs is a tricategory, some parts of the proof (which are listed in Subsection 3.6) we carried out in [13, Section 4], which is the first part of a previous version of this paper.

Contrarily to *LxDbl*, the 1-cells of the 2-category *PsDbl* are particular cases of double pseudo functors. Having in mind the above Strictification Theorem and adding only the trivial 3-cells to *PsDbl*, we also prove that thus obtained tricategory $PsDbl_3^*$ embeds in our tricategory DblPs. As a byproduct to this proof we obtain a more general result: supposing that there is a connection ([5]) on 1v-components of strong vertical transformations ([22, Section 7.4]), there is a bijection between strong vertical transformations and those strong horizontal transformations whose 1h-cell components are 1h-companions of some 1v-cells. This we prove in Corollary 4.3. (The abbreviations 1h and 1v stand for horizontal, respectively vertical, 1-cells.)

The general idea of considering categories internal in higher dimensional categories was present in the literature, see for example [30]. In *loc.cit* a category internal in a tricategory is called a (1×2)-category. Concretely, though, the notion of a category internal in a Gray category was introduced in [10]. For our announced purpose we introduce a notion of a category internal in a tricategory V which is of a similar type as DblPs. First of all, we introduce tricategorical pullbacks, which we call simply 3-pullbacks. The ambient tricategory V for internalization needs to have 3-pullbacks, an underlying 1-category, and apart from the interchange law, the associativity on the 2-cells holds only up to isomorphism, which makes it weaker than a Gray category. The latter two properties mean that we need to work with an *iconic tricategory*, see [32]. We then describe the structure of a category internal in DblPs, and, similarly to intercategories, we give a geometric interpretation of it in the form of cubes. Moreover, we upgrade the monoidal category *Dbl* to a 2-category Dbl_2, and this one to an iconic tricategory Dbl_3, and show how pseudomonoids in Dbl_2 and "weak pseudomonoids" in Dbl_3, are categories internal in DblPs. The examples of intercategories treated in [21] are also examples of categories internal in DblPs, except from duoidal categories and Gray categories, whose compositions on the pullback induce lax (double) functors on the Cartesian product, rather than pseudo ones.

As we mentioned above, bicategories (which are categories enriched over the 2-category Cat_2 of categories) embed into double categories (which are categories internal in Cat_2). We examine if the analogous happens in one dimension higher: under which conditions a category enriched in a tricategory V is a category internal in V. We have in mind the standard example of the bicategory of algebras and their bimodules, and its well-known analogue in one dimension higher, the tricategory Tens of tensor categories, bimodule categories over the latter, bimodule functors and bimodule natural transformations. In order to prove our conjectured result, we first introduce the notions of tricategorical products and coproducts (which we call simply 3-(co)products), and of a category enriched over an iconic tricategory with 3-products. Then we show that Tens is a category enriched over the tricategory $2\text{-}\mathrm{Cat}_{wk}$, of 2-categories, pseudofunctors, weak natural transformations and modifications. Moreover, we show that Tens is a part of the structure of a category internal in $2\text{-}\mathrm{Cat}_{wk}$ (of a certain (1×2)-category). This responds to [10, Example 2.14], where it was conjectured that Tens is a category internal in the Gray 3-category $2CAT_{nwk}$, which differs from $2\text{-}\mathrm{Cat}_{wk}$ in that its 1-cells are 2-functors, rather than pseudofunctors as in $2\text{-}\mathrm{Cat}_{wk}$. Motivated by this example, we prove in Proposition 8.5 that under mild conditions categories enriched over iconic tricategories can be made into categories internal in them. This generalizes to tricategories analogous results from [11] and [7], studied among 1-categories. Since $2\text{-}\mathrm{Cat}_{wk}$ embeds into DblPs, the (1×2)-category assigned to Tens is also an example of our alternative notion to intercategories.

Let Bicat_3 be the tricategory of bicategories, pseudofunctors, pseudonatural transformations and modifications. In [15] we used the construction of a category internal in an iconic tricategory from the present article to create two internal categories in Bicat_3: one is $\mathcal{S}$ of spans in a generic iconic tricategory V with 3-pullbacks, and the other is $\mathcal{M}$ of matrices in a generic iconic tricategory V with 3-products. (This means that, among other data, we constructed a bicategory whose 0-cells are spans in V, and a bicategory whose 0-cells are matrices in V.) We used then $\mathcal{S}$ and $\mathcal{M}$ to deduce equivalence of the category of categories internal in V and the category of categories enriched in V.

The composition of the paper is the following. In Section 2 we give a description of the monoidal structure of *Dbl* of Böhm, we define cubical double functors and prove that *Gray* embeds into *Dbl*. Section 3 is dedicated to the construction of our tricategory DblPs. In Section 4 we prove that the tricategory $PsDbl_3^*$ embeds into DblPs and prove the bijection between vertical and horizontal strong transformations, supposing the mentioned connection. In Section 5 we define tricategorical pullbacks and (co)products, and in the next one we define categories internal in iconic tricategories with 3-pullbacks. In the subsequent section we describe the structure of a category internal in DblPs, we show here that monoids in Böhm's *Dbl*, pseudomonoids in Dbl_2 and weak pseudomonoids in Dbl_3 fit this setting, and we present the announced geometric interpretation on cubes. In Section 8 we define categories enriched over iconic tricategories with 3-products, we prove that categories enriched over certain type of iconic tricategories V are special cases of categories internal in V, and we discuss examples in dimensions 2 and 1. In the last section we show the enrichment and internal structures of Tens in $2\text{-}\mathrm{Cat}_{wk}$, illustrating the mentioned result. In the Appendix the definition of a tricategory is recalled and discussed.

The reader is supposed to know double categories, distinct versions of double functors, transformations and modifications. For reference we recommend [18]. For tricategories we recommend [17] and [25].

2. Monoidal double categories into which monoidal bicategories embed

Although bicategories embed into pseudo double categories, this embedding is not monoidal, if one takes for a definition of a monoidal double category any of the ones in [6] (a monoid in the category of strict double categories and strict double functors) and in [30] (a pseudomonoid in the 2-category of pseudo double categories, pseudo double functors and say vertical transformations). Namely, a monoidal bicategory is a one-object tricategory, so its 0-cells have a product associative up to an *equivalence*. This is far from what happens in the mentioned two definitions of a monoidal double category. Even if we consider the triequivalence due to [17] of a monoidal bicategory with a one-object Gray-category, that is, a Gray monoid, one does not have monoidal embeddings, as we will show. Nevertheless, a Gray monoid, which is in fact a monoid in the monoidal category $(Gray, \otimes)$ of 2-categories, 2-functors with the monoidal product due to Gray [23], can be seen as a monoid in the monoidal category $(Dbl, \otimes)$ of strict double categories and strict double functors with the monoidal product constructed in [3, Section 4.3]. We will show in this section that $(Gray, \otimes)$ embeds monoidally into $(Dbl, \otimes)$.

2.1. *The monoidal structure in* $(Dbl, \otimes)$

The monoidal structure in $(Dbl, \otimes)$ is constructed in the analogous way as in [23]. For two double categories $\mathbb{A}, \mathbb{B}$ a double category $[\![\mathbb{A}, \mathbb{B}]\!]$ is defined in [3, Section 2.2] which induces a functor $[\![-,-]\!] : Dbl^{op} \times Dbl \longrightarrow Dbl$. Representability of the functor $Dbl(\mathbb{A}, [\![\mathbb{B}, -]\!]) : Dbl \longrightarrow Set$ is proved, which induces a functor $- \otimes - : Dbl \times Dbl \longrightarrow Dbl$. For two double categories $\mathbb{A}, \mathbb{B}$ we will give a full description of the double category $\mathbb{A} \otimes \mathbb{B}$. We will do this using the natural isomorphism

$$Dbl(\mathbb{A} \otimes \mathbb{B}, \mathbb{C}) \cong Dbl(\mathbb{A}, [\![\mathbb{B}, \mathbb{C}]\!]), \tag{2.1}$$

that is, characterizing a double functor $F : \mathbb{A} \longrightarrow [\![\mathbb{B}, \mathbb{C}]\!]$ for another double category $\mathbb{C}$ and reading off the structure of the image double category $F(\mathbb{A})(\mathbb{B})$, setting $\mathbb{C} = \mathbb{A} \otimes \mathbb{B}$.

Let us fix the notation in a double category $\mathbb{D}$. Objects we denote by $A, B, \dots$, horizontal 1-cells we will call for brevity 1h-cells and denote them by $f, f', g, F, \dots$, vertical 1-cells we will call 1v-cells and denote by $u, v, U, \dots$, and squares we will call just 2-cells and denote them by $\omega, \zeta, \dots$. In this section, we denote the horizontal identity 1-cell by 1_A, vertical identity 1-cell by 1^A for an object $A \in \mathbb{D}$, horizontal identity 2-cell on a 1v-cell u by Id^u, and vertical identity 2-cell on a 1h-cell f by Id_f (with subindices we denote those identity 1- and 2-cells which come from the horizontal 2-category lying in $\mathbb{D}$). The composition of 1h-cells as well as the horizontal composition of 2-cells we will denote by $\odot$ in this section, while the composition of 1v-cells as well the vertical composition of 2-cells we will denote by juxtaposition.

We start by noticing that a strict double functor $F : \mathbb{C} \longrightarrow \mathbb{D}$ is given by 1) the data: images on objects, 1h-, 1v- and 2-cells of $\mathbb{C}$, and 2) rules (in $\mathbb{D}$):
$F(u'u) = F(u')F(u), \quad F(1^A) = 1^{F(A)}$,
$F(\omega\zeta) = F(\omega)F(\zeta), \quad F(1_f) = 1_{F(f)}$,
$F(g \odot f) = F(g) \odot F(f), \quad F(\omega \odot \zeta) = F(\omega) \odot F(\zeta)$,
$F(1_A) = 1_{F(A)}, \quad F(Id^u) = Id^{F(u)}$.

Having in mind the definition of a double category $[\![\mathbb{A}, \mathbb{B}]\!]$ from [3, Section 2.2], writing out the list of the data and relations that determine a double functor $F : \mathbb{A} \longrightarrow [\![\mathbb{B}, \mathbb{C}]\!]$, one gets the following characterization of it:

Proposition 2.1 *A double functor* $F : \mathbb{A} \longrightarrow [\![\mathbb{B}, \mathbb{C}]\!]$ *of double categories consists of the following:*
1. double functors

$$(-, A) : \mathbb{B} \longrightarrow \mathbb{C} \quad \text{and} \quad (B, -) : \mathbb{A} \longrightarrow \mathbb{C}$$

such that $(-,A)|_B = (B,-)|_A = (B,A)$, *for objects* $A \in \mathbb{A}, B \in \mathbb{B}$,

2. *given 1h-cells* $A \xrightarrow{F} A'$ *and* $B \xrightarrow{f} B'$ *and 1v-cells* $A \xrightarrow{U} \tilde{A}$ *and* $B \xrightarrow{u} \tilde{B}$ *there are 2-cells*

$$\begin{array}{ccccc} (B,A) & \xrightarrow{(B,F)} & (B,A') & \xrightarrow{(f,A')} & (B',A') \\ \downarrow = & & \boxed{(f,F)} & & \downarrow = \\ (B,A) & \xrightarrow{(f,A)} & (B',A) & \xrightarrow{(B',F)} & (B',A') \end{array}$$

$$\begin{array}{ccc} (B,A) & \xrightarrow{(B,F)} & (B,A') \\ (u,A)\downarrow & \boxed{(u,F)} & \downarrow (u,A') \\ (\tilde{B},A) & \xrightarrow{(\tilde{B},F)} & (\tilde{B},A') \end{array} \qquad \begin{array}{ccc} (B,A) & \xrightarrow{(f,A)} & (B',A) \\ (B,U)\downarrow & \boxed{(f,U)} & \downarrow (B',U) \\ (B,\tilde{A}) & \xrightarrow{(f,\tilde{A})} & (B',\tilde{A}) \end{array}$$

$$\begin{array}{ccc} (B,A) & \xrightarrow{=} & (B,A) \\ (B,U)\downarrow & & \downarrow (u,A) \\ (B,\tilde{A}) & \boxed{(u,U)} & (\tilde{B},A) \\ (u,\tilde{A})\downarrow & & \downarrow (\tilde{B},U) \\ (\tilde{B},\tilde{A}) & \xrightarrow{=} & (\tilde{B},\tilde{A}) \end{array}$$

of which (f,F) *is vertically invertible and* (u,U) *is horizontally invertible, which satisfy:*

a) (11) $(1_B,F) = Id_{(B,F)}$ *and* $(f,1_A) = Id_{(f,A)}$

(21) $(1^B,F) = Id_{(B,F)}$ *and* $(u,1_A) = Id^{(u,A)}$

(12) $(1_B,U) = Id^{(B,U)}$ *and* $(f,1^A) = Id_{(f,A)}$

(22) $(1^B,U) = Id^{(B,U)}$ *and* $(u,1^A) = Id^{(u,A)}$;

b) (11) $(f' \odot f, F) =$

$$\begin{array}{ccccccc} (B,A) & \xrightarrow{(B,F)} & (B,A') & \xrightarrow{(f,A')} & (B',A') & & \\ \downarrow = & & \boxed{(f,F)} & & \downarrow = & & \\ (B,A) & \xrightarrow{(f,A)} & (B',A) & \xrightarrow{(B',F)} & (B',A') & \xrightarrow{(f',A')} & (B'',A') \\ & & = \downarrow & & \boxed{(f',F)} & & \downarrow = \\ & & (B',A) & \xrightarrow{(f',A)} & (B'',A) & \xrightarrow{(B'',F)} & (B'',A') \end{array}$$

and

$$(f, F' \odot F) = \begin{array}{ccccccc} & & (B,A') & \xrightarrow{(B,F')} & (B,A'') & \xrightarrow{(f,A'')} & (B',A'') \\ & & \downarrow = & & \boxed{(f,F')} & & \downarrow = \\ (B,A) & \xrightarrow{(B,F)} & (B,A') & \xrightarrow{(f,A')} & (B',A') & \xrightarrow{(B',F')} & (B',A'') \\ = \downarrow & & \boxed{(f,F)} & & \downarrow = & & \\ (B,A) & \xrightarrow{(f,A)} & (B',A) & \xrightarrow{(B',F)} & (B',A') & & \end{array}$$

(21) $(u'u, F) = (u', F)(u, F)$ *and* $(u, F' \odot F) = (u, F') \odot (u, F)$

(12) $(f' \odot f, U) = (f', U) \odot (f, U)$ *and* $(f, U'U) = (f, U')(f, U)$

(22)

$$(u, U'U) = \begin{array}{ccccc} & & (B,A) & \xrightarrow{=} & (B,A) \\ & & {\scriptstyle (B,U)}\downarrow & & \downarrow{\scriptstyle (u,A)} \\ (B,\tilde{A}) & \xrightarrow{=} & (B,\tilde{A}) & \boxed{(u,U)} & (\tilde{B},A) \\ {\scriptstyle (B,U')}\downarrow & & {\scriptstyle (u,\tilde{A})}\downarrow & & \downarrow{\scriptstyle (\tilde{B},U)} \\ (B,\tilde{A}') & & (\tilde{B},\tilde{A}) & \xrightarrow{=} & (\tilde{B},\tilde{A}) \\ {\scriptstyle (u,\tilde{A}')}\downarrow & \boxed{(u,U')} & \downarrow{\scriptstyle (\tilde{B},U')} & & \\ (\tilde{B},\tilde{A}') & \xrightarrow{=} & (\tilde{B},\tilde{A}') & & \end{array}$$

and

$$(u'u, U) = \begin{array}{ccccc} (B,A) & \xrightarrow{=} & (B,A) & & \\ {\scriptstyle (B,U)}\downarrow & & \downarrow{\scriptstyle (u,A)} & & \\ (B,\tilde{A}) & \boxed{(u,U)} & (\tilde{B},A) & \xrightarrow{=} & (\tilde{B},A) \\ {\scriptstyle (u,\tilde{A})}\downarrow & & \downarrow{\scriptstyle (\tilde{B},U)} & & \downarrow{\scriptstyle (u',A)} \\ (\tilde{B},\tilde{A}) & \xrightarrow{=} & (\tilde{B},\tilde{A}) & & (\tilde{B}',A) \\ & & {\scriptstyle (U',\tilde{A})}\downarrow & \boxed{(u',U)} & \downarrow{\scriptstyle (\tilde{B}',U)} \\ & & (\tilde{B}',\tilde{A}) & \xrightarrow{=} & (\tilde{B}',\tilde{A}) \end{array}$$

c) *(11)*

$$\begin{array}{ccccc} (B,A) & \xrightarrow{(B,F)} & (B,A') & \xrightarrow{(f,A')} & (B',A') \\ \downarrow{\scriptstyle =} & & \boxed{(f,F)} & & \downarrow{\scriptstyle =} \\ (B,A) & \xrightarrow{(f,A)} & (B',A) & \xrightarrow{(B',F)} & (B',A') \\ {\scriptstyle (u,A)}\downarrow & \boxed{(\omega,A)} & {\scriptstyle (v,A)}\downarrow & \boxed{(v,F)} & \downarrow{\scriptstyle (v,A')} \\ (\tilde{B},A) & \xrightarrow{(g,A)} & (\tilde{B}',A) & \xrightarrow{(\tilde{B}',F)} & (\tilde{B}',A') \end{array} = \begin{array}{ccccc} (B,A) & \xrightarrow{(B,F)} & (B,A') & \xrightarrow{(f,A')} & (B',A') \\ {\scriptstyle (u,A)}\downarrow & \boxed{(u,F)} & {\scriptstyle (u,A')}\downarrow & \boxed{(\omega,A')} & \downarrow{\scriptstyle (v,A')} \\ (\tilde{B},A) & \xrightarrow{(\tilde{B},F)} & (\tilde{B},A') & \xrightarrow{(g,A')} & (\tilde{B}',A') \\ {\scriptstyle =}\downarrow & & \boxed{(g,F)} & & \downarrow{\scriptstyle =} \\ (\tilde{B},A) & \xrightarrow{(g,A)} & (\tilde{B}',A) & \xrightarrow{(\tilde{B}',F)} & (\tilde{B},A') \end{array}$$

and

$$\begin{array}{ccccc} (B,A) & \xrightarrow{(B,F)} & (B,A') & \xrightarrow{(f,A')} & (B',A') \\ \downarrow{\scriptstyle =} & & \boxed{(f,F)} & & \downarrow{\scriptstyle =} \\ (B,A) & \xrightarrow{(f,A)} & (B',A) & \xrightarrow{(B',F)} & (B',A') \\ {\scriptstyle (B,U)}\downarrow & \boxed{(f,U)} & {\scriptstyle (B',U)}\downarrow & \boxed{(B',\zeta)} & \downarrow{\scriptstyle (B',V)} \\ (B,\tilde{A}) & \xrightarrow{(f,\tilde{A})} & (B',\tilde{A}) & \xrightarrow{(B',G)} & (B',\tilde{A}') \end{array} = \begin{array}{ccccc} (B,A) & \xrightarrow{(B,F)} & (B,A') & \xrightarrow{(f,A')} & (B',A') \\ {\scriptstyle (B,U)}\downarrow & \boxed{(B,\zeta)} & {\scriptstyle (B,V)}\downarrow & \boxed{(F,V)} & \downarrow{\scriptstyle (B',V)} \\ (B,\tilde{A}) & \xrightarrow{(B,G)} & (B,\tilde{A}') & \xrightarrow{(f,\tilde{A}')} & (B',\tilde{A}') \\ {\scriptstyle =}\downarrow & & \boxed{(f,G)} & & \downarrow{\scriptstyle =} \\ (B,\tilde{A}) & \xrightarrow{(f,\tilde{A})} & (B',\tilde{A}) & \xrightarrow{(B',G)} & (B',\tilde{A}') \end{array}$$

(22)

$$\begin{array}{ccccc}
(B,A) & \xrightarrow{=} & (B,A) & \xrightarrow{(f,A)} & (B',A) \\
{\scriptstyle (B,U)}\big\downarrow & & {\scriptstyle (u,A)}\big\downarrow & \boxed{(\omega,A)} & \big\downarrow{\scriptstyle (v,A)} \\
(B,\tilde A) & \boxed{(u,U)} & (\tilde B,A) & \xrightarrow{(g,A)} & (\tilde B',A) \\
{\scriptstyle (u,\tilde A)}\big\downarrow & & {\scriptstyle (\tilde B,U)}\big\downarrow & \boxed{(g,U)} & \big\downarrow{\scriptstyle (\tilde B',U)} \\
(\tilde B,\tilde A) & \xrightarrow{=} & (\tilde B,\tilde A) & \xrightarrow{(g,\tilde A)} & (\tilde B',\tilde A)
\end{array}
=
\begin{array}{ccccc}
(B,A) & \xrightarrow{(f,A)} & (B',A) & \xrightarrow{=} & (B',A) \\
{\scriptstyle (B,U)}\big\downarrow & \boxed{(f,U)} & \big\downarrow{\scriptstyle (B',U)} & & \big\downarrow{\scriptstyle (v,A)} \\
(B,\tilde A) & \xrightarrow{(f,\tilde A)} & (B',\tilde A) & \boxed{(v,U)} & (\tilde B',A) \\
{\scriptstyle (u,\tilde A)}\big\downarrow & \boxed{(\omega,\tilde A)} & \big\downarrow{\scriptstyle (v,\tilde A)} & & \big\downarrow{\scriptstyle (\tilde B',U)} \\
(\tilde B,\tilde A) & \xrightarrow{(g,\tilde A)} & (\tilde B',\tilde A) & \xrightarrow{=} & (\tilde B',\tilde A)
\end{array}$$

and

$$\begin{array}{ccccc}
(B,A) & \xrightarrow{=} & (B,A) & \xrightarrow{(B,F)} & (B,A') \\
{\scriptstyle (B,U)}\big\downarrow & & {\scriptstyle (u,A)}\big\downarrow & \boxed{(u,F)} & \big\downarrow{\scriptstyle (u,A')} \\
(B,\tilde A) & \boxed{(u,U)} & (\tilde B,A) & \xrightarrow{(\tilde B,F)} & (\tilde B,A') \\
{\scriptstyle (u,\tilde A)}\big\downarrow & & {\scriptstyle (\tilde B,U)}\big\downarrow & \boxed{(\tilde B,\zeta)} & \big\downarrow{\scriptstyle (\tilde B,V)} \\
(\tilde B,\tilde A) & \xrightarrow{=} & (\tilde B,\tilde A) & \xrightarrow{(\tilde B,G)} & (\tilde B,\tilde A')
\end{array}
=
\begin{array}{ccccc}
(B,A) & \xrightarrow{(B,F)} & (B,A') & \xrightarrow{=} & (B,A') \\
{\scriptstyle (B,U)}\big\downarrow & \boxed{(B,\zeta)} & \big\downarrow{\scriptstyle (B,V)} & & \big\downarrow{\scriptstyle (u,A')} \\
(B,\tilde A) & \xrightarrow{(B,G)} & (B,\tilde A') & \boxed{(u,V)} & (\tilde B,A') \\
{\scriptstyle (u,\tilde A)}\big\downarrow & \boxed{(u,G)} & \big\downarrow{\scriptstyle (u,\tilde A')} & & \big\downarrow{\scriptstyle (\tilde B,V)} \\
(\tilde B,\tilde A) & \xrightarrow{(\tilde B,G)} & (\tilde B,\tilde A') & \xrightarrow{=} & (\tilde B,\tilde A')
\end{array}$$

for any 2-cells

$$\begin{array}{ccc}
B & \xrightarrow{f} & B' \\
{\scriptstyle u}\big\downarrow & \boxed{\omega} & \big\downarrow{\scriptstyle v} \\
\tilde B & \xrightarrow[g]{} & \tilde B'
\end{array}
\quad \textit{and} \quad
\begin{array}{ccc}
A & \xrightarrow{F} & A' \\
{\scriptstyle U}\big\downarrow & \boxed{\zeta} & \big\downarrow{\scriptstyle V} \\
\tilde A & \xrightarrow[G]{} & \tilde A'.
\end{array}
\qquad (2.2)$$

in $\mathbb{B}$*, respectively* $\mathbb{A}$*.*

In analogy to [17, Section 4.2] we set:

Definition 2.2 *A* cubical double functor $H : \mathbb{A} \times \mathbb{B} \longrightarrow \mathbb{C}$ *consists of:*

1. *two families of double functors*

$$(-,A) : \mathbb{B} \longrightarrow \mathbb{C} \quad \textit{and} \quad (B,-) : \mathbb{A} \longrightarrow \mathbb{C}$$

   such that $H(A,-) = (-,A), H(-,B) = (B,-)$ *and* $(-,A)|_B = (B,-)|_A = (B,A)$*, for objects* $A \in \mathbb{A}, B \in \mathbb{B}$*, and*

2. *four families of 2-cells* $(f,F),(u,F),(f,U),(u,U)$ *in* $\mathbb{C}$ *for 1h-cells* F *of* $\mathbb{A}$ *and* f *of* $\mathbb{B}$*, and 1v-cells* U *of* $\mathbb{A}$ *and* u *of* $\mathbb{B}$*,*

satisfying the conditions listed in part 2. of Proposition 2.1.

We may now describe a double category $\mathbb{A}\otimes\mathbb{B}$ by reading off the structure of the image double category $F(\mathbb{A})(\mathbb{B})$ for any double functor $F : \mathbb{A} \longrightarrow [\![\mathbb{B}, \mathbb{A}\otimes\mathbb{B}]\!]$ in the right hand-side of (2.1) using the characterization of a double functor before Proposition 2.1. With the notation $F(x)(y) = (y,x) =: x \otimes y$ for any 0-, 1h-, 1v- or 2-cells x of $\mathbb{A}$ and y of $\mathbb{B}$ we get to the following:

Proposition 2.3 *By the construction in [3, Section 2]* $\mathbb{A}\otimes\mathbb{B}$ *is the double category generated by the following data:*
ŏbjects: $A \otimes B$ *for objects* $A \in \mathbb{A}, B \in \mathbb{B}$*;*
Ĭh-cells: $A \otimes f, F \otimes B$ *modulo the following relations:*

$$(A \otimes f') \odot (A \otimes f) = A \otimes (f' \odot f), \quad (F' \otimes B) \odot (F \otimes B) = (F' \odot F) \otimes B, \quad A \otimes 1_B = 1_{A\otimes B} = 1_A \otimes B$$

where f, f' are 1h-cells of $\mathbb{B}$ and F, F' 1h-cells of $\mathbb{A}$;
$\check{1}$v-cells: $A \otimes u, U \otimes B$ obeying the following rules:

$$(A \otimes u')(A \otimes u) = A \otimes u'u, \quad (U' \otimes B)(U \otimes B) = U'U \otimes B, \quad A \otimes 1^B = 1^{A \otimes B} = 1^A \otimes B$$

where u, u' are 1v-cells of $\mathbb{B}$ and U, U' 1v-cells of $\mathbb{A}$;
$\check{2}$-cells: $A \otimes \omega, \zeta \otimes B$:

$$\begin{array}{ccc} A \otimes B & \xrightarrow{A \otimes f} & A \otimes B' \\ {\scriptstyle A \otimes u}\downarrow & \boxed{A \otimes \omega} & \downarrow{\scriptstyle A \otimes v} \\ A \otimes \tilde{B} & \xrightarrow{A \otimes g} & A' \otimes \tilde{B}' \end{array} \qquad \begin{array}{ccc} A \otimes B & \xrightarrow{F \otimes B} & A' \otimes B \\ {\scriptstyle U \otimes B}\downarrow & \boxed{\zeta \otimes B} & \downarrow{\scriptstyle V \otimes B} \\ \tilde{A} \otimes B & \xrightarrow{G \otimes B} & \tilde{A}' \otimes B \end{array}$$

where ω and ζ are as in (2.2), and four types of 2-cells coming from the 2-cells of point 2. in Proposition 2.1: vertically invertible globular 2-cell $F \otimes f : (A' \otimes f) \odot (F \otimes B) \Rightarrow (F \otimes B') \odot (A \otimes f)$, horizontally invertible globular 2-cell $U \otimes u : (\tilde{A} \otimes u)(U \otimes B) \Rightarrow (U \otimes \tilde{B})(A \otimes u)$, 2-cells $F \otimes u$ and $U \otimes f$ subject to the rules induced by a), b) and c) of point 2. in Proposition 2.1 and the following ones:

$$A \otimes (\omega' \odot \omega) = (A \otimes \omega') \odot (A \otimes \omega), \quad (\zeta' \odot \zeta) \otimes B = (\zeta' \otimes B) \odot (\zeta \otimes B),$$

$$A \otimes (\omega'\omega) = (A \otimes \omega')(A \otimes \omega), \quad (\zeta'\zeta) \otimes B = (\zeta' \otimes B)(\zeta \otimes B),$$

$$A \otimes \mathrm{Id}_f = \mathrm{Id}_{A \otimes f}, \quad \mathrm{Id}_F \otimes B = \mathrm{Id}_{F \otimes B}, \quad A \otimes \mathrm{Id}^f = \mathrm{Id}^{A \otimes f}, \quad \mathrm{Id}^F \otimes B = \mathrm{Id}^{F \otimes B}.$$

2.2. *A monoidal embedding of $(Gray, \otimes)$ into $(Dbl, \otimes)$*

Let $E : (Gray, \otimes) \hookrightarrow (Dbl, \otimes)$ denote the embedding functor which to a 2-category assigns a strict double category whose all vertical 1-cells are identities and whose 2-cells are vertically globular cells. Then E is a left adjoint to the functor that to a strict double category assigns its underlying horizontal 2-category. Let us denote by $C = (Gray, \otimes)$ and by $\mathcal{D} = \mathrm{Im}\,(E) \subseteq (Dbl, \otimes)$, the image category by E, then the corestriction of E to $\mathcal{D}$ is the identity functor

$$F : C \longrightarrow \mathcal{D}. \tag{2.3}$$

In order to examine the monoidality of F let us first consider an assignment $t : F(\mathcal{A} \otimes \mathcal{B}) \longrightarrow F(\mathcal{A}) * F(\mathcal{B})$ for two 2-categories $\mathcal{A}$ and $\mathcal{B}$, where $*$ denotes some monoidal product in the category of strict double categories which a priori could be the Cartesian one or the one from the monoidal category $(Dbl, \otimes)$.

Observe that given 1-cells $f : A \longrightarrow A'$ in $\mathcal{A}$ and $g : B \longrightarrow B'$ in $\mathcal{B}$ the composition 1-cells $(f \otimes B') \odot (A \otimes g)$ and $(A' \otimes g) \odot (f \otimes B)$ in $\mathcal{A} \otimes \mathcal{B}$ are not equal both in $(Gray, \otimes)$ and in $(Dbl, \otimes)$. This means that their images $F\big((f \otimes B') \odot (A \otimes g)\big)$ and $F\big((A' \otimes g) \odot (f \otimes B)\big)$ are different as 1h-cells of the double category $F(\mathcal{A} \otimes \mathcal{B})$. Now if we map these two images by t into the Cartesian product $F(\mathcal{A}) \times F(\mathcal{B})$, we will get in both cases the 1h-cell (f, g). Then t with the codomain in the Cartesian product is a bad candidate for the monoidal structure of the identity functor F. This shows that the Cartesian monoidal product on the category of strict double categories is not a good choice for a monoidal structure if one wants to embed the Gray category of 2-categories into the latter category. In contrast, if the codomain of t is the monoidal product of $(Dbl, \otimes)$, we see that t is identity on these two 1-cells.

Similar considerations and comparing the monoidal product from [23, Theorem I.4.9] in $(Gray, \otimes)$ to the one after Definition 2.2 above in $(Dbl, \otimes)$, show that for the candidate for (the one part of) a monoidal structure on the identity functor F we may take the identity $s = \mathrm{Id} : F(\mathcal{A} \otimes \mathcal{B}) \longrightarrow F(\mathcal{A}) \otimes F(\mathcal{B})$, and that it is indeed a strict double functor of strict double categories. For the other part of a monoidal structure on F, namely $s_0 : F(*_2) \longrightarrow *_{Dbl}$, where $*_2$ is the trivial 2-category with a single object, and similarly $*_{Dbl}$ is the trivial double category, it is clear that we again may take identity. The hexagonal and two square relations for the monoidality of the functor (F, s, s_0) come down to checking if

$$F(\alpha_1) = \alpha_2, \quad F(\lambda_1) = \lambda_2, \quad \text{and} \quad F(\rho_1) = \rho_2$$

where the monoidal constraints with indexes 1 are those from C and those with indexes 2 from $\mathcal{D}$.

Given any monoidal closed category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ in [3, Section 4.1] the author constructs a mate

$$a^C_{A,B} : [A \otimes B, C] \longrightarrow [A, [B, C]] \tag{2.4}$$

for α under the adjunctions $(- \otimes X, [X, -])$, for X taking to be A, B and $A \otimes B$, and then she constructs a mate of a:

$$l^C_{A,B} : \left([A, B] \xrightarrow{[\varepsilon^C_A, 1]} [[C, A] \otimes C, B] \xrightarrow{a^B} [[C, A], [C, B]]\right) \tag{2.5}$$

where ε is the counit of the adjunction. By the mate correspondence one gets:

$$a^C_{A,B} : \left([A \otimes B, C] \xrightarrow{l^B_{A\otimes B,C}} [[B, A \otimes B], [B, C]] \xrightarrow{[\eta^B_A, 1]} [A, [B, C]]\right) \tag{2.6}$$

where η is the unit of the adjunction. As above for the monoidal constraints, let us write l_i, a_i, ε_i and η_i with $i = 1, 2$ for the corresponding 2-functors in C (with $i = 1$), respectively double functors in $\mathcal{D}$ (with $i = 2$). Comparing the description of l_1 from [3, Section 4.7], obtained as indicated above: α_1 determines a_1, which in turn determines l_1 by (2.5), to the construction of l_2 in [3, Section 2.4], on one hand, and the well-known 2-category $[\mathcal{A}, \mathcal{B}] = \mathrm{Fun}(\mathcal{A}, \mathcal{B})$ of 2-functors between 2-categories $\mathcal{A}$ and $\mathcal{B}$, pseudo natural transformations and modifications (see *e.g.* [21, Section 5.1], [2]) to the definition of the double category $[\![\mathbb{A}, \mathbb{B}]\!]$ from [3, Section 2.2] for double categories $\mathbb{A}, \mathbb{B}$, on the other hand, one immediately obtains:

Lemma 2.4 *For two 2-categories $\mathcal{A}$ and $\mathcal{B}$, functor F from (2.3) and l_1 and l_2 as above, it is:*

- $F(l_1) = l_2$,
- $F([\mathcal{A}, \mathcal{B}]) = [\![F(\mathcal{A}), F(\mathcal{B})]\!]$.

