Filomat 38:8 (2024), 2673–2690
https://doi.org/10.2298/FIL2408673B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Fourier transform of stock asset returns uncertainty under Covid-19 surge

**Philip Ajibola Bankole[a], Victoria O. Olisama[a], Ezekiel K. Ojo[a], Ini Adinya[b]**

[a]*Department of Mathematics Education, College of Science Education, Lagos State University of Education, Oto/Ijanikin, Lagos, Nigeria*
[b]*Department of Mathematics, Faculty of Science, University of Ibadan, Ibadan, Nigeria*

**Abstract.** This study presents Fourier Transform of Stock asset returns uncertainty under Covid-19 surge. Computational issue with respect to Covid-19 induced payoff uncertainty of option on an underlying stock is mathematically studied. A newly modified Black-Scholes model (MBSM) is proposed incorporating Factorial function for generalization of Black-Scholes model (BSM) and random variable volatility function as an alternative to constant volatility in the BSM. A closed form integral formula for stock price is obtained for MBSM. The variable volatility function is to account for uncertainty effect owing to Covid-19 on the stock price in the MBSM. Option prices are compared based on the selected methods. The option output via the MBSM and Fast Fourier transform (FFT) method varies on replicating simulation which is more realistic in financial market compared to the BSM which returns constant option price all-through. Hence, the study findings show that the MBSM assumption of non-constancy in the volatility term suits the stock markets behaviour over the BSM.

## 1. Introduction

In this study, Fourier transform methodology of a class of partial differential equation applicable for option price computation on stock asset under Covid-19 induced price uncertainty is studied. The book written by [20] discusses the Fourier transform methodology in option pricing and volatility modeling in detail. Cont [14] applies the Fourier transform method to analyze the empirical properties of asset returns, including distributional characteristics and serial correlations in his study while [12] presents a fast Fourier transform-based methodology for pricing options and demonstrates its efficiency compared to traditional methods. In [2], the use of Fourier transform techniques in estimating stochastic volatility models, which are widely used in pricing options and risk management were discussed. In similar enthusiasm, [1] explores the application of wavelet-based Fourier methods in analyzing the self-similarity and long-range dependence in financial time series. A comprehensive treatment of derivative pricing and risk management using Fourier transform techniques, specifically focusing on models with stochastic volatility was presented by [19]. Just to mention few, [23] applies the Fourier transform method within a Bayesian framework to estimate stochastic volatility models with fat-tailed and correlated errors. Several studies have been rolled out on Fourier transform technique and applications in finance. The definitions and useful axioms

2020 *Mathematics Subject Classification*. Primary 91G30; Secondary 62M20, 42B10
*Keywords*. Fourier Transform, Variable volatility function, Modified Black-Scholes model (MBSM).
Received: 21 March 2023; Revised: 30 August 2023; Accepted: 18 September 2023
Communicated by Miljana Jovanović
*Email addresses:* bankolepa@lasued.edu.ng (Philip Ajibola Bankole), vicolisama@yahoo.com (Victoria O. Olisama), kadjoj@yahoo.com (Ezekiel K. Ojo), i.adinya@mail.ui.edu.ng (Ini Adinya)

on Fourier transform method are reported in the next section vividly. Also, [12] presents a fast Fourier transform-based methodology for pricing options and demonstrates its efficiency compared to traditional methods.

In this present study, one will see that the statement of the problem follows from the fact that stock prices are affected by uncertainties in financial markets. Various sources of uncertainty are in existence which are responsible for price fluctuations of stock asset. In recent time, Bankole and Ugbebor [8] incorporated the concept of economic recession induced volatility uncertainties in option price computation. They considered option price instability subject to *information flow* (filtration) from recessed economy into financial markets and the uncertainties such filtration pose on stock assets' returns determined by the volatility term in financial models. Some studies in the past have been concentrated on stochastic volatility term structure with respect to stock price characteristics in real life ([13], [22]). The shortcomings of the classical Black - Scholes model [10] in fitting volatility smile of stock price owing to assumption of constant volatility in their model, led to the development of various forms of stochastic volatility models.

However, despite the observable shortcomings in the Black - Scholes model, the model is still in use as a benchmark for option price comparison among some other option pricing models. In an attempt to unravel the shortcomings of constant volatility term in the Black - Scholes model, the main objective of this study is to incorporate random variable volatility function $\tilde{\sigma}_t$ in the deterministic Black - Scholes model (BSM). The new model is termed Modified Black - Scholes Model (MBSM). Our attention is devoted to uncertainties on Nigerian stock payoff with respect to Covid-19 information flow from the economy to stock markets. In the next paragraph, we consider the existing literature on Covid-19 and its effect on the output of financial assets.

Corbet et al.[15] gave an analysis on the effect of corona virus on stocks' return and volatility behaviour during the Covid-19 pandemic outbreak. Their results revealed that companies went through negative hourly returns, exceptional large increase in hourly volatility and trading volumes owing to the announcement of Covid-19 pandemic outbreak. Recently, some scholars studied "the impact of Covid-19 pandemic on the global economy and international financial markets" [6]. Eichenbaum et al. [16] used a method based on canonical epidemiology model to study an interaction between economic decisions and pandemic. Pagnottoni, Famà and Kim [28] in their research on cryptocurrency interactions in frequency domain revealed that Covid-19 pandemic outflow of information had substantial impact on the cryptocurrency market. The research studies of [3] reported that there was loss in the return of stocks under Covid-19 period using the historical Nigeria stock Exchange data from $2^{nd}$ January, 2020 through 16th April, 2020. Their findings further documented high in volatility of stocks during the Covid-19 period. It was also stressed in the research carried out by [4] that the outbreak of Covid-19 caused poor performance of Ghana Stock Exchange market. The authors reported that the Ghana stock exchange experienced better returns prior to the Covid-19 period. Quite number of literature show evidence of negative effect Covid-19 has on financial asset returns.

The inference from the above literature are evidences to show that financial markets are sensitive to Covid-19 information inflow to the market especially stock markets. In addition to the aforemention citations, one can deduce that the economy state affects the price of goods and services generally. In Bankole & Ugbebor [8], [9], it was discovered that "assets' value are affected by the state of the economy". There exists uncertainty in the price of financial assets based on the information inflow to the market. The price of Stock assets undergo stochastic movement in response to every filtration of bad news inflow to the market from the economy [29]. The Covid-19 pandemic news is one of the sources for stock's value stochastic movement in the stock markets during Covid-19 period.