Because of the extent of the definitions and the detailed proofs we will omit them, we only record that the counits of the adjunctions $\varepsilon_i, i = 1, 2$ are basically given as evaluations and it is $F(\varepsilon_1) = \varepsilon_2$. The counits $\eta_i, i = 1, 2$ are defined in the natural way and it is also clear that $F(\eta_1) = \eta_2$. Now by the above Lemma and (2.6) we get: $F(a_1) = a_2$. Then from the next Lemma we get that $F(\alpha_1) = \alpha_2$:

Lemma 2.5 *Suppose that there is an embedding functor $F : C \longrightarrow \mathcal{D}$ between monoidal closed categories which fulfills:*

a) $F(X) \otimes F(Y) = F(X \otimes Y)$ *for objects* $X, Y \in C$,

b) $F([X, Y]) = [F(X), F(Y)]$,

c) $F(a_C) = a_{\mathcal{D}}$, *where the respective a's are given through (2.4),*

then it is $F(\alpha_C) = \alpha_{\mathcal{D}}$, being α's the respective associativity constraints.

Proof. By the mate construction in (2.4) we have a commuting diagram:

$$\begin{array}{ccc}
C(A \otimes (B \otimes C), D) & \xrightarrow{\cong} & C(A, [B \otimes C, D]) \\
{\scriptstyle C(\alpha_C, id)} \downarrow & \boxed{(1)} & \downarrow {\scriptstyle C(id, a_C)} \\
C((A \otimes B) \otimes C, D) & \xleftarrow{\cong} & C(A, [B, [C, D]])
\end{array}$$

Applying F to it, by the assumptions *a*) and *b*) we obtain a commuting diagram:

$$\begin{array}{ccc}
\mathcal{D}(F(A) \otimes (F(B) \otimes F(C)), F(D)) & \xrightarrow{\cong} & \mathcal{D}(F(A), [F(B) \otimes F(C), F(D)]) \\
{\scriptstyle \mathcal{D}(F(\alpha_C), id)} \downarrow & \boxed{(2)} & \downarrow {\scriptstyle \mathcal{D}(id, F(a_C))} \\
\mathcal{D}((F(A) \otimes F(B)) \otimes F(C), F(D)) & \xleftarrow{\cong} & \mathcal{D}(F(A), [F(B), [F(C), F(D)]])
\end{array}$$

Now by the assumption *c*) and the mate construction in (2.4) it follows $F(\alpha_C) = \alpha_{\mathcal{D}}$. □

So far we have proved that for the categories $\mathcal{C}$ and $\mathcal{D}$ as in (2.3) we have $F(\alpha_{\mathcal{C}}) = \alpha_{\mathcal{D}}$. For the unity constraints $\rho_i, \lambda_i, i = 1, 2$ in the cases of both categories (see Sections 3.3 and 4.7 of [3]) it is:

$$\rho_i^A = \varepsilon_{i,A}^{1_i} \circ (c_i \otimes id_{1_i}) \quad \text{and} \quad \lambda_i^A = \varepsilon_{i,A}^{A} \circ (1_A \otimes id_A)$$

where $c_i : A \longrightarrow [1_i, A]$ is the canonical isomorphism and $1_A : 1_i \longrightarrow [A, A]$ the 2-functor (pseudofunctor) sending the single object of the terminal 2-category 1_1 (double category 1_2) to the identity 2-functor (pseudofunctor) $A \longrightarrow A$ (here we have used the same notation for objects A and inner home objects both in $\mathcal{C}$ and in $\mathcal{D}$). Then it is clear that also $F(\lambda_1) = \lambda_2$ and $F(\rho_1) = \rho_2$, which finishes the proof that the functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a monoidal embedding.

Proposition 2.6 *The category* $(Gray, \otimes)$ *monoidally embeds into* $(Dbl, \otimes)$*, where the respective monoidal structures are those from [23] and [3]. Consequently, a monoid in* $(Gray, \otimes)$ *is a monoid in* $(Dbl, \otimes)$*, and a monoidal bicategory can be seen as a monoidal double category with respect to Böhm's tensor product.*

2.3. *A monoid in* $(Dbl, \otimes)$

In [3, Section 4.3] a complete list of data and conditions defining the structure of a monoid $\mathbb{A}$ in $(Dbl, \otimes)$ is given. As a part of this structure we have the following occurrence. As a monoid in $(Dbl, \otimes)$, we have that $\mathbb{A}$ is equipped with a strict double functor $m : \mathbb{A} \otimes \mathbb{A} \longrightarrow \mathbb{A}$. Since in the monoidal product $\mathbb{A} \otimes \mathbb{A}$ horizontal and vertical 1-cells of the type $(f \otimes 1)(1 \otimes g)$ and $(1 \otimes g)(f \otimes 1)$ are not equal (here juxtaposition denotes the corresponding composition of the 1-cells), one can fix a choice for how to define an image 1-cell $f \circledast g$ by m (either $m\big((f \otimes 1)(1 \otimes g)\big)$ or $m\big((1 \otimes g)(f \otimes 1)\big)$). Any of the two choices yields a *double pseudo* functor from the *Cartesian* product double category

$$\circledast : \mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{A}. \tag{2.7}$$

Let us see this. If we take two pairs of horizontal 1-cells $(h, k), (h', k')$ in $A \times A$, for the images under $\circledast$, fixing the second choice above, we get $(h'h) \circledast (k'k) = m\big((1 \otimes k'k)(h'h \otimes 1)\big) = m(1 \otimes k')m(1 \otimes k)m(h' \otimes 1)m(h \otimes 1)$, whereas $(h' \circledast k')(h \circledast k) = m\big((1 \otimes k')(h' \otimes 1)\big)m\big((1 \otimes k)(h \otimes 1)\big) = m(1 \otimes k')m(h' \otimes 1)m(1 \otimes k)m(h \otimes 1)$. So, the two images differ in the flip on the middle factors. The analogous situation happens on the vertical level, thus the functor $\circledast$ preserves both vertical and horizontal 1-cells only up to an isomorphism 2-cell. This makes it a double pseudo functor due to [31, Definition 6.1].

As outlined at the end of [3, Section 4.3], monoids in (the non-Cartesian monoidal category) $(Dbl, \otimes)$ are monoids in the Cartesian monoidal category $(Dbl, \circledast)$ of strict double categories and double pseudo functors (in the sense of [31]).

2.4. *Monoidal double categories as intercategories and beyond*

A monoidal double category in [30] is a pseudomonoid in the 2-category *PsDbl* of pseudo double categories, pseudo double functors and vertical transformations, seen as a monoidal 2-category with the Cartesian product. As such it is a particular case of an intercategory [19].

An intercategory is a pseudocategory (*i.e.* weakly internal category) in the 2-category *LxDbl* of pseudo double categories, lax double functors and horizontal transformations. It consists of pseudodouble categories $\mathbb{D}_0$ and $\mathbb{D}_1$ and pseudo double functors $S, T : \mathbb{D}_1 \longrightarrow \mathbb{D}_0, U : \mathbb{D}_0 \longrightarrow \mathbb{D}_1, M : \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \longrightarrow \mathbb{D}_1$ (where S and T are strict) satisfying the corresponding properties. One may denote this structure formally by

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \overrightarrow{\overrightarrow{\longrightarrow}} \mathbb{D}_1 \overrightarrow{\overleftarrow{\longrightarrow}} \mathbb{D}_0$$

where $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1$ is a certain 2-pullback and the additional two arrows $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \longrightarrow \mathbb{D}_1$ stand for the two projections. When $\mathbb{D}_0$ is the trivial double category 1 (the terminal object in *LxDbl*, consisting of a single object *), setting $\mathbb{D}_1 = \mathbb{D}$ one has that $\mathbb{D} \times \mathbb{D}$ is the Cartesian product of pseudo double categories.

As a pseudomonoid in *PsDbl*, a monoidal double category of Shulman consists of a pseudo double category $\mathbb{D}$ and pseudo double functors $M : \mathbb{D}\times\mathbb{D} \longrightarrow \mathbb{D}$ and $U : 1 \longrightarrow \mathbb{D}$ which satisfy properties that make $\mathbb{D}$ precisely an intercategory

$$\mathbb{D}\times\mathbb{D} \overset{\textstyle\rightarrow}{\rightrightarrows} \mathbb{D} \overset{\textstyle\rightarrow}{\leftrightarrows} 1,$$

as explained in [21, Section 3.1].

Nevertheless, if one would try to make a monoid in $(Dbl, \otimes)$, which is a monoid in $(Dbl, \circledast)$, into an intercategory, one would need a lax double functor on the Cartesian product $\mathbb{D}\times\mathbb{D}$ (the pullback). However, as we showed in the last subsection, on the Cartesian product one has a *double pseudo* functor $\circledast$. So, as observed at the end of [3, Section 4.3], there seems to be no easy way to regard a monoid in $(Dbl, \otimes)$ as a suitably degenerated intercategory. Motivated by this and Proposition 2.6, we want now to upgrade the Cartesian monoidal category $(Dbl, \circledast)$ from the end of Subsection 2.3 to a 2-category, so to obtain an intercategory-type notion which would include monoidal double categories due to Böhm.

In the next section we introduce 2-cells and "unfortunately" rather than a 2-category we will obtain a tricategory of strict double categories whose 1-cells are double pseudo functors of Shulman. Since 1-cells of the 2-category *LxDbl* (considered by Grandis and Paré to define intercategories) are lax double functors (they are lax in one and strict in the other direction), and our 1-cells are double pseudo functors, that is, they are given by isomorphisms in both directions, we can not generalize intercategories this way, rather, we will propose an alternative notion to intercategories which will include most of the examples of intercategories treated in [21], but not duoidal categories, as they induce lax functors on Cartesian product, rather than pseudo ones.

3. Tricategory of strict double categories and double pseudo functors

For the notion of a tricategory we refer the reader to the first part of the Appendix. Let us denote the tricategory from the title of this section by DblPs. As we are going to use double pseudo functors of [31], which preserve compositions of 1-cells and identity 1-cells in both horizontal and vertical direction up to an isomorphism, we have to introduce accordingly horizontal and vertical transformations. A pair consisting of a horizontal and a vertical pseudonatural transformation, which we define next, will be a part of the data constituting a 2-cell of the tricategory DblPs. Note that while *PsDbl* usually denotes a category or a 2-category of *pseudo* double categories and pseudo double functors, that is, in which 0- and 1-cells are weakened, in the notation DblPs we wish to stress that both 1- and 2-cells are weakened in both directions so to deal with *double pseudo* functors. Because of the extensiveness, in this section we only spell out the definitions and the structure of a tricategory for DblPs. The easier checks are left to the reader, while the detailed proofs that require more involved computations are carried out in [13].

3.1. Towards the 2-cells

For the structure of a double pseudo functor we use the same notation as in [31, Definition 6.1] with the only difference that 0-cells we denote by A, B... and 1v-cells by u, v.... To simplify the notation, we will denote by juxtaposition the compositions of both 1h- and 1v-cells, from the notation of the 1-cells it will be clear which kind of 1-cells and therefore composition is meant. Let $\mathbb{A}, \mathbb{B}, \mathbb{C}$ be strict double categories throughout.

Definition 3.1 *A* horizontal pseudonatural transformation *between double pseudo functors $F, G : \mathbb{A} \longrightarrow \mathbb{B}$ consists of the following:*

- *for every 0-cell A in $\mathbb{A}$ a 1h-cell $\alpha(A) : F(A) \longrightarrow G(A)$ in $\mathbb{B}$,*
- *for every 1v-cell $u : A \longrightarrow A'$ in $\mathbb{A}$ a 2-cell in $\mathbb{B}$:*

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha(A)} & G(A) \\ {\scriptstyle F(u)}\downarrow & \boxed{\alpha_u} & \downarrow{\scriptstyle G(u)} \\ F(A') & \xrightarrow{\alpha(A')} & G(A') \end{array}$$

- *for every 1h-cell $f : A \longrightarrow B$ in $\mathbb{A}$ there is a 2-cell in $\mathbb{B}$:*

$$\begin{array}{ccccc} F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{\alpha(B)} & G(B) \\ \big\downarrow = & & \boxed{\delta_{\alpha,f}} & & \big\downarrow = \\ F(A) & \xrightarrow{\alpha(A)} & G(A) & \xrightarrow{G(f)} & G(A) \end{array}$$

so that the following are satisfied:

1. *pseudonaturality of 2-cells: for every 2-cell in $\mathbb{A}$* $\begin{array}{ccc} A & \xrightarrow{f} & B \\ u\big\downarrow & \boxed{a} & \big\downarrow v \\ A' & \xrightarrow[g]{} & B' \end{array}$ *the following identity in $\mathbb{B}$ must hold:*

$$\begin{array}{ccccc} F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{\alpha(B)} & G(B) \\ F(u)\big\downarrow & \boxed{F(a)} & F(v)\big\downarrow & \boxed{\alpha_v} & \big\downarrow G(v) \\ F(A') & \xrightarrow{F(g)} & F(B') & \xrightarrow{\alpha(B')} & G(B') \\ =\big\downarrow & & \boxed{\delta_{\alpha,g}} & & \big\downarrow = \\ F(A') & \xrightarrow{\alpha(A')} & G(A') & \xrightarrow{G(g)} & G(B') \end{array} = \begin{array}{ccccc} F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{\alpha(B)} & G(B) \\ \big\downarrow = & & \boxed{\delta_{\alpha,f}} & & \big\downarrow = \\ F(A) & \xrightarrow{\alpha(A)} & G(A) & \xrightarrow{G(f)} & G(B) \\ F(u)\big\downarrow & \boxed{\alpha_u} & G(u)\big\downarrow & \boxed{G(a)} & \big\downarrow G(v) \\ F(A') & \xrightarrow{\alpha(A')} & G(A') & \xrightarrow{G(g)} & G(B') \end{array}$$

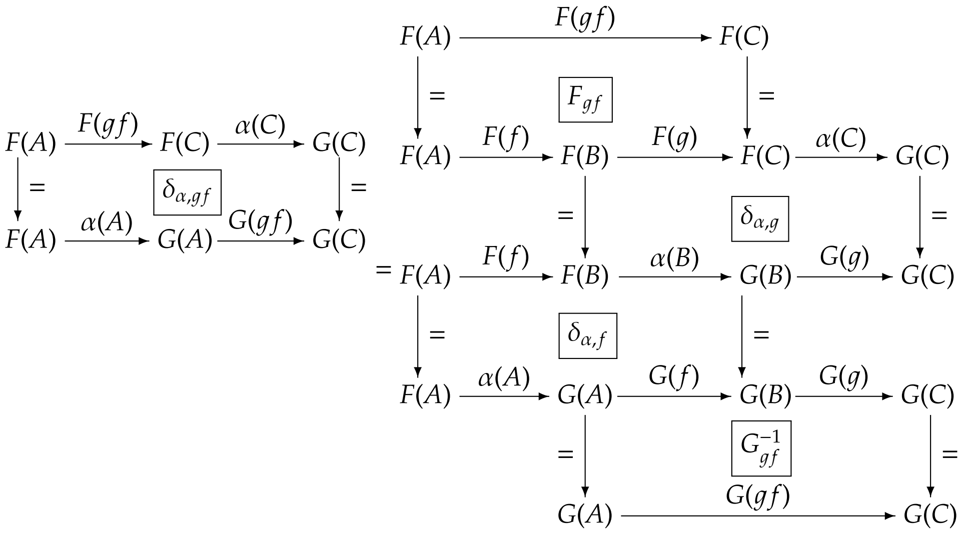
2. *vertical functoriality: for any composable 1v-cells u and v in $\mathbb{A}$:*

$$\begin{array}{ccccc} F(A) & \xrightarrow{=} & F(A) & \xrightarrow{\alpha(A)} & G(A) \\ F(u)\big\downarrow & \boxed{F^{vu}} & \Big| & & \Big| \\ F(A') & & F(vu) & \boxed{\alpha^{vu}} & G(vu) \\ F(v)\big\downarrow & & \Big\downarrow & & \Big\downarrow \\ F(A'') & \xrightarrow{=} & F(A'') & \xrightarrow{\alpha(A'')} & G(A'') \end{array} = \begin{array}{ccccc} F(A) & \xrightarrow{\alpha(A)} & G(A) & \xrightarrow{=} & G(A) \\ F(u)\big\downarrow & \boxed{\alpha_u} & \big\downarrow G(u) & & \Big| \\ F(A') & \xrightarrow{\alpha(A')} & G(A') & \boxed{G^{vu}} & G(vu) \\ F(v)\big\downarrow & \boxed{\alpha_v} & \big\downarrow G(v) & & \Big\downarrow \\ F(A'') & \xrightarrow{\alpha(A'')} & G(A'') & \xrightarrow{=} & G(A'') \end{array}$$

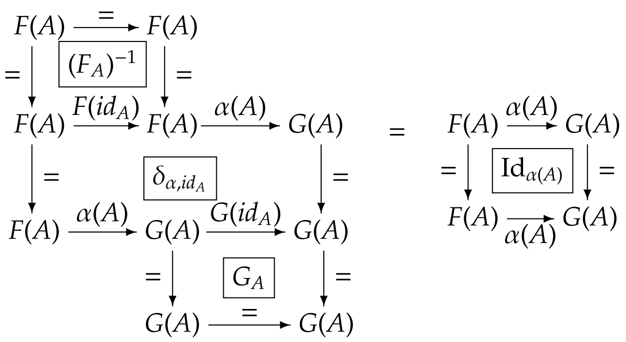
and

$$\begin{array}{ccccc} F(A) & \xrightarrow{=} & F(A) & \xrightarrow{\alpha(A)} & G(A) \\ \big\downarrow = & \boxed{F^A} & F(id_A)\big\downarrow & \boxed{\alpha_{id_A}} & \big\downarrow G(id_A) \\ F(A) & \xrightarrow{=} & F(A) & \xrightarrow{\alpha(A)} & G(A) \end{array} = \begin{array}{ccccc} F(A) & \xrightarrow{\alpha(A)} & G(A) & \xrightarrow{=} & G(A) \\ \big\downarrow = & \boxed{\mathrm{Id}_{\alpha(A)}} & =\big\downarrow & \boxed{G^A} & \big\downarrow G(id_A) \\ F(A) & \xrightarrow{\alpha(A)} & G(A) & \xrightarrow{=} & G(A) \end{array}$$

3. *horizontal functoriality for $\delta_{\alpha,-}$: for any composable 1h-cells f and g in $\mathbb{A}$ the 2-cell $\delta_{\alpha,gf}$ is given by:*



and unit:



Remark 3.2 Recall horizontal transformations from [20, Section 2.2] and their version when $R = S = \mathrm{Id}, \mathbb{A} = \mathbb{B}, \mathbb{C} = \mathbb{D}$, which constitute the 2-cells of the 2-category $\mathcal{L}x\mathcal{D}bl$ from [19]. Considering them as acting between pseudo double functors of strict double categories, one has that they are particular cases of our horizontal pseudonatural transformations so that the 2-cells $\delta_{\alpha,f}$ and F_{gf}, F_A are identities. Similarly, vertical transformations from [30] and [16] are particular cases of the vertical analogon of Definition 3.1.

Remark 3.3 Our above definition generalizes also horizontal pseudotransformations from [3, Section 2.2] to the case of *double pseudo functors* instead of strict double functors. In [3, Section 2.2] horizontal pseudotransformations appear as 1h-cells of the double category $[\![\mathbb{A}, \mathbb{B}]\!]$ defined therein, which we mentioned in Subsection 2.1. On the other hand, our definition of horizontal pseudotransformations differs from *strong horizontal transformations* from [22, Section 7.4] in that therein the authors work with *pseudo* double categories (whereas we work here with strict ones), and they work with double functors which are lax in one direction and strict in the other, whereas we work with double functors which are pseudo in both directions.

The following results are straightforwardly proved:

Lemma 3.4 *For a double pseudofunctor $H : \mathbb{B} \longrightarrow \mathbb{C}$ and a horizontal pseudonatural transformation $\alpha : F \longrightarrow G$ of double pseudofunctors $F, G : \mathbb{A} \longrightarrow \mathbb{B}$, $H(\alpha)$ is a horizontal pseudonatural transformation with $(H(\alpha))_u = H(\alpha_u)$ and*

$\delta_{H(\alpha),f}$ satisfying:

$$\begin{array}{ccccc} HF(A) & & \xrightarrow{H(\alpha(B)F(f))} & & HG(B) \\ = \downarrow & & \boxed{H_{\alpha(B),F(f)}} & & \downarrow = \\ HF(A) & \xrightarrow{HF(f)} & HF(B) & \xrightarrow{H(\alpha(B))} & HG(B) \\ = \downarrow & & \boxed{\delta_{H(\alpha),f}} & & \downarrow = \\ HF(A) & \xrightarrow{H(\alpha(A))} & HG(A) & \xrightarrow{HG(f)} & HG(B) \end{array} \quad = \quad \begin{array}{ccccccc} HF(A) & \xrightarrow{=} & HF(A) & \xrightarrow{H(\alpha(B)F(f))} & HG(B) & \xrightarrow{=} & HG(B) \\ = \downarrow & \boxed{H^{F(A)}} & \downarrow H(id) & \boxed{H(\delta_{\alpha,f})} & H(id) \downarrow & \boxed{(H^{G(B)})^{-1}} & \downarrow = \\ HF(A) & \xrightarrow{=} & HF(A) & \xrightarrow{H\big(G(f)\alpha(A)\big)} & HG(B) & \xrightarrow{=} & HG(B) \\ & & = \downarrow & \boxed{H_{G(f),\alpha(A)}} & \downarrow = & & \\ & & HF(A) \xrightarrow{H(\alpha(A))} & HG(A) & \xrightarrow{HG(f)} HG(B). & & \end{array}$$

Lemma 3.5 *Horizontal composition of two horizontal pseudonatural transformations $\alpha_1 : F \Rightarrow G : \mathbb{A} \longrightarrow \mathbb{B}$ and $\beta_1 : F' \Rightarrow G' : \mathbb{B} \longrightarrow \mathbb{C}$, denoted by $\beta_1 \circ \alpha_1$, is well-given by:*

- *for every 0-cell A in $\mathbb{A}$ a 1h-cell in $\mathbb{C}$:*

$$(\beta_1 \circ \alpha_1)(A) = \Big(F'F(A) \stackrel{F'(\alpha_1(A))}{\longrightarrow} F'G(A) \stackrel{\beta_1(G(A))}{\longrightarrow} G'G(A)\Big),$$

- *for every 1v-cell $u : A \longrightarrow A'$ in $\mathbb{A}$ a 2-cell in $\mathbb{C}$:*

$$(\beta_1 \circ \alpha_1)_u = \begin{array}{ccccc} F'F(A) & \xrightarrow{F'(\alpha_1(A))} & F'G(A) & \xrightarrow{\beta_1(G(A))} & G'G(A) \\ F'F(u) \downarrow & \boxed{F'((\alpha_1)_u)} & \downarrow F'G(u) & \boxed{(\beta_1)_{G(u)}} & \downarrow G'G(u) \\ F'F(A') & \xrightarrow{F'(\alpha_1(A'))} & F'G(A') & \xrightarrow{\beta_1(G(A'))} & G'G(A') \end{array}$$

- *for every 1h-cell $f : A \longrightarrow B$ in $\mathbb{A}$ a 2-cell in $\mathbb{C}$:*

$$\delta_{\beta_1 \circ \alpha_1, f} = \begin{array}{ccccccc} F'F(A) & \xrightarrow{F'F(f)} & F'F(B) & \xrightarrow{F'(\alpha_1(B))} & F'G(B) & & \\ \downarrow = & & \boxed{\delta_{F'(\alpha_1),f}} & & \downarrow = & & \\ F'F(A) & \xrightarrow{F'(\alpha_1(A))} & F'G(A) & \xrightarrow{F'G(f)} & F'G(B) & \xrightarrow{\beta_1(G(B))} & G'G(B) \\ & & = \downarrow & & \boxed{\delta_{\beta_1,G(f)}} & & \downarrow = \\ & & F'G(A) & \xrightarrow{\beta_1(G(A))} & G'G(A) & \xrightarrow{G'G(f)} & G'G(B) \end{array}$$

where $\delta_{F'(\alpha_1),f}$ is from Lemma 3.4.

Vertical pseudonatural transformations between double pseudo functors $F, G : \mathbb{A} \longrightarrow \mathbb{B}$ are defined in an analogous way, consisting of a 1v-cell $\alpha(A) : F(A) \longrightarrow G(A)$ in $\mathbb{B}$ for every 0-cell A in $\mathbb{A}$, for every 1h-cell $f : A \longrightarrow B$ in $\mathbb{A}$ a 2-cell on the left hand-side below and for every 1v-cell $u : A \longrightarrow A'$ in $\mathbb{A}$ a 2-cell on the right hand-side below, both in $\mathbb{B}$:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \alpha(A) \downarrow & \boxed{\alpha_f} & \downarrow \alpha(B) \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array} \qquad \begin{array}{ccc} F(A) & \xrightarrow{=} & F(A) \\ F(u) \downarrow & & \downarrow \alpha(A) \\ F(A') & \boxed{\delta_{\alpha,u}} & G(A) \\ \alpha(A') \downarrow & & \downarrow G(u) \\ G(A') & \xrightarrow{=} & G(A') \end{array}$$

Observe that we use the same notation for the 2-cells $\alpha_\bullet$ and $\delta_{\alpha,\bullet}$ both for a horizontal and a vertical pseudonatural transformation α, the difference is indicated by the notation for the respective 1-cell, recall that horizontal ones are denoted by f, g.. and vertical ones by u, v....

For vertical pseudonatural transformations results analogous to Lemma 3.4 and Lemma 3.5 hold, the analogon of the latter one we state here in order to fix the structures that we use:

Lemma 3.6 *Horizontal composition of two vertical pseudonatural transformations $\alpha_0 : F \Rightarrow G : \mathbb{A} \longrightarrow \mathbb{B}$ and $\beta_0 : F' \Rightarrow G' : \mathbb{B} \longrightarrow \mathbb{C}$, denoted by $\beta_0 \circ \alpha_0$, is well-given by:*

- *for every 0-cell A in $\mathbb{A}$ a 1v-cell on the left below, and for every 1h-cell $f : A \longrightarrow B$ in $\mathbb{A}$ a 2-cell on the right below, both in $\mathbb{C}$:*

$$
(\beta_0 \circ \alpha_0)(A) = \begin{array}{c} F'F(A) \\ \Big\downarrow {\scriptstyle F'(\alpha_0(A))} \\ F'G(A) \\ \Big\downarrow {\scriptstyle \beta_0(G(A))} \\ G'G(A) \end{array}
\qquad
(\beta_0 \circ \alpha_0)_f = \begin{array}{ccc} F'F(A) & \xrightarrow{F'F(f)} & F'F(B) \\ {\scriptstyle F'(\alpha_0(A))}\Big\downarrow & \boxed{F'((\alpha_0)_f)} & \Big\downarrow{\scriptstyle F'(\alpha_0(B))} \\ F'G(A) & \xrightarrow{F'G(f)} & F'G(B) \\ {\scriptstyle \beta_0(G(A))}\Big\downarrow & \boxed{(\beta_0)_{G(f)}} & \Big\downarrow{\scriptstyle \beta_0(G(B))} \\ G'G(A) & \xrightarrow{G'G(f)} & G'G(B) \end{array}
$$

- *for every 1v-cell $u : A \longrightarrow A'$ in $\mathbb{A}$ a 2-cell in $\mathbb{C}$:*

$$
\delta_{\beta_0 \circ \alpha_0, u} = \begin{array}{ccccc} F'F(A) & \xrightarrow{=} & F'F(A) & & \\ {\scriptstyle F'F(u)}\Big\downarrow & & \Big\downarrow{\scriptstyle F'(\alpha_0(A))} & & \\ F'F(A') & & F'G(A) & \xrightarrow{=} & F'G(A) \\ {\scriptstyle F'(\alpha_0(A'))}\Big\downarrow & \boxed{\delta_{F'(\alpha_0),u}} & \Big\downarrow{\scriptstyle F'G(u)} & & \Big\downarrow{\scriptstyle \beta_0(G(A))} \\ F'G(A') & \xrightarrow{=} & F'G(A') & & G'G(A) \\ & & {\scriptstyle \beta_0(G(A'))}\Big\downarrow & \boxed{\delta_{\beta_0,G(u)}} & \Big\downarrow{\scriptstyle G'G(u)} \\ & & G'G(A') & \xrightarrow{=} & G'G(A') \end{array}
$$

where $\delta_{F'(\alpha_0),u}$ is defined analogously as in Lemma 3.4.

Horizontal compositions of horizontal and of vertical pseudonatural transformations are not strictly associative.

We proceed by defining vertical compositions of horizontal and of vertical pseudonatural transformations. From the respective definitions it will be clear that these vertical compositions are strictly associative.

Lemma 3.7 *Vertical composition of two horizontal pseudonatural transformations $\alpha_1 : F \Rightarrow G : \mathbb{A} \longrightarrow \mathbb{B}$ and $\beta_1 : G \Rightarrow H : \mathbb{A} \longrightarrow \mathbb{B}$, denoted by $\frac{\alpha_1}{\beta_1}$, is well-given by:*

- *for every 0-cell A in $\mathbb{A}$ a 1h-cell in $\mathbb{B}$:*

$$
(\frac{\alpha_1}{\beta_1})(A) = \big(F(A) \stackrel{\alpha_1(A)}{\longrightarrow} G(A) \stackrel{\beta_1(A)}{\longrightarrow} H(A)\big),
$$

- *for every 1v-cell $u : A \longrightarrow A'$ in $\mathbb{A}$ a 2-cell in $\mathbb{B}$:*

$$(\frac{\alpha_1}{\beta_1})(u) = \begin{array}{ccccc} F(A) & \xrightarrow{\alpha_1(A)} & G(A) & \xrightarrow{\beta_1(A)} & H(A) \\ F(u)\downarrow & \boxed{(\alpha_1)_u} & \downarrow G(u) & \boxed{(\beta_1)_u} & \downarrow H(u) \\ F(A') & \xrightarrow{\alpha_1(A')} & G(A') & \xrightarrow{\beta_1(A')} & H(A') \end{array}$$

- *for every 1h-cell $f : A \longrightarrow B$ in $\mathbb{A}$ a 2-cell in $\mathbb{B}$:*

$$\delta_{\frac{\alpha_1}{\beta_1},f} = \begin{array}{ccccccc} F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{\alpha_1(B)} & G(B) & & \\ \downarrow = & & \boxed{\delta_{\alpha_1,f}} & & \downarrow = & & \\ F(A) & \xrightarrow{\alpha_1(A)} & G(A) & \xrightarrow{G(f)} & G(B) & \xrightarrow{\beta_1(B)} & H(B) \\ & & = \downarrow & & \boxed{\delta_{\beta_1,f}} & & \downarrow = \\ & & G(A) & \xrightarrow{\beta_1(A)} & H(A) & \xrightarrow{H(f)} & G(B). \end{array}$$

Lemma 3.8 *Vertical composition of two vertical pseudonatural transformations $\alpha_0 : F \Rightarrow G : \mathbb{A} \longrightarrow \mathbb{B}$ and $\beta_0 : G \Rightarrow H : \mathbb{A} \longrightarrow \mathbb{B}$, denoted by $\frac{\alpha_0}{\beta_0}$, is well-given by:*

- *for every 0-cell A in $\mathbb{A}$ a 1v-cell on the left below, and for every 1h-cell $f : A \longrightarrow B$ in $\mathbb{A}$ a 2-cell on the right below, both in $\mathbb{B}$:*

$$(\frac{\alpha_0}{\beta_0})(A) = \begin{array}{c} F(A) \\ \alpha_0(A)\downarrow \\ G(A) \\ \beta_0(A)\downarrow \\ H(A) \end{array} \qquad (\frac{\alpha_0}{\beta_0})(f) = \begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \alpha_0(A)\downarrow & \boxed{(\alpha_0)_f} & \downarrow \alpha_0(B) \\ G(A) & \xrightarrow{G(f)} & G(B) \\ \beta_0(A)\downarrow & \boxed{(\beta_0)_f} & \downarrow \beta_0(B) \\ H(A) & \xrightarrow{H(f)} & H(B) \end{array}$$

- *for every 1v-cell $u : A \longrightarrow A'$ in $\mathbb{A}$ a 2-cell in $\mathbb{B}$:*

$$\delta_{\frac{\alpha_0}{\beta_0},u} = \begin{array}{ccccc} F(A) & \xrightarrow{=} & F(A) & & \\ F(u)\downarrow & \boxed{\delta_{\alpha_0,u}} & \downarrow \alpha_0(A) & & \\ F(A') & & G(A) & \xrightarrow{=} & G(A) \\ \alpha_0(A')\downarrow & & G(u)\downarrow & \boxed{\delta_{\beta_0,u}} & \downarrow \beta_0(A) \\ G(A') & \xrightarrow{=} & G(A') & & H(A) \\ & & \beta_0(A')\downarrow & & \downarrow H(u) \\ & & H(A') & \xrightarrow{=} & H(A'). \end{array}$$

3.2. 2-cells of the tricategory and their compositions

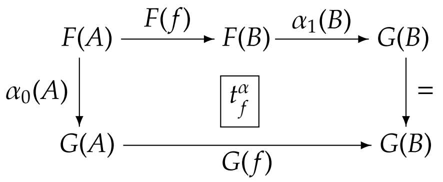
Now we may define what will be the 2-cells of our tricategory DblPs of strict double categories and double pseudofunctors.