Several financial models are in existence in valuation of stock price, option on an underlying stock price such as: the classical Black - Scholes model, a single factor Heston model, Double Heston model, to mention few. The classical Black - Scholes model [10] has been criticized based on its inability to described the real life situation of stock price . Hence, the purpose of this study is to relax the assumption of constant volatility in the Black - Scholes model which is the major deficiency, and propose a random variable volatility term. In other words, a form of random-volatility $\tilde{\sigma}_t$, is proposed to replace the constant volatility parameter. The new model is however referred to as modified Black - Scholes Model (MBSM) in this study. The paper is structured as follows: Fourier Transform methodology, Black - Scholes Model Partial Differential Equation for Nigerian Stock asset, Fourier Transform and computation of modified Black - Scholes Model (MBSM), Numerical results and analysis, Volatility surface for Stock with respect to MBSM, and Conclusion.

## 2. Fourier Transform methodology preliminaries

Fourier Transform based computation is adopted in this study. There exists an algorithm on Fourier transform called Fast Fourier transform (FFT) which is a mathematical algorithm for computing Discrete Fourier Transform (DFT) of a sequence. FFT has been proven to be effective in the valuation of option prices ([7],[8],[12]). For more details on fast Fourier transform in option pricing, the reader could check the following references ([12],[24],[27],[30]). However, in this paper, we demonstrated simplicity approach of using differential axiom of Fourier transform to solve a modified Black - Scholes partial differential equation for stock asset value valuation. The novel methodology coupled with introduction of factorial function and random volatility function are in our model formulation for asset valuation is the major contribution to the existing literature. The steps are discussed vividly in the next section.

**Definition 2.1 ([31]).** *Let $f(x)$ be a Lebesgue -measurable function of $x \in \mathbb{R}$, then the $L^2$-norm of $f$ is defined as:*

$$\|f\| = \left(\int_{-\infty}^{\infty} \left|f(x)\right|^2 dx\right)^{\frac{1}{2}} \tag{1}$$

*where*

$$L^2 = \{f : \|f\| < \infty\}. \tag{2}$$

*Equivalently, $f(x)$ is a piecewise integrable real function over the entire real line satisfying the condition*

$$\int_{-\infty}^{\infty} \left|f(x)\right| dx < \infty. \tag{3}$$

*For Fourier transform and inverse Fourier transform of a function to exist, it is necessary that the function is absolutely integrable over $\mathbb{R}$ with finite value as defined in (3) above. An absolutely integrable functions on a given interval $(a, b)$ are said to be on the space of $L^1(a, b)$. The concept of Fourier transform can be extended to the square integrable functions.*

**Definition 2.2 ([31]).** *A square integrable functions is defined as:*

$$\int_{-\infty}^{\infty} \left|f(x)\right|^2 dx < \infty. \tag{4}$$

*The space of square integrable functions is represented by $L^2(\mathbb{R})$ over the real line or simply an interval $(a, b)$.*

### 2.1. *Fourier Transform and Inverse Fourier Transform*

There are some conventional representations of Fourier transform found in use depending on various fields of application. For instance, in Physics, Fourier transform of a given function say, $f(t)$, is carried out by moving from time $t$-domain into an angular frequency $\omega$-domain measured in (*radians*/*second*), while in the case of signal processing, Fourier transform is done moving from time $t$-domain into frequency (cycles/second)-domain [26]. The transition from one domain of operation to another is dependent on the area of application that one is considering and what one intends to measure. The relationship between the frequency $f$, and angular frequency $\omega$, is given by $\omega = 2\pi f$.

**Definition 2.3 ([26], [31]).** *Let $f : \mathbb{R} \to \mathbb{R}$ be a real-value function for $x \in \mathbb{R}$, then the Fourier transform of $f$ is defined as:*

$$\mathcal{F}(f(x); w) = \hat{f}(w) = \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx, \ \omega \in \mathbb{R}, \tag{5}$$

*where $i = \sqrt{-1}$ and $\omega$ is a parameter.*

We can recover $f(x)$ from $\hat{f}(\omega)$ by the *inverse Fourier transform*.

**Definition 2.4 ([31], page 13).** *The* inverse Fourier transform *of the real valued function $f(x)$ is defined as:*

$$\mathcal{F}^{-1}\Big(\hat{f}(\omega);x\Big) = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{-i\omega x}d\omega,\ x\in\mathbb{R} \tag{6}$$

*which belongs to either $L^1$ or $L^2$ - spaces.*

**Definition 2.5 ([8],[12],[31]).** *FFT is a highly efficient implementation of discrete Fourier transform which maps a vector $\mathbf{h} = (h_j)_{j=0}^{N-1}$ onto some vector $D_k(\mathbf{h})$ such that*

$$D_k(\mathbf{h}) = \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N}jk}h_j, \quad k = 0,...,N-1. \tag{7}$$

*In application, N is usually chosen as power of 2 so that the complexity of the FFT algorithm reduces from an order of $N^2$ for direct numerical integration methods to that of $N\log_2 N$ operations to compute N values faster.*

### *2.2. Some Axioms of Fourier Transform*

In this subsection, we highlight some useful properties of Fourier transform applied in this study. For more details, please see ([26], [31]).

- **Differentiation axiom:** Let $f$ be a piecewise $n$-times continuously differentiable function such that $f^{(n)}$ is the derivative of order $n \in \mathbb{N}$. Suppose each of the derivative is integrable absolutely on the whole real line, then the Fourier transform of the derivative is given by

$$\begin{aligned}\mathcal{F}\Big(f^{(n)}(x)\,;\omega\Big) &= \int_{-\infty}^{\infty} f^{(n)}(x)e^{i\omega x}dx\\ &= (i\omega)^n\int_{-\infty}^{\infty} f(x)e^{i\omega x}dx\\ &= (i\omega)^n\mathcal{F}\Big(f(x)\,;\omega\Big)\\ &= (i\omega)^n\hat{f}(\omega),\end{aligned} \tag{8}$$

- **Symmetric axiom:**

$$\mathcal{F}\Big(f(x)\,;\omega\Big) = \hat{f}(\omega) \Longrightarrow \mathcal{F}\Big(f(-x)\,;\omega\Big) = \hat{f}(-\omega). \tag{9}$$

There is a very close relation between Fourier transform and characteristic function.

**Definition 2.6 ([31]).** *Let $\mathbb{P}(x)$ be the probability density function with $x \in \mathbb{R}$. Then a characteristic function $\phi(\omega), \omega \in \mathbb{R}$, is defined as the Fourier transform of $\mathbb{P}(x)$ by:*

$$\mathcal{F}\Big(\mathbb{P}(x)\,;\omega\Big) \equiv \hat{\phi}(\omega) \equiv \int_{-\infty}^{\infty}\mathbb{P}(x)\,e^{i\omega x}d\omega = \mathbb{E}[e^{i\omega x}] \tag{10}$$

*We can recover the probability density function $\mathbb{P}(x)$ via inverse Fourier transform of the characteristic function.*

This we define as:

$$\mathbb{P}(x) = \mathcal{F}^{-1}\left(\hat{\phi}(\omega)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\phi}(\omega)\,e^{-i\omega x}d\omega. \tag{11}$$

**Example 2.7.** *Consider a piecewise function $f(x)$ defined below:*

$$f(x) = \begin{cases} \frac{1}{4}, & for \quad -2 \le x < 0; \\ -\frac{1}{4}, & for \quad 0 < x \le 2. \\ 0; & otherwise \end{cases} \tag{12}$$

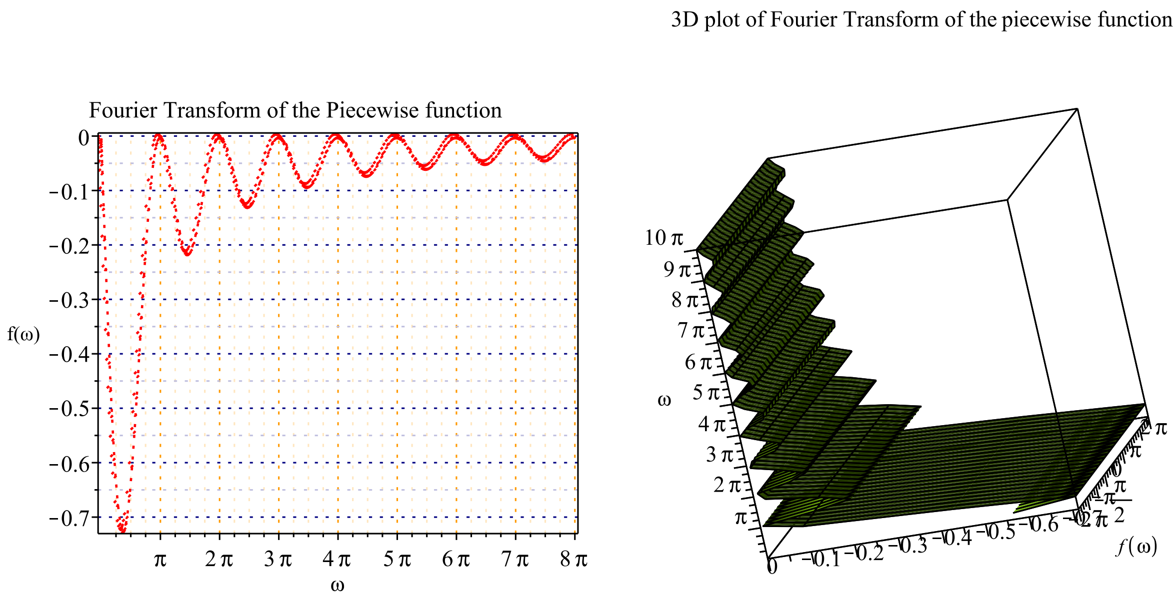
*The Fourier transform for $f(x)$ is computed as follows.*
***Solution:***

$$\mathcal{F}\left(f(x)\,;\omega\right) = \hat{f}(\omega) = \int_{-2}^{0} \frac{1}{4} e^{i\omega x} dx - \int_{0}^{2} \frac{1}{4} e^{i\omega x} dx \tag{13}$$

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{4}\left(\frac{e^{i\omega x}}{i\omega}\right)\Bigg|_{-2}^{0} - \frac{1}{4}\left(\frac{e^{i\omega x}}{i\omega}\right)\Bigg|_{0}^{2} = \frac{1}{4i\omega}\left(e^{i\omega(0)} - e^{i\omega(-2)}\right) - \frac{1}{4i\omega}\left(e^{i\omega(2)} - e^{i\omega(0)}\right) \\ &= \frac{1}{4i\omega}\left(1 - e^{-2i\omega}\right) - \frac{1}{4i\omega}\left(e^{2i\omega} - 1\right) \\ &= \frac{1}{4i\omega}\left(-e^{2i\omega} - e^{-2i\omega} + 2\right) \\ &= -\frac{1}{4i\omega}\left(e^{i\omega} - e^{-i\omega}\right)^2 \end{aligned}$$

*Applying Euler's identity* $e^{i\omega} = \cos\omega + i\sin\omega, \quad e^{-i\omega} = \cos\omega - i\sin\omega.$
*We have:* $\hat{f}(\omega) = -\frac{1}{4i\omega}\left(2i\sin\omega\right)^2 \quad = -\frac{i\sin^2\omega}{\omega}.$

Intuitively, the solution steps of example 2.7 demonstrates the simple way of performing Fourier transform of piecewise function, using equation (5). It is a straightforward hint to execution of Fourier transform of real-valued functions. We give the graphical representation in 2-dimension and 3-dimension of $\hat{f}(\omega) = -\frac{i\sin^2\omega}{\omega}$ respectively.
The figure 1(a) is the 2-dimensional graph of the Fourier transform $\hat{f}(\omega)$ of the piecewise function in equation (12). It was observed that the graph behaves like signal processing. At a uniform spacing of the Fourier-based argument $\omega$ taking interval of 20 each up to 120 points, the corresponding Fourier transform $\hat{f}(\omega)$ were obtained from $0.0 - 0.7$. From the graph, $\hat{f}(\omega)$ was at its peak value when the argument $\omega = 0$. There was contraction in value for increase in the argument $\omega$ up to 120. In other words, the signal plots converges to zero for sufficient $\omega$-values $\ge 120$.




(a)

(b)

Figure 1: (a) 2D plot of the Fourier transform $\hat{f}(\omega)$ of the piecewise function $f(x)$ and (b) describes the 3D-plot of the Fourier transform $\hat{f}(\omega)$ of the piecewise function $f(x)$.

## 3. The Model

### *3.1. Black - Scholes model (BSM) Partial Differential Equation [10]*

The general Black-Scholes Partial Differential Equation is of the form:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0, \tag{14}$$

subject to initial condition $V(S,0) = \max(S-K,0)$, $S \geq 0$, and boundary condition $V(0,t) = 0$, $V(S,t) = S$ as $S \to \infty$, $0 \leq t \leq T$, where $V(S,t)$ is the price for an option, $S$ is the current option price, $K$ is the strike price, $r$ is a risk-neutral interest rate assumed constant, and $\sigma$ is the volatility rate which is also assumed to be constant by Black-Scholes [10]. The call option value $V(S,t)$ is exactly $S$ when the underlying asset value $S$ becomes very large and therefore the strike price becomes negligible. The real life interpretation of this condition that $S \to \infty$ (the underlying price becomes extremely large) causes the call option price to approach the exercise value $S$ implies the call option value is extremely "in the money" (ITM) and it shows that the call option fully worths exercising in the market. For detail, see equation (26) – (27) in [32].

Some solution steps of solving the BSM were demonstrated in [25] and [32] among other methods of solution. However, we give another formulation and demonstrate differential axiom of Fourier transform of solution other than those used in most studies and arrived at the generalization of BSM as follows.