Definition 3.9 *A* double pseudonatural transformation $\alpha : F \longrightarrow G$ *between double pseudofunctors is a quadruple* $(\alpha_0, \alpha_1, t^\alpha, r^\alpha)$, *where:*

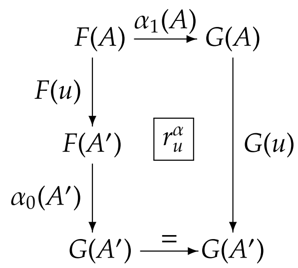
(T1) $\alpha_0 : F \Rightarrow G$ *is a vertical pseudonatural transformation, and*
$\alpha_1 : F \Rightarrow G$ *is a horizontal pseudonatural transformation,*

(T2) the 2-cells $\delta_{\alpha_1,f}$ *and* $\delta_{\alpha_0,u}$ *are invertible when* f *is a 1h-cell component of a horizontal pseudonatural transformation, and* u *is a 1v-cell component of a vertical pseudonatural transformation;*

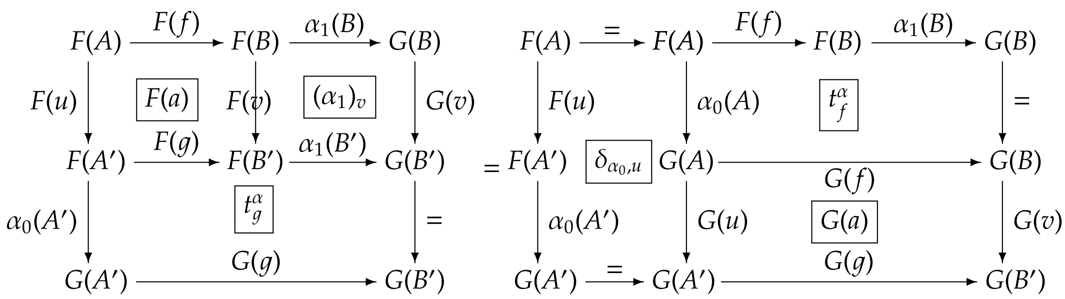
(T3) for every 1h-cell $f : A \longrightarrow B$ *and 1v-cell* $u : A \longrightarrow A'$ *in* $\mathbb{A}$ *there are 2-cells in* $\mathbb{B}$*:*



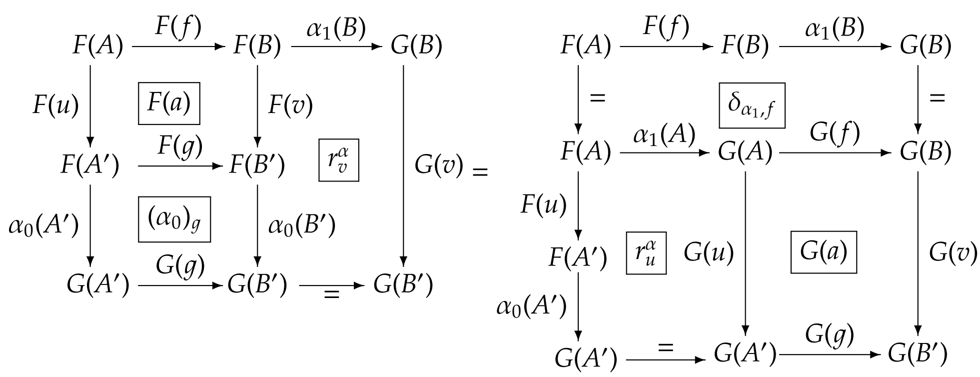
and



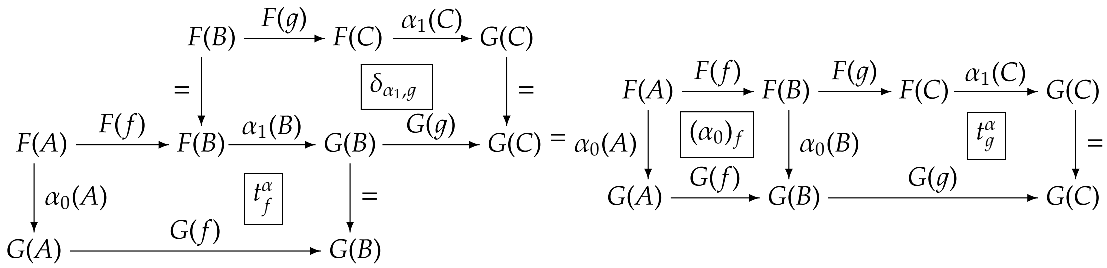
satisfying:
(T3-1)



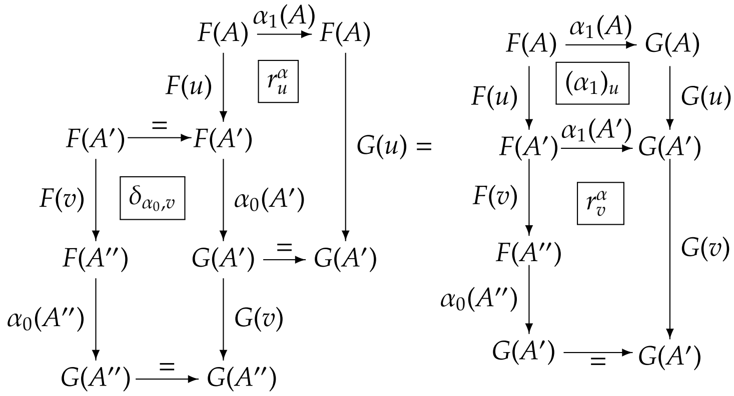
and



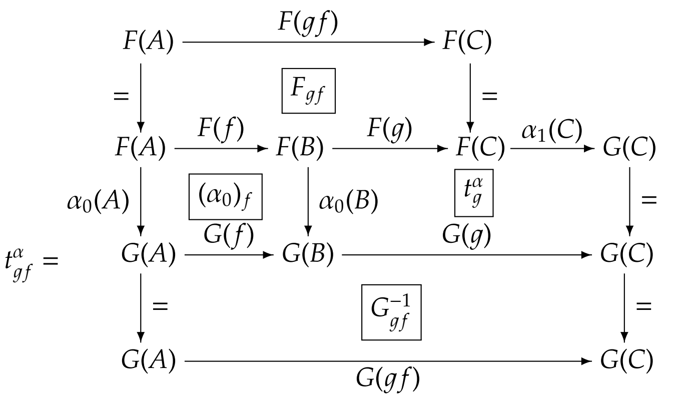
for every 2-cell a *in* $\mathbb{A}$, *(T3-2) for every composable 1h-cells f and g and every composable 1v-cells u and v it is:*



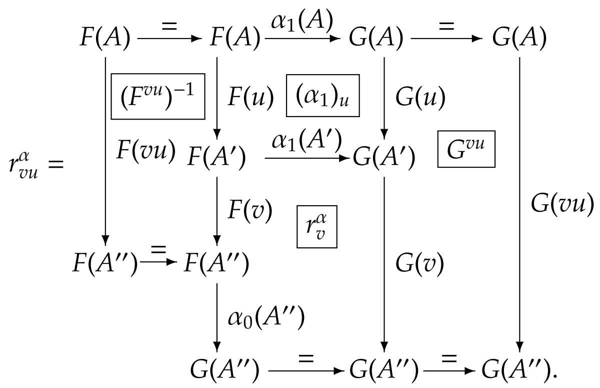
and



(T3-3) for every composable 1h-cells f and g and every composable 1v-cells u and v it is:



and



By the axiom (v) of a double pseudofunctor in [31, Definition 6.1] one has:

Lemma 3.10 *Given three composable 1h-cells f, g, h and three composable 1v-cells u, v, w for a double pseudonatural transformation α it is: $t^{\alpha}_{(hg)f} = t^{\alpha}_{h(gf)}$ and $r^{\alpha}_{(wv)u} = r^{\alpha}_{w(vu)}$.*

Remark 3.11 The horizontal and vertical compositions of t's and r's are defined in the next two Propositions below. Axiom (T3-2) in the above definition is introduced in order for t's to satisfy the interchange law (up to isomorphism).

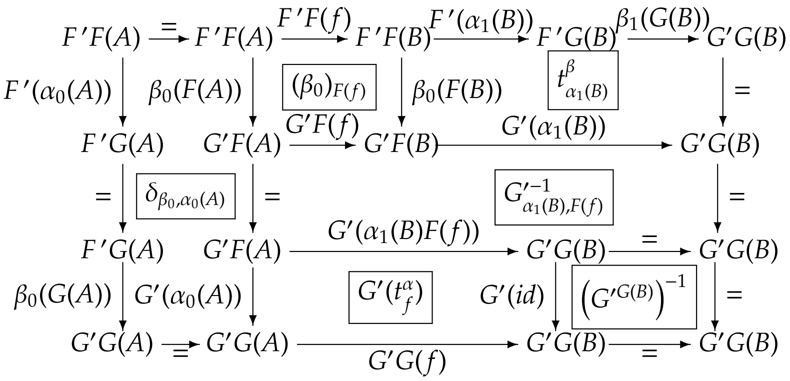
For every 1-cell F of DblPs, the identity 2-cell $\mathrm{Id}_F : F \Rightarrow F$ is given by the 2-cells: $((\mathrm{Id}_F)_0)_f = \mathrm{Id}_{F(f)} = t^{\mathrm{Id}_F}_f$, $((\mathrm{Id}_F)_1)_u = \mathrm{Id}_{F(u)} = r^{\mathrm{Id}_F}_u$, $\delta_{(\mathrm{Id}_F)_0,u} = \mathrm{Id}_{F(u)}$ and $\delta_{(\mathrm{Id}_F)_1,f} = \mathrm{Id}_{F(f)}$, with $(\mathrm{Id}_F)_0(A)$ and $(\mathrm{Id}_F)_1(A)$ being the identity 1v- and 1h-cells on $F(A)$, respectively, f an arbitrary 1h-cell and u an arbitrary 1v-cell.

For the horizontal and vertical compositions of double pseudonatural transformations we have:

Proposition 3.12 *A horizontal composition of two double pseudonatural transformations acting between double pseudo functors $(\alpha_0, \alpha_1, t^{\alpha}, r^{\alpha}) : F \Rightarrow G : \mathbb{A} \longrightarrow \mathbb{B}$ and $(\beta_0, \beta_1, t^{\beta}, r^{\beta}) : F' \Rightarrow G' : \mathbb{B} \longrightarrow \mathbb{C}$, denoted by $\beta \circ \alpha$, is well-given by:*

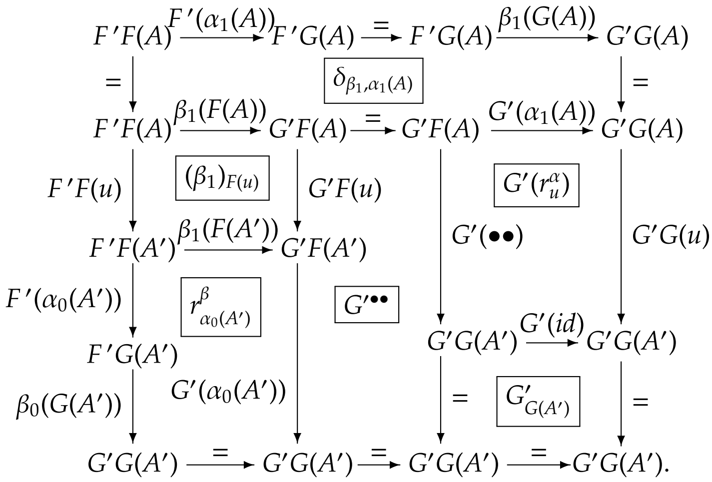
- *the horizontal pseudonatural transformation $\beta_1 \circ \alpha_1$ from Lemma 3.5,*
- *the vertical pseudonatural transformation $\beta_0 \circ \alpha_0$ from Lemma 3.6,*
- *for every 1h-cell $f : A \longrightarrow B$ and 1v-cell $u : A \longrightarrow A'$ in $\mathbb{A}$: 2-cells in $\mathbb{B}$:*

$t^{\beta\circ\alpha}_f := t^{\beta}_f \circ t^{\alpha}_f =$



and

$r^{\beta\circ\alpha}_u := r^{\beta}_u \circ r^{\alpha}_u =$



From the axioms of Definition 3.1 and Lemma 3.4 identities (9) and (10) in [13, Section 4.1] are deduced, of which the vertical version of (10) is used to prove that the horizontal composition of t's satisfies the axiom (T3-1), and the vertical version of (9) is used in order to show for this composition to be associative. Identities after [13, Remark 4.11] are used in order to prove that the horizontal composition of t's (and r's) satisfies the axiom (T3-2).

Proposition 3.13 *A vertical composition of two double pseudonatural transformations acting between double pseudo functors* $(\alpha_0, \alpha_1, t^\alpha, r^\alpha) : F \Rightarrow G : \mathbb{A} \longrightarrow \mathbb{B}$ *and* $(\beta_0, \beta_1, t^\beta, r^\beta) : G \Rightarrow H : \mathbb{A} \longrightarrow \mathbb{B}$*, denoted by* $\frac{\alpha}{\beta}$*, is well-given by:*

- *the horizontal pseudonatural transformation* $\frac{\alpha_1}{\beta_1}$ *from Lemma 3.7,*
- *the vertical pseudonatural transformation* $\frac{\alpha_0}{\beta_0}$ *from Lemma 3.8,*
- *for every 1h-cell* $f : A \longrightarrow B$ *and 1v-cell* $u : A \longrightarrow A'$ *in* $\mathbb{A}$*: 2-cells in* $\mathbb{B}$*:*

$$t_f^{\frac{\alpha}{\beta}} := \frac{t_f^\alpha}{t_f^\beta} = \begin{array}{ccccc}
F(A) & \xrightarrow{F(f)} & F(B) \xrightarrow{\alpha_1(B)} & G(B) & \\
\downarrow{\scriptstyle \alpha_0(A)} & & \boxed{t_f^\alpha} & \downarrow{\scriptstyle =} & \\
G(A) & & \xrightarrow{G(f)} & G(B) & \xrightarrow{\beta_1(B)} H(B) \\
\downarrow{\scriptstyle \beta_0(A)} & & & \boxed{t_f^\beta} & \downarrow{\scriptstyle =} \\
H(A) & & \xrightarrow{H(f)} & & H(B)
\end{array}$$

and

$$r_u^{\frac{\alpha}{\beta}} := \frac{r_u^\alpha}{r_u^\beta} = \begin{array}{ccccc}
F(A) & \xrightarrow{\alpha_1(A)} & G(A) & \xrightarrow{\beta_1(A)} & H(A) \\
{\scriptstyle F(u)}\downarrow & \boxed{r_u^\alpha} & \big| & & \big| \\
F(A') & & {\scriptstyle G(u)}\big| & \boxed{r_u^\beta} & {\scriptstyle H(u)}\big| \\
{\scriptstyle \alpha_0(A')}\downarrow & & \downarrow & & \big| \\
G(A') & \xrightarrow{=} & G(A') & & \big| \\
& & {\scriptstyle \beta_0(A')}\downarrow & & \downarrow \\
& & H(A') & \xrightarrow{=} & H(A').
\end{array}$$

This composition is clearly strictly associative. The unity constraint 3-cells for the vertical composition of 2-cells will be identities. The unity constraints for the horizontal composition we will discuss in Subsection 3.6.

3.3. A subclass of the class of 2-cells

In [4, Definition 6.3] *double natural transformations* between strict double functors were used, as a particular case of *generalized natural transformations* from [6, Definition 3]. Adapting the former to the case of *double pseudo functors* of Shulman, we get the following weakening of [4, Definition 6.3]:

Definition 3.14 *A* Θ-double pseudonatural transformation *between two double pseudofunctors* $F \Rightarrow G : \mathbb{A} \longrightarrow \mathbb{B}$ *is a tripple* $(\alpha_0, \alpha_1, \Theta^\alpha)$*, which we will denote shortly by* Θ^α*, where:*

- α_0 *is a vertical and* α_1 *a horizontal pseudonatural transformation (from Definition 3.1 and the analogous one),*
- *the axiom (T2) from Definition 3.9 holds, and*

- *for every 0-cell A in $\mathbb{A}$ there are 2-cells in $\mathbb{B}$:*

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_1(A)} & G(A) \\ {\scriptstyle \alpha_0(A)}\downarrow & \boxed{\Theta^\alpha_A} & \downarrow{\scriptstyle =} \\ G(A) & \xrightarrow{=} & G(A) \end{array}$$

so that for every 1h-cell $f: A \longrightarrow B$ and every 1v-cell $u: A \longrightarrow A'$ in $\mathbb{A}$ the following identities hold:
(Θ0)

$$\begin{array}{ccccc} F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{\alpha_1(B)} & G(B) \\ {\scriptstyle \alpha_0(A)}\downarrow & \boxed{(\alpha_0)_f} & {\scriptstyle \alpha_0(B)}\downarrow & \boxed{\Theta^\alpha_B} & \downarrow{\scriptstyle =} \\ G(A) & \xrightarrow{G(f)} & G(B) & \xrightarrow{=} & G(B) \end{array} \quad = \quad \begin{array}{ccccc} F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{\alpha_1(B)} & G(B) \\ {\scriptstyle =}\downarrow & & \boxed{\delta_{\alpha_1,f}} & & \downarrow{\scriptstyle =} \\ F(A) & \xrightarrow{\alpha_1(A)} & G(A) & \xrightarrow{G(f)} & G(B) \\ {\scriptstyle \alpha_0(A)}\downarrow & \boxed{\Theta^\alpha_A} & \downarrow{\scriptstyle =} & & \\ F(A') & \xrightarrow{=} & G(A') & & \end{array}$$

and
(Θ1)

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_1(A)} & G(A) \\ {\scriptstyle F(u)}\downarrow & \boxed{(\alpha_1)_u} & \downarrow{\scriptstyle G(u)} \\ F(A') & \xrightarrow{\alpha_1(A')} & G(A') \\ {\scriptstyle \alpha_0(A')}\downarrow & \boxed{\Theta^\alpha_{A'}} & \downarrow{\scriptstyle =} \\ G(A') & \xrightarrow{=} & G(A') \end{array} \quad = \quad \begin{array}{ccccc} F(A) & \xrightarrow{=} & F(A) & \xrightarrow{\alpha_1(A)} & G(A) \\ {\scriptstyle F(u)}\downarrow & & {\scriptstyle \alpha_0(A)}\downarrow & \boxed{\Theta^\alpha_A} & \downarrow{\scriptstyle =} \\ F(A') & & G(A) & \xrightarrow{=} & G(A) \\ {\scriptstyle \alpha_0(A')}\downarrow & \boxed{\delta_{\alpha_0,u}} & \downarrow{\scriptstyle G(u)} & & \\ G(A') & \xrightarrow{=} & G(A'). & & \end{array}$$

Let us denote a Θ-double pseudonatural transformation Θ^α by $\mathbb{A} \underset{G}{\overset{F}{\Downarrow\Theta^\alpha}} \mathbb{B}$.

Horizontal composition of Θ-double pseudonatural transformations $\mathbb{A} \underset{G}{\overset{F}{\Downarrow\Theta^\alpha}} \mathbb{B} \underset{G'}{\overset{F'}{\Downarrow\Theta^\beta}} \mathbb{C}$ is given by

$$\Theta^{\beta\circ\alpha}_A := \Theta^\beta_A \circ \Theta^\alpha_A = \begin{array}{ccccccc} F'F(A) & \xrightarrow{F'(\alpha_1(A))} & F'G(A) & \xrightarrow{=} & F'G(A) & & \\ {\scriptstyle F'(\alpha_0(A))}\downarrow & \boxed{F'(\Theta^\alpha_A)} & {\scriptstyle F'(id)}\downarrow & \boxed{F'^{\bullet -1}} & \downarrow{\scriptstyle =} & & \\ F'G(A) & \xrightarrow{F'(id)} & F'G(A) & \xrightarrow{=} & F'G(A) & & \\ {\scriptstyle =}\downarrow & \boxed{F'_{G(A)}} & \downarrow{\scriptstyle =} & \boxed{1} & \downarrow{\scriptstyle =} & & \\ F'G(A) & \xrightarrow{=} & F'G(A) & \xrightarrow{=} & F'G(A) & \xrightarrow{\beta_1(G(A))} & G'G(A) \\ & & & & {\scriptstyle \beta_0(G(A))}\downarrow & \boxed{\Theta^\beta_{G(A)}} & \downarrow{\scriptstyle =} \\ & & & & G'G(A) & \xrightarrow{=} & G'G(A) \end{array}$$

and vertical composition of Θ-double pseudonatural transformations $\mathbb{A} \begin{array}{c} \xrightarrow{F} \\ \Downarrow\Theta^{\alpha} \\ \xrightarrow{G} \\ \Downarrow\Theta^{\beta} \\ \xrightarrow{H} \end{array} \mathbb{B}$ is given by

$$\frac{\Theta^{\alpha}_A}{\Theta^{\beta}_A} = \begin{array}{ccccc} F(A) & \xrightarrow{\alpha_1(A)} & G(A) & & \\ \alpha_0(A)\downarrow & \boxed{\Theta^{\alpha}_A} & \downarrow = & & \\ G(A) & \xrightarrow{=} & G(A) & \xrightarrow{\beta_1(A)} & H(A) \\ & & \beta_0(A)\downarrow & \boxed{\Theta^{\beta}_A} & \downarrow = \\ & & H(A) & \xrightarrow{=} & H(A). \end{array}$$

The following result is directly proved:

Proposition 3.15 *A Θ-double pseudonatural transformation Θ^{α} gives rise to a double pseudonatural transformation $(\alpha_0, \alpha_1, t^{\alpha}, r^{\alpha})$, where*

$$t^{\alpha}_f = \begin{array}{ccccc} F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{\alpha_1(B)} & G(B) \\ \alpha_0(A)\downarrow & \boxed{(\alpha_0)_f} & \alpha_0(B)\downarrow & \boxed{\Theta^{\alpha}_B} & \downarrow = \\ G(A) & \xrightarrow{G(f)} & G(B) & \xrightarrow{=} & G(B) \end{array} \quad \text{and} \quad r^{\alpha}_u = \begin{array}{ccc} F(A) & \xrightarrow{\alpha_1(A)} & G(A) \\ F(u)\downarrow & \boxed{(\alpha_1)_u} & \downarrow G(u) \\ F(A') & \xrightarrow{\alpha_1(A')} & G(A') \\ \alpha_0(A')\downarrow & \boxed{\Theta^{\alpha}_{A'}} & \downarrow = \\ G(A') & \xrightarrow{=} & G(A') \end{array}$$

for every 1h-cell $f : A \longrightarrow B$ and 1v-cell $u : A \longrightarrow A'$. Moreover, the class of all Θ-double pseudonatural transformations is a subclass of the class of double pseudonatural transformations.

Proof. By the axiom (Θ1), axiom 1. for the horizontal pseudonatural transformation α_1 implies axiom (T3-1) for t^{α}_f. □

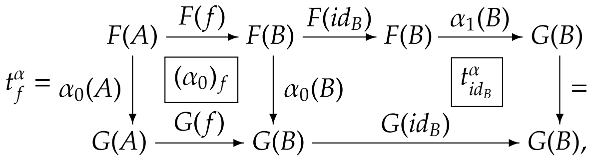
Thus, from the point of view of Θ-double pseudonatural transformations, the axioms (T3-2) and (T3-3) of double pseudonatural transformations become redundant.

Observe also that given a double pseudonatural transformation $\alpha : F \Rightarrow G$ acting between *strict* double functors, the 2-cells $t^{\alpha}_{id_A}$ obey the conditions (Θ0) and (Θ1) for every 0-cell A.

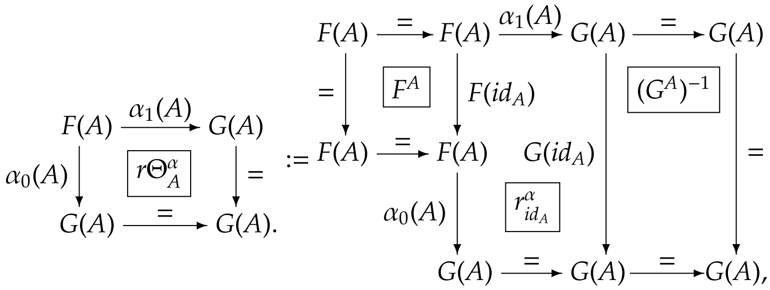
The other way around, observe that setting

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_1(A)} & G(A) \\ \alpha_0(A)\downarrow & \boxed{t\Theta^{\alpha}_A} & \downarrow = \\ G(A) & \xrightarrow{=} & G(A). \end{array} := \begin{array}{ccccc} F(A) & \xrightarrow{=} & F(A) & & \\ =\downarrow & \boxed{(F_A)^{-1}} & \downarrow = & & \\ F(A) & \xrightarrow{F(id_A)} & F(A) & \xrightarrow{\alpha_1(A)} & G(A) \\ \downarrow \alpha_0(A) & & & \boxed{t^{\alpha}_{id_A}} & \downarrow = \\ F(A) & & \xrightarrow{G(id_A)} & & G(A) \\ =\downarrow & & \boxed{G_A} & & \downarrow = \\ G(A) & & \xrightarrow{=} & & G(A), \end{array}$$

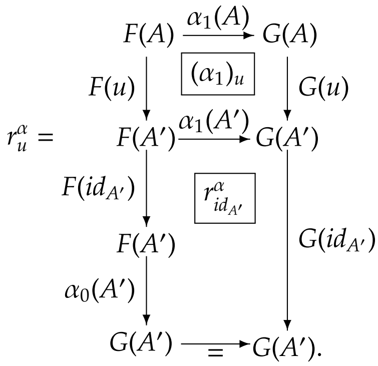
by (T3-3) we get



and analogously, setting



one gets

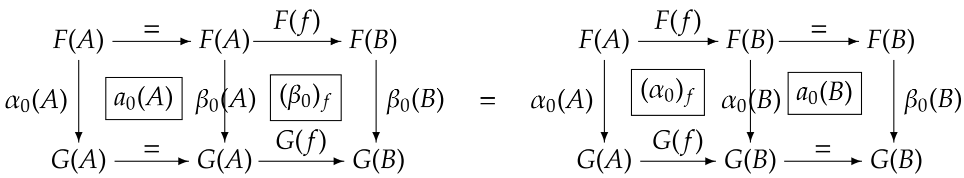


By successive applications of (T3-2) and axiom 2. for α_0 one gets that $t\Theta^{\alpha}_{\bullet}$ satisfies the axiom (Θ0), and similarly $r\Theta^{\alpha}_{\bullet}$ satisfies the axiom (Θ1) of Definition 3.14. Since the 2-cells t^{α}_{f} and r^{α}_{f} are not related, we can not claim that all double pseudonatural transformations are Θ-double pseudotransformations.

3.4. 3-cells of the tricategory

We first define modifications for horizontal and vertical pseudonatural transformations. Since we will then define modifications for double pseudonatural transformations, for mnemonic reasons we will denote vertical pseudonatural transformations with index 0 and horizontal ones with index 1.

Definition 3.16 *A modification between two vertical pseudonatural transformations α_0 and β_0 which act between double psuedofunctors $F \Rightarrow G$ is an application $a : \alpha_0 \Rrightarrow \beta_0$ such that for each 0-cell A in $\mathbb{A}$ there is a horizontally globular 2-cell $a_0(A) : \alpha_0(A) \Rightarrow \beta_0(A)$ which for each 1h-cell $f : A \longrightarrow B$ satisfies:*



and

$$
\begin{array}{ccccc}
 & & F(A) & \xrightarrow{=} & F(A) \\
 & & {\scriptstyle F(u)}\downarrow & \boxed{\delta_{\beta_0,u}} & \downarrow{\scriptstyle \beta_0(A)} \\
F(A') & \xrightarrow{=} & F(A') & & G(A) \\
{\scriptstyle \alpha_0(A')}\downarrow & \boxed{a_0(A')} & \downarrow{\scriptstyle \beta_0(A')} & & \downarrow{\scriptstyle G(u)} \\
G(A') & \xrightarrow{=} & G(A') & \xrightarrow{=} & G(A')
\end{array}
\quad = \quad
\begin{array}{ccccc}
F(A) & \xrightarrow{=} & F(A) & \xrightarrow{=} & F(A) \\
{\scriptstyle F(u)}\downarrow & & {\scriptstyle \alpha_0(A)}\downarrow & \boxed{a_0(A)} & \downarrow{\scriptstyle \beta_0(A)} \\
F(A') & & G(A) & \xrightarrow{=} & G(A) \\
{\scriptstyle \alpha_0(A')}\downarrow & \boxed{\delta_{\alpha_0,u}} & \downarrow{\scriptstyle G(u)} & & \\
G(A') & \xrightarrow{=} & G(A'). & &
\end{array}
$$

A modification between two horizontal pseudonatural transformations α_1 and β_1 which act between double pseudofunctors $F \Rightarrow G$ is an application $a : \alpha_1 \Rrightarrow \beta_1$ such that for each 0-cell A in $\mathbb{A}$ there is a vertically globular 2-cell $a_1(A) : \alpha_1(A) \Rightarrow \beta_1(A)$ which for each 1v-cell $u : A \longrightarrow A'$ satisfies two conditions analogous to those of the above definition.

Now, 3-cells for our tricategory DblPs will be modifications which we define here:

Definition 3.17 *A modification between two double pseudonatural transformations $\alpha = (\alpha_0, \alpha_1, t^\alpha, r^\alpha)$ and $\beta = (\beta_0, \beta_1, t^\beta, r^\beta)$ which act between double pseudofunctors $F \Rightarrow G$ is an application $a : \alpha \Rrightarrow \beta$ consisting of a modification a_0 for vertical pseudonatural transformations and a modification a_1 for horizontal pseudonatural transformations, such that for each 0-cell A in $\mathbb{A}$ it holds:*

$$
\begin{array}{ccccccc}
 & & & & F(B) & \xrightarrow{\alpha_1(B)} & G(B) \\
 & & & & =\downarrow & \boxed{a_1(B)} & \downarrow = \\
F(A) & \xrightarrow{=} & F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{\beta_1(B)} & G(B) \\
{\scriptstyle \alpha_0(A)}\downarrow & \boxed{a_0(A)} & \downarrow{\scriptstyle \beta_0(A)} & & \boxed{t^\beta_f} & & \downarrow = \\
G(A) & \xrightarrow{=} & G(A) & & \xrightarrow{G(f)} & & G(B)
\end{array}
\quad = \quad
\begin{array}{ccccc}
F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{\alpha_1(B)} & G(B) \\
\downarrow{\scriptstyle \alpha_0(A)} & & \boxed{t^\alpha_f} & & \downarrow = \\
F(A) & & \xrightarrow[G(f)]{} & & G(A)
\end{array}
\tag{3.8}
$$

and

$$
\begin{array}{ccccc}
 & & F(A) & \xrightarrow{\alpha_1(A)} & G(A) \\
 & & =\downarrow & \boxed{a_1(A)} & \downarrow = \\
 & & F(A) & \xrightarrow{\beta_1(A)} & G(A) \\
 & & {\scriptstyle F(u)}\downarrow & \boxed{r^\beta_u} & \\
F(A') & \xrightarrow{=} & F(A') & & \downarrow{\scriptstyle G(u)} \\
{\scriptstyle \alpha_0(A')}\downarrow & \boxed{a_0(A')} & \downarrow{\scriptstyle \beta_0(A')} & & \\
G(A') & \xrightarrow{=} & G(A') & \xrightarrow{=} & G(A')
\end{array}
\quad = \quad
\begin{array}{ccc}
F(A) & \xrightarrow{\alpha_1(A)} & G(A) \\
{\scriptstyle F(u)}\downarrow & & \\
F(A') & \boxed{r^\alpha_u} & \downarrow{\scriptstyle G(u)} \\
{\scriptstyle \alpha_0(A')}\downarrow & & \\
G(A') & \xrightarrow{=} & G(A').
\end{array}
$$

Horizontal composition of the modifications $a : \alpha \Rrightarrow \beta : F \Rightarrow G$ and $b : \alpha' \Rrightarrow \beta' : F' \Rightarrow G'$, acting between horizontally composable double pseudonatural transformations $\alpha' \circ \alpha \Rrightarrow \beta' \circ \beta : F' \circ F \Rightarrow G' \circ G$ is given for every 0-cell A in $\mathbb{A}$ by pairs consisting of

$$(b\circ a)_0(A)=\begin{array}{ccc} F'F(A) & \xrightarrow{=} & F'F(A)\\ {\scriptstyle =}\downarrow & \boxed{(F'_\bullet)^{-1}} & \downarrow{\scriptstyle =}\\ F'F(A) & \xrightarrow{F'(id)} & F'F(A)\\ {\scriptstyle F'(\alpha_0(A))}\downarrow & \boxed{F'(a_0(A))} & \downarrow{\scriptstyle F'(\beta_0(A))}\\ F'G(A) & \xrightarrow[F'(id)]{} & F'G(A)\\ {\scriptstyle =}\downarrow & \boxed{F'_\bullet} & \downarrow{\scriptstyle =}\\ F'G(A) & \xrightarrow[=]{} & F'G(A)\\ {\scriptstyle \alpha'_0(G(A))}\downarrow & \boxed{b_0(G(A))} & \downarrow{\scriptstyle \beta'_0(G(A))}\\ G'G(A) & \xrightarrow[=]{} & G'G(A) \end{array}$$

and

$$(b\circ a)_1(A)=\begin{array}{ccccccccccc} F'F(A) & \xrightarrow{=} & F'F(A) & \xrightarrow{F'(\alpha_1(A))} & F'G(A) & \xrightarrow{=} & F'G(A) & \xrightarrow{\alpha'_1(G(A))} & F'G(A)\\ {\scriptstyle =}\downarrow & \boxed{F'^{\bullet}} & \downarrow{\scriptstyle F'(id)} & \boxed{F'(a_1(A))} & \downarrow{\scriptstyle F'(id)} & \boxed{(F'^{\bullet})^{-1}} & \downarrow{\scriptstyle =} & \boxed{b_1(G(A))} & \downarrow{\scriptstyle =}\\ F'F(A) & \xrightarrow{=} & F'F(A) & \xrightarrow{F'(\beta_1(A))} & F'G(A) & \xrightarrow{=} & F'G(A) & \xrightarrow{\beta'_1(G(A))} & F'G(A). \end{array}$$

Vertical composition of the modifications $a:\alpha\Rrightarrow\beta:F\Rightarrow G$ and $b:\alpha'\Rrightarrow\beta':G\Rightarrow H$, acting between vertically composable double pseudonatural transformations $F\overset{\alpha}{\Rightarrow}G\overset{\alpha'}{\Rightarrow}H$ and $F\overset{\beta}{\Rightarrow}G\overset{\beta'}{\Rightarrow}H$, is given for every 0-cell A in $\mathbb{A}$ by pairs consisting of

$$(\frac{a}{b})_0(A)=\begin{array}{ccc} F(A) & \xrightarrow{=} & F(A)\\ {\scriptstyle \alpha_0(A)}\downarrow & \boxed{a_0(A)} & \downarrow{\scriptstyle \beta_0(A)}\\ G(A) & \xrightarrow{=} & G(A)\\ {\scriptstyle \alpha'_0(A)}\downarrow & \boxed{a'_0(A)} & \downarrow{\scriptstyle \beta'_0(A)}\\ H(A) & \xrightarrow[\beta_1(A')]{} & H(A) \end{array}\quad\text{and}\quad (\frac{a}{b})_1(A)=\begin{array}{ccccc} F(A) & \xrightarrow{\alpha_1(A)} & G(A) & \xrightarrow{\alpha'_1(A)} & H(A)\\ {\scriptstyle =}\downarrow & \boxed{a_1(A)} & \downarrow{\scriptstyle =} & \boxed{a'_1(A)} & \downarrow{\scriptstyle =}\\ F(A) & \xrightarrow{\beta_1(A)} & G(A) & \xrightarrow{\beta'_1(A)} & H(A). \end{array}$$

Transversal composition of the modifications $\alpha\overset{a}{\Rrightarrow}\beta\overset{b}{\Rrightarrow}\gamma:F\Rightarrow G$ is given for every 0-cell A in $\mathbb{A}$ by pairs consisting of

$$(b\cdot a)_0(A)=\begin{array}{ccccc} F(A) & \xrightarrow{=} & G(A) & \xrightarrow{=} & H(A)\\ \downarrow{\scriptstyle \alpha_0(A)} & \boxed{a_0(A)} & \downarrow{\scriptstyle \beta_0(A)} & \boxed{b_0(A)} & \downarrow{\scriptstyle \gamma_0(A)}\\ F(A) & \xrightarrow{=} & G(A) & \xrightarrow{=} & H(A) \end{array}\quad\text{and}\quad (b\cdot a)_1(A)=\begin{array}{ccc} F(A) & \xrightarrow{\alpha_1(A)} & G(A)\\ {\scriptstyle =}\downarrow & \boxed{a_1(A)} & \downarrow{\scriptstyle =}\\ F(A) & \xrightarrow{\beta_1(A)} & G(A)\\ {\scriptstyle =}\downarrow & \boxed{b_1(A)} & \downarrow{\scriptstyle =}\\ F(A) & \xrightarrow[\gamma_1(A)]{} & G(A). \end{array}$$

From the definitions it is clear that vertical and transversal composition of the 3-cells is strictly associative. That the associativity in the horizontal direction is also strict we proved in [13, Section 4.7].

3.5. A subclass of the 3-cells

For Θ-double pseudonatural transformations we define modifications as follows:

Definition 3.18 *A modification between two Θ-double pseudonatural transformations* $\Theta^\alpha \equiv (\alpha_0, \alpha_1, \Theta^\alpha)$ *and* $\Theta^\beta \equiv (\beta_0, \beta_1, \Theta^\beta)$ *which act between double psuedofunctors* $F \Rightarrow G$ *is an application* $a : \alpha \Rrightarrow \beta$ *consisting of a modification* a_0 *for vertical pseudonatural transformations and a modification* a_1 *for horizontal pseudonatural transformations, such that for each 0-cell* A *in* $\mathbb{A}$ *it holds:*

$$
\begin{array}{ccccc}
 & & F(A) & \xrightarrow{\alpha_1(A)} & G(A) \\
 & & = \big\downarrow & \boxed{a_1(A)} & \big\downarrow = \\
F(A) & \xrightarrow{=} & F(A) & \xrightarrow{\beta_1(A)} & G(A) \\
\alpha_0(A) \big\downarrow & \boxed{a_0(A)} & \big\downarrow \beta_0(A) & \boxed{\Theta^\beta_A} & \big\downarrow = \\
G(A) & \xrightarrow{=} & G(A) & \xrightarrow{=} & G(A)
\end{array}
\quad = \quad
\begin{array}{ccc}
F(A) & \xrightarrow{\alpha_1(A)} & G(A) \\
\alpha_0(A) \big\downarrow & \boxed{\Theta^\alpha_A} & \big\downarrow = \\
G(A) & \xrightarrow{=} & G(A).
\end{array}
$$

It is directly proved that modifications between Θ-double pseudonatural transformations are particular cases of modifications between double pseudonatural transformations. This gives a sub-tricategory DblPs^Θ of the tricategory DblPs.

3.6. The obtained tricategory

In Sections 4.6 and 4.9 of [13] we proved that horizontal associativity of the 2-cells of DblPs and the interchange law for 2-cells, respectively, hold up to isomorphisms, which we gave explicitly. In Section 4.7 of *loc.cit.* we proved the strict associativity of the 3-cells, and in Section 4.8 we showed that left unity constraints on 2-cells is identity, but for the right one we gave an isomorphism. In Section 4.10 we showed that the distinguished modifications from the axiom (TD8) of [17] fulfill the required identities, which concludes the construction of the tricategory DblPs.

4. The 2-category *PsDbl* embeds into our tricategory DblPs

As we want to propose an alternative notion to intercategories as *categories internal to the tricategory* DblPs, which we do in the next two sections, so that monoids in the Cartesian monoidal category $(Dbl, \circledast)$ fit in it, in this section we compare the 2-category *LxDbl* and our tricategory DblPs. As we explained, we can not embed *LxDbl* (whose 1-cells are lax double functors) into DblPs, instead we will embed the 2-category *PsDbl* of pseudo double categories, pseudo double functors and vertical transformations, used in [30]. Apart from 1-cells, it differs from *LxDbl* in that the horizontal direction is weak and the vertical one is strict, while in the approach of Grandis and Paré and in *LxDbl* it is the other way around. Moreover, 2-cells in *PsDbl* are vertical rather than horizontal transformations, as in *LxDbl*. Thus the 2-category *PsDbl* is the closest one to *LxDbl* in the presented context which we could embed into our tricategory DblPs.

The 0-cells of *PsDbl* are pseudo double categories and not *strict* double categories as in DblPs. Though, by Strictification Theorem of [22, Section 7.5] every pseudo double category is equivalent by a pseudodouble functor to a strict double category. Let $PsDbl_3^*$ be the 3-category defined by adding only the identity 3-cells to the 2-category equivalent to *PsDbl* having strict double categories for 0-cells. Thus $PsDbl_3^*$ consists of strict double categories, pseudo double functors, vertical transformations and identity modifications among the latter. Pseudo double functors are in particular double pseudo functors, so the only thing it remains to check is how to make a vertical transformation a double pseudonatural transformation, that is, embed 2-cells of *PsDbl* into those of DblPs.

Before doing this, we prove some more general results.

4.1. Bijectivity between strong vertical and strong horizontal transformations

Recall that a *companion* for a 1v-cell $u : A \longrightarrow A'$ is a 1h-cell $u_* : A \longrightarrow A'$ together with certain 2-cells ε and η satisfying $[\eta|\varepsilon] = \mathrm{Id}_{u_*}$ and $\frac{\eta}{\varepsilon} = \mathrm{Id}_u$, [20, Section 1.2], [30, Section 3]. (Here $[\eta|\varepsilon]$ denotes the horizontal composition of 2-cells, where η acts first, and the fraction denotes their vertical composition.) We will say that u_* is a *1h-companion* of u. Companions are unique up to a unique globular isomorphism [30, Lemma 3.8] and a *connection* on a double category is a functorial choice of a companion for each 1v-cell, [5]. We will need a functorial choice of companions only for 1v-cell components of vertical pseudonatural transformations, accordingly we will speak about a *connection on those 1v-cells*.

Proposition 4.1 *Let $\alpha_0 : F \Rightarrow G$ be a strong vertical transformation between pseudo double functors acting between strict double categories $\mathbb{A} \longrightarrow \mathbb{B}$ ([22, Section 7.4]). The following data define a horizontal pseudonatural transformation $\alpha_1 : F \Rightarrow G$:*

- *a fixed choice of a 1h-companion of $\alpha_0(A)$, for every 0-cell A of $\mathbb{A}$ (with corresponding 2-cells ε^α_A and η^α_A), we denote it by $\alpha_1(A)$;*
- *the 2-cell*

$$
(\alpha_1)_u = \begin{array}{ccccccc}
 & & F(A) & \xrightarrow{=} & F(A) & \xrightarrow{\alpha_1(A)} & G(A) \\
 & & {\scriptstyle F(u)}\downarrow & & \downarrow{\scriptstyle \alpha_0(A)}\ \boxed{\varepsilon^\alpha_A} & & \downarrow{\scriptstyle =} \\
F(A') & \xrightarrow{=} & F(A') & \boxed{\delta_{\alpha_0,u}} & G(A) & \xrightarrow{=} & G(A) \\
{\scriptstyle =}\downarrow & \boxed{\eta^\alpha_{A'}} & \downarrow{\scriptstyle \alpha_0(A')} & & \downarrow{\scriptstyle G(u)} & & \\
F(A') & \xrightarrow{\alpha_1(A')} & G(A') & \xrightarrow{=} & G(A') & &
\end{array}
$$

for every 1v-cell $u : A \longrightarrow A'$;

- *the 2-cell*

$$
\delta_{\alpha_1,f} = \begin{array}{ccccccc}
F(A) & \xrightarrow{=} & F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{\alpha_1(B)} & G(B) \\
{\scriptstyle =}\downarrow & \boxed{\eta^\alpha_A}\ {\scriptstyle \alpha_0(A)} & \downarrow & \boxed{(\alpha_0)_f} & \downarrow{\scriptstyle \alpha_0(B)}\ \boxed{\varepsilon^\alpha_B} & & \downarrow{\scriptstyle =} \\
F(A) & \xrightarrow{\alpha_1(A)} & G(A) & \xrightarrow[G(f)]{} & G(B) & \xrightarrow{=} & G(B)
\end{array}
$$

for every 1h-cell $f : A \longrightarrow B$.

Proof. To prove axiom 1), the axiom 1) of α_0 is used; the first part of the axiom 2) works directly, and in the second one use second part of the axiom 3) for α_0; the first part of the axiom 3) works directly, and in the second one use second part of the axiom 2) for α_0; in checking of all the three axioms also the rules ε-η are used. □

Observe that there is a way in the other direction:

Proposition 4.2 *Let $\alpha_1 : F \Rightarrow G$ be a strong horizontal transformation between pseudo double functors acting between strict double categories $\mathbb{A} \longrightarrow \mathbb{B}$. Suppose that for every 0h-cell A the 1h-cell $\alpha_1(A)$ is a 1h-companion of some 1v-cell (with corresponding 2-cells ε^α_A and η^α_A). Fix a choice of such 1v-cells for each A and denote them by $\alpha_0(A)$. The following data define a vertical pseudonatural transformation $\alpha_0 : F \Rightarrow G$:*

- *the 1v-cell $\alpha_0(A)$, for every 0-cell A of $\mathbb{A}$;*

- *the 2-cell*

$$
(\alpha_0)_f = \quad
\begin{array}{ccccccc}
 & & F(B) & \xrightarrow{=} & F(B) & & \\
 & & {\scriptstyle =}\downarrow & \boxed{\eta^\alpha_B} & \downarrow{\scriptstyle \alpha_0(B)} & & \\
F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{\alpha_1(B)} & G(B) & & \\
\downarrow{\scriptstyle =} & & \boxed{\delta_{\alpha_1,f}} & & \downarrow{\scriptstyle =} & & \\
F(A) & \xrightarrow{\alpha_1(A)} & G(A) & \xrightarrow{G(f)} & G(B) & & \\
{\scriptstyle \alpha_0(A)}\downarrow & \boxed{\varepsilon^\alpha_A} & \downarrow{\scriptstyle =} & & & & \\
G(A) & \xrightarrow{=} & G(A) & & & &
\end{array}
$$

for every 1h-cell $f : A \longrightarrow B$;

- *the 2-cell*

$$
\delta_{\alpha_0,u} = \quad
\begin{array}{ccc}
F(A) & \xrightarrow{=} & F(A) \\
{\scriptstyle =}\downarrow & \boxed{\eta^\alpha_A} & \downarrow{\scriptstyle \alpha_0(A)} \\
F(A) & \xrightarrow{\alpha_1(A)} & G(A) \\
{\scriptstyle F(u)}\downarrow & \boxed{(\alpha_1)_u} & \downarrow{\scriptstyle G(u)} \\
F(A') & \xrightarrow{\alpha_1(A')} & G(A') \\
{\scriptstyle \alpha_0(A')}\downarrow & \boxed{\varepsilon^\alpha_{A'}} & \downarrow{\scriptstyle =} \\
G(A') & \xrightarrow[=]{} & G(A')
\end{array}
$$

for every 1v-cell $u : A \longrightarrow A'$.

By ε-η-relations, there is a 1-1 correspondence between those strong vertical transformations whose 1v-cell components have 1h-companions and those strong horizontal transformations whose 1h-cell components are 1h-companions of some 1v-cells.

Corollary 4.3 *Suppose that there is a connection on 1v-components of strong vertical transformations. Then there is a bijection between strong vertical transformations and those strong horizontal transformations whose 1h-cell components are 1h-companions of some 1v-cells.*

As a direct corollary of Proposition 4.1 we get:

Corollary 4.4 *Suppose that the 1v-components of a strong vertical transformation $\alpha_0 : F \Rightarrow G$ have 1h-companions $\alpha_1(A)$, for every 0-cell A (with corresponding 2-cells ε^α_A and η^α_A), and define the 2-cells $(\alpha_1)_u$ and $\delta_{\alpha_1,f}$ as in Proposition 4.1. The following identities then follow:*

$$
\begin{array}{ccccc}
F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{\alpha_1(B)} & G(B) \\
{\scriptstyle \alpha_0(A)}\downarrow & \boxed{(\alpha_0)_f} & \downarrow{\scriptstyle \alpha_0(B)} & \boxed{\varepsilon^\alpha_B} & \downarrow{\scriptstyle =} \\
G(A) & \xrightarrow{G(f)} & G(B) & \xrightarrow{=} & G(B)
\end{array}
\quad = \quad
\begin{array}{ccccc}
F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{\alpha_1(B)} & G(B) \\
\downarrow{\scriptstyle =} & & \boxed{\delta_{\alpha_1,f}} & & \downarrow{\scriptstyle =} \\
F(A) & \xrightarrow{\alpha_1(A)} & G(A) & \xrightarrow{G(f)} & G(B) \\
{\scriptstyle \alpha_0(A)}\downarrow & \boxed{\varepsilon^\alpha_A} & \downarrow{\scriptstyle =} & & \\
F(A') & \xrightarrow{=} & G(A'). & &
\end{array}
$$

and

$$
\begin{array}{ccccc}
F(A) & \xrightarrow{=} & F(A) & \xrightarrow{F(f)} & G(B) \\
{\scriptstyle =}\Big\downarrow & \boxed{\eta^{\alpha}_{A}} & \Big\downarrow{\scriptstyle \alpha_0(A)} & \boxed{(\alpha_0)_f} & \Big\downarrow{\scriptstyle \alpha_0(B)} \\
F(A) & \xrightarrow{\alpha_1(A)} & G(A) & \xrightarrow{G(f)} & G(B)
\end{array}
\quad = \quad
\begin{array}{ccccc}
 & & F(B) & \xrightarrow{=} & F(B) \\
 & & {\scriptstyle =}\Big\downarrow & \boxed{\eta^{\alpha}_{B}} & \Big\downarrow{\scriptstyle \alpha_0(B)} \\
F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{\alpha_1(B)} & G(B) \\
\Big\downarrow{\scriptstyle =} & & \boxed{\delta_{\alpha_1,f}} & & \Big\downarrow{\scriptstyle =} \\
F(A) & \xrightarrow{\alpha_1(A)} & G(A) & \xrightarrow{G(f)} & G(B)
\end{array}
$$

for every 1h-cell $f : A \longrightarrow B$;

$$
\begin{array}{ccc}
F(A) & \xrightarrow{\alpha_1(A)} & G(A) \\
{\scriptstyle F(u)}\Big\downarrow & \boxed{(\alpha_1)_u} & \Big\downarrow{\scriptstyle G(u)} \\
F(A') & \xrightarrow{\alpha_1(A')} & G(A') \\
{\scriptstyle \alpha_0(A')}\Big\downarrow & \boxed{\varepsilon^{\alpha}_{A'}} & \Big\downarrow{\scriptstyle =} \\
G(A') & \xrightarrow{=} & G(A')
\end{array}
\quad = \quad
\begin{array}{ccccc}
F(A) & \xrightarrow{=} & F(A) & \xrightarrow{\alpha_1(A)} & G(A) \\
{\scriptstyle F(u)}\Big\downarrow & & {\scriptstyle \alpha_0(A)}\Big\downarrow & \boxed{\varepsilon^{\alpha}_{A}} & \Big\downarrow{\scriptstyle =} \\
F(A') & & G(A) & \xrightarrow{=} & G(A) \\
{\scriptstyle \alpha_0(A')}\Big\downarrow & \boxed{\delta_{\alpha_0,u}} & \Big\downarrow{\scriptstyle G(u)} & & \\
F(A'') & \xrightarrow{=} & F(A'') & &
\end{array}
$$

and

$$
\begin{array}{ccc}
F(A) & \xrightarrow{=} & F(A) \\
{\scriptstyle =}\Big\downarrow & \boxed{\eta^{\alpha}_{A}} & \Big\downarrow{\scriptstyle \alpha_0(A)} \\
F(A) & \xrightarrow{\alpha_1(A)} & G(A) \\
{\scriptstyle F(u)}\Big\downarrow & \boxed{(\alpha_1)_u} & \Big\downarrow{\scriptstyle G(u)} \\
F(A') & \xrightarrow{\alpha_1(A')} & G(A')
\end{array}
\quad = \quad
\begin{array}{ccccc}
 & & F(A) & \xrightarrow{=} & F(A) \\
 & & {\scriptstyle F(u)}\Big\downarrow & \boxed{\delta_{\alpha_0,u}} & \Big\downarrow{\scriptstyle \alpha_0(A)} \\
F(A') & \xrightarrow{=} & F(A') & & G(A) \\
\Big\downarrow{\scriptstyle =} & \boxed{\eta^{\alpha}_{A'}} & \Big\downarrow{\scriptstyle \alpha_0(A')} & & \Big\downarrow{\scriptstyle G(u)} \\
F(A') & \xrightarrow{\alpha_1(A')} & G(A') & \xrightarrow{=} & G(A')
\end{array}
$$

for every 1v-cell $u : A \longrightarrow A'$.

In the following Proposition $\eta_{F(u)}$ and $\varepsilon_{G(u)}$ are given as in [30, Lemma 3.16].

Proposition 4.5 *Given a strong vertical transformation* α_0 *under conditions of Proposition 4.1. Let* $u : A \longrightarrow A'$ *be a 1v-cell with a 1h-companion* $f = u_*$*. Then the inverse of the 2-cell* δ_{α_1,u_*} *is given by:*

$$
\delta^{-1}_{\alpha_1,u_*} =
\begin{array}{ccccccccccccc}
F(A) & \xrightarrow{=} & F(A) & & F(A) & \xrightarrow{=} & F(A) & \xrightarrow{\alpha_1(A)} & G(A) & & & & \\
{\scriptstyle =}\Big\downarrow & \boxed{\eta_{F(u)}} & \Big\downarrow{\scriptstyle F(u)} & & {\scriptstyle F(u)}\Big\downarrow & & \Big\downarrow{\scriptstyle \alpha_0(A)} & \boxed{\varepsilon^{\alpha}_{A}} & \Big\downarrow{\scriptstyle =} & & & & \\
F(A) & \xrightarrow[F(u)_*]{} & F(A') & \xrightarrow{=} & F(A') & \boxed{\delta_{\alpha_0,u}} & G(A) & \xrightarrow{=} & G(A) & \xrightarrow{G(u)_*} & G(A') & & \\
 & & {\scriptstyle =}\Big\downarrow & \boxed{\eta^{\alpha}_{A'}} & \Big\downarrow{\scriptstyle \alpha_0(A')} & & \Big\downarrow{\scriptstyle G(u)} & & {\scriptstyle G(u)}\Big\downarrow & \boxed{\varepsilon_{G(u)}} & \Big\downarrow{\scriptstyle =} & & \\
 & & F(A') & \xrightarrow{\alpha_1(A')} & G(A') & \xrightarrow{=} & G(A') & & G(A') & \xrightarrow{=} & G(A'). & &
\end{array}
$$

Proof. Use axiom 1) for α_0 and (6.3) of [31, Definition 6.1], together with ε-η-relations. □

Note that the above inverse of δ_{α_1,u_*} is in fact the image of the known pseudofunctor $V\mathbb{D} \longrightarrow H\mathbb{D}$ from the vertical 2-category to the horizontal one of a given double category $\mathbb{D}$ in which all 1v-cells have 1h-companions. For a horizontally globular 2-cell a with a left 1v-cell u and a right 1v-cell v, the image by this functor of a is given by $[\eta_u|a|\varepsilon_v]$ (horizontal composition of 2-cells).

Remark 4.6 One could start with a vertical transformation α_0 (for which $\delta_{\alpha_0,u} = \mathrm{Id}$ for all 1v-cells $u : A \longrightarrow A'$) and define a horizontal transformation α_1 setting $\delta_{\alpha_1,f} = \mathrm{Id}$ for all 1h-cells $f : A \longrightarrow B$ and defining $(\alpha_1)_u$ as in Proposition 4.1. Though, in order for α_1 to satisfy the corresponding axiom 1), one needs to assume the first two identities of Corollary 4.4.

4.2. *Embedding* $PsDbl_3^*$ *into DblPs*

It remains to show how to turn vertical transformations into double pseudonatural transformations. We will assume that 1v-cell components of vertical transformations have 1h-companions.

Observe that for vertical transformations the 2-cells $\delta_{\alpha_0,u}$ are identities. Moreover, we know that vertical transformations are particular cases of vertical pseudonatural transformations (Remark 3.2), and that strong horizontal transformations are particular cases of horizontal pseudonatural transformations. By Proposition 4.1 we have that a vertical transformation α_0 determines a strong horizontal transformation. So far we have axiom (T1) of Definition 3.9. Furthermore, by Proposition 4.5 we have in particular that for all 1v-cell components $\alpha_0(A)$ of vertical transformations the 2-cells $\delta_{\alpha_1,\alpha_1(A)}$ are invertible. Then we have that the axiom (T2) is fulfilled. Observe that setting $\Theta_A^\alpha = \varepsilon_{A'}^\alpha$, by the first and third identities in Corollary 4.4 we have a Θ-double pseudonatural transformation between pseudo double functors. Due to Proposition 3.15 we have indeed a double pseudonatural transformation, as we wanted. (Actually, thanks to the ε-η-relations, by the first identity in Corollary 4.4, axiom 1) for the horizontal pseudonatural transformation α_1 holds if and only if axiom (T3-1) for t_f^α in Proposition 3.15 holds.)

Moreover, we may deduce the following bijective correspondence $t_f^\alpha \leftrightarrow \delta_{\alpha_1,f}$:

$$t_f^\alpha = \begin{array}{ccccc} F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{\alpha_1(B)} & G(B) \\ \downarrow = & & \boxed{\delta_{\alpha_1,f}} & & \downarrow = \\ F(A) & \xrightarrow{\alpha_1(A)} & G(A) & \xrightarrow{G(f)} & G(B), \\ \downarrow \alpha_0(A) & \boxed{\varepsilon_A^\alpha} & \downarrow = & & \\ F(A') & \xrightarrow{=} & G(A') & & \end{array} \qquad \delta_{\alpha_1,f} = \begin{array}{ccccccc} F(A) & \xrightarrow{=} & F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{\alpha_1(B)} & G(B) \\ \downarrow = & \boxed{\eta_A^\alpha} & \downarrow \alpha_0(A) & & \boxed{t_f^\alpha} & & \downarrow = \\ F(A) & \xrightarrow[\alpha_1(A)]{} & G(A) & \xrightarrow[G(f)]{} & G(B) & \xrightarrow[=]{} & G(B), \end{array}$$

and complete the bijection $t_f^\alpha \leftrightarrow (\alpha_0)_f$:

$$(\alpha_0)_f = \begin{array}{ccccc} & & F(B) & \xrightarrow{=} & F(B) \\ & & = \downarrow & \boxed{\eta_B^\alpha} & \downarrow \alpha_0(B) \\ F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{\alpha_1(B)} & G(B) \\ \downarrow = & & \boxed{t_f^\alpha} & & \downarrow = \\ F(A) & & \xrightarrow{G(f)} & & G(A). \end{array}$$

Remark 4.7 Given the ε-η-relations, by the properties developed in this and the previous subsection, axiom 1) for the horizontal pseudonatural transformation α_1 holds if and only if axiom (T3-1) for t_f^α in Proposition 3.15 holds, if and only if axiom 1) for the vertical pseudonatural transformation α_0 holds.

5. Tricategorical pullbacks and (co)products

For a notion of enrichment over (an iconic) tricategory V we need some notion of a monoidal structure on V, while for a notion of an internal category in V we need some notion of tricategorical pullbacks. We define both such notions in this section, where for the monoidal structure we consider tricategorical products. As justified in Remark 9.3 we denote by fractions the vertical composition of 2- and 3-cells.

5.1. Tricategorical pullbacks

In this subsection we define tricategorical pullbacks, that is pullbacks in tricategories. We will also call them shortly 3-pullbacks.

Definition 5.1 *A 3-pullback over a 0-cell S with respect to 1-cells $f : M \longrightarrow S$ and $g : N \longrightarrow S$ in a tricategory V is given by: a 0-cell P, 1-cells $p_1 : P \longrightarrow M, p_2 : P \longrightarrow N$ and an equivalence 2-cell $\omega : gp_2 \Rightarrow fp_1$ so that*

- *for every 0-cell T, 1-cells $q_1 : T \longrightarrow M, q_2 : T \longrightarrow N$ and equivalence 2-cell $\sigma : gq_2 \Rightarrow fq_1$ there are a 1-cell $u : T \longrightarrow P$, equivalence 2-cells $\zeta_1 : p_1u \Rightarrow q_1$ and $\zeta_2 : q_2 \Rightarrow p_2u$ and an isomorphism 3-cell*

$$\Sigma : \frac{\dfrac{\mathrm{Id}_g \otimes \zeta_2}{\omega \otimes \mathrm{Id}_u}}{\mathrm{Id}_f \otimes \zeta_1} \Rrightarrow \sigma;$$

- *for all 1-cells $u, v : T \longrightarrow P$, 2-cells $\alpha : p_1u \Rightarrow p_1v, \beta : p_2u \Rightarrow p_2v$ and a 3-cell $\kappa : \frac{\mathrm{Id}_g \otimes \beta}{\omega \otimes \mathrm{Id}_v} \Rrightarrow \frac{\omega \otimes \mathrm{Id}_u}{\mathrm{Id}_f \otimes \alpha}$ such that*

$$\begin{array}{ccccc}
\dfrac{(\mathrm{Id}_g \otimes \mathrm{Id}_{p_2}) \otimes \gamma}{\omega \otimes \mathrm{Id}_v} & \overset{\xi}{\Rrightarrow} & \dfrac{\omega \otimes \mathrm{Id}_u}{(\mathrm{Id}_f \otimes \mathrm{Id}_{p_1}) \otimes \gamma} & \overset{\frac{\mathrm{Id}}{a^{-1}}}{\Rrightarrow} & \dfrac{\omega \otimes \mathrm{Id}_u}{\mathrm{Id}_f \otimes (\mathrm{Id}_{p_1} \otimes \gamma)} \\
\Downarrow \dfrac{a^{-1}}{\mathrm{Id}} & & & & \Downarrow \dfrac{\mathrm{Id}}{\mathrm{Id} \otimes \Gamma_1} \\
\dfrac{\mathrm{Id}_g \otimes (\mathrm{Id}_{p_2} \otimes \gamma)}{\omega \otimes \mathrm{Id}_v} & \overset{\frac{\mathrm{Id} \otimes \Gamma_1}{\mathrm{Id}}}{\Rrightarrow} & \dfrac{\mathrm{Id}_g \otimes \beta}{\omega \otimes \mathrm{Id}_v} & \overset{\kappa}{\Rrightarrow} & \dfrac{\omega \otimes \mathrm{Id}_u}{\mathrm{Id}_f \otimes \alpha}
\end{array}$$

commutes, there are a 2-cell $\gamma : u \Rightarrow v$ and isomorphism 3-cells $\Gamma_1 : \mathrm{Id}_{p_1} \otimes \gamma \Rrightarrow \alpha, \Gamma_2 : \mathrm{Id}_{p_2} \otimes \gamma \Rrightarrow \beta$;

- *for all 2-cells $\gamma, \gamma' : u \Rightarrow v$ and 3-cells $\chi_1 : \mathrm{Id}_{p_1} \otimes \gamma \Rrightarrow \mathrm{Id}_{p_1} \otimes \gamma'$ and $\chi_2 : \mathrm{Id}_{p_2} \otimes \gamma \Rrightarrow \mathrm{Id}_{p_2} \otimes \gamma'$ such that*

$$\begin{array}{ccccc}
\dfrac{\mathrm{Id}_g \otimes (\mathrm{Id}_{p_2} \otimes \gamma)}{\omega \otimes \mathrm{Id}_v} & \overset{\xi}{\Rrightarrow} & \dfrac{\omega \otimes \mathrm{Id}_u}{\mathrm{Id}_f \otimes (\mathrm{Id}_{p_1} \otimes \gamma)} & \overset{\frac{\mathrm{Id}}{\mathrm{Id} \otimes \chi_1}}{\Rrightarrow} & \dfrac{\omega \otimes \mathrm{Id}_u}{\mathrm{Id}_f \otimes (\mathrm{Id}_{p_1} \otimes \gamma')} \\
\Downarrow \dfrac{\mathrm{Id} \otimes \chi_2}{\mathrm{Id}} & & & & \Downarrow \dfrac{\mathrm{Id}}{a} \\
\dfrac{\mathrm{Id}_g \otimes (\mathrm{Id}_{p_2} \otimes \gamma')}{\omega \otimes \mathrm{Id}_v} & \overset{\frac{a}{\mathrm{Id}}}{\Rrightarrow} & \dfrac{(\mathrm{Id}_g \otimes \mathrm{Id}_{p_2}) \otimes \gamma'}{\omega \otimes \mathrm{Id}_v} & \overset{\xi}{\Rrightarrow} & \dfrac{\omega \otimes \mathrm{Id}_u}{(\mathrm{Id}_f \otimes \mathrm{Id}_{p_1}) \otimes \gamma'}
\end{array}$$

commutes, there exists a unique 3-cell $\chi : \gamma \Rrightarrow \gamma'$ such that $\chi_1 = \mathrm{Id}_{\mathrm{Id}_{p_1}} \otimes \chi$ and $\chi_2 = \mathrm{Id}_{\mathrm{Id}_{p_2}} \otimes \chi$.

A 3-pullback with notations as in the above Definition we will denote shortly by $(P, M, N, S, p_1, p_2; f, g)$, or $(M \times_S N, f, g)$.

5.2. Tricategorical (co)products

In the literature there are bicategorical (co)products, that is, (co)products in bicategories. In this section we propose a definition for their tricategorical companions, we will shortly also call them 3-(co)products. Before defining them let us remark what data comprise a 2-product in a bicategory $\mathcal{K}$. A 2-product consists of: 1) a 0-cell $A \times B$ for 0-cells $A, B \in \mathcal{K}$ and 1-cells $p_1 : A \times B \longrightarrow A, p_2 : A \times B \longrightarrow B$, and 2) for every $X \in \mathcal{K}$ a natural equivalence of categories: $F : \mathcal{K}(X, A \times B) \longrightarrow \mathcal{K}(X, A) \times \mathcal{K}(X, B)$. Observe that the point 2) means that F is an equivalence 1-cell in the 2-category Cat_2 of categories, and that the 2-functor $\mathcal{K}(X, -) : \mathcal{K} \longrightarrow \mathrm{Cat}_2$ sends the product 0-cell $A \times B$ in $\mathcal{K}$ to the 1-product in the 1-category of categories Cat_1.

With this in mind we define:

Definition 5.2 *A 3-product of 0-cells A and B in a tricategory V consists of:*

- *a 0-cell $A \times B$ and 1-cells $p_1 : A \times B \longrightarrow A, p_2 : A \times B \longrightarrow B$, such that*
- *for every $X \in V$ there is a biequivalence of bicategories*

$$V(X, A \times B) \simeq V(X, A) \times V(X, B)$$

where on the right hand-side the 2-product in the 2-category Bicat_2 *of bicategories, pseudofunctors and icons [27] is meant.*

The second point in the above definition says that there is an equivalence 1-cell in the tricategory Bicat_3 of bicategories (bicategories, pseudofunctors, pseudonatural transformations and modifications) up to which the "trifunctor" $V(X, -) : V \longrightarrow \text{Bicat}_3$ sends the product 0-cell $A \times B$ to the 2-product of bicategories.

For 3-products of $k > 2$ 0-cells the projection 1-cells to the first i and last j components we will write by $p^k_{1...i}$ and $p^k_{k-j+1...k}$, respectively.

It is useful to unpack the above definition. We will do it for the dual notion of a 3-coproduct in V. In this case the natural biequivalence of bicategories in question takes the form $V(A \amalg B, X) \simeq V(A, X) \amalg V(B, X)$ and the analogous "trifunctor" $V(-, X) : V \longrightarrow \text{Bicat}$ is now contravariant.

Definition 5.3 *A 3-coproduct of 0-cells A and B in a tricategory V consists of: a 0-cell $A \amalg B$ and 1-cells $\iota_1 : A \longrightarrow A \amalg B, \iota_2 : B \longrightarrow A \amalg B$, such that*

- *for every 0-cell T and 1-cells $f_1 : A \longrightarrow T, f_2 : B \longrightarrow T$ there are a 1-cell $u : A \amalg B \longrightarrow T$ and equivalence 2-cells $\zeta_i : u\iota_i \Rightarrow f_i, i = 1, 2$;*
- *for all 1-cells $u, v : A \amalg B \longrightarrow T$ and 2-cells $\alpha : u\iota_1 \Rightarrow v\iota_1$ and $\beta : u\iota_2 \Rightarrow v\iota_2$, there are a 2-cell $\gamma : u \Rightarrow v$ and 3-cells $\Gamma_1 : \gamma \otimes \text{Id}_{\iota_1} \Rrightarrow \alpha$ and $\Gamma_2 : \gamma \otimes \text{Id}_{\iota_2} \Rrightarrow \beta$;*
- *for every two 2-cells $\gamma, \gamma' : u \Rightarrow v$ and every two 3-cells $\chi_i : \gamma \otimes \iota_i \Rrightarrow \gamma' \otimes \iota_i, i = 1, 2$ there is a unique 3-cell $\Gamma : \gamma \Rrightarrow \gamma'$ such that $\chi_i = \Gamma \otimes \text{Id}_{\text{Id}_{\iota_i}}, i = 1, 2$.*

We say that a tricategory V has small 3-(co)products if it has them for any family of 0-cells indexed by (elements of) a set.