### *3.2. Modified Black - Scholes Model (MBSM) Partial Differential Equation for Stock Asset*

In order to give the proposed MBSM model, we first consider the following definitions 3.1- 3.2.

**Definition 3.1 ([11]).** *Let $m \in \mathbb{N}$ be the set of natural numbers. The Factorial function denoted by $m!$ is the function that computes the product of the first $m$ natural numbers defined by:*

$$m! = m(m-1)\cdots 3\cdot 2\cdot 1. \tag{15}$$

*For large values of m, $m! \sim m^m e^{-m}$, equation (15) is equivalent to the restriction of the gamma function to positive integers. The restriction $f_{\mathcal{H}}(x)$ of a function $f(x)$ to a given nonempty set $\mathcal{H}$ is the set of pairs $\langle x, y\rangle$ such that $y = f(x)$ and $x \in \mathcal{H}$. For integral m, the Factorial function of m is given as*

$$m! = \Gamma(m+1) \quad \equiv \int_0^\infty t^m e^{-t} dt. \tag{16}$$

*From (16), it implies that for any $m \in \mathbb{N}$, $\Gamma(m) = \int_0^\infty t^{m-1}e^{-t} = (m-1)!$. Also, using integration by parts in Calculus, it holds that $\Gamma(m+1) = m\Gamma(m)$ which corresponds to (16). The gamma function generalizes the factorial function to positive real values. It was introduced by the "Swiss mathematician Leonhard Euler in the 18th century" [5]. It is further stressed in [5] that "the numerical values of factorial function between any two non-negative integers must lie between the range of their numerical values.*

**Definition 3.2.** *Let $(\Omega, \mathcal{F}, \mathbb{Q}, \mathbb{F})$ be a filtered probability space with $\mathbb{F} = \mathcal{F}_t$, $t \in [0, \infty)$. We define a random-variable volatility function as:*

$$\tilde{\sigma}_t = \Big\{ randn(a,b) \Big| a < \tilde{\sigma}_t < b, \quad \forall\, a, b \in \mathbb{R} \Big\} \tag{17}$$

*where $randn(\cdot,\cdot)$ is a function for generating random numbers within a specified interval. The choice of the value of a and b depends on the information outflow from the economy to the financial markets based on the state of the COVID-19 pandemic. Therefore, $\tilde{\sigma}_t = \Big\{\tilde{\sigma}_\tau \Big| \mathcal{F}(t)\Big\}$ where $\mathcal{F}(t)$ is the filtration of the random volatility function at a time t, up to the maturity time T of the option.*

Intuitively, the value of $a$ and $b$ respectively denote the lower bound and upper bound for the volatility function. That is, $a$ is the probable minimum value of the stock volatility and $b$ is the probable maximum value of the stock volatility in an unstable stock market. The prediction of the value of $a$ and $b$ is constrained on the historical volatility or the observable market price movement of the asset under consideration in application.

We incorporate the factorial function $m!$ equation (15) and variable volatility function $\tilde{\sigma}_t$ equation (17) into the BSM (14) and referred to the new model as modified Black-Scholes Model (MBSM) as follow. Consider an option on a Nigerian stock, $N\big(S(t), t = T\big)$, in the form of European style option where the operating rule is to leave the option until its maturity time $T$ before exercising it. As a call and put option on $S(t)$, setting $h = N^{call}\big(S(t), T\big) \equiv N^{call}$ and $N^{put}\big(S(t), T\big) = N^{put}$ respectively gives

$$\frac{\partial N^{call}}{\partial t} + m!\Big(r + \frac{1}{2}(m-1)\tilde{\sigma}_t{}^2\Big)S(t)\frac{\partial N^{call}}{\partial S(t)} + \frac{1}{2}\left(m!\,\tilde{\sigma}_t S(t)\right)^2 \frac{\partial^2 N^{call}}{\partial S^2(t)} - rN^{call} = 0, \tag{18}$$

subject to initial conditions (ICs) and boundary condition (BCs):

$$\begin{cases} IC_1: & \lim\limits_{S(t)\to\infty} N^{call}\big(S(t), T\big) & = \infty, \ \text{ on } [0,T], S > 0, \\ IC_2: & N^{call}\left(S(t), T\right) & = \phi\left(S(t)\right) = (S(t) - K)^+ \ \text{ on } [0,T], \ \text{(the call option payoff function)}. \\ BC: & N^{call}\left(0, T\right) & = 0, \ \text{ on } \ [0,T]. \end{cases} \tag{19}$$

Similarly, for put option, we have

$$\frac{\partial N^{put}}{\partial t} + m!\left(r + \frac{1}{2}(m-1)\tilde{\sigma}_t^2\right)S(t)\frac{\partial N^{put}}{\partial S(t)} + \frac{1}{2}\left(m!\,\tilde{\sigma}_t S(t)\right)^2 \frac{\partial^2 N^{put}}{\partial S^2(t)} - rN^{put} = 0 \tag{20}$$

subject to boundary conditions

$$\begin{cases} IC_1: & \lim\limits_{S(t)\to\infty} N^{put}\left(S(t), T\right) & = \ 0, \ \text{ on } [0,T], S(t) > 0, \\ IC_2: & N^{put}\left(S(t), T\right) & = \phi\left(S(t)\right) = (K - S(t))^+ \ \text{ on } [0,T], \ \text{(the put option payoff function)}. \\ BC: & N^{put}\left(0, T\right) & = Ke^{-r(T-t)} \ \text{ on } [0,T]. \end{cases} \tag{21}$$

The equations (18) and (20) are the generalization of Black and Scholes PDE for European-style call and put option respectively where specified values for $m \in \mathbb{N} \cup \{0\}$ are endowed with the Factorial function. For example, if $m = 1$, then (18) and (20) will be exactly the BSM of the form (14) except that the volatility in our case is not constant. The boundary conditions (19) and (21) are necessary as they determine the type of derivative evaluated. In addition, the boundary conditions help in obtaining closed - form formular for the FBS-PDEs which translate to the option pricing simulated values. Among the influences of the factorial function $m!$ on (18) and (20) is the generalizaton possibility of the BSM. Also, the influence of the factorial function will be more felt in further studies that have to do with statistical analysis (such as skewness, Kurtosis, etc) of financial models with respect to financial securities live data analysis especially via simulations. For instance, the figure 4(d) shows that an increase in $m!$-values results to fat tail of the density function but decrease in the density amplitude and this verifies risky asset return behaviour during financial crisis compare to the projected returns in normalcy period ([17], [18]). Hence, one could infer from here that incorporation of factorial function concept is not a waste of effort especially from financial data Scientist analysis perception which we hope to explore more in further studies.

## 4. Main results

### *4.1. Fourier Transform and computation of modified Black-Scholes Model (MBSM)*

We consider the MBSM presented in the section 3.2 with random-variable volatility function in place of constant volatility for Black - Scholes model. The intuition behind the introduction of variable volatility is to account for Covid-19 induced uncertainty effect on the volatility term on the underlying stock asset being a risky form of asset that its price changes stochastically in the financial markets.

**Theorem 4.1.** *Let the Stock asset $S(t)$ be driven by the Stochastic Differential Equation*

$$dS(t) = \left(m!r + \frac{1}{2}m!(m-1)\tilde{\sigma}_t^2 - q\right)S(t)dt + m!\tilde{\sigma}_t S(t)dW_t \tag{22}$$

*whose pde representation is given as*

$$\frac{\partial N^{put}}{\partial t}(S(t),T) + \frac{1}{2}\tilde{\sigma}_t^2 S^2 \frac{\partial^2 N^{put}}{\partial S^2(t)}(S(t),T) + rS\frac{\partial N^{put}(S(t),T)}{\partial S(t)} = rN^{put}(S(t),T), \quad m \in [0,1], \tag{23}$$

*An integral analytic formula for the stock is expressed as*

$$N(S,\tau) = \frac{1}{\tilde{\sigma}_t\sqrt{2\pi\tau}}e^{-r\tau}\int_0^\infty N_0(S_T)\frac{1}{S_T}\exp\left(-\frac{1}{2}\left(\frac{\ln\left(\frac{S_T}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right)^2\right)dS_T. \tag{24}$$

*Proof.* The following change of variable are made at constant interest rate $r$, and volatility $\sigma$.
Let $\tau : t \mapsto T - t$ be a mapping such that $\frac{\partial N}{\partial t} \mapsto -\frac{\partial N}{\partial \tau}$. The pde (23) in backward time becomes:

$$-\frac{\partial N^{put}}{\partial \tau}(S(t),T) + \frac{1}{2}\tilde{\sigma}^2 S^2 \frac{\partial^2 A^{put}}{\partial S^2(t)}(S(t),T) + rS\frac{\partial N^{put}(S(t),T)}{\partial S(t)} - rN^{put}(S(t),T) = 0. \tag{25}$$

Equivalently given as:

$$\frac{\partial N^{put}}{\partial \tau}(S(t),T) = \frac{1}{2}\tilde{\sigma}^2 S^2 \frac{\partial^2 N^{put}}{\partial S^2(t)}(S(t),T) + rS\frac{\partial N^{put}(S(t),T)}{\partial S(t)} - rN^{put}(S(t),T) \tag{26}$$

Let the log stock asset be defined by $y := \ln(S(t))$ as $S(t) \mapsto \ln S(t)$ whose option value is denoted by $N(S,t)$ in the partial differential equation at a time $t$ base on the new variables. The derivatives are now transformed as follows:

$$\frac{\partial N(S,t)}{\partial S(t)} = \frac{\partial N(S,t)}{\partial y}\cdot\frac{\partial y}{\partial S(t)} = \frac{\partial N(S,t)}{\partial y}\cdot\frac{\partial \ln S(t)}{\partial S(t)} \tag{27}$$

which reduces to:

$$\frac{\partial N(S,t)}{\partial S(t)} = \frac{1}{S(t)}\frac{\partial N(S,t)}{\partial y}. \tag{28}$$

Similarly,

$$\begin{aligned}\frac{\partial^2 N(S,t)}{\partial S^2(t)} &= \frac{\partial}{\partial S(t)}\cdot\frac{\partial N(S,t)}{\partial S(t)} = \frac{\partial}{\partial S(t)}\left(\frac{1}{S(t)}\frac{\partial N(S,t)}{\partial y}\right)\\ &= \frac{1}{S(t)}\frac{\partial}{\partial S(t)}\frac{\partial N(S,t)}{\partial y} + -\frac{1}{S^2(t)}\frac{\partial N(S,t)}{\partial y}\\ &= \frac{1}{S(t)}\left(\frac{\partial y}{\partial S(t)}\cdot\frac{\partial}{\partial y}\right)\frac{\partial N(S,t)}{\partial y} - \frac{1}{S^2(t)}\frac{\partial N(S,t)}{\partial y}.\end{aligned} \tag{29}$$

Substituting for $\frac{\partial y}{\partial S(t)} = \frac{1}{S(t)}$ yields the following

$$\begin{aligned}\frac{\partial^2 N(S,t)}{\partial S^2(t)} &= \frac{1}{S(t)}\left(\frac{1}{S(t)}\right)\cdot\frac{\partial}{\partial y}\frac{\partial N(S,t)}{\partial y} - \frac{1}{S^2(t)}\frac{\partial N(S,t)}{\partial y}\\ &= \frac{1}{S^2(t)}\cdot\frac{\partial^2 N(S,t)}{\partial y^2} - \frac{1}{S^2(t)}\frac{\partial N(S,t)}{\partial y}\end{aligned} \tag{30}$$

We factorize the common term $\frac{1}{S^2(t)}$ in the right hand side of (30) to obtain

$$\frac{\partial^2 N(S,t)}{\partial S^2(t)} = \frac{1}{S^2(t)}\left(\frac{\partial^2 N(S,t)}{\partial y^2} - \frac{\partial N(S,t)}{\partial y}\right). \tag{31}$$

Next, we substitute equation (28) and (31) in the pde representation given in (26) as follow:

$$\frac{\partial N^{put}}{\partial \tau}(S(t),T) = \frac{1}{2}\tilde{\sigma}^2 S^2\frac{\partial^2 N^{put}}{\partial S^2(t)}(S(t),T) + rS\frac{\partial N^{put}(S(t),T)}{\partial S(t)} - rN^{put}(S(t),T) \tag{32}$$

Based on the initial dynamics of the stock given as

$$dS(t) = \left(m!r + \frac{1}{2}m!(m-1)\tilde{\sigma}^2 - q\right)S(t)dt + m!\tilde{\sigma}S(t)dW_t \tag{33}$$

We first revert the pde (32) to the form:

$$\frac{\partial N^{put}}{\partial \tau}(S(t),T) = \frac{1}{2}(m!)^2\tilde{\sigma}_t^2 S^2\frac{\partial^2 N^{put}}{\partial S^2(t)}(S(t),T) + m!\left(r - \frac{1}{2}(m-1)\tilde{\sigma}^2\right)S(t)\frac{\partial N^{put}(S(t),T)}{\partial S(t)} - rN^{put}(S(t),T) \tag{34}$$

and then make substitution for the derivatives so that we have the following equations.