6. Categories internal in iconic tricategories

We are interested in internalization in ambient weak n-categories, for $n = 1, 2, 3$, that have an underlying 1-category. Such ambient weak n-categories can be various "categories of categories".

6.1. Iconic tricategories

For $n = 2$ a folklore example of internal categories are pseudo double categories for which the above condition is fulfilled: they are internal categories in the 2-category of categories. For $n = 3$ such ambient categories are Gray categories and *iconic tricategories* from [32]. We recall that Gray categories, introduced in [17], are categories enriched in the category of 2-categories and 2-functors equipped with the monoidal product constructed by Gray in [23]. As such, Gray categories are tricategories with: strictly associative and unital composition on 1-cells, horizontal composition on 2-cells is given by whiskering and it turns out to be also strictly associative and unital, the only non-strict isomorphism is the interchange (between 2-cells).

On the other hand, iconic tricategories, introduced in [32], are categories enriched over Cartesian monoidal 2-category Bicat_2 of bicategories, pseudofunctors and icons from [27]. Being the pseudonatural equivalences for associativity and units icons, *i.e.* their 1-cell components are identities, associativity and unitality of 1-cells is strict, but it is not necessarily so (in the horizontal direction) on 2-cells. The associativity and units icons must satisfy the pentagon and the triangle identities strictly (from the enrichment). As the edges of the pentagonal and triangular diagrams are the 1-cell components of the icons, they are identity 2-cells in the iconic tricategory, and what forces the pentagon and the triangle to commute strictly are uniquely determined 3-cells (that determine modifications π and μ), which satisfy any axiom, including the tricategory axioms. Similarly, one has modifications λ and ρ whose 1-cell components are uniquely determined 3-cells, and all the modifications π, μ, λ and ρ fulfill the desired axioms for a tricategory from [17]. One has that every Gray category is an iconic tricategory, and consequently every tricategory is triequivalent to an iconic tricategory (see *e.g.* [32, Remark 6.7]).

A very well-known example of an iconic tricategory is the tricategory Bicat_3 of bicategories, pseudofunctors, pseudonatural transformations and modifications. In particular, in this paper we are interested in iconic tricategories DblPs and $2Cat_{wk}$, the latter being the tricategory of 2-categories, pseudofunctors, pseudonatural transformations and modifications. Observe that none of these three iconic tricategories is a Gray category. Namely, we saw in Subsection 3.6 that the horizontal associativity of 2-cells in DblPs is not strict. As for 2-cells of Bicat_3 and $2Cat_{wk}$, that the associativity of pseudonatural transformations is not strict, see *e.g.* in [34, Lemma 11.5.9].

From now on we fix throughout this section V to be an iconic tricategory.

6.2. *Internalization*

We want to define a category internal in V. In [10, Definition 2.11] an internal category in a Gray-category was defined. Therein, the definition of a Gray-category is based on whisker, so that instead of a full interchange law there appears an isomorphism 3-cell sw (with an additional rule for whiskering). From the point of view of V, the 3-cell sw can be defined as the following transversal composition of 3-cells:

$$\Big(\frac{[\alpha|\,\mathrm{Id}]}{[\mathrm{Id}\,|\beta]}\Big) \stackrel{\xi}{\Rrightarrow} \Big[\Big(\frac{\alpha}{\mathrm{Id}}\Big)\Big|\Big(\frac{\mathrm{Id}}{\beta}\Big)\Big] \cong \Big[\Big(\frac{\mathrm{Id}}{\alpha}\Big)\Big|\Big(\frac{\beta}{\mathrm{Id}}\Big)\Big] \stackrel{\xi^{-1}}{\Rrightarrow} \Big(\frac{[\mathrm{Id}\,|\beta]}{[\alpha|\,\mathrm{Id}]}\Big),$$

for 2-cells α and β, where the middle isomorphism stands for the composition of one "vertical" unity constraint with the inverse of the other in the appropriate order, in both coordinates. Here $[\alpha|\beta]$ denotes the horizontal composition $\beta \otimes \alpha$, and the fractions denote the vertical one. We consider by the coherence Theorem [17, Theorem 1.5] that these unity constraints are identities, so sw will be identity. Another difference with respect to [10, Definition 2.11] is that therein the authors work with 1-pullbacks (a Gray-category is an iconic tricategory), while we are working with 3-pullbacks introduced in Section 5.

As a matter of fact, we will need only certain 3-pullbacks. For this reason we define iterated 3-pullbacks, analogously to iterated 2-pullbacks from [19]. Let $B_1 \overset{s}{\underset{t}{\rightrightarrows}} B_0$ be 1-cells in an iconic tricategory V. Iterated n-fold composition of the span $B_1 \overset{s}{\underset{t}{\rightrightarrows}} B_0$ in V can be defined via 3-pullbacks. Such n-fold composition we call *iterated 3-pullbacks*. We denote them by (any of the distributions of the parentheses on) $B_1 \times_{B_0} B_1 \times_{B_0} \cdots \times_{B_0} B_1$ (n times). We will write this shortly as $B_1^{(n)_0}$, regardless the choice of the distributions, which will be clear from the context. Let $B_1^{(0)_0}$ be B_0. For the projections $p_i : B_1^{(n)_0} \longrightarrow B_1$ for $i = 1, 2 \dots n$ we will use lexicographical order.

Remark 6.1 The 3-pullback $(B_1 \times_{B_0} B_1, s, t)$ we will consider with the following order of factors:

$$\begin{array}{ccc} B_1 \times_{B_0} B_1 & \xrightarrow{p_2} & B_1 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle t} \\ B_1 & \xrightarrow[s]{} & B_0. \end{array}$$

The labels s and t are suggestive for the case when B_1 is a hom-set, then as the diagram indicates, the 3-pullback $B_1 \times_{B_0} B_1$ is read from right to left, although the projections are labeled in the lexicographical order.

In the next definition, to simplify the notation, the unsubscribed symbol $\times$ will stand for $\times_{B_0}$ at many places.

Definition 6.2 *Let V be an iconic tricategory. A category internal in V consists of:*

1. *1-cells* $B_1 \overset{s}{\underset{t}{\rightrightarrows}} B_0$ *in V, which we call source and target morphisms, for which the iterated 3-pullbacks* $B_1^{(n)_0}, n \in \mathbb{N}$ *exist;*
2. *1-cells:* $B_1 \times_{B_0} B_1 \xrightarrow{c} B_1$ *composition and* $u : B_0 \longrightarrow B_1$ *unit (or identity) morphism in V;*
3. *equivalence 2-cells* $a^* : c\otimes(id_{B_1} \times_{B_0} c) \Rightarrow c\otimes(c\times_{B_0} id_{B_1})$, $l^* : c\otimes(u\times_{B_0} id_{B_1}) \Rightarrow id_{B_1}$ *and* $r^* : c\otimes(id_{B_1} \times_{B_0} u) \Rightarrow id_{B_1}$ *in V;*
4. *3-cells in V:*

$$\pi^* : \frac{\frac{\mathrm{Id}_c \otimes(\mathrm{Id}_{id_{B_1}} \times a^*)}{a^* \otimes \mathrm{Id}_{1\times c\times 1}}}{\mathrm{Id}_c \otimes(a^* \times \mathrm{Id}_{id_{B_1}})} \Rrightarrow \frac{\frac{a^* \otimes \mathrm{Id}_{1\times 1\times c}}{\mathrm{Id}_c \otimes \mathrm{Nat}_{(c\times 1)(1\times 1\times c)}}}{a^* \otimes \mathrm{Id}_{c\times 1\times 1}}$$

$$\mu^* : \mathrm{Id}_c \otimes(\mathrm{Id}_{id_{B_1}} \times r^*) \Rrightarrow \frac{a^* \otimes \mathrm{Id}_{id_{B_1}\times u\times id_{B_1}}}{\mathrm{Id}_c \otimes(l^* \times id_{B_1})}$$

$$\lambda^* : \mathrm{Id}_c \otimes(\mathrm{Id}_{id_{B_1}} \times l^*) \Rrightarrow \frac{\frac{a^* \otimes \mathrm{Id}_{id_{B_1}\times id_{B_1}\times u}}{\mathrm{Id}_c \otimes \mathrm{Nat}_{(c\times id_{B_1})\otimes(id_{B_1}\times id_{B_1}\times u)}}}{l^* \otimes \mathrm{Id}_c}$$

$$\rho^* : \frac{a^* \otimes \mathrm{Id}_{u\times id_{B_1}\times id_{B_1}}}{\mathrm{Id}_c \otimes(r^* \times id_{B_1})} \Rrightarrow \frac{\frac{\mathrm{Id}_c \otimes \mathrm{Nat}_{(c\times id_{B_1})\otimes(u\times id_{B_1}\times id_{B_1})}}{r^* \otimes \mathrm{Id}_c}}{\mathrm{Nat}_{id_{B_1}\otimes c}}$$

$$\epsilon^* : l^* \otimes \mathrm{Id}_u \Rrightarrow \frac{\mathrm{Id}_c \otimes \mathrm{Nat}_{(id_{B_1}\times u)\otimes u}}{r^* \otimes \mathrm{Id}_u}$$

   which satisfy axioms (IT-1) - (IT-5) in the second part of the Appendix and symmetric versions of (IT-1), (IT-3) and (IT-4) (here the 2-cells v are all identities, see the Remark below);

the above data should moreover satisfy the following compatibility conditions:

$$sp_2 = tp_1, \quad su = id_{B_0} = tu, \quad sc = sp_1, \quad tc = tp_2,$$

$$id_s \otimes l^* = id_s = id_s \otimes r^*, \quad id_t \otimes l^* = id_t = id_t \otimes r^*, \quad id_s \otimes a^* = id_{sp_1}, \quad id_t \otimes a^* = id_{tp_3},$$

$$\mathrm{Id}_{id_s} \otimes\pi^* = \mathrm{Id}_{id_{sp_1}}, \quad \mathrm{Id}_{id_t} \otimes\pi^* = \mathrm{Id}_{id_{tp_4}},$$

$$\mathrm{Id}_{id_s} \otimes\mu^* = \mathrm{Id}_{id_s} \otimes\lambda^* = \mathrm{Id}_{id_s} \otimes\rho^* = \mathrm{Id}_{id_{sp_1}}, \quad \mathrm{Id}_{id_t} \otimes\mu^* = \mathrm{Id}_{id_t} \otimes\lambda^* = \mathrm{Id}_{id_t} \otimes\rho^* = \mathrm{Id}_{id_{tp_2}},$$

where $p_i, i = 1, 2, 3, 4$, *are 1-cells projections from the corresponding pullbacks in V.*

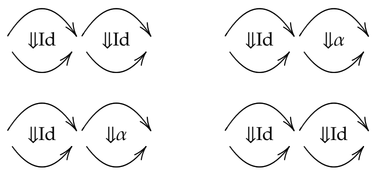
Remark 6.3 When writing out the 3-cells (and the axioms) in our definition, the following should be kept in mind.

a) We will identify 1-cells $(id_{B_1} \times u) \otimes c$ (acting on $B_1 \times_{B_0} B_1$) and $c \times u$ (acting on $(B_1 \times_{B_0} B_1) \times_{B_0} B_0$), by suppressing the isomorphism 1-cells (for the associativity and unity of the 3-pullback) between their domains. We do similar for u and their symmetric counterparts. Recall that since V is iconic, one has $c \otimes id_{B_1} = c$.

b) We explain the (co)domains of the 2-cells $\mathrm{Id}_\bullet \times a^*, \mathrm{Id}_\bullet \times r^*, \mathrm{Id}_\bullet \times l^*$ and their symmetric counterparts in item 4 in the definition above. Given a 2-cell $\alpha : G\otimes F \Rightarrow G' \otimes F'$, by abuse of notation, by $\alpha \times id_{B_1}$ we will mean the induced 2-cell $(G \times 1) \otimes (F \times 1) \Rightarrow (G' \times 1) \otimes (F' \times 1)$. (Observe that by the 3-pullback property, between $(G \times 1) \otimes (F \times 1)$ and $(G \otimes F) \times 1$ there exists a (possibly non-isomorphism) 2-cell γ.)

c) The naturality identity 2-cells we will sometimes draw explicitly and denote them all by ν, or we will just write "=" between two equal compositions of 1-cells. Here we refer to the 1-cells of the form $G \times F = (G \otimes 1) \times (1 \otimes F) = (1 \otimes F) \times (G \otimes 1)$.

d) In order to simplify the diagrams and the definition, we could want the following two vertical compositions of horizontal compositions of 2-cells to be equal:



When V = DblPs, applying Proposition 3.12 and Proposition 3.13 one can see that the two compositions above differ by a modification given by the globular 2-cells $\delta_{\alpha_i,id}$ for $i = 1, 2$. Thus one could restrict to a full sub tricategory of V whose 1-cells are double pseudofunctors F which applied to the identity 1h- and 1v-cells give identities. Then one could also consider that their distinguished 2-cells F^A and F_A (see the next section) are identities (for all 0-cells A of the domain strict double category of F), thus the unity constraints for the horizontal composition would both be identities (see [13, Section 4.8]), and one could also consider that 2-cells of the sub tricategory are those double pseudonatural tranformations α of V whose associated globular 2-cells $\delta_{\alpha_i,id}$ for $i = 1, 2$ are identities (see the end of Definition 3.1).

Remark 6.4 Let us comment the axioms (IT-1) - (IT-5). We do it for the case of the full sub tricategory of V from point e) in the above Remark, let us denote it by V^*. Although the 3-cell sw is identity in our context, we will mention it, as it helps to better understand technically how the compositions of 3-cells are made in the axioms.

By n-fold fractions we denote vertical composition of n 3-cells (observe that we consider vertical associativity of 2-cells as identity). All the drawings of 2-cells (bicategory diagrams), and accordingly the 3-cells acting between them, are read from top to bottom and from left to right, including the horizontal composition of 3-cells $\alpha \otimes \beta$, (first acts α, then β) which otherwise is read from right to left. In one entry of an n-fraction vertical lines present transversal composition of 3-cells (read from left to right). Moreover, in one such entry may appear: $\frac{\alpha}{\alpha'}|\beta|\beta'|\beta''$ where all the named cells are 3-cells. This means that instead of writing separate drawings for four transversally composed 3-cells, we condense them into one 3-cell written this way. We usually do this when applying the distinguished 3-cells sw, a, ξ from the ambient tricategory V^* (associativity of 2-cells and interchangers).

(IT-1) comprises of $\lambda^*_u, \varepsilon^*, \mu^*_u, sw$ and the first 1-cell in its domain 2-cell is $id_{B_1} \otimes u \otimes u$ (in the symmetric version it is $u \otimes u \otimes id_{B_1}$).

(IT-2) comprises of $\lambda^*, \rho^*, sw, a, \xi$ and the first 1-cell in its domain 2-cell is $u \otimes id_{B_1} \otimes u$.

(IT-3) comprises of $\lambda^*, \pi^*, sw, a, \xi$ and the first 1-cell in its domain 2-cell is $id_{B_1} \otimes id_{B_1} \otimes id_{B_1} \otimes u$ (in the symmetric version it is $u \otimes id_{B_1} \otimes id_{B_1} \otimes id_{B_1}$). It corresponds to the *normalization in the first and the fourth coordinate*.

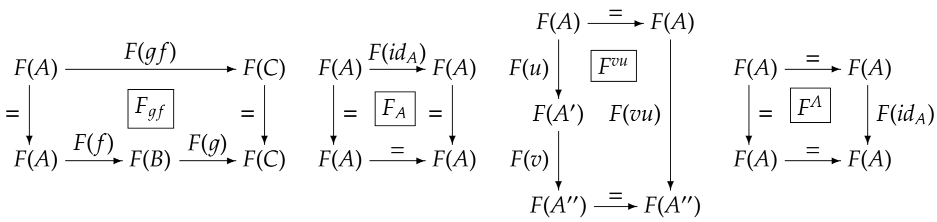
(IT-4) comprises of $\mu^*, \lambda^*, \pi^*, sw, a, \xi$ and the first 1-cell in its domain 2-cell is $id_{B_1} \otimes id_{B_1} \otimes u \otimes id_{B_1}$ (in the symmetric version it is $id_{B_1} \otimes u \otimes id_{B_1} \otimes id_{B_1}$). It corresponds to the *normalization in the second and the third coordinate*.

(IT-5) comprises of π^*, sw, ξ. It corresponds to the *4-cocycle condition on* a^*.

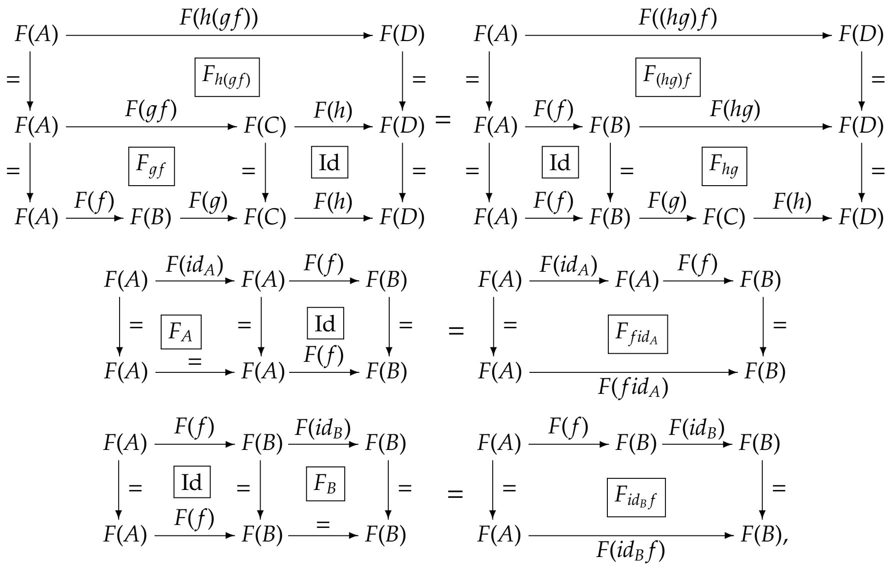
Observe in these axioms that the 3-cells sw, a, ξ are the distinguished 3-cells from the ambient tricategory V^*.

7. Categories internal in DblPs

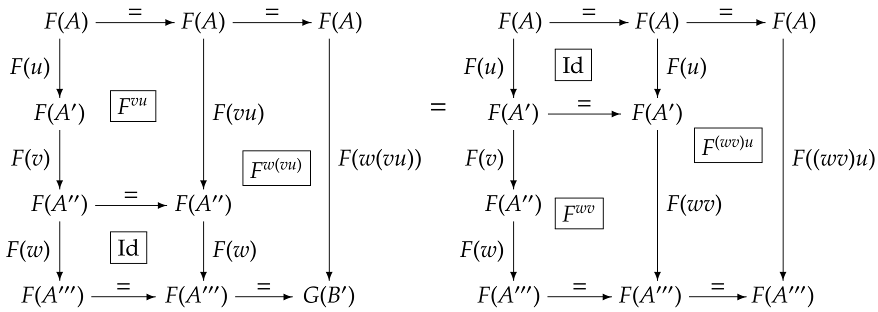
For the iterated 3-pullbacks $B_1^{(n)_0} = B_1 \times_{B_0} B_1 \times_{B_0} \cdots \times_{B_0} B_1$ (n times) in DblPs to exist we consider the source and target double functors to be strict. Under this assumption, an internal category in DblPs consists of strict double categories $\mathbb{D}_0$ and $\mathbb{D}_1$, strict double functors $S, T : \mathbb{D}_1 \longrightarrow \mathbb{D}_0$, double pseudo functors $U : \mathbb{D}_0 \longrightarrow \mathbb{D}_1, M : \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \longrightarrow \mathbb{D}_1$, double pseudonatural transformations a^*, l^*, r^* and double modifications $\pi^*, \mu^*, \lambda^*, \rho^*, \varepsilon^*$, satisfying the corresponding axioms from the previous section. Both double pseudo functors U and M are equipped with distinguished globular 2-cells (set F for any of U and M):

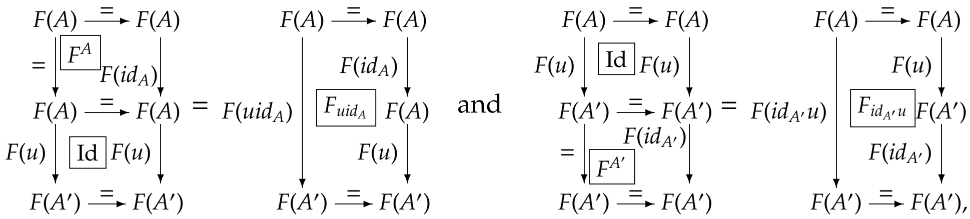


satisfying the following axioms, where f, g, h are composable 1h-cells, u, v, w are composable 1v-cells, a and b are 2-cells composable horizontally and a and a' are 2-cells composable vertically (note that here, for simplicity of the notation, we are denoting both 1h- and 1v-composition by juxtaposition, the difference is clear from the letters denoting 1-cells). Coherence in the 1h-direction:

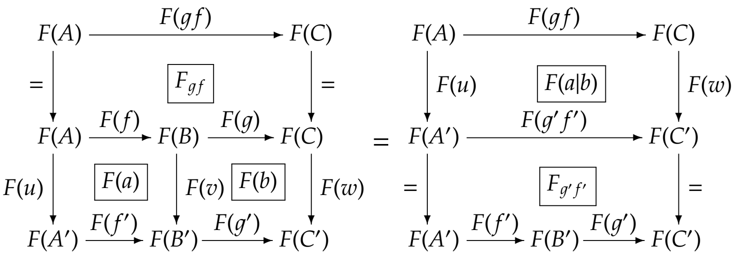


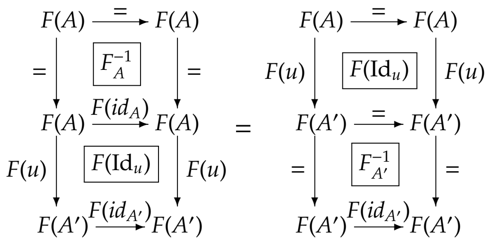
coherence in the 1v-direction:



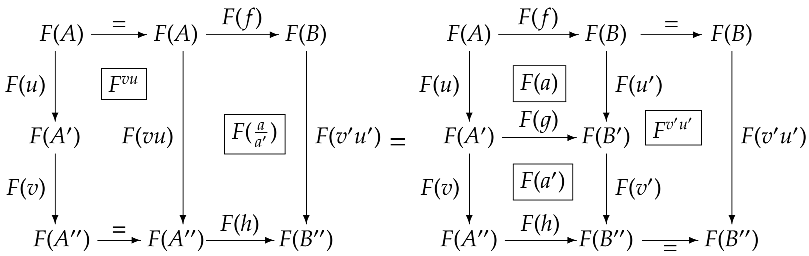


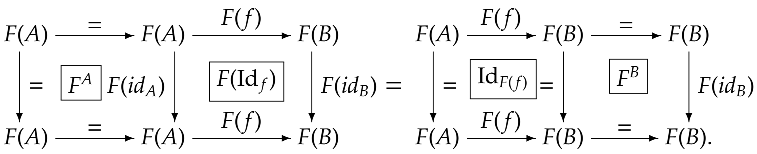
coherence for the composition of and unity 2-cells, horizontally:





and vertically:





The above three coherences in the 1v-direction for U and M correspond to axioms (21)-(26) of [19, Section 3], respectively. The analogous six coherences in the 1h-direction do not appear there. The two (horizontally) globular 2-cells F^{vu} and F^A for U and M correspond to natural transformations (17)-(20): $U^A = \tau, U^{vu} = \mu, M^A = \delta, M^{vu} = \chi$, and the above two coherences for the composition of and unity 2-cells in the vertical direction for U and M correspond to naturalities of (17)-(20). One can analogously formulate natural transformations in the horizontal direction, introducing additional two (vertically) globular 2-cells F_{gf} and F_A for U and M and the above two coherences for the composition of and unity 2-cells in the horizontal direction, which correspond to their naturalities. (To formulate these natural transformations in the horizontal direction change the roles of vertical and horizontal cells in the definition of two categories determining a strict double category.) For the sake of comparing this structure to intercategories, for mnemotechnical reasons we could denote these distinguished (vertically) globular 2-cells as follows: $U_A = \tau', U_{gf} = \mu', M_A = \delta', M_{gf} = \chi'$.

Summing up, for the double pseudo functors U and M we have eight globular 2-cells:

$$U_{gf}, U_A, U^{vu}, U^A, \quad M_{gf}, M_A, M^{vu}, M^A,$$

which satisfy in total 20 axioms named above. We will denote their actions as follows. Let us denote the image under $M : \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \longrightarrow \mathbb{D}_1$ of (y, x) by $(x|y)$ for any of the four types of cells $(y, x) \in \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1$.

Moreover, let us denote by Id^h_x the image under $U : \mathbb{D}_0 \longrightarrow \mathbb{D}_1$ of any of the four types of cells x in $\mathbb{D}_0$. Now for 1h-cells g, g', f, f' and 1v-cells u, u', v, v' of $\mathbb{D}_1$ (for the action of M), respectively of $\mathbb{D}_0$ (for the action of U) we will write:

$$\chi : \frac{(u|u')}{(v|v')} \Rightarrow (\frac{u}{v}|\frac{u'}{v'}), \quad \delta : id^v_{(A|A')} \Rightarrow (id^v_A|id^v_{A'}), \quad \mu : \frac{\mathrm{Id}^h_u}{\mathrm{Id}^h_v} \Rightarrow \mathrm{Id}^h_{\frac{u}{v}}, \quad \tau : id^v_{\mathrm{Id}^h_A} \Rightarrow \mathrm{Id}^h_{id^v_A} \tag{7.9}$$

$$\begin{cases} (gf|g'f') \\ \Downarrow \chi' \\ (g|g')(f|f') \end{cases} \qquad \begin{cases} (id^h_A|id^h_{A'}) \\ \Downarrow \delta' \\ id^h_{(A|A')} \end{cases} \qquad \begin{cases} \mathrm{Id}^h_{gf} \\ \Downarrow \mu' \\ \mathrm{Id}^h_g \, \mathrm{Id}^h_f \end{cases} \qquad \begin{cases} \mathrm{Id}^h_{id^h_A} \\ \Downarrow \tau' \\ id^h_{\mathrm{Id}^h_A} \end{cases} \tag{7.10}$$

here id^v_A denotes the identity 1v-cell on A (observe that the composition in the juxtapositions is read from right to left, while in $(-|-)$ it is done the other way around!).

A double pseudonatural transformation $\alpha : F \Rightarrow G$ between double pseudo functors F and G consists of a vertical pseudonatural transformation $\alpha_0 : F \Rightarrow G$ and a horizontal pseudonatural transformation $\alpha_1 : F \Rightarrow G$, both of which by Definition 3.1 are given by two distinguished globular 2-cells $\delta_{\alpha_0,u}$ and $\delta_{\alpha_1,f}$ and satisfy 5 axioms (two of them are trivial and one is simplified in the context of intercategories), two distinguished 2-cells t^α_f and r^α_u for every 1v-cell u and 1h-cell f, which have to satisfy 6 axioms in total, by Definition 3.9. Comparing such a structure of a double pseudonatural transformation with the context of intercategories, that is, comparing 2-cells of the tricategory DblPs and the 2-category *LxDbl*, one finds that in the latter context only α_1 appears (with $\delta_{\alpha_1,f}$ trivial), being the resting data α_0, four 2-cells and 6 axioms new in our context.

Thus each of double pseudonatural transformations $a^* : M(\mathrm{Id} \times_{\mathbb{D}_0} M) \longrightarrow M(M \times_{\mathbb{D}_0} \mathrm{Id}), l^* : M(U \times_{\mathbb{D}_0} -) \longrightarrow -$ and $r^* : M(- \times_{\mathbb{D}_0} U) \longrightarrow -$ is equipped with 6 distinguished 2-cells for every 1v-cell u and 1h-cell f and satisfies 16 axioms. This makes 18 distinguished 2-cells and 48 axioms. As commented in Subsection 3.3, if double pseudonatural transformations come from Θ-double pseudonatural transformations (the 2-cells t^α_f and r^α_u come from a 2-cell Θ^α_A), as indicated in Proposition 3.15, then two axioms become trivially fulfilled for each double pseudonatural transformation, reducing the amount of axioms to 42. The 6 conditions (27)-(32) from [19, Section 3] for horizontal transformations, corresponding to our a^*, l^*, r^*, together with the corresponding three naturality conditions, so 9 in total, are substituted by 42 or 48 axioms in our context.

Instead of writing out all the axioms for all of the transformations here, let us just record the following. For the double pseudonatural transformation $a^* : M(\mathrm{Id} \times_{\mathbb{D}_0} M) \longrightarrow M(M \times_{\mathbb{D}_0} \mathrm{Id})$, which we can also write as $a^* : ((-|-)|-) \Rightarrow (-|(-|-))$, let us shorten: $L = (-|-)|- = ((-|-)|-)$ and $R = -|(-|-) = (-|(-|-))$. The 1v- and 1h-composition in $\mathbb{D}_1$ we will denote by fractions and juxtapositions: $\frac{u}{v}$ and gf, respectively. Then the distinguished globular 2-cells for the double pseudonatural transformations L and R are given by:

$$L^{vu} = \Big(\frac{(u|u')|u''}{(v|v')|v''} \overset{\chi\bullet\bullet\bullet}{\Rightarrow} \frac{(u|u')}{(v|v')}|\frac{u''}{v''} \overset{\chi|1}{\Rightarrow} \big(\frac{u}{v}|\frac{u'}{v'}\big)|\frac{u''}{v''}\Big), \quad R^{vu} = \Big(\frac{u|(u'|u'')}{v|(v'|v'')} \overset{\chi\bullet\bullet\bullet}{\Rightarrow} \frac{u}{v}|\frac{(u'|u'')}{(v'|v'')} \overset{1|\chi}{\Rightarrow} \frac{u}{v}|\big(\frac{u'}{v'}|\frac{u''}{v''}\big)\Big)$$

$$L^A = \Big(\mathrm{Id}^v_{(A|A')|A''} \overset{\delta\bullet\bullet\bullet}{\Rightarrow} [\mathrm{Id}^v_{(A|A')} \,|\, \mathrm{Id}^v_{A''}] \overset{[\delta|1]}{\Rightarrow} [\mathrm{Id}^v_A \,|\, \mathrm{Id}^v_{A'}] \, \mathrm{Id}^v_{A''}\Big)$$

$$R^A = \Big(\mathrm{Id}^v_{A|(A'|A'')} \overset{\delta\bullet\bullet\bullet}{\Rightarrow} [\mathrm{Id}^v_A \,|\, \mathrm{Id}^v_{(A'|A'')}] \overset{[1|\delta]}{\Rightarrow} \mathrm{Id}^v_A [\mathrm{Id}^v_{A'} \,|\, \mathrm{Id}^v_{A''}]\Big)$$

$$L_{gf} = \begin{cases} \big((f|g)|(f'|g')\big)|(f''|g'') \\ \Downarrow \chi'|\,\mathrm{Id} \\ (f'f|g'g)|(f''|g'') \\ \Downarrow \chi' \\ f''(f'f)|g''(g'g) \end{cases} \qquad R_{gf} = \begin{cases} (f|g)|\big((f'|g')|(f''|g'')\big) \\ \Downarrow \mathrm{Id}\,|\chi' \\ (f|g)|(f''f'|g''g') \\ \Downarrow \chi' \\ f''(f'f)|g''(g'g) \end{cases}$$

$$L_A = \begin{cases} (id^h_A|id^h_{A'})|id^h_{A''} \\ \qquad \Downarrow \delta'|\,\mathrm{Id} \\ id^h_{A|A'}|id^h_{A''} \\ \quad \Downarrow \delta' \\ id^h_{(A|A')|A''} \end{cases} \qquad R_A = \begin{cases} id^h_A|(id^h_{A'}|id^h_{A''}) \\ \qquad \Downarrow \mathrm{Id}\,|\delta' \\ id^h_A|id^h_{A'|A''} \\ \quad \Downarrow \delta' \\ id^h_{A|(A'|A'')}. \end{cases}$$

In [14, Section 4.2] we wrote out a half of the axioms for the double pseudonatural transformation a^*.

7.1. *(Pseudo)monoid in Böhm's* $(Dbl, \otimes)$ *as a category internal in* DblPs

From our discussion from the end of Subsection 2.4 we see that in order to view a monoid $\mathbb{A}$ in $(Dbl, \otimes)$ as a category internal in DblPs, the double pseudo functor $\circledast : \mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{A}$ is a good candidate for a desired composition on the pullback ($M : \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \longrightarrow \mathbb{D}_1$, with $\mathbb{D}_1 = \mathbb{A}$ and $\mathbb{D}_0 = 1$).

Recall that $m : \mathbb{A} \otimes \mathbb{A} \longrightarrow \mathbb{A}$ is a strict double functor on the Gray type monoidal product on $(Dbl, \otimes)$, while $\circledast : \mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{A}$ is a double pseudo functor on the Cartesian product of double categories. Let us set $f \circledast g = m\big((1 \otimes g)(f \otimes 1)\big)$ (recall the discussion from Subsection 2.3). Since m is strictly multiplicative in both directions, we find $m\big((1 \otimes g)(f \otimes 1)\big) = m(1 \otimes g)m(f \otimes 1)$, which yields $(1 \circledast g)(f \circledast 1) = f \circledast g$ (taking $h' = 1, k = 1$ in the computation in Subsection 2.3 we recover the same identity).

Now direct computation shows: $h \circledast (g \circledast f) = (h \circledast g) \circledast f$ in both vertical and horizontal direction of 1-cells: use the distributive law of the tensor with respect to the composition of 1-cells in the Gray type tensor product $\mathbb{A}\otimes\mathbb{A}$ (see the description of this tensor product after Definition 2.2), the fact that associativity of the latter compositions is strict and that m is strictly associative [3, (iii) of Section 4.3]). This yields an analogous result on 0- and double cells, then for the double pseudonatural transformation $a^* : \circledast(\mathrm{Id} \times \circledast) \longrightarrow \circledast(\circledast \times \mathrm{Id})$ we may set to be identity: $(a^*_0)_{C,B,A} = id^v_{(A|B)|C}$ and $(a^*_1)_{C,B,A} = id^h_{(A|B)|C}$, $(a^*_0)_{f'',f',f} = \mathrm{Id}_{(f|f')|f''} = t^{a^*}_f = t^{a^*}_{f'',f',f}$ and $(a^*_1)_{u'',u',u} = \mathrm{Id}_{(u|u')|u''} = r^{a^*}_u = r^{a^*}_{u'',u',u}$, and $L^A = L^{A'',A',A} = 1_{(A|A')|A''}$, the same for R^A, here C, B, A are 0-cells, u'', u', u 1v-cells, and f'', f', f 1h-cells of $\mathbb{A}$. Observe that it is $M = \circledast, M(y, x) = y \circledast x = (x|y)$.