$$\frac{\partial N}{\partial \tau}(S_t,T) = \frac{1}{2}(m!)^2\tilde{\sigma}^2 S_t^2\frac{1}{S_t^2}\left(\frac{\partial^2 N(S,t)}{\partial y^2} - \frac{\partial N(S,t)}{\partial y}\right) + m!\left(r - \frac{1}{2}(m-1)\sigma^2\right)S_t\frac{1}{S_t}\frac{\partial N(S,t)}{\partial y} - rN(S_t,T) \tag{35}$$

which reduces to:

$$\frac{\partial N}{\partial \tau}(S_t,T) = \frac{1}{2}(m!)^2\tilde{\sigma}^2\left(\frac{\partial^2 N(S,t)}{\partial y^2} - \frac{\partial N(S,t)}{\partial y}\right) + m!\left(r - \frac{1}{2}(m-1)\tilde{\sigma}^2\right)\frac{\partial N(S,t)}{\partial y} - rN(S_t,T). \tag{36}$$

$$\frac{\partial N}{\partial \tau}(S_t,T) = \frac{1}{2}(m!)^2\sigma^2\frac{\partial^2 N(S,t)}{\partial y^2} + m!\left(-\frac{m!\tilde{\sigma}^2}{2} + \left(r - \frac{(m-1)\sigma^2}{2}\right)\right)\frac{\partial N(S,t)}{\partial y} - rN(S_t,T). \tag{37}$$

We are now set to take the Fourier transform of the derivatives with respect to the independent variable $y := \ln S(t)$ as follows

$$\mathcal{F}\left(\frac{\partial N}{\partial \tau}(S_t, T)\right) = \frac{(m!)^2}{2}\tilde{\sigma}^2 \mathcal{F}\left(\frac{\partial^2 N(S,t)}{\partial y^2}\right) + m!\left(-\frac{m!\tilde{\sigma}^2}{2} + \left(r - \frac{(m-1)\tilde{\sigma}^2}{2}\right)\right)\mathcal{F}\left(\frac{\partial N(S,t)}{\partial y}\right) - \mathcal{F}\left(rN(S_t,T)\right). \quad (38)$$

To this end, we apply the derivative property of Fourier transform (8) on the differential operators and the function $N(S_t, t)$ in (38) as follows:

$$\begin{cases} \mathcal{F}\left(\frac{\partial N}{\partial y}(S_t,T)\right) = i\omega\widehat{N}(S,t), \qquad ....(*) \\ \mathcal{F}\left(\frac{\partial^2 N(S,t)}{\partial y^2}\right) = \mathcal{F}\left(\frac{\partial}{\partial y}\left(\frac{\partial N}{\partial y}(S_t,T)\right)\right) = \mathcal{F}\left(\frac{\partial}{\partial y}\underbrace{\left(i\omega\widehat{N}(S,t)\right)}_{\text{from } (*)}\right) = (i\omega)^2\widehat{N}(S,t) = -\omega^2\widehat{N}(S,t), \\ \mathcal{F}\left(\frac{\partial N(S,t)}{\partial \tau}\right) = \frac{\partial \widehat{N}(S,t)}{\partial \tau}. \end{cases} \quad (39)$$

Substituting (39) into (38) yields the first order ordinary differential equation (40)

$$\frac{\partial \widehat{N}}{\partial \tau}(S_t,T) = -\frac{1}{2}(m!)^2\tilde{\sigma}^2\,\omega^2\widehat{N}(S_t,T) + m!\left(-\frac{m!\tilde{\sigma}^2}{2} + \left(r - \frac{(m-1)\tilde{\sigma}^2}{2}\right)\right)i\omega\widehat{N}(S_t,T) - r\widehat{N}(S_t,T) \quad (40)$$

We integrate (38) with respect to $\tau$ as follows:

$$\widehat{N} = \widehat{N}_0\, e^{-r\tau} \exp\left[-\frac{1}{2}(m!)^2\sigma^2\,\omega^2\tau + i\omega m!\left(-\frac{m!\sigma^2}{2} + \left(r - \frac{(m-1)\tilde{\sigma}^2}{2}\right)\right)\tau\right], \qquad \widehat{N_0} \neq 0. \quad (41)$$

With values of $m \in \mathbb{N} \cup \{0\}$, various solution will emanate from the generalized solution (41). For an illustration, setting $m = 0$ with $0! = 1$, we have a solution of the form:

$$\widehat{N} = \widehat{N}_0\, e^{-r\tau} \exp\left[-\frac{\tilde{\sigma}^2}{2}\omega^2\tau + i\omega\left(r + \frac{\tilde{\sigma}^2}{2}\right)\tau\right] \quad (42)$$

For $m = 1,\ 1! = 1$, the solution in equation (41) reduces to the form:

$$\widehat{N} = \widehat{N}_0\, e^{-r\tau} \exp\left[-\frac{\tilde{\sigma}^2}{2}\omega^2\tau + i\omega\left(r - \frac{\tilde{\sigma}^2}{2}\right)\tau\right] \quad (43)$$

Next by considering Fourier transform such that we set $\mu = \left(\frac{\tilde{\sigma}^2}{2} - r\right)\tau$ and $s = \tilde{\sigma}\sqrt{\tau}$. The term

$$e^{-r\tau}\exp\left(-\frac{\tilde{\sigma}^2}{2}\omega^2\tau + i\omega\left(r - \frac{\tilde{\sigma}^2}{2}\right)\tau\right) = \mathcal{F}\left[\frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left(-\frac{1}{2}\left(\frac{y - \left(\frac{\tilde{\sigma}^2}{2} - r\right)\tau}{\tilde{\sigma}\sqrt{\tau}}\right)^2\right)\right] \quad (44)$$

Choosing the solution in equation (43) with respect to (44) yields

$$\widehat{N}(y,\tau) = \widehat{N}_0\,\frac{1}{\sigma\sqrt{2\pi\tau}}\,e^{-r\tau}\mathcal{F}\left[N_0 \cdot \exp\left(-\frac{1}{2}\left(\frac{y - \left(\frac{\tilde{\sigma}^2}{2} - r\right)\tau}{\tilde{\sigma}\sqrt{\tau}}\right)^2\right)\right] \quad (45)$$

In the next step of solution, we take the *inverse Fourier transform* in order to revert to the normal form as follows:

$$N(y,\tau) = \frac{1}{\tilde{\sigma}\sqrt{2\pi\tau}}\,e^{-r\tau}\int_{-\infty}^{\infty} N_0(z)\exp\left(-\frac{1}{2}\left(\frac{y - z - \left(\frac{\tilde{\sigma}^2}{2} - r\right)\tau}{\tilde{\sigma}\sqrt{\tau}}\right)^2\right)dz \quad (46)$$

Equivalently stated as:

$$N(y,\tau) = \frac{1}{\tilde{\sigma}\sqrt{2\pi\tau}} e^{-r\tau} \int_{-\infty}^{\infty} N_0(z) \exp\left(-\frac{1}{2}\left(\frac{z - \left(y + \left(r - \frac{\tilde{\sigma}^2}{2}\right)\tau\right)}{\tilde{\sigma}\sqrt{\tau}}\right)^2\right) dz \tag{47}$$