Let I denote the image 0-cell of the strict double functor $u : * \longrightarrow \mathbb{A}$. Observe that: $m(A, I) = \circledast(A, I) = A \circledast I$ and similarly the other way around, for any 0-cell $A \in \mathbb{A}$. Now by [3, (iii) of Section 4.3] we deduce that left and right unity constraints l^* and r^* for $\circledast : \mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{A}$ are identities. As a matter of fact, as a monoid in a 1-category it can not have 2- and 3-cells for the constraints, so we have that a monoid $\mathbb{A}$ in $(Dbl, \otimes)$ is not only a category internal in DblPs, but even a category internal in the underlying 1-category of DblPs, which is the category from [31, Section 6].

Let us consider a monoidal 2-category made out of the monoidal category $(Dbl, \otimes)$ from [3] by adding as 2-cells vertical transformations, whose 1v-cell components have 1h-companions (recall Subsection 4.1). We denote this 2-category by $(Dbl_2, \otimes)$. Let us now consider peudomonoids in this 2-category. We repeat the analogous arguments as in the above computations. The difference appears when computing associativity on the 1-cells: now m is not strictly associative, rather there is an isomorphism $a^m_0 : \otimes(id \times \otimes) \longrightarrow \otimes(\otimes \times id)$. We have to take into account the form of (horizontal and vertical) 1-cells in $\mathbb{A} \otimes \mathbb{A} \otimes \mathbb{A}$, we find: $h \circledast (g \circledast f) = \big(h \circledast (1 \circledast 1)\big)[\big(1 \circledast (g \circledast 1)\big) \cdot (1 \circledast (1 \circledast f))]$ and $(h \circledast g) \circledast f = [\big((h \circledast 1) \circledast 1\big) \cdot \big((1 \circledast g) \circledast 1\big)]\big((1 \circledast 1) \circledast f\big)$, where the square brackets may be omitted, and the dot denotes the composition of 1-cells (in the corresponding direction). Then we define the 2-cell $(a^*_0)_{h,g,f}$ as the following 2-cell:

$$(a^*_0)_{h,g,f} = (a^m_0)_{A,B,C} \; \begin{array}{c} \xrightarrow{(f|1)|1} \\ \boxed{(a^m_0)_{f,1,1}} \\ \xrightarrow[f|(1|1)]{} \end{array} (a^m_0)_{A',B,C} \begin{array}{c} \xrightarrow{(1|g)|1} \\ \boxed{(a^m_0)_{1,g,1}} \\ \xrightarrow[1|(g|1)]{} \end{array} (a^m_0)_{A',B',C} \begin{array}{c} \xrightarrow{(1|1)|h} \\ \boxed{(a^m_0)_{1,1,h}} \\ \xrightarrow[1|(1|h)]{} \end{array} (a^m_0)_{A',B',C'} \tag{7.11}$$

so that on 0-cells we have: $(a^*_0)_{A,B,C} = (a^m_0)_{A,B,C}$. (On the right hand-side of the identity (7.11) the indexes are read from the left to the right, to accompany the notation of the 1h-cells used here.) In Subsection 4.2 we proved for what here are 2-cells of $(Dbl_2, \otimes)$ that they can be turned into 2-cells in the tricategory DblPs. Let a_1^m denote the obtained (strong) horizontal transformation, and $t^m_{h,g,f}$ and $r^m_{w,v,u}$ the obtained distinguished 2-cells making $a^m = (a_0^m, a_1^m, t^m, r^m)$ a double pseudonatural transformation. We define the 2-cell $(a_1^*)_{w,v,u}$, for 1v-cells u, v, w, in the analogous way as we did for $(a_0^*)_{h,g,f}$ above. The 2-cells t^m, r^m are constructed due to Proposition 3.15 as follows:

$$t^m_{\tilde f} = \begin{array}{ccccc} F(\tilde A) & \xrightarrow{F(\tilde f)} & F(\tilde B) & \xrightarrow{a_1^m(\tilde B)} & G(\tilde B) \\ {\scriptstyle a_0^m(\tilde A)}\downarrow & \boxed{(a_0^m)_{\tilde f}} & \downarrow{\scriptstyle a_0^m(\tilde B)} & \boxed{\varepsilon^m_{\tilde B}} & \downarrow = \\ G(\tilde A) & \xrightarrow[G(\tilde f)]{} & G(\tilde B) & \xrightarrow{=} & G(\tilde B) \end{array} \quad \text{and} \quad r^m_{\tilde u} = \begin{array}{ccc} F(\tilde A) & \xrightarrow{a_1^m(\tilde A)} & G(\tilde A) \\ {\scriptstyle F(\tilde u)}\downarrow & \boxed{(a_1^m)_{\tilde u}} & \downarrow{\scriptstyle G(\tilde u)} \\ F(\tilde A') & \xrightarrow{a_1^m(\tilde A')} & G(\tilde A') \\ {\scriptstyle a_0^m(\tilde A')}\downarrow & \boxed{\varepsilon^m_{\tilde A'}} & \downarrow = \\ G(\tilde A') & \xrightarrow{=} & G(\tilde A') \end{array} \tag{7.12}$$

where $F = \otimes(id \times \otimes)$ and $G = \otimes(\otimes \times id)$, $\tilde f$ and $\tilde u$ are 1h- and 1v-cell in $\mathbb{A} \times \mathbb{A} \times \mathbb{A}$, respectively, and ε^m_A is the 2-cell from the data that $a_0^m(A)$ is a companion of $a_1^m(A)$. We construct t^* and r^* by the same recipe: substitute $(a_0^m)_{\tilde f}$ from (7.12) by $(a_0^*)_{h,g,f}$ from (7.11), and set $\varepsilon^*_{C',B',A'} = \varepsilon^m_{C',B',A'}$ to define $t^*_{h,g,f}$, analogously for $r^*_{w,v,u}$. Then $a^* = (a_0^*, a_1^*, t^*, r^*)$ constitutes a 2-cell in DblPs.

For the unity constraints l^*, r^* the argument is simpler. Since $A \circledast I$ is an image both by $m : \mathbb{A} \otimes \mathbb{A} \longrightarrow \mathbb{A}$ and by $\circledast : \mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{A}$, as we argued above, we just set $l^* = l^m$ and $r^* = r^m$, being the right hand-sides unity constraints for m. Analogously as above, these vertical transformations can be made into double pseudonatural transformations, hence l^* and r^* are indeed 2-cells in DblPs.

For the 3-cells in Definition 6.2 we take to be identities and get that a pseudomonoid in $(Dbl_2, \otimes)$ is indeed a category internal in DblPs.

In order to have an example with non-trivial 3-cells from Definition 6.2, one can take a "weak pseudomonoid" in the tricategory $(Dbl_3, \otimes)$, which is obtained from the 2-category $(Dbl_2, \otimes)$ by adding invertible vertical modifications as 3-cells, *i.e.* invertible modifications of vertical transformations.

Let us now prove that invertible vertical modifications give rise to invertible horizontal modifications, so that together they make (invertible) 3-cells in the tricategory DblPs. Then the 3-cells constraints for m, which are $\pi^m, \mu^m, \lambda^m, \rho^m$, can be upgraded to 3-cells $\pi^*, \mu^*, \lambda^*, \rho^*$ corresponding to the desired 3-cells in Definition 6.2, and we would have this desired example.

Recall that vertical modifications are given by 2-cells $b_0(A)$ as on the left hand-side below, then let the inverses of horizontal modifications be given via the 2-cells $b_1^{-1}(A)$ on the right hand-side below:

$$\begin{array}{ccc} F(A) & \xrightarrow{=} & F(A) \\ {\scriptstyle \alpha_0(A)}\downarrow & \boxed{b_0(A)} & \downarrow{\scriptstyle \beta_0(A)} \\ G(A) & \xrightarrow{=} & G(A) \end{array} \qquad b_1^{-1}(A) = \begin{array}{ccccccc} F(A) & \xrightarrow{=} & F(A) & \xrightarrow{=} & F(A) & \xrightarrow{\beta_1(A)} & G(A) \\ =\downarrow & \boxed{\eta^\alpha(A)} & \downarrow{\scriptstyle \alpha_0(A)} & \boxed{b_0(A)} & \downarrow{\scriptstyle \beta_0(A)} & \boxed{\varepsilon^\beta_A} & \downarrow = \\ F(A) & \xrightarrow{\alpha_1(A)} & G(A) & \xrightarrow[=]{} & G(A) & \xrightarrow[=]{} & G(A) \end{array}$$

(in the obvious way $b_1(A)$ is given via $b_0^{-1}(A)$; recall that η and ε come from the data of companions). It is straightforward to prove that this defines horizontal modifications (one uses ε-η-properties and the construction of a horizontal transformation out of a vertical one from Proposition 4.1; recall that for vertical transformations α_0 the distinguished 2-cells $\delta_{\alpha_0,u}$ are identities, for 1v-cells u). Finally, the two compatibility conditions between a horizontal and a vertical modification from Definition 3.17 are directly proved. In the second condition one uses the third identity in Corollary 4.4 which is fulfilled in this context. This finishes the proof that a "weak pseudomonoid" in the tricategory $(Dbl_3, \otimes)$ is a category internal in the tricategory DblPs.

Among the examples of intercategories from [21] duoidal categories and Gray categories are such that their composition functor on the pullback (when they are seen as internal categories) induces a lax (double) functor on the Cartesian product. For this reason they do not fit our construction, of a category internal in DblPs. The rest of the examples do (so that 3-cells for the internal structure are trivial). These are *e.g.* monoidal double categories of [30], cubical bicategories of [16], Verity double bicategories from [33].

7.2. *Geometric interpretation of a category internal in* DblPs

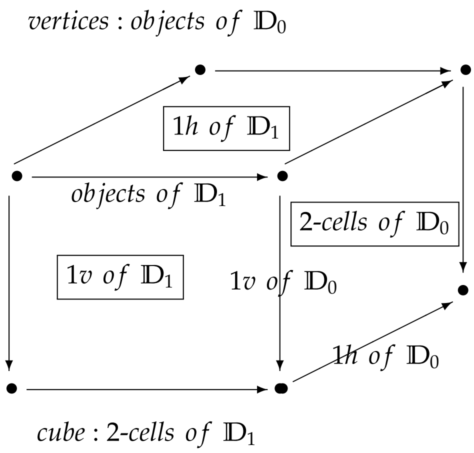
Let us denote this structure formally by

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \mathbb{D}_1 \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} \mathbb{D}_0$$

and the functor components of the double pseudo functors $U : \mathbb{D}_0 \longrightarrow \mathbb{D}_1$ and $M : \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \longrightarrow \mathbb{D}_1$ by $U_i, M_i, i = 0, 1$. Then we may obtain a similar grid of categories and functors to (∗) from [19, Section 4], the difference is that now the three columns in the grid are *strict* double categories and the rows differ in that not only the functors U_1 and M_1, but now also U_0 and M_0, are equipped with natural transformations expressing their lax multiplicativity and lax unitality.

Let us see a geometrical representation of this alternative notion to intercategories on a cube. Considering source and target functors, as well as arrows from morphisms to objects in the categories $(\mathbb{D}_0)_0, (\mathbb{D}_0)_1, (\mathbb{D}_1)_0, (\mathbb{D}_1)_1$ constituting double categories $\mathbb{D}_0$ and $\mathbb{D}_1$, one sees that the objects of $(\mathbb{D}_0)_0$ are the lowest and morphisms of $(\mathbb{D}_1)_1$ are the highest in this hierarchy, so we may present the former by vertices of a cube and the latter by the whole cube. For the rest of gadgets there is a choice, we will fix the one as in [19, Section 4], so that we have:

vertices - objects of $\mathbb{D}_0$
horizontal arrows - objects of $\mathbb{D}_1$,
vertical arrows - 1v-cells of $\mathbb{D}_0$,
transversal arrows - 1h-cells of $\mathbb{D}_0$,
horizontal cells - 1h-cells of $\mathbb{D}_1$,
lateral cells - 2-cells of $\mathbb{D}_0$,
basic cells - 1v-cells of $\mathbb{D}_1$ and
cube - 2-cells of $\mathbb{D}_1$.



From here we see that vertical and transversal arrows compose in their respective directions, horizontal cells compose in the transversal direction, basic cells compose in the vertical direction, and lateral cells both in vertical and transversal directions. All of them compose strictly associatively and unitary. The pullback $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1$ can be represented by horizontal connecting of cubes, and accordingly the functor $M : \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \longrightarrow \mathbb{D}_1$ corresponds to the horizontal composition of cubes.

The globular 2-cells (7.9) of $\mathbb{D}_1$ are thus cubes whose only non-identity cells are the basic ones, and we will consider that they map from the back towards the front. They compose in the transversal direction. On the other hand, the globular 2-cells (7.10) of $\mathbb{D}_1$ are cubes whose only non-identity cells are the horizontal ones, they map from top to bottom, and compose in the vertical direction.

The double pseudofunctor U applied to a 2-cell a of $\mathbb{D}_0$ gives a cube Id^h_a which is horizontal identity cube on the lateral cell a, and the rest of the cells are identities on the corresponding 1h- and 1v-cells at the borders of a.

A 2-cell in $\mathbb{D}_1$ is a cube whose lateral cells are identities, top and bottom correspond to its source and target 1h-cells, while front and back basic cells correspond to its source and target 1v-cells.

For all the laws described in Section 7 observe that horizontal composition of 2-cells in $\mathbb{D}_1$ corresponds to the transversal composition of cubes, and that vertical composition of 2-cells in $\mathbb{D}_1$ corresponds to the vertical composition of cubes.

8. Enriched categories as internal categories in iconic tricategories

In the first subsection of this section we introduce categories enriched over iconic tricategories. In the second subsection we prove the result from the above title and in the third one we discuss examples in lower dimensions that can be seen as its consequences. The next section we dedicate to illustrate this result on the tricategory of tensor categories.

8.1. Categories enriched over iconic tricategories

For enrichment we need some kind of a monoidal product in the ambient tricategory. We will consider tricategorical products from Subsection 5.2 (with the terminal object). By a terminal object in a tricategory we mean a 0-cell I so that for any 0-cell T there is a unique 1-cell $t : T \longrightarrow I$, and all the 2-cells $t \Rightarrow t$ are the identity one. In the following definition for the terminal object in the ambient tricategory V we will write just $*$.

Definition 8.1 *Let V be an iconic tricategory with 3-products. We say that $\mathcal{T}$ is a category enriched over V if it consists of:*

1. *a set of objects $Ob\mathcal{T}$ of $\mathcal{T}$;*
2. *for all $A, B \in Ob\mathcal{T}$ a 0-cell $\mathcal{T}(A, B)$ in V;*
3. *for all $A, B, C \in Ob\mathcal{T}$ a 1-cell $\circ : \mathcal{T}(B, C) \times \mathcal{T}(A, B) \longrightarrow \mathcal{T}(A, C)$ in V called* composition*;*
4. *for all $A \in Ob\mathcal{T}$ a 1-cell $I_A : * \longrightarrow \mathcal{T}(A, A)$ in V called* unit*;*
5. *equivalence 2-cells in V: $a^\dagger : - \circ (- \circ -) \longrightarrow (- \circ -) \circ -$, and for all $A, B \in Ob\mathcal{T}$: $l^\dagger : I_B \cdot 1_{\mathcal{T}(A,B)} \longrightarrow 1_{\mathcal{T}(A,B)}$ and $r^\dagger : 1_{\mathcal{T}(A,B)} \cdot I_A \longrightarrow 1_{\mathcal{T}(A,B)}$;*
6. *3-cells $\pi^\dagger, \mu^\dagger, l^\dagger, r^\dagger$ and $\varepsilon^\dagger$ analogous to those in item 4. of Definition 6.2 and which satisfy the analogous Axioms as the latter ones.*

The formal differences in the cells and Axioms in Definition 6.2 and the above one are the following. In the vertices of the diagrams the iterated 3-pullbacks $B_1^{(n)_0}$ are replaced by $\mathcal{T}(\bullet, \bullet)^{\times n}$ for natural numbers n, 1-cells c and u are replaced by $\circ$ and $I_\bullet$, respectively, and supraindeces * are replaced by supraindeces †.

Lemma 8.2 *There exist equivalence 1-cells in V between the following 3-coproducts:*

$$\amalg_{A\in Ob\mathcal{T}} \amalg_{B\in Ob\mathcal{T}} \mathcal{T}(A, B) \simeq \amalg_{B\in Ob\mathcal{T}} \amalg_{A\in Ob\mathcal{T}} \mathcal{T}(A, B) \simeq \amalg_{A,B\in Ob\mathcal{T}} \mathcal{T}(A, B).$$

Example 8.3 Let Bicat_3 denote the iconic tricategory of bicategories, pseudofunctors, pseudonatural transformations and modifications. A tricategory from [17, Definition 2.2] is a category enriched in Bicat_3 such that $\varepsilon^\dagger$ is an identity. Of course, the latter can not be used as a definition of the notion of a tricategory. Rather, one could say that a tricategory is a category *weakly enriched over the category* Bicat_1 of bicategories and pseudofunctors. More general, instead of saying "a category enriched over an iconic tricategory V" one could say "a category *weakly enriched over the underlying category* of V".

8.2. Enriched categories as internal categories in iconic tricategories

Let V be a tricategory with a terminal object $\mathcal{I}$, finite tricategorical products and tricategorical pullbacks. Then observe that a 3-product $X \times Y$ is in particular a 3-pullback $\big(X \times_{\mathcal{I}} Y; t_X, t_Y\big)$, where t_X, t_Y are the unique morphisms into $\mathcal{I}$. Moreover, a 3-product $X \times Y \times Z$ is a 3-pullback $\big((X \times Y) \times_Y (Y \times Z); p_2, p_1\big)$. In particular, for $Y = Y_1 \times \cdots \times Y_k$ for any natural k, a 3-product $X \times Y_1 \times \cdots \times Y_k \times Z$ is a 3-pullback $\big((X \times Y_1 \times \cdots \times Y_k) \times_{(Y_1 \times \cdots \times Y_k)} (Y_1 \times \cdots \times Y_k \times Z); p_2, p_1\big)$.

In this section we deal with "hands on enrichment" and for this we found it easier to use lexicographical order when writing 3-products and 3-pullbacks (contrary to Remark 6.1).

Proposition 8.4 *Let V be an iconic tricategory with finite 3-products, a terminal object $\mathcal{I}$ and small tricategorical coproducts. Assume that $\mathcal{T}$ is a category enriched over V, set $T_0 = \amalg_{A\in Ob\mathcal{T}}\mathcal{I}_A$ - the coproduct of copies of the terminal object indexed by the objects of $\mathcal{T}$, and $T_1 = \amalg_{A,B\in Ob\mathcal{T}}\mathcal{T}(A,B)$, and suppose that V has iterated 3-pullbacks $T_1^{(n)_0}$. If additionally the following conditions are fulfilled:*

1. *for every natural $n \geq 2$ the trifunctors $\amalg_{B_1,\dots,B_{n-1}}$ preserve the following 3-pullbacks: $\big(\amalg_A\mathcal{T}(A,B_1)\big)\times\mathcal{T}(B_1,B_2)\times\cdots\times\mathcal{T}(B_{n-2},B_{n-1})\times\big(\amalg_C\mathcal{T}(B_{n-1},C)\big)$;*
2. *the trifunctors $X\times -$ and $-\times X$, for $X \in Ob\mathcal{T}$, preserve the coproducts $\amalg_A\mathcal{T}(A,B)$ and $\amalg_B\mathcal{T}(A,B)$;*

then the resulting 3-pullbacks in 1. are $\amalg_{A,B_1\dots,B_{n-1},C\in Ob\mathcal{T}}\mathcal{T}(A,B_1)\times\cdots\times\mathcal{T}(B_{n-1},C)$ and for every natural $n \geq 2$ there are equivalence 1-cells in V:

$$a^n_{A_1,\dots,A_{n+1}} : \amalg_{A_1,\dots,A_{n+1}\in Ob\mathcal{T}}\mathcal{T}(A_1,A_2)\times\cdots\times\mathcal{T}(A_n,A_{n+1}) \xrightarrow{\simeq} \underbrace{T_1\times_{T_0}\cdots\times_{T_0} T_1}_{n}$$

(with all possible distributions of parentheses).

Proof. We will do the proof for the cases when n equals 2 and 3, the higher cases are proven in analogous way. For $n = 2$ we start by a 3-pullback $\big(\amalg_A\mathcal{T}(A,B)\big)\times\big(\amalg_C\mathcal{T}(B,C)\big)$ (over $\mathcal{I}$), and act by the trifunctor $\amalg_B$ on it. By the assumption 1. we get the following 3-pullback, where in the first coordinate we apply the preservation assumption 2., and in the second and third the corresponding equivalences of the coproducts (in the rest of coordinates by abuse of notation we do not change the notation of the 1-cells for simplicity reasons):

$$(\amalg_{A,B,C}\mathcal{T}(A,B)\times\mathcal{T}(B,C),\ \amalg_{A,B}\mathcal{T}(A,B),\ \amalg_{B,C}\mathcal{T}(B,C),\ \amalg_B\mathcal{I}_B,\ \amalg_B p_1,\ \amalg_B p_2;\ \amalg_B t,\ \amalg_B t).$$

On the other hand, by construction this 3-pullback is $(T_1\times_{T_0}T_1, s, t)$. Thus there is an equivalence

$$a^2_{A,B,C} : \amalg_{A,B,C}\mathcal{T}(A,B)\times\mathcal{T}(B,C) \xrightarrow{\simeq} T_1\times_{T_0}T_1.$$

For $n = 3$ we start with a 3-pullback

$$\Big(\Big[\big(\amalg_A\mathcal{T}(A,B)\big)\times\mathcal{T}(B,C)\Big]\times_{\mathcal{T}(B,C)}\Big[\mathcal{T}(B,C)\times\big(\amalg_D\mathcal{T}(C,D)\big)\Big], p_2, p_1\Big), \tag{8.13}$$

which can be rewritten as the 3-product:

$$\Big(\big(\amalg_A\mathcal{T}(A,B)\big)\times\mathcal{T}(B,C)\times\big(\amalg_D\mathcal{T}(C,D)\big),\ \big(\amalg_A\mathcal{T}(A,B)\big)\times\mathcal{T}(B,C),\ \mathcal{T}(B,C)\times\big(\amalg_D\mathcal{T}(C,D)\big),\ \mathcal{T}(B,C), p^3_{12}, p^3_{23}; p_2, p_1\Big).$$

We act on it by the trifunctor $\amalg_B\amalg_C \simeq \amalg_{B,C} \simeq \amalg_C\amalg_B$ and get by the part 1. a 3-pullback, which by the assumption 2. has the form:

$$\Big(\amalg_{A,B,C,D}\big(\mathcal{T}(A,B)\times\mathcal{T}(B,C)\times\mathcal{T}(C,D)\big),\ \amalg_{A,B,C}\big(\mathcal{T}(A,B)\times\mathcal{T}(B,C)\big),\ \amalg_{B,C,D}\big(\mathcal{T}(B,C)\times\mathcal{T}(C,D)\big),\ \amalg_{B,C}\mathcal{T}(B,C),\ \amalg_{B,C}p_{12},\ \amalg_{B,C}p_{23},\ \amalg_{B,C}p_2,\ \amalg_{B,C}p_1\Big)$$

and by construction (see (8.13)) it is indeed the 3-pullback $T_1\times_{T_0}T_1\times_{T_0}T_1$ (we differentiate the two distributions of the parentheses). This yields equivalences

$$a^{3,L}_{A,B,C,D} : \amalg_{A,B,C,D}\big(\mathcal{T}(A,B)\times\mathcal{T}(B,C)\big)\times\mathcal{T}(C,D) \xrightarrow{\simeq} (T_1\times_{T_0}T_1)\times_{T_0}T_1$$

and

$$a^{3,R}_{A,B,C,D} : \amalg_{A,B,C,D}\mathcal{T}(A,B)\times\Big(\mathcal{T}(B,C)\times\mathcal{T}(C,D)\Big) \xrightarrow{\simeq} T_1\times_{T_0}(T_1\times_{T_0}T_1).$$

For general n one obtains that by construction $\amalg_{A,B_1...,B_{n-1},C\in Ob\mathcal{T}}\mathcal{T}(A,B_1)\times\cdots\times\mathcal{T}(B_{n-1},C)$ is $\Big(\underbrace{T_1\times_{T_0}\cdots\times_{T_0}T_1}_{n-1}\Big)\times_{\underbrace{T_1\times_{T_0}\cdots\times_{T_0}T_1}_{n-2}}\Big(\underbrace{T_1\times_{T_0}\cdots\times_{T_0}T_1}_{n-1}\Big)$, which is equivalent to $\underbrace{T_1\times_{T_0}\cdots\times_{T_0}T_1}_{n}$. □

Let us fix the notation for the associated equivalence 2-cells for the above equivalences:

$$\alpha^{3,L} : (a^{3,L})^{-1}\circ a^{3,L} \stackrel{\simeq}{\Rightarrow} \mathrm{Id},\ \ \alpha^{3,R} : (a^{3,R})^{-1}\circ a^{3,R} \stackrel{\simeq}{\Rightarrow} \mathrm{Id},\ \ \alpha^{2} : (a^{2})^{-1}\circ a^{2} \stackrel{\simeq}{\Rightarrow} \mathrm{Id} \tag{8.14}$$

(by $(\bullet)^{-1}$ here we denote a quasi-inverse).

Proposition 8.5 *In the conditions of the previous Proposition, a category $\mathcal{T}$ enriched over V is a particular case of a category internal in V.*

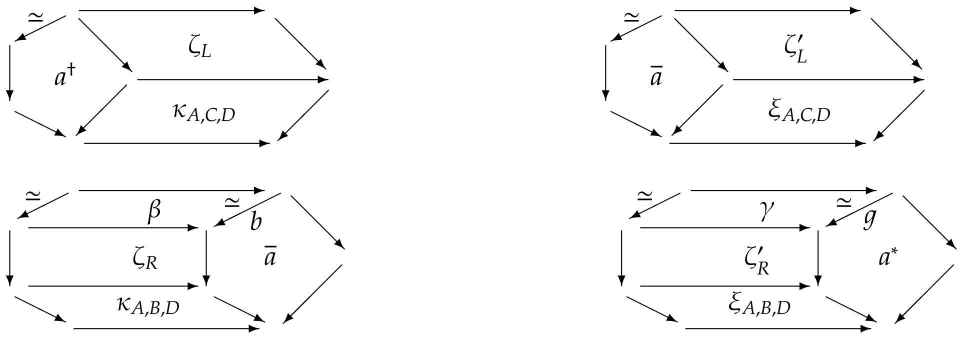
Proof. By the 3-coproduct property, the composition from the enrichment $\circ : \mathcal{T}(A,B)\times\mathcal{T}(B,C)\longrightarrow\mathcal{T}(A,C)$ induces a 1-cell $\overline{\circ}$ up to an equivalence 2-cell $\kappa_{A,B,C}$, and similarly the 1-cell $\circ\times id$ induces a 1-cell $\overline{\circ}_{12}$ up to an equivalence 2-cell ζ_L, as indicated in the two left squares in the diagram:

$$\begin{array}{ccccc}
\Big(\mathcal{T}(A,B)\mathrm{x}\mathcal{T}(B,C)\Big)\mathrm{x}\mathcal{T}(C,D) & \xrightarrow{\iota^{4,L}_{A,B,C,D}} & \amalg_{A,B,C,D}\Big(\mathcal{T}(A,B)\mathrm{x}\mathcal{T}(B,C)\Big)\mathrm{x}\mathcal{T}(C,D) & \xrightarrow{a^{3,L}_{A,B,C,D}} & (T_1\times_{T_0}T_1)\times_{T_0}T_1 \\
\circ\times id\downarrow & \swarrow\!\!\!\swarrow\ \zeta_L & \downarrow\overline{\circ}_{12} & \swarrow\!\!\!\swarrow\ \zeta'_L & \downarrow c\times_{T_0} id \\
\mathcal{T}(A,C)\times\mathcal{T}(C,D) & \xrightarrow{\iota^{3}_{A,C,D}} & \amalg_{A,C,D\in Ob\mathcal{T}}\,\mathcal{T}(A,C)\times\mathcal{T}(C,D) & \xrightarrow{a^{2}_{A,C,D}} & T_1\times_{T_0}T_1 \\
\circ\downarrow & \swarrow\!\!\!\swarrow\ \kappa_{A,C,D} & \downarrow\overline{\circ} & \swarrow\!\!\!\swarrow\ \xi & \downarrow c \\
\mathcal{T}(A,D) & \xrightarrow{\iota^{2}_{A,D}} & \amalg_{A,D\in Ob\mathcal{T}}\,\mathcal{T}(A,D) & \xrightarrow{=} & T_1.
\end{array}$$

Using the equivalences $a^{3,L}, a^2$ and their quasi-inverses, the 1-cells $\overline{\circ}$ and $\overline{\circ}_{12}$ induce 1-cells $c : T_1\times_{T_0}T_1 \longrightarrow T_1$ and $c\times_{T_0} id$ up to equivalence 2-cells ξ and ζ'_L in V, respectively, in the two right squares above. Here $a^2, a^{3,L}$ are biequivalences from the above Proposition, and ι's are the corresponding 3-coproduct structure embeddings. Observe that ξ and ζ'_L are given through the 2-cells (8.14) (composed with suitable identity 2-cells).

Now one may draw an analogous diagram to the above one in a parallel plane above it, where now $id\times\circ$ induces a 1-cell $\overline{\circ}_{23}$ up to an equivalence 2-cell ζ_R, and $\overline{\circ}_{23}$ induces a 1-cell $id\times_{T_0} c$ up to an equivalence 2-cell ζ'_R. From the enrichment we have an equivalence 2-cell $a^\dagger$ up to which the pentagon for the associativity of $\circ$ commutes. One can draw this pentagon transversally to the plane of the paper so that it connects the two diagrams, in the two planes, at their extreme left edges, adding a fifth edge for the associativity in the top 0-cell. The latter associativity 1-cell induces an associativity 1-cell b between 3-coproduct 0-cells by the property of a 3-coproduct and via some equivalence 2-cell β. Similarly, the latter associativity 1-cell b induces an associativity 1-cell g between 3-pullback 0-cells, via the equivalence 1-cells $a^{3,L}, a^{3,R}$, and some equivalence 2-cell γ. (We draw below the 1-cells b, g and 2-cells β,γ.) Now the 2-cell $a^\dagger$ induces an equivalence 2-cell $\overline{a}$, up to which $\overline{\circ}$ is associative, so to say. This 2-cell $\overline{a}$ connects the two diagrams in two planes transversally at the level of the 3-coproduct vertices. Finally, $\overline{a}$ induces an equivalence 2-cell a^* up to which c is associative, connecting the two diagrams in two planes transversally at their extreme right edges.

To understand how $\overline{a}$ and a^* are defined, observe that the three 2-cells $a^\dagger$ and yet-to-be-defined $\overline{a}$ and a^* divide this three-dimensional diagram into two horizontal prisms with pentagonal bases in the transversal direction. These two prisms have the following faces (we draw on the left hand-side first upper, then lower view of the left prism, and on the right hand-side first the upper, then the lower view of the right prism):



Now we define a 2-cell:

$$\delta : \overline{\circ} \otimes \overline{\circ}_{12} \otimes \iota^{4,L} \Rrightarrow \overline{\circ} \otimes \overline{\circ}_{23} \otimes b \otimes \iota^{4,L}$$

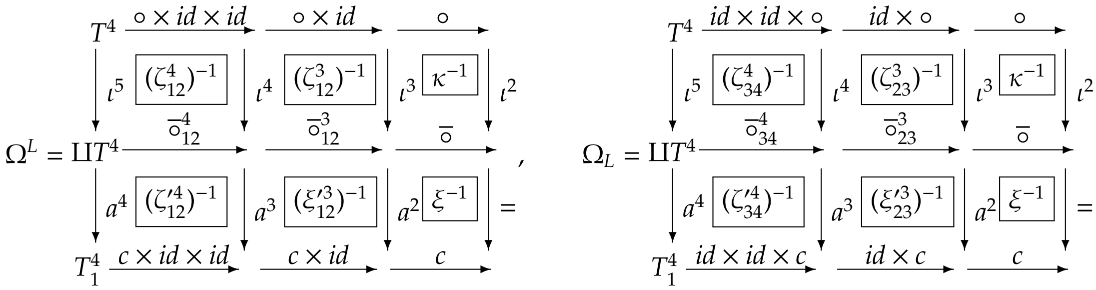
as the composition of six of the seven faces of the left prism, seen as (whiskerings of) equivalence 2-cells or their quasi-inverses, namely: $a^\dagger, \zeta_L, \kappa_{A,C,D}, \kappa^{-1}_{A,B,D}, \zeta_R$ and β. Then by the second property of the 3-coproduct there exists a 2-cell $\overline{a}$ at the desired place, namely such that $\overline{a} \otimes \mathrm{Id}_{\iota^{4,L}}$ is isomorphic to the 2-cell δ via an invertible 3-cell $\Gamma_1 : \delta \Rrightarrow \overline{a} \otimes \mathrm{Id}_{\iota^{4,L}}$. The 2-cell a^* is obtained similarly via an invertible 3-cell Γ_2, with the difference that instead of the 3-coproduct property one uses the fact that all horizontal 1-cells in the right prism above are equivalences, thus a^* is given as a suitable composition of the 2-cells $\alpha^2, \overline{a}, \xi^{-1}$ and $(\zeta'_R)^{-1}$ (recall that by $(\bullet)^{-1}$ we denote the corresponding quasi-inverse 2-cells).

This means that the invertible 3-cell Γ_1 connects the upper and lower composition of faces in the left prism above, and that the invertible 3-cell Γ_2 connects the upper and lower composition of faces in the right prism. Informally, we think of Γ_1 as "a prism from $a^\dagger$ to $\overline{a}$", and Γ_2 as "a prism from $\overline{a}$ to a^*".