The final step of solution which entails switching back to the original variables suitable for the initial setting such that logarithm of the initial stock price is defined by $y := \ln S(t_0)$ and $S(t) \mapsto \ln S(t)$ whose value was initially denoted by $N(S,t)$. It remains to change the variable $z$ back such that the stock price $S_\tau = e^z$ implies $z := \ln S_T$ and the known initial stock price at time $t_0 = 0$ is $S_0$. With the possible change of variables highlighted, we have an integral analytic formula for an asset price as

$$N(S,\tau) = \frac{1}{\tilde{\sigma}\sqrt{2\pi\tau}} e^{-r\tau} \int_{0}^{\infty} N_0(S_T) \frac{1}{S_T} \exp\left(-\frac{1}{2}\left(\frac{\ln S_T - \left(\ln S_0 + \left(r - \frac{\tilde{\sigma}^2}{2}\right)\tau\right)}{\tilde{\sigma}\sqrt{\tau}}\right)^2\right) dS_T. \tag{48}$$

This is further expressed as:

$$N(S,\tau) = \frac{1}{\tilde{\sigma}\sqrt{2\pi\tau}} e^{-r\tau} \int_{0}^{\infty} N_0(S_T) \frac{1}{S_T} \exp\left(-\frac{1}{2}\left(\frac{\ln\left(\frac{S_T}{S_0}\right) - \left(r - \frac{\tilde{\sigma}^2}{2}\right)\tau}{\tilde{\sigma}\sqrt{\tau}}\right)^2\right) dS_T. \tag{49}$$

□

*4.2. European Call & Put option price formula for the MBSM.*

**Lemma 4.2.** *Let $S(t)$ be an underlying stock asset whose dynamics is given by*

$$\frac{\partial N^{call}}{\partial t} + m!\left(r + \frac{1}{2}(m-1)\tilde{\sigma}_t{}^2\right)S(t)\frac{\partial N^{call}}{\partial S(t)} + \frac{1}{2}\left(m!\,\tilde{\sigma}_t S(t)\right)^2 \frac{\partial^2 N^{call}}{\partial S^2(t)} - rN^{call} = 0, \tag{50}$$

*with initial condition $N_0(S(t),T) = \phi(S(t)) = (S(t) - K)^+$ on $[0,T]$. Then an analytical integral formula for plain vanilla call option price is given as*

$$N^{call}(S,T) = \frac{1}{\tilde{\sigma}\sqrt{2\pi\tau}} e^{-r\tau} \int_{K}^{\infty} (S_T - K) \frac{1}{S_T} \exp\left(-\frac{1}{2}\left(\frac{\ln S_T - \left(\ln S_0 + \left(r - \frac{\tilde{\sigma}^2}{2}\right)\tau\right)}{\tilde{\sigma}\sqrt{\tau}}\right)^2\right) dS_T. \tag{51}$$

*Proof.* Given the dynamics of a stock asset price as

$$\frac{\partial N^{call}}{\partial t} + m!\left(r + \frac{1}{2}(m-1)\tilde{\sigma}_t{}^2\right)S(t)\frac{\partial N^{call}}{\partial S(t)} + \frac{1}{2}\left(m!\,\tilde{\sigma}_t S(t)\right)^2 \frac{\partial^2 N^{call}}{\partial S^2(t)} - rN^{call} = 0. \tag{52}$$

Using the derivative property of Fourier transform (8) on the differential operators and the function $N(S_t,t)$ in (52) gives

$$\frac{\partial \widehat{N}}{\partial \tau}(S_t,T) = -\frac{1}{2}(m!)^2\tilde{\sigma}^2\,\omega^2\widehat{N}(S_t,T) + m!\left(-\frac{m!\tilde{\sigma}^2}{2} + \left(r - \frac{(m-1)\tilde{\sigma}^2}{2}\right)\right) i\omega\widehat{N}(S_t,T) - r\widehat{N}(S_t,T). \tag{53}$$

Integrating (53) on the interval $[0,T]$ leads to

$$\widehat{N}^{call}(S,T) = \widehat{N}_0\, e^{-r\tau} \exp\left[-\frac{1}{2}(m!)^2\sigma^2\,\omega^2\tau + i\omega m!\left(-\frac{m!\sigma^2}{2} + \left(r - \frac{(m-1)\tilde{\sigma}^2}{2}\right)\right)\tau\right], \quad \widehat{N_0}(S,T) \neq 0. \tag{54}$$

Applying the procedures in (43) through (47) on (53) yields

$$N(S,\tau) = \frac{1}{m!\tilde{\sigma}\sqrt{2\pi\tau}}\, e^{-r\tau} \int_0^\infty N_0(S_T)\,\frac{1}{S_T}\exp\left[-\frac{1}{2}\left(\frac{\ln S_T - \left(\ln S_0 + m!\left(r - \frac{\tilde{\sigma}^2}{2}\right)\tau\right)}{m!\tilde{\sigma}\sqrt{\tau}}\right)^2\right] dS_T. \tag{55}$$

We then use the initial condition such that $N_0(S_T) = \max(S_T - K, 0) = (S_T - K)$, if $S_T > K$, we have the desired analytical formula for a plain vanilla call option price as

$$\begin{aligned} N^{call}(S,T) &= \frac{1}{m!\tilde{\sigma}\sqrt{2\pi\tau}}\, e^{-r\tau}\int_K^\infty (S_T - K)\,\frac{1}{S_T}\exp\left[-\frac{1}{2}\left(\frac{\ln S_T - \left(\ln S_0 + m!\left(r-\frac{\tilde{\sigma}^2}{2}\right)\tau\right)}{m!\tilde{\sigma}\sqrt{\tau}}\right)^2\right] dS_T, \\ &= e^{-r\tau}\int_K^\infty (S_T - K)\,\frac{1}{S_T\tilde{\sigma}\sqrt{2\pi\tau}}\exp\left[-\frac{1}{2}\left(\frac{\ln S_T - \left(\ln S_0 + m!\left(r-\frac{\tilde{\sigma}^2}{2}\right)\tau\right)}{m!\tilde{\sigma}\sqrt{\tau}}\right)^2\right] dS_T. \end{aligned} \tag{56}$$

where $\tau = t - T$ is the time to the expiration of the option. For $m = 0$, we arrived at

$$N^{call}(S,T) = e^{-r\tau}\int_K^\infty (S_T - K)\,\frac{1}{S_T\tilde{\sigma}\sqrt{2\pi\tau}}\exp\left[-\frac{1}{2}\left(\frac{\ln S_T - \left(\ln S_0 + \left(r-\frac{\tilde{\sigma}^2}{2}\right)\tau\right)}{\tilde{\sigma}\sqrt{\tau}}\right)^2\right] dS_T \tag{57}$$

□

The equation (57) verifies the analytical call option pricing formula given as equation 2.7 on page 7 in [31], except that our volatility here is not constant and the methodology adopted in this paper is different from that of [31]. It proven that the Fourier transform methodology applied on the MBSM is able to yield a desirable result and the MBSM generalizes the BSM.