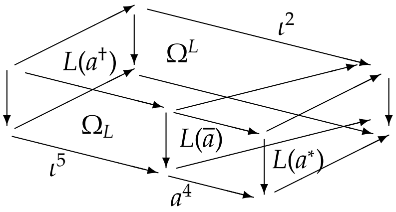
From the enrichment we have an invertible 3-cell

$$\pi^\dagger : \frac{\dfrac{\mathrm{Id}_\circ \otimes(\mathrm{Id}_{id_T} \times a^\dagger)}{a^\dagger \otimes \mathrm{Id}_{1\times\circ\times 1}}}{\mathrm{Id}_\circ \otimes(a^\dagger \times \mathrm{Id}_{id_T})} \Rrightarrow \frac{\dfrac{a^\dagger \otimes \mathrm{Id}_{1\times 1\times\circ}}{\mathrm{Id}_\circ \otimes \mathrm{Nat}_{(\circ\times 1)(1\times 1\times\circ)}}}{a^\dagger \otimes \mathrm{Id}_{\circ\times 1\times 1}}$$

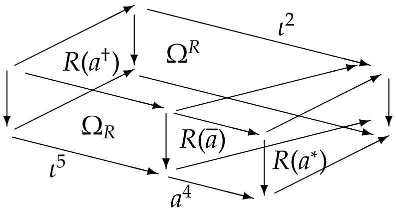
satisfying a septagonal identity (here T stands for 0-cells of the form $\mathcal{T}(A,B), A, B \in V$). Let us denote the domain and the codomain 2-cells of $\pi^\dagger$ by $L(a^\dagger)$ and $R(a^\dagger)$, respectively. We next show that $\pi^\dagger$ induces a 3-cell $\pi^* : L(a^*) \Rrightarrow R(a^*)$, where now L, R have the obvious adjusted meaning. Consider the following two 2-cells in V:



The upper and lower 1-cell of Ω^L are the domain of $L(a^\dagger)$ and $L(a^*)$, respectively, and the upper and lower 1-cell of Ω_L are the codomain of $L(a^\dagger)$ and $L(a^*)$, respectively. Analogously, one defines Ω^R and Ω_R for the corresponding 2-cells for $R(a^\dagger)$ and $R(a^*)$. Then we can consider the following two prisms determined by invertible 3-cells that we will denote as $L(\Gamma_2 \cdot \Gamma_1)$ and $R(\Gamma_2 \cdot \Gamma_1)$ and refer to informally as a prism "from $L(a^\dagger)$ to $L(a^*)$" and another prism "from $R(a^\dagger)$ to $R(a^*)$" (recall that $\cdot$ denotes the transversal composition of 3-cells):



$L(\Gamma_2 \cdot \Gamma_1)$



$R(\Gamma_2 \cdot \Gamma_1)$

All the vertical 1-cells in the above picture are identities. The top 2-cell of $L(\Gamma_2 \cdot \Gamma_1)$ is Ω^L and the bottom one Ω_L, while the top 2-cell of $R(\Gamma_2 \cdot \Gamma_1)$ is Ω^R and the bottom one Ω_R. The 2-cells $L(a^\dagger), L(\overline{a}), L(a^*)$ and $R(a^\dagger), R(\overline{a}), R(a^*)$ are meant as vertically transversal faces (of which $L(\overline{a})$ devides $L(\Gamma_2 \cdot \Gamma_1)$ into $L(\Gamma_1)$ and $L(\Gamma_2)$, and similarly $R(\overline{a})$ devides $R(\Gamma_2 \cdot \Gamma_1)$ into $R(\Gamma_1)$ and $R(\Gamma_2)$). The vertical 2-cells with edges 1-cells ι^2, ι^5, a^4 are meant as the identity 2-cells on these 1-cells. Then $\pi^\dagger$ induces a 3-cell $\overline{\pi}$, and the latter induces the wanted 3-cell π^*. Namely, consider the following transversal composition of 3-cells that we call χ:

$$\chi : \Big([\mathrm{Id}_{\iota^5} | L(\overline{a})] \stackrel{L(\Gamma_1)^{-1}}{\Rrightarrow} [L(a^\dagger) | \mathrm{Id}_{\iota^2}] \stackrel{[\pi^\dagger | \mathrm{Id}]}{\Rrightarrow} [R(a^\dagger) | \mathrm{Id}_{\iota^2}] \stackrel{R(\Gamma_1)}{\Rrightarrow} [\mathrm{Id}_{\iota^5} | R(\overline{a})] \Big).$$

Then χ, which is a certain conjugation of $[\pi^\dagger | \mathrm{Id}_{\mathrm{Id}_{\iota^2}}]$, determines a unique 3-cell $\overline{\pi}$, by the third 3-coproduct property, so that $\chi = [\mathrm{Id}_{\mathrm{Id}_{\iota^5}} | \overline{\pi}]$.

Finally, we define π^* as the following transversal composition of 3-cells:

$$\pi^* := \Big(L(a^*) \stackrel{L(\Gamma_2)^{-1}}{\Rrightarrow} [\mathrm{Id}_{(a^4)^{-1}} | L(\overline{a})] \stackrel{[\mathrm{Id} | \overline{\pi}]}{\Rrightarrow} [\mathrm{Id}_{(a^4)^{-1}} | R(\overline{a})] \stackrel{R(\Gamma_2)}{\Rrightarrow} R(a^*) \Big).$$

(Recall that $(a^4)^{-1}$ denotes a quasi-inverse.) Since $\pi^\dagger$ satisfies a septagonal identity, so does π^* in its desired form.

This proves the existence of a composition $c : T_1 \times_{T_0} T_1 \longrightarrow T_1$ associative up to an equivalence for a structure of a category internal in V.

In an analogous way the unit 1-cell $I_A : \mathcal{I} \longrightarrow \mathcal{T}(A, A)$ from enrichment induces a unit 1-cell $u : T_0 \longrightarrow T_1$, and the 2-cells $l^\dagger, r^\dagger$ for the unity law in the enrichment induce 2-cells l^*, r^* for the unity law in an internal category in V. The induction of the associated 3-cells λ^* and ρ^*, but also of μ^* and ε^*, from the 3-cells from the enrichment $\lambda^\dagger, \rho^\dagger, \mu^\dagger$ and $\varepsilon^\dagger$, respectively, goes the analogous way as we proved it above for $\pi^\dagger$. □

Observe that by the construction of T_0 in the above proof, if V is an iconic tricategory whose 0-cells are bicategories, then 0-cells of T_0 are the same as 0-cells of $\mathcal{T}$. Moreover, 1-cells of T_0 are the identities on its 0-cells and 2-cells are identities on the latter, *i.e.* the object of objects T_0 is discrete.

8.3. *Examples of enrichment and internalization in lower dimensions*

In the examples where V is some kind of "category of categories", for the existence of the iterated n-pullbacks, $n = 1, 2, 3$, it is sufficient to require that source and target 1-cells s, t be *strict* functors. We illustrate this by showing it for the 2-category $PsDbl_2$ of pseudodouble categories, pseudodouble functors and vertical transformations, [16, 30] (then it also applies to the 2-category Cat_2 of categories, functors and natural transformations). Namely, as in [19, Proposition 2.1], we have:

Proposition 8.6 *The set-theoretical pullback of strict double functors $F : \mathcal{A} \longrightarrow C$ and $G : \mathcal{B} \longrightarrow C$ determines on objects, 1-cells and 2-cells a pseudo double category $\mathcal{A} \times_C \mathcal{B}$ which is the 2-pullback of F and G in $PsDbl_2$. The projections onto $\mathcal{A}$ and $\mathcal{B}$ are strict double functors.*

The construction in the proposition from the last subsection can be carried out in 2-categories: consider 3-cells to be identities, and consider equivalence 2-cells to be bijective. Then we obtain:

Corollary 8.7 *Let V be a Cartesian monoidal 2-category with a terminal object $\mathcal{I}$, finite 2-products and small 2-coproducts preserved by the pseudofunctors $- \times X, X \times - : V \longrightarrow V$ for every $X \in V$. Let $\mathcal{T}$ be a category enriched over V, set $T_0 = \amalg_{A \in \mathcal{T}} \mathcal{I}_A$ and $T_1 = \amalg_{A,B \in \mathcal{T}} \mathcal{T}(A, B)$, suppose that the iterated 2-pullbacks $T_1^{(n)_0}$ exist and that the pseudofunctors $\amalg_{B_1,\dots,B_{n-1}}$ map the 2-products: $\big(\amalg_A \mathcal{T}(A, B_1)\big) \times \mathcal{T}(B_1, B_2) \times \cdots \times \mathcal{T}(B_{n-2}, B_{n-1}) \times \big(\amalg_C \mathcal{T}(B_{n-1}, C)\big)$ to iterated 2-pullbacks: $\underbrace{T_1 \times_{T_0} \cdots \times_{T_0} T_1}_{n}$. Then $\mathcal{T}$ is a particular case of a category internal to V.*

Example 8.8 A category enriched over the 2-category $V = Cat_2$ is a bicategory, and it is well-known that a bicategory embeds into a pseudodouble category, which is a category internal in Cat_2.

Example 8.9 A category enriched over $PsDbl_2$ is a locally cubical bicategory from [16, Definition 11]. A category internal to $PsDbl_2$ is a version of an intercategory. Corollary 8.7 applied to $V = PsDbl_2$ uses the argumentation similar to [21, Section 3.5], where a locally cubical bicategory is shown to be a particular case of an intercategory.

Truncating the result of Corollary 8.7 to 1-categories one recovers a version of the results in [11, Appendix] and [7]. As a particular case of this we have the following. A Gray-category is a category enriched over the monoidal category *Gray* with the Gray monoidal product. This product is defined as an image of a cubical functor defined on the Cartesian product of two 2-categories. In [21, Section 5.2] it is shown how a Gray-category can be seen as an intercategory, a category internal in *LxDbl*. As an intermediate step one can see how a Gray-category is made a category internal in *Gray*.

In the above three examples we can embed the 1-category *Gray* and the 2-categories Cat_2 and $PsDbl_2$ to our tricategory DblPs and we get three examples of categories internal in DblPs.

9. Tricategory of tensor categories: enrichment and internalization

Apart from our search for an alternative framework to intercategories and what we developed in Section 7, we had another motivating example to introduce internal categories in a tricategory in Section 6. Namely, analogously to the double category of rings, in one dimension higher we have a (1×2)-category of tensor categories (for this name see *e.g.* [30]). It is an internal category in a suitable tricategory V, so that the category of objects consists of tensor categories, tensor functors and tensor natural transformations (thus the vertical direction is strictly associative and unital), while the category of morphisms consists of bimodule categories, bimodule functors, and bimodule natural transformations. Since the associativity for the relative tensor product of bimodule categories is an equivalence (and not an isomorphism!), the horizontal direction of this (1×2)-category is tricategorical in nature. Clearly, the tricategory Tens of tensor categories, with 0-cells tensor categories, 1-cells bimodule categories, 2-cells bimodule functors, and 3-cells bimodule natural transformations lies in this (1×2)-category.

In the first subsection below we will show that the tricategory Tens is enriched over the tricategory $2Cat_{wk}$, of 2-categories, pseudofunctors, weak natural transformations and modifications. Recall from Subsection 6.1 that 2- Cat_{wk} is iconic. In the second subsection we will present an internal category structure for Tens in $2Cat_{wk}$ richer than the one coming from Proposition 8.5, that is, where the object of objects T_0 is not discrete.

9.1. Tens *as an enriched category*

Let us recall and discuss the structure of a tricategory Tens of tensor categories.

1. For every two tensor categories $\mathcal{C}$ an $\mathcal{D}$ we have a 2-category $\text{Bimod}(\mathcal{C},\mathcal{D})$;

2. given two $\mathcal{C}$-$\mathcal{D}$-bimodule categories $\mathcal{M},\mathcal{N}$ there is a category $\text{Bimod}(\mathcal{C},\mathcal{D})\big(\mathcal{M},\mathcal{N}\big) = {}_{\mathcal{C}}\text{Fun}_{\mathcal{D}}(\mathcal{M},\mathcal{N})$ whose composition of morphisms is given by the vertical composition of $\mathcal{C}$-$\mathcal{D}$-bimodule natural transformations, which we denote by $\cdot$ (this is the transversal composition of 3-cells in Tens);

3. given a third $\mathcal{C}$-$\mathcal{D}$-bimodule category $\mathcal{L}$ there is a functor $\circ : {}_{\mathcal{C}}\text{Fun}_{\mathcal{D}}(\mathcal{N},\mathcal{L}) \times {}_{\mathcal{C}}\text{Fun}_{\mathcal{D}}(\mathcal{M},\mathcal{N}) \longrightarrow {}_{\mathcal{C}}\text{Fun}_{\mathcal{D}}(\mathcal{M},\mathcal{L})$ given by the composition of $\mathcal{C}$-$\mathcal{D}$-bimodule functors and $\mathcal{C}$-$\mathcal{D}$-bimodule natural transformations; thus the horizontal composition of 2-cells in $\text{Bimod}(\mathcal{C},\mathcal{D})$ is given by the usual horizontal composition of natural transformations (this is the vertical composition of 2- and 3-cells in Tens); by the functor properties of $\circ$, on this 2-category level we have the strict interchange law: $(\zeta'\cdot\omega')\circ(\zeta\cdot\omega) = (\zeta'\circ\zeta)\cdot(\omega'\circ\omega)$ for accordingly composable natural transformations $\omega,\omega',\zeta,\zeta'$;

4. given a third tensor category $\mathcal{E}$ there is a pseudofunctor $\boxtimes_{\mathcal{D}} : \text{Bimod}(\mathcal{D},\mathcal{E}) \times \text{Bimod}(\mathcal{C},\mathcal{D}) \longrightarrow \text{Bimod}(\mathcal{C},\mathcal{E})$, so the composition of 1-cells and the horizontal composition of 2- and 3-cells in Tens is given by the relative tensor product of bimodule categories. Let $(\mathcal{M},\mathcal{N}),(\mathcal{M}',\mathcal{N}') \in \text{Bimod}(\mathcal{D},\mathcal{E}) \times \text{Bimod}(\mathcal{C},\mathcal{D})$, then set for the hom-set

$$\text{Bimod}(\mathcal{D},\mathcal{E}) \times \text{Bimod}(\mathcal{C},\mathcal{D})\Big((\mathcal{M},\mathcal{N}),(\mathcal{M}',\mathcal{N}')\Big) = {}_{\mathcal{C}}\text{Fun}_{\mathcal{E}}^{\mathcal{D}\text{-}bal}\Big((\mathcal{M},\mathcal{N}),(\mathcal{M}',\mathcal{N}')\Big),$$

which is the category of $\mathcal{D}$-balanced $\mathcal{C}$-$\mathcal{E}$-bimodule functors and natural transformations. Then there is a functor $\widetilde{\ }: {}_{\mathcal{C}}\text{Fun}_{\mathcal{E}}^{\mathcal{D}\text{-}bal}\Big((\mathcal{M},\mathcal{N}),(\mathcal{M}',\mathcal{N}')\Big) \longrightarrow {}_{\mathcal{C}}\text{Fun}_{\mathcal{E}}(\mathcal{N}\boxtimes_{\mathcal{D}}\mathcal{M},\mathcal{N}'\boxtimes_{\mathcal{D}}\mathcal{M}')$ and there are natural isomorphisms $\widetilde{G}\circ\widetilde{F} \cong \widetilde{G\circ F}$ and $\text{Id}_{\mathcal{N}\boxtimes_{\mathcal{D}}\mathcal{M}} \cong \widetilde{\text{Id}_{(\mathcal{M},\mathcal{N})}}$ for all $F \in {}_{\mathcal{C}}\text{Fun}_{\mathcal{E}}^{\mathcal{D}\text{-}bal}\Big((\mathcal{M},\mathcal{N}),(\mathcal{M}',\mathcal{N}')\Big)$ and $G \in {}_{\mathcal{C}}\text{Fun}_{\mathcal{E}}^{\mathcal{D}\text{-}bal}\Big((\mathcal{M}',\mathcal{N}'),(\mathcal{M}'',\mathcal{N}'')\Big)$ (this corresponds to the bimodule case of [29, Proposition 3.3.2]). In particular, the latter natural isomorphisms imply that we have the interchange law at this level holding up to an isomorphism: $(\mathcal{F}'\boxtimes_{\mathcal{D}}\mathcal{G}')\circ(\mathcal{F}\boxtimes_{\mathcal{D}}\mathcal{G}) \cong (\mathcal{F}'\circ\mathcal{F})\boxtimes_{\mathcal{D}}(\mathcal{G}'\circ\mathcal{G})$ for according bimodule functors, and also: $\text{Id}_{\mathcal{N}\boxtimes_{\mathcal{D}}\mathcal{M}} \cong \text{Id}_{\mathcal{N}}\boxtimes_{\mathcal{D}}\text{Id}_{\mathcal{M}}$. The above functor property implies in particular: $(\zeta'\circ\omega')\boxtimes_{\mathcal{D}}(\zeta\circ\omega) = (\zeta'\boxtimes_{\mathcal{D}}\zeta)\circ(\omega'\boxtimes_{\mathcal{D}}\omega)$ for according bimodule natural transformations, and $\text{Id}_{\mathcal{G}\boxtimes_{\mathcal{D}}\mathcal{F}} = \text{Id}_{\mathcal{G}}\boxtimes_{\mathcal{D}}\text{Id}_{\mathcal{F}}$;

5. for 0-, 1- and 2-cells $\mathcal{C},\mathcal{M}$ and $\mathcal{F}$ respectively there are identity 1-, 2- and 3-cells $\mathcal{C}, id_{\mathcal{M}}$ and $\text{Id}_{\mathcal{F}}$, respectively;

6. there are pseudonatural equivalences a,l,r so that concretely for the corresponding bimodule categories one has equivalence functors: $a_{\mathcal{M},\mathcal{N},\mathcal{L}} : (\mathcal{M}\boxtimes_{\mathcal{C}}\mathcal{N})\boxtimes_{\mathcal{D}}\mathcal{L} \xrightarrow{\cong} \mathcal{M}\boxtimes_{\mathcal{C}}(\mathcal{N}\boxtimes_{\mathcal{D}}\mathcal{L}), l_{\mathcal{N}} : \mathcal{C}\boxtimes_{\mathcal{C}}\mathcal{N} \xrightarrow{\cong} \mathcal{N}$ and $r_{\mathcal{N}} : \mathcal{N}\boxtimes_{\mathcal{D}}\mathcal{D} \xrightarrow{\cong} \mathcal{N}$ (observe that the respective naturalities hold up to natural isomorphisms);

7. there are modifications π,μ,λ and ρ which evaluated at bimodule categories give natural isomorphisms

$$\pi : (id_{\mathcal{K}}\boxtimes_{\mathcal{C}} a_{\mathcal{N},\mathcal{M},\mathcal{L}})\circ a_{\mathcal{K},\mathcal{N}\boxtimes_{\mathcal{D}}\mathcal{M},\mathcal{L}}\circ(a_{\mathcal{K},\mathcal{N},\mathcal{M}}\boxtimes_{\mathcal{E}} id_{\mathcal{L}}) \Rrightarrow a_{\mathcal{K},\mathcal{N},\mathcal{M}\boxtimes_{\mathcal{E}}\mathcal{L}}\circ a_{\mathcal{K}\boxtimes_{\mathcal{C}}\mathcal{N},\mathcal{M},\mathcal{L}},$$

$$\mu_{\mathcal{M},\mathcal{D},\mathcal{L}} : r_{\mathcal{M}}\boxtimes_{\mathcal{D}} id_{\mathcal{N}} \Rrightarrow (id_{\mathcal{M}}\boxtimes_{\mathcal{D}} l_{\mathcal{N}})\circ a_{\mathcal{M},\mathcal{D}\mathcal{N}},$$

$$\lambda_{\mathcal{C},\mathcal{M},\mathcal{N}} : l_{\mathcal{M}}\boxtimes_{\mathcal{D}} id_{\mathcal{N}} \Rrightarrow l_{\mathcal{M}\boxtimes_{\mathcal{D}}\mathcal{N}}\circ a_{\mathcal{C},\mathcal{M},\mathcal{N}},$$

$$\rho_{\mathcal{C},\mathcal{M},\mathcal{E}} : (id_{\mathcal{M}}\boxtimes_{\mathcal{D}} r_{\mathcal{N}})\circ a_{\mathcal{M},\mathcal{N},\mathcal{E}} \Rrightarrow r_{\mathcal{M}\boxtimes_{\mathcal{D}}\mathcal{N}},$$

similar to those in (vi)-(ix) of [29, Theorem 3.6.1] and they satisfy three axioms analogous to those in (x) of *loc.cit.*.

Remark 9.1 To see that the functors on the two sides in the isomorphism $(\mathcal{F}'\boxtimes_{\mathcal{D}}\mathcal{G}')\circ(\mathcal{F}\boxtimes_{\mathcal{D}}\mathcal{G})\cong(\mathcal{F}'\circ\mathcal{F})\boxtimes_{\mathcal{D}}(\mathcal{G}'\circ\mathcal{G})$ are a priori not equal, observe the following. As functors acting on the relative tensor product, they both are given up to an isomorphism by the defining functors $(\mathcal{F}'\times\mathcal{G}')\circ(\mathcal{F}\times\mathcal{G})$ and $(\mathcal{F}'\circ\mathcal{F})\times(\mathcal{G}'\circ\mathcal{G})$, respectively, which are clearly equal between themselves. Since both functors are determined up to an isomorphism by the same functor, they only can be isomorphic between themselves, and one can not claim that they are equal. This applies to the point 4. above. By the same reason naturalities in the point 6. above hold only up to an isomorphism.

Remark 9.2 For a fixed tensor category $\mathcal{C}$ it was proved in [24] that Bimod$(\mathcal{C},\mathcal{C})$ forms a monoidal 2-category in the sense of [26], which is a non-semistrict monoidal bicategory, namely, it is weaker than a Gray monoid. Though, [24] follows the approach of [12] where the relative tensor product of bimodule categories is defined in such a way that a functor from such tensor product is defined *uniquely* by a balanced functor, whereas in [9] it is defined *up to a unique isomorphism*. This has for a consequence that many of the structure isomorphisms in Bimod$(\mathcal{C},\mathcal{C})$ in [24] result to be identities (coherence 3-cells: π for the associativity constraint, [24, Proposition 4.9], and λ and ρ for the left and right unity constraints [24, Proposition 4.11]), and moreover the associativity constraint a itself is an isomorphism instead of being an equivalence (see the proof of [24, Proposition 4.4]). Substituting tensor categories by fusion categories (semisimple tensor categories), in [29] it was proved that these form a tricategory (in a weaker sense than in [24], as we just pointed out). Semisimplicity does not influence the arguments of the proof, so we may take it as a proof that Tens is a tricategory. Note that the author uses the term "2-functor" for a pseudofunctor, [29, Definition A 3.6].

From the items 1, 4, 5, 6 and 7 above it is clear that Tens is a category enriched over the tricategory of 2-categories, pseudofunctors, weak natural transformations and modifications, which we denoted earlier by $2\text{-}\mathrm{Cat}_{wk}$.

9.2. *Internal category structure for* Tens

Now let us explain the (1×2)-category structure for tensor categories, *i.e.* of a category internal in $2\text{-}\mathrm{Cat}_{wk}$. To do so we will give 2-categories C_0 and C_1, pseudofunctors s,t,u and c, weak natural equivalences α^*,λ^* and ρ^* and modifications $\pi^*,\mu^*,\lambda^*,\rho^*,\varepsilon^*$. As we announced at the beginning of this section, let C_0 be the 2-category of tensor categories, tensor functors and tensor natural transformations, and let C_1 be the 2-category of bimodule categories, bimodule functors and bimodule natural transformations. Fix tensor categories $\mathcal{C}$ and $\mathcal{D}$. To give a source and target 2-functors $s,t:C_1\longrightarrow C_0$, let $\mathcal{M}$ be a $\mathcal{C}$-$\mathcal{D}$-bimodule category, $\mathcal{F}$ a $\mathcal{C}$-$\mathcal{D}$-bimodule functor, and ω a $\mathcal{C}$-$\mathcal{D}$-bimodule natural transformation. Set $s(\mathcal{M})=\mathcal{C},t(\mathcal{M})=\mathcal{D},s(\mathcal{F})=id_{\mathcal{C}},t(\mathcal{F})=id_{\mathcal{D}}$ and $s(\omega)=\mathrm{Id}_{id_{\mathcal{C}}}$ and $t(\omega)=\mathrm{Id}_{id_{\mathcal{D}}}$ - the identity functors on $\mathcal{C}$ and $\mathcal{D}$ are obviously tensor functors, and the identity natural transformations on these two identity functors are obviously tensor ones. It is also clear that thus defined source and target functors are strict 2-functors. To define the identity 2-functor $u:C_0\longrightarrow C_1$, take tensor categories $\mathcal{C},\mathcal{D}$, tensor functors $F,G:\mathcal{C}\longrightarrow\mathcal{D}$ and a tensor natural transformation $\zeta:F\longrightarrow G$, and for $C,C',C''\in\mathcal{C}$ let $C\triangleright C'$ denote the left action of C on C' and $C'\triangleleft C''$ the right action of C'' on C'. Set $u(\mathcal{C})=\mathcal{C}$ as a $\mathcal{C}$-bimodule category, $u(F)=F$ as a $\mathcal{C}$-bimodule functor where $\mathcal{D}$ is a $\mathcal{C}$-bimodule category through F, that is: $C\triangleright D\triangleleft C'=F(C)\otimes D\otimes F(C')$ for an object $D\in\mathcal{D}$ and where $\otimes$ denotes the tensor product in $\mathcal{D}$ (a well-known fact), then F is clearly $\mathcal{C}$-bilinear. Finally, set $u(\zeta)=\zeta$, then similarly as for functors, ζ is a $\mathcal{C}$-bilinear natural transformation. To see that u is indeed a 2-functor, take a further tensor category $\mathcal{E}$ and a tensor functor $G:\mathcal{D}\longrightarrow\mathcal{E}$, then it is clear that GF as a $\mathcal{C}$-bimodule functor is equal to the composition of G as a $\mathcal{D}$-bimodule functor and F as a $\mathcal{C}$-bimodule functor.

The rest of the structure (a pseudofunctor c, pseudonatural equivalences $\alpha^*,\lambda^*,\rho^*$ and modifications $\pi^*,\mu^*,\lambda^*,\rho^*,\varepsilon^*$) are given as in Proposition 8.5. That c is a pseudofunctor and not a 2-functor follows from Remark 9.1. For this reason the tricategory Tens is an internal category in the iconic tricategory $2\text{-}\mathrm{Cat}_{wk}$, rather than in the Gray 3-category $2CAT_{nwk}$, as conjectured in [10, Example 2.14] (1-cells in $2CAT_{nwk}$ are 2-functors, while in $2\text{-}\mathrm{Cat}_{wk}$ these are pseudofunctors).

Observe that the tricategory of 2-categories 2- Cat_{wk} embeds into the tricategory DblPs. Thus the (1×2)-category of tensor categories is also an example of our alternative notion to intercategories (with non-trivial 3-cells involved in the internalization).

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Appendix: tricategories

We summarize now the definition of a tricategory from [17] with slight changes (in the direction of r in (TD6) and accordingly in μ and ρ in (TD8)).

A tricategory $\mathcal{T}$ consists of the following data (TDi) and axioms (TAj):

(TD1) a set $Ob\mathcal{T}$ of objects of $\mathcal{T}$;

(TD2) a bicategory $\mathcal{T}(p,q)$ for objects p,q of $\mathcal{T}$;

(TD3) a pseudofunctor $\otimes : \mathcal{T}(q,r) \times \mathcal{T}(p,q) \longrightarrow \mathcal{T}(p,r)$, for $p,q,r \in Ob\mathcal{T}$, called composition;

(TD4) a pseudofunctor $I_p : 1 \longrightarrow \mathcal{T}(p,p)$, for $p \in Ob\mathcal{T}$, where 1 is the unit bicategory;

(TD5) a pseudo natural equivalence $a : \otimes \circ (\otimes \times 1) \Rightarrow \otimes \circ (1 \times \otimes)$, where the respective pseudofunctors act between bicategories $\mathcal{T}(r,s) \times \mathcal{T}(q,r) \times \mathcal{T}(p,q) \longrightarrow \mathcal{T}(p,s)$, for $p,q,r,s \in Ob\mathcal{T}$;

(TD6) pseudo natural equivalences $l : \otimes \circ (I_q \times \mathrm{Id}_{\mathcal{T}(p,q)}) \longrightarrow \mathrm{Id}_{\mathcal{T}(p,q)}$ and $r : \otimes \circ (\mathrm{Id}_{\mathcal{T}(p,q)} \times I_p) \longrightarrow \mathrm{Id}_{\mathcal{T}(p,q)}$ for objects p,q in $\mathcal{T}$;

(TD7) an invertible modification π up to which the pentagon for a commutes;

(TD8) invertible modifications μ, λ and ρ relating a with l and r, then a with l and a with r, respectively;

(TA1) non abelian 4-cocycle condition for π;

(TA2) left normalization for the 4-cocycle π, and

(TA3) right normalization for the 4-cocycle π.

We unpack now the conditions (TD2) – (TD8). Observe that in (TD5-TD8) we give a description more detailed than in [25, Section 4.4].

(TD2): That $\mathcal{L} := \mathcal{T}(p,q)$ is a bicategory for every $p,q \in Ob\mathcal{T}$ comprises the following items:

1) 1-cells $A, B...$ acting from p to q (which we will write as horizontal simple arrows, straight or arched $p \xrightarrow{A} q$, $p \overset{B}{\frown\!\!\!\to} q$);

2) 2-cells $F, G...$ (we will denote them by a double arrow in the vertical direction: $p \underset{A'}{\overset{A}{\Downarrow F}} q$, or in the form of a rectangular diagram whose vertical arrows are identities), 3-cells $\alpha, \beta...$ (we will think them in the direction perpendicular to the plane of the paper, in the transversal direction, and will denote them by a triple arrow);

   a strictly associative *transversal composition* of 3-cells denoted by $\cdot$, and an identity 3-cell Id_F (which is the strict unit for $\cdot$);

3) *vertical composition* of 2- and 3-cells denoted by $\odot$ such that:

$$\mathrm{Id}_G \odot \mathrm{Id}_F = \mathrm{Id}_{G\odot F} \qquad \text{and} \qquad (\beta' \cdot \beta) \odot (\alpha' \cdot \alpha) = (\beta' \odot \alpha') \cdot (\beta \odot \alpha);$$

4) for each 1-cell A of $\mathcal{T}$ the identity 2-cell id_A;

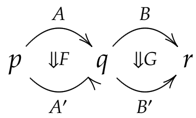
5) associativity isomorphism 3-cell $\alpha : (H \odot G) \odot F \Rrightarrow H \odot (G \odot F)$ for the vertical composition of 2-cells, natural in them;

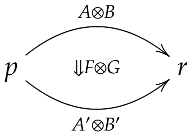
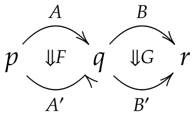
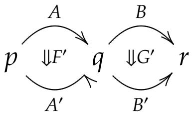
6) left and right unity isomorphism 3-cells $\lambda_F : id_B \odot F \Rrightarrow F$ and $\rho_F : F \odot id_A \Rrightarrow F$, natural in any 2-cell $A \overset{F}{\Rightarrow} B$;

7) the pentagon constraint for α and triangle constraint for α-λ-ρ commute;

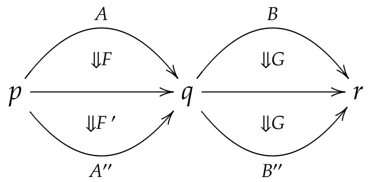
(TD3): That $\otimes : \mathcal{T}(q,r) \times \mathcal{T}(p,q) \longrightarrow \mathcal{T}(p,r)$ is a pseudofunctor, for every $p,q,r \in Ob\mathcal{T}$, it comprises the following items:

1) for 1-cells $p \xrightarrow{A} q \xrightarrow{B} r$ there is a composition 1-cell $p \xrightarrow{B\otimes A} r$;

2) *horizontal composition* of 2- and 3-cells denoted by $\otimes$ such that (a) $\mathrm{Id}_G \otimes \mathrm{Id}_F = \mathrm{Id}_{G\otimes F}$ and (b) $(\beta' \odot \beta)\otimes(\alpha' \odot \alpha) = (\beta' \otimes \alpha')\odot(\beta\otimes\alpha)$ (concretely, for two 2-cells F, G between two composable pairs of 1-cells 

there is a horizontal composition 2-cell  and for two 3-cells α, β between two horizontally composable pairs of 2-cells  and  there is a horizontal composition 3-cell $\beta \otimes \alpha : G \otimes F \Rrightarrow G' \otimes F'$);

3) for 2-cells in $\mathcal{T}$ composable pairwise horizontally and vertically:



there are natural isomorphism 3-cells

$$\xi : (G' \otimes F') \odot (G \otimes F) \Rrightarrow (G' \odot G) \otimes (F' \odot F)$$

$$\xi_0 : id_{B\otimes A} \Rrightarrow id_B \otimes id_A,$$

so that the corresponding hexagonal constraint for ξ, the square for ξ-ξ_0-λ and the square for ξ-ξ_0-ρ commute;

Remark 9.3 By the coherence Theorem 1.5 of [17] the items 5)-7) in TD2) can be ignored. This justifies the assumption that the 2-cells compose striclty associatively and unitary in the vertical direction. When convenient, in the equations we adopt the notation $\frac{\alpha}{\beta}$ for the vertical composition $\beta \odot \alpha$ of 2- and 3-cells.

The hexagonal and two square constraints in item 3) of TD3) can be ignored by the coherence Theorem 1.6 of [17].

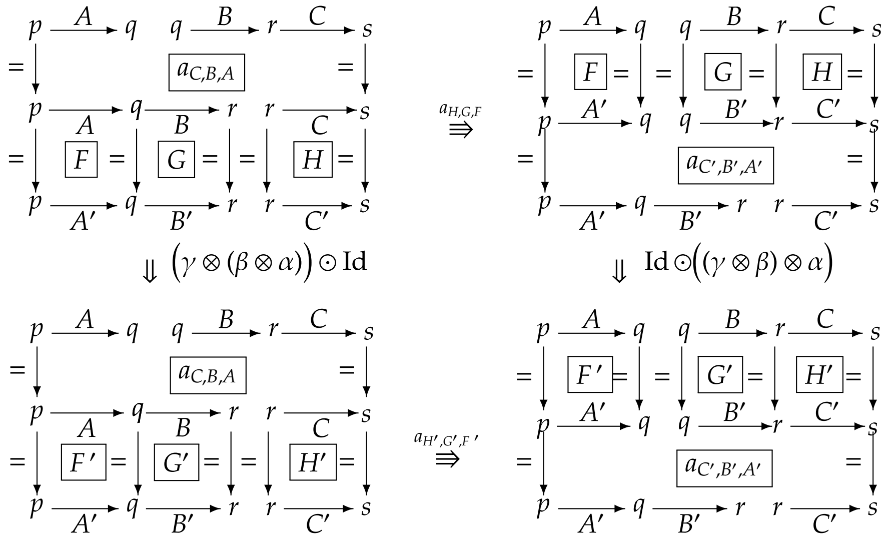
(TD4): states that for each 0-cell p in $\mathcal{T}$ there is a 1-cell I_p and a 2-cell $\iota_P : I_p \longrightarrow I_p$ so that there is an isomorphism 3-cell $Id_{I_p} \cong \iota_p$;

(TD5): states that we have associativity of $\otimes$ on the levels of 1-, 2- and 3-cells;
for three horizontally composable 1-cells C, B, A there are 2-cells $a_{C,B,A} : (C \otimes B) \otimes A \Rightarrow C \otimes (B \otimes A)$ and its quasi-inverse $a'_{C',B',A'}$ so that there are invertible 3-cells $a_{C,B,A} \odot a'_{C,B,A} \Rrightarrow \mathrm{Id}_{(C\otimes B)\otimes A}$ and $\mathrm{Id}_{C\otimes(B\otimes A)} \Rrightarrow a'_{C,B,A} \odot a_{C,B,A}$;
for each pair of triples of composable 1-cells C, B, A and C', B', A' and three 2-cells acting between them H, G, F, there are invertible 3-cells (natural in H, G, F):

$$a_{C',B',A'} \odot \Big((H \otimes G) \otimes F\Big) \overset{a_{H,G,F}}{\Rrightarrow} \Big(H \otimes (G \otimes F)\Big) \odot a_{C,B,A}$$

$$a'_{C',B',A'} \odot \Big(H \otimes (G \otimes F)\Big) \overset{a'_{H,G,F}}{\Rrightarrow} \Big((H \otimes G) \otimes F\Big) \odot a'_{C,B,A};$$

for three horizontally composable 3-cells γ, β, α the following diagram of (the transversal composition of) 3-cells commutes:



(in terms of equation: $a_{H',G',F'} \cdot \Big(\big(\gamma \otimes (\beta \otimes \alpha)\big) \odot \mathrm{Id}\Big) = \Big(\mathrm{Id} \odot \big((\gamma \otimes \beta) \otimes \alpha\big)\Big) \cdot a_{H,G,F}$);

(TD6): gives unity laws for $\otimes$;

for a 1-cell $A : p \longrightarrow q$ there are 2-cells $l_A : I_q \otimes A \Rightarrow A$ and $r_A : A \otimes I_p \Rightarrow A$ and their quasi-inverses l'_A and r'_A so that there are invertible 3-cells $l_A \odot l'_A \Rrightarrow id_A$, $id_{I_q \otimes A} \Rrightarrow l'_A \odot l_A$, $r_A \odot r'_A \Rrightarrow id_A$, $id_{A \otimes I_p} \Rrightarrow l'_A \odot l_A$; moreover, for any 2-cell $F : A \Rightarrow B$ there are invertible 3-cells (natural in F):

$$l_B \odot (\mathrm{Id}_{I_q} \otimes F) \overset{l_F}{\Rrightarrow} F \odot l_A, \quad l'_B \odot F \overset{l'_F}{\Rrightarrow} (\mathrm{Id}_{I_q} \otimes F) \odot l'_A \ ,$$

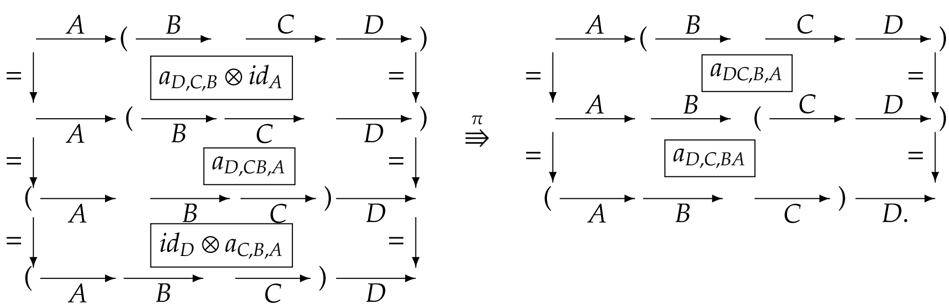
$$r_B \odot (F \otimes \mathrm{Id}_{I_p}) \overset{r_F}{\Rrightarrow} F \odot r_A, \quad r'_B \odot F \overset{r'_F}{\Rrightarrow} (F \otimes \mathrm{Id}_{I_p}) \odot r'_A;$$

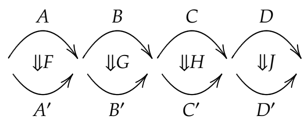
for a 3-cell $\alpha : F \Rrightarrow G$ one has the identities:

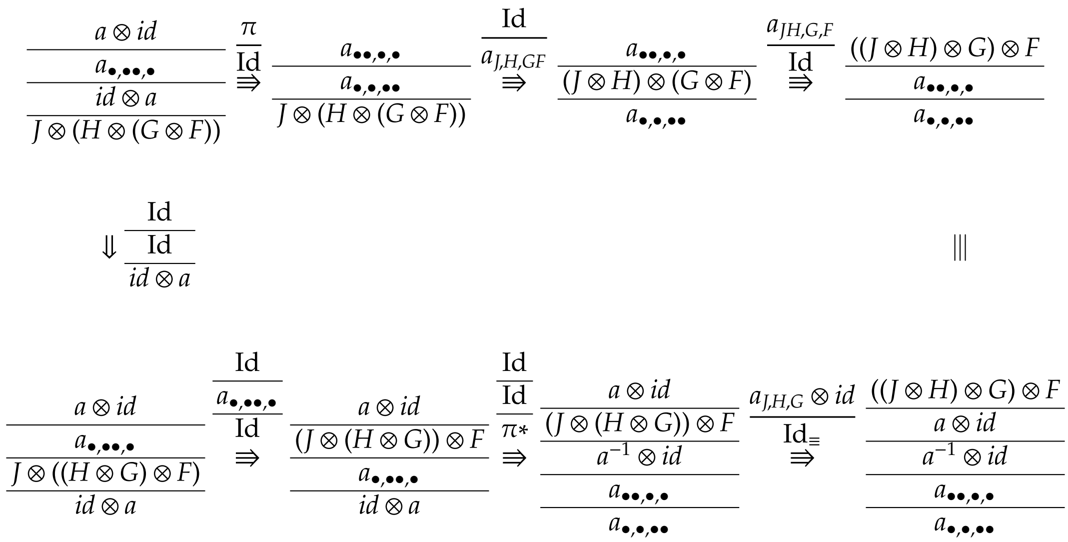
$$l_G \cdot \Big(\mathrm{Id}_{l_B} \odot (\mathrm{Id}_{\mathrm{Id}_{I_q}} \otimes \alpha)\Big) = (\alpha \odot \mathrm{Id}_{l_A}) \cdot l_F$$

$$r_G \cdot \Big(\mathrm{Id}_{r_B} \odot (\alpha \otimes \mathrm{Id}_{\mathrm{Id}_{I_p}})\Big) = (\alpha \odot \mathrm{Id}_{r_A}) \cdot r_F;$$

(TD7): for every four composable 1-cells D, C, B, A there is an invertible 3-cell

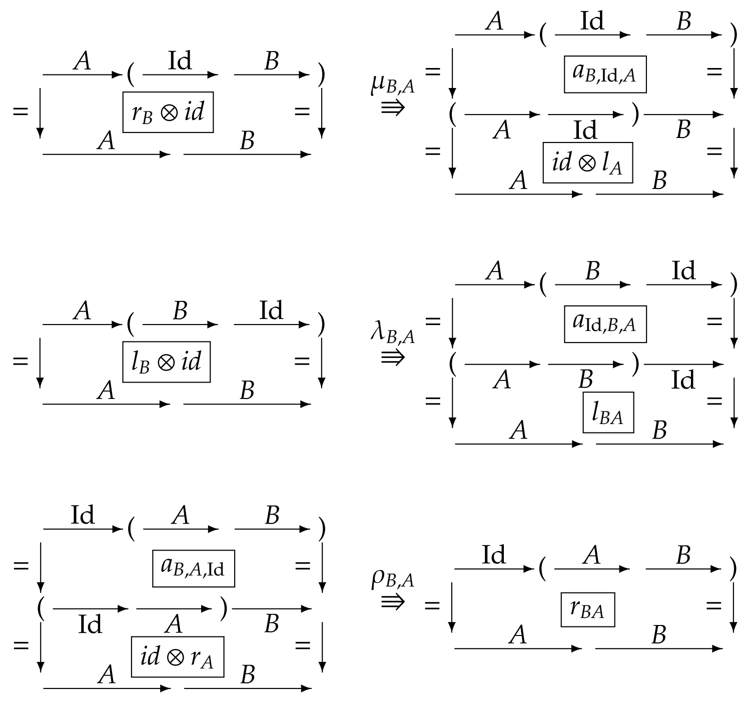


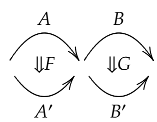
so that for four horizontally composable 2-cells  the following two transversal compositions of 3-cells coincide:



where the fractions denote vertical compositions of both 2- and 3-cells and the 2-cells $a_{\bullet,\bullet,\bullet}$ are evaluated at 1-cells A, B, C, D;

(TD8): for composable 1-cells $p \xrightarrow{A} q \xrightarrow{B} r$ there exist 3-cells:



so that for two horizontally composable 2-cells  the following three pairs of transversal compositions of 3-cells coincide, the first one involving μ:

$$
\begin{array}{ccccccc}
\left\{\begin{array}{c}(G\otimes \mathrm{Id})\otimes F\\ \hline r_{B'}\otimes id_{A'}\end{array}\right\} & \overset{\frac{\mathrm{Id}}{\mu_{B',A'}}}{\Rrightarrow} & \left\{\begin{array}{c}(G\otimes \mathrm{Id})\otimes F\\ \hline a_{\bullet,id,\bullet}\\ \hline id_{B'}\otimes l_{A'}\end{array}\right\} & \overset{\frac{a^{-1}_{G,\mathrm{Id},F}}{\mathrm{Id}}}{\Rrightarrow} & \left\{\begin{array}{c}a_{\bullet,id,\bullet}\\ \hline G\otimes(\mathrm{Id}\otimes F)\\ \hline id_{B'}\otimes l_{A'}\end{array}\right\} & \overset{\frac{\mathrm{Id}}{\xi}}{\Rrightarrow} & \left\{\begin{array}{c}a_{\bullet,id,\bullet}\\ \hline [\frac{G}{id_{B'}}]\otimes[\frac{\mathrm{Id}\otimes F}{l_{A'}}]\end{array}\right\}\\
\Downarrow \xi & & & & & & \Downarrow \frac{\mathrm{Id}}{\omega_G\otimes l_F}\\
\left\{[\frac{G\otimes \mathrm{Id}}{r_{B'}}]\otimes[\frac{F}{id_{A'}}]\right\} \overset{r_G\otimes\omega_F}{\Rrightarrow} \left\{[\frac{r_B}{G}]\otimes[\frac{id_A}{F}]\right\} & \overset{\xi^{-1}}{\Rrightarrow} & \left\{\begin{array}{c}r_B\otimes id_A\\ \hline G\otimes F\end{array}\right\} & \overset{\frac{\mu_{B,A}}{\mathrm{Id}}}{\Rrightarrow} & \left\{\begin{array}{c}a_{\bullet,id,\bullet}\\ \hline id_B\otimes l_A\\ \hline G\otimes F\end{array}\right\} & \overset{\frac{\mathrm{Id}}{\xi}}{\Rrightarrow} & \left\{\begin{array}{c}a_{\bullet,id,\bullet}\\ \hline [\frac{id_B}{G}]\otimes[\frac{l_A}{F}]\end{array}\right\}
\end{array}
$$

the second one involving λ:

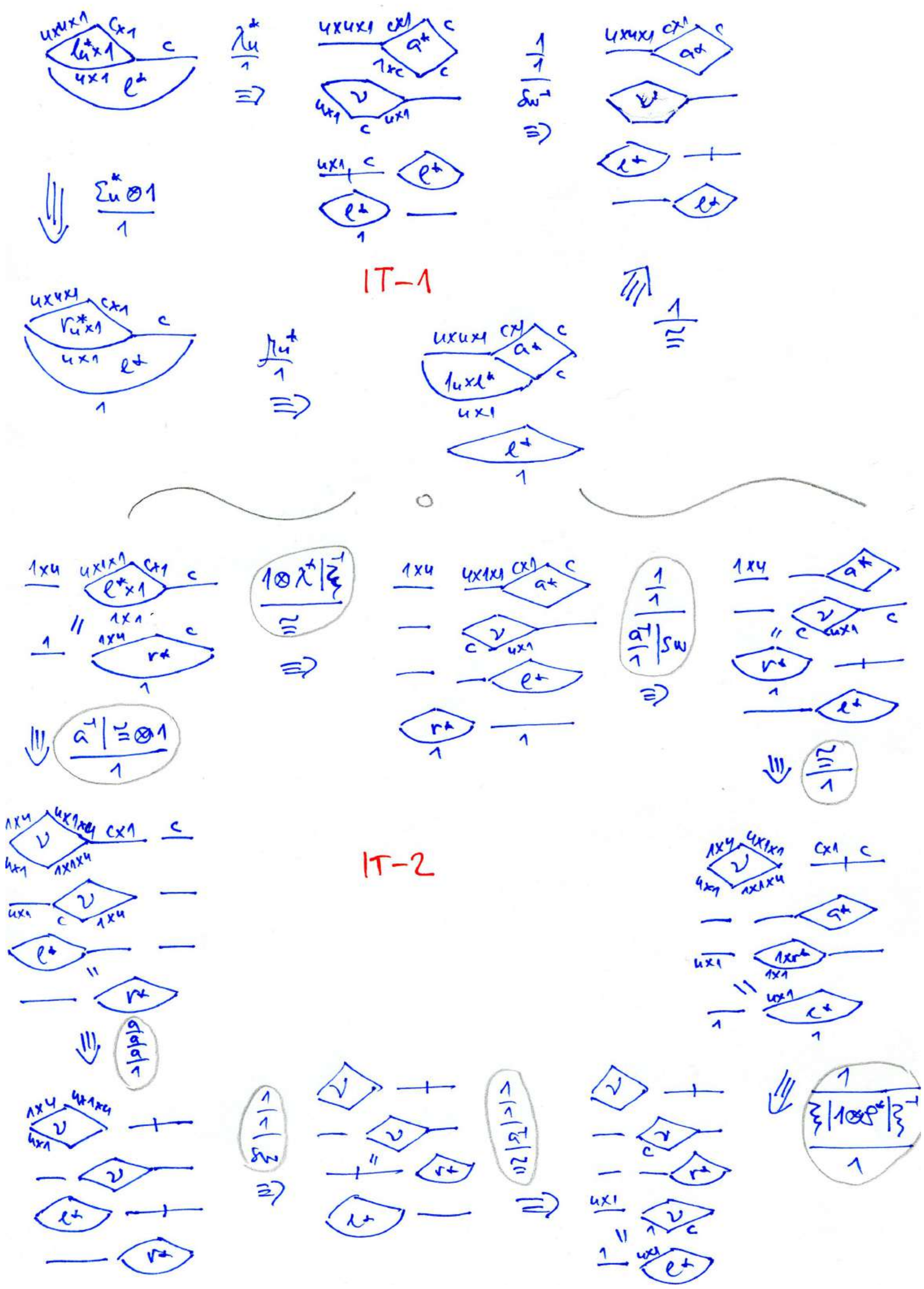
$$
\begin{array}{ccccccc}
\left\{\begin{array}{c}a_{id,\bullet,\bullet}\\ \hline \mathrm{Id}\otimes(G\otimes F)\\ \hline a^{-1}_{id,\bullet,\bullet}\\ \hline l_{B'}\otimes id_{A'}\end{array}\right\} & \overset{\frac{a_{\mathrm{Id},G,F}}{\mathrm{Id}_=}}{\Rrightarrow} & \left\{\begin{array}{c}(\mathrm{Id}\otimes G)\otimes F\\ \hline a_{id,\bullet,\bullet}\\ \hline a^{-1}_{id,\bullet,\bullet}\\ \hline l_{B'}\otimes id_{A'}\end{array}\right\} & \overset{\equiv}{\Rrightarrow} & \left\{\begin{array}{c}(\mathrm{Id}\otimes G)\otimes F\\ \hline l_{B'}\otimes id_{A'}\end{array}\right\} & \overset{\xi}{\Rrightarrow} & \left\{[\frac{\mathrm{Id}\otimes G}{l_{B'}}]\otimes[\frac{F}{id_{A'}}]\right\}\\
\Downarrow \frac{\mathrm{Id}_=}{\lambda^*_{B'A'}} & & & & & & \Downarrow l_G\otimes\omega_F\\
\left\{\begin{array}{c}a_{id,\bullet,\bullet}\\ \hline \mathrm{Id}\otimes(G\otimes F)\\ \hline l_{B'A'}\end{array}\right\} & \overset{\frac{\mathrm{Id}}{l_{GF}}}{\Rrightarrow} & \left\{\begin{array}{c}a_{id,\bullet,\bullet}\\ \hline a^{-1}_{id,\bullet,\bullet}\\ \hline l_B\otimes id_A\\ \hline G\otimes F\end{array}\right\} & \equiv & \left\{\begin{array}{c}l_B\otimes id_A\\ \hline G\otimes F\end{array}\right\} & \overset{\xi}{\Rrightarrow} & \left\{[\frac{l_B}{G}]\otimes[\frac{id_A}{F}]\right\}
\end{array}
$$

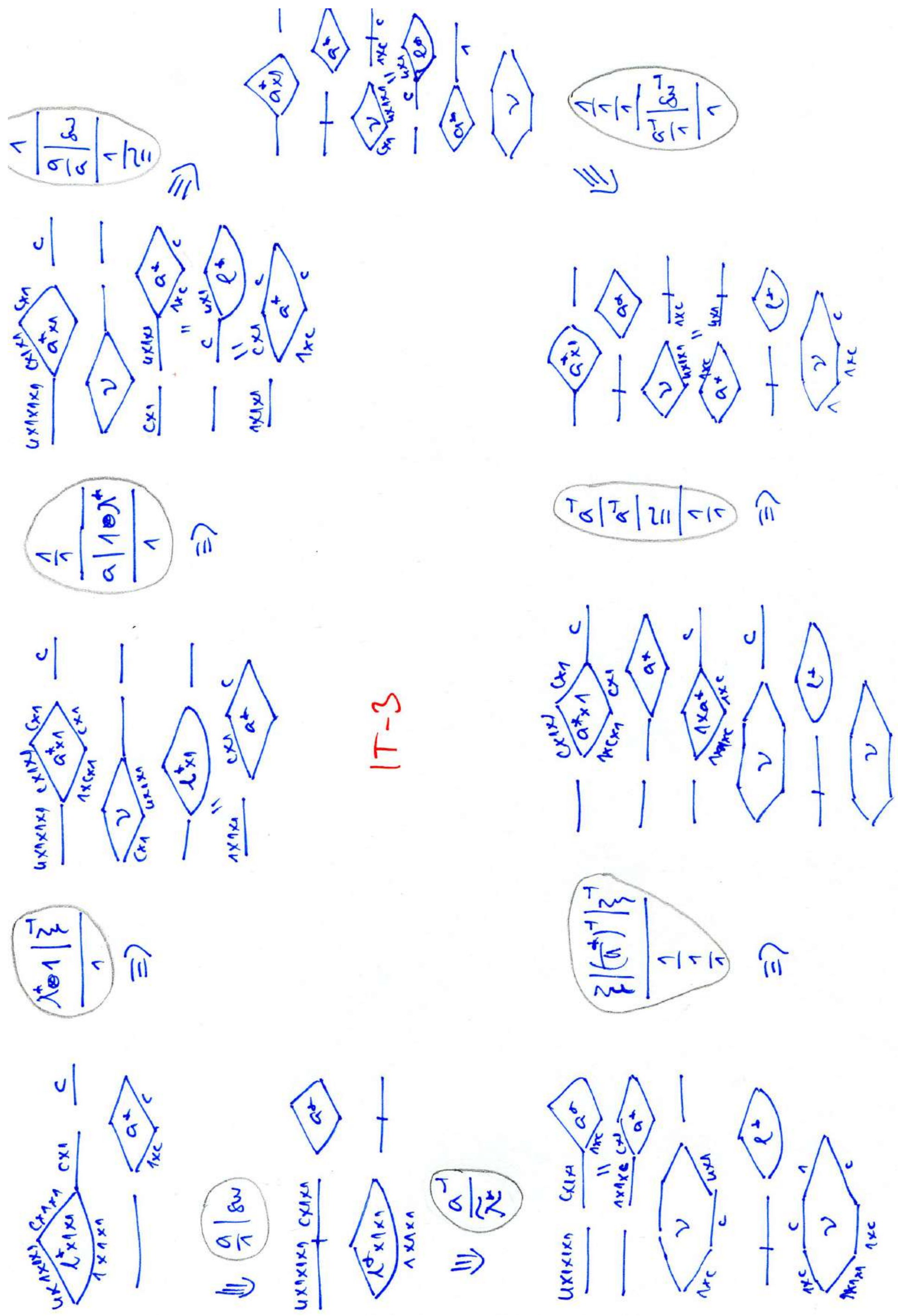
the third one involving ρ:

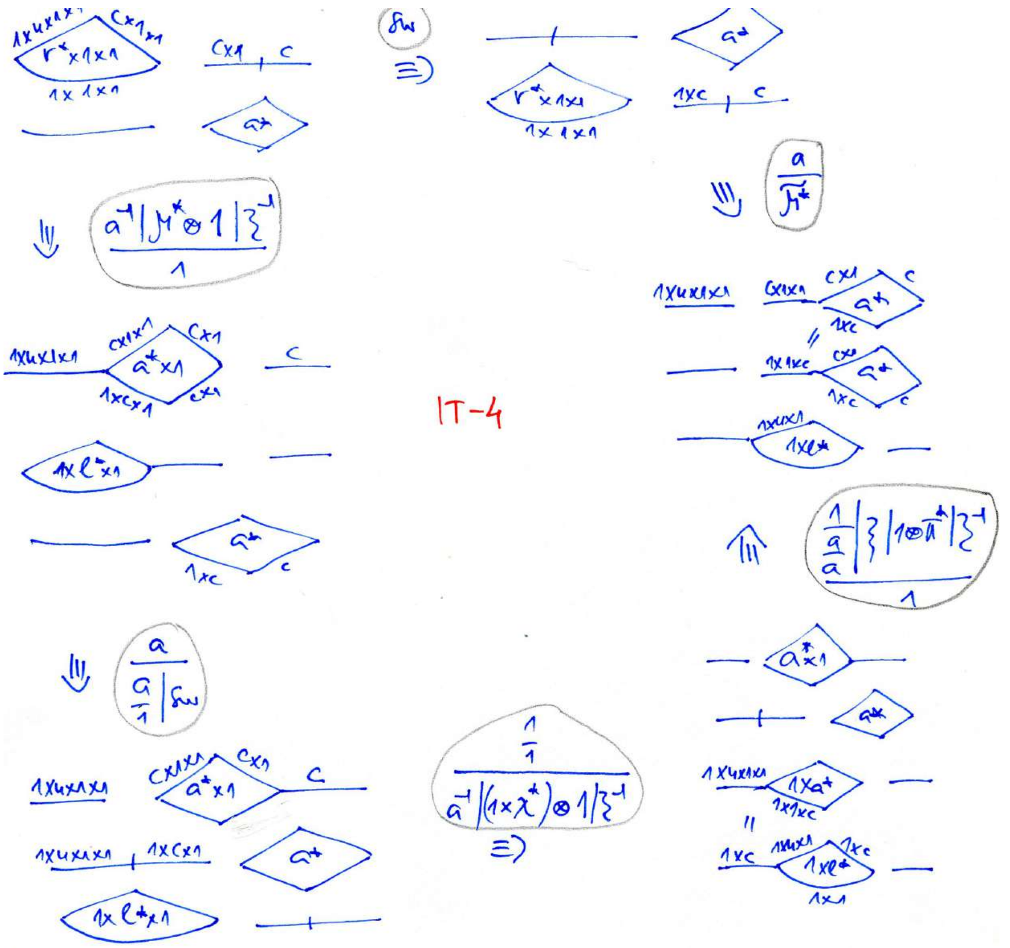
$$
\begin{array}{ccccccc}
\left\{\begin{array}{c}(G\otimes F)\otimes \mathrm{Id}\\ \hline a_{\bullet,\bullet,id}\\ \hline id_{B'}\otimes r_{A'}\end{array}\right\} & \overset{\frac{a^{-1}_{G,F,\mathrm{Id}}}{\mathrm{Id}}}{\Rrightarrow} & \left\{\begin{array}{c}a_{\bullet,\bullet,id}\\ \hline G\otimes(F\otimes \mathrm{Id})\\ \hline id_{B'}\otimes r_{A'}\end{array}\right\} & \overset{\frac{\mathrm{Id}}{\xi}}{\Rrightarrow} & \left\{\begin{array}{c}a_{\bullet,\bullet,id}\\ \hline [\frac{G}{id_{B'}}]\otimes[\frac{F\otimes \mathrm{Id}}{r_{A'}}]\end{array}\right\} & & \\
\Downarrow \frac{\mathrm{Id}}{\rho_{B'A'}} & & & & \Downarrow \frac{\mathrm{Id}}{\omega_G\otimes r_F} & & \\
\left\{\begin{array}{c}(G\otimes F)\otimes \mathrm{Id}\\ \hline \mathrm{r}_{B'A'}\end{array}\right\} \overset{r_{GF}}{\Rrightarrow} \left\{\begin{array}{c}r_{BA}\\ \hline G\otimes F\end{array}\right\} & \overset{\frac{\rho^{-1}_{BA}}{\mathrm{Id}}}{\Rrightarrow} & \left\{\begin{array}{c}a_{\bullet,\bullet,id}\\ \hline id_B\otimes r_A\\ \hline G\otimes F\end{array}\right\} & \overset{\frac{\mathrm{Id}}{\xi}}{\Rrightarrow} & \left\{\begin{array}{c}a_{\bullet,\bullet,id}\\ \hline [\frac{id_B}{G}]\otimes[\frac{r_A}{F}]\end{array}\right\}, & &
\end{array}
$$

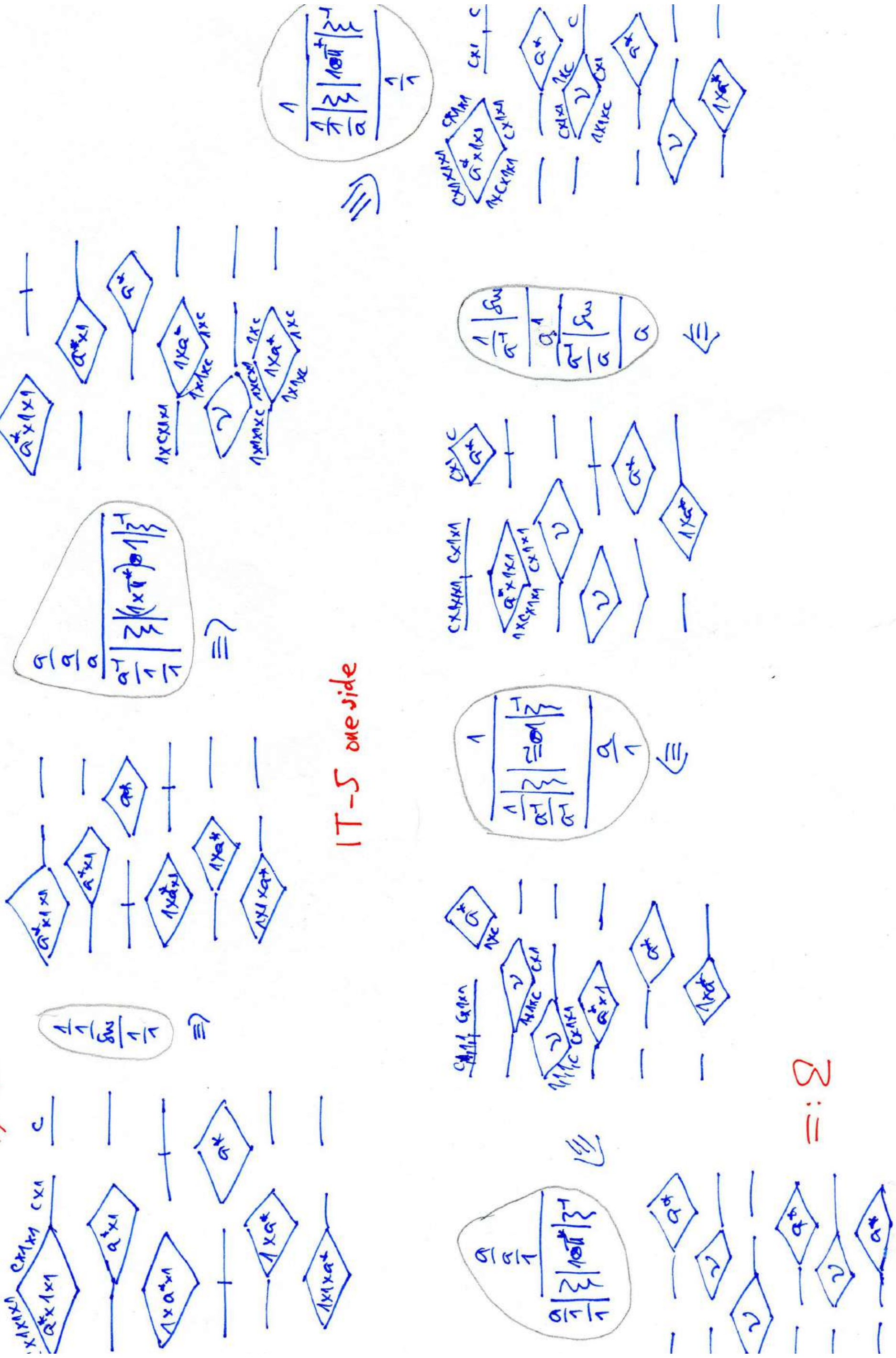
here ω is the appropriate composition of unity constraints for the vertical composition mapping $\omega_F : \frac{F}{id_{A'}} \longrightarrow \frac{id_A}{F}$ and similarly for G.

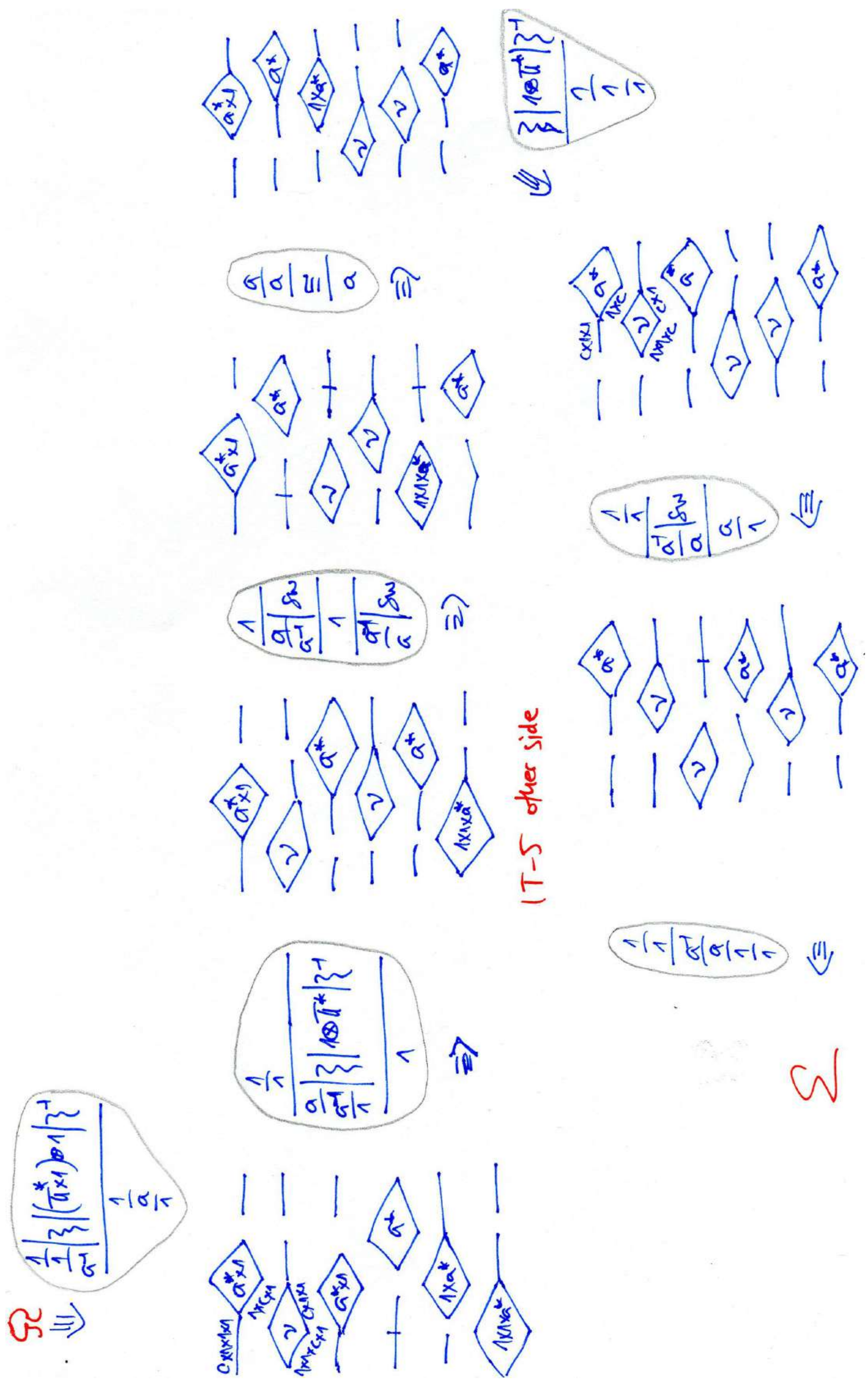
AXIOMS FOR $\pi^*, \mu^*, \lambda^*, \rho^*, \Sigma^*$






IT-4


IT-5 one side


IT-5 other side