**Lemma 4.3.** *Let $S(t)$ be an underlying stock asset with dynamics as the MBSM given by*

$$\frac{\partial N^{put}}{\partial t} + m!\left(r + \frac{1}{2}(m-1)\tilde{\sigma}_t{}^2\right)S(t)\frac{\partial N^{put}}{\partial S(t)} + \frac{1}{2}\left(m!\,\tilde{\sigma}_t S(t)\right)^2\frac{\partial^2 N^{put}}{\partial S^2(t)} - rN^{call} = 0, \tag{58}$$

*with initial condition $N_0\left(S(t), T\right) = \phi\left(S(t)\right) = (K - S_t)^+$ on $[0,T]$, for $K > S_t$. Then an analytical integral formula for plain vanilla put option price is given as*

$$N^{put}(S,T) = e^{-r\tau}\int_K^\infty (K - S_T)\,\frac{1}{S_T\tilde{\sigma}\sqrt{2\pi\tau}}\exp\left[-\frac{1}{2}\left(\frac{\ln S_T - \left(\ln S_0 + \left(r-\frac{\tilde{\sigma}^2}{2}\right)\tau\right)}{\tilde{\sigma}\sqrt{\tau}}\right)^2\right] dS_T. \tag{59}$$

*Proof.* The proof is similar to the proof of 4.2 except that the payoff function is different for put option. Let the dynamics of the stock asset price be given as

$$\frac{\partial N^{put}}{\partial t} + m!\left(r + \frac{1}{2}(m-1)\tilde{\sigma}_t{}^2\right)S(t)\frac{\partial N^{put}}{\partial S(t)} + \frac{1}{2}\left(m!\,\tilde{\sigma}_t S(t)\right)^2\frac{\partial^2 N^{put}}{\partial S^2(t)} - rN^{put} = 0. \tag{60}$$

Using the derivative property of Fourier transform (8) on the differential operators and the function $N(S_t, t)$ in (60) gives

$$\frac{\partial\widehat{N}}{\partial\tau}(S_t,T) = -\frac{1}{2}(m!)^2\tilde{\sigma}^2\,\omega^2\widehat{N}(S_t,T) + m!\left(-\frac{m!\tilde{\sigma}^2}{2} + \left(r - \frac{(m-1)\tilde{\sigma}^2}{2}\right)\right)i\omega\widehat{N}(S_t,T) - r\widehat{N}(S_t,T). \tag{61}$$

Integrating (61) on the interval $[0,T]$ leads to

$$\widehat{N}^{put}(S,T) = \widehat{N}_0\, e^{-r\tau}\exp\left[-\frac{1}{2}(m!)^2\sigma^2\,\omega^2\tau + i\omega m!\left(-\frac{m!\sigma^2}{2} + \left(r - \frac{(m-1)\tilde{\sigma}^2}{2}\right)\right)\tau\right], \quad \widehat{N_0}(S,T) \neq 0. \tag{62}$$

Applying the procedures in (43) through (47) on (61) yields

$$N^{\text{put}}(S,\tau) = \frac{1}{m!\tilde{\sigma}\sqrt{2\pi\tau}} e^{-r\tau} \int_0^\infty N_0(S_T) \frac{1}{S_T} \exp\left(-\frac{1}{2}\left(\frac{\ln S_T - \left(\ln S_0 + m!\left(r - \frac{\tilde{\sigma}^2}{2}\right)\tau\right)}{m!\tilde{\sigma}\sqrt{\tau}}\right)^2\right) dS_T. \tag{63}$$

Setting $N_0(S_T) = \max(K - S_T, 0) = (K - S_T)$, for $K > S_T$ and for $m = 0$, an analytical formula for a plain vanilla European put option price is given as

$$\begin{aligned} N^{put}(S,T) &= \frac{1}{\tilde{\sigma}\sqrt{2\pi\tau}} e^{-r\tau} \int_K^\infty (K - S_T) \frac{1}{S_T} \exp\left(-\frac{1}{2}\left(\frac{\ln S_T - \left(\ln S_0 + \left(r - \frac{\tilde{\sigma}^2}{2}\right)\tau\right)}{\tilde{\sigma}\sqrt{\tau}}\right)^2\right) dS_T, \\ &= e^{-r\tau} \int_K^\infty (K - S_T) \frac{1}{S_T\tilde{\sigma}\sqrt{2\pi\tau}} \exp\left(-\frac{1}{2}\left(\frac{\ln S_T - \left(\ln S_0 + \left(r - \frac{\tilde{\sigma}^2}{2}\right)\tau\right)}{\tilde{\sigma}\sqrt{\tau}}\right)^2\right) dS_T. \end{aligned} \tag{64}$$

□

**Remark 4.4.** *The probability density function for a normal distribution of the stock price from our result in the equation (64) is hereby denoted by* $\mathbf{q}(S_T)$ *and given as*

$$\mathbf{q}(S_T) = \frac{1}{S_T\tilde{\sigma}\sqrt{2\pi\tau}} \exp\left(-\frac{1}{2}\left(\frac{\ln S_T - \left(\ln S_0 + \left(r - \frac{\tilde{\sigma}^2}{2}\right)\tau\right)}{\tilde{\sigma}\sqrt{\tau}}\right)^2\right). \tag{65}$$

*The MBSM stock price dynamics following normal distribution has mean value as* $\mu = \ln S_0 + \left(r - \frac{\sigma_t^2}{2}\right)T$ *and variance* $\sigma^2 T$. *We state the following call and put option formula in closed form respectively for the MBSM analogous to the Black-Scholes [10] formula for option pricing as follow.*

*(i) Let the stock asset $S(t)$ evolves in accordance to the Stochastic differential equation:*

$$dS(t) = \left(m!r + \frac{1}{2}m!(m-1)\tilde{\sigma}_t^2 - q\right)S(t)dt + m!\tilde{\sigma}_t S(t)dW_t, \quad m \in [0,1], \tag{66}$$

*satisfying the density function (65). Then The plain vanilla call option price formula on the underlying stock price $S$ at exercising time $T$ for the MBSM is expressed as*

$$\begin{aligned} Call_{MBSM}(S,t) &= \Phi(d_1)S(t) - \Phi(d_2)K e^{-r(T-t)} \\ d_1 &= \frac{\ln\left(\frac{S(t)}{K}\right) + \left(r - q + \frac{\tilde{\sigma}_t^2}{2}\right)(T-t)}{\tilde{\sigma}_t\sqrt{T-t}} \\ d_2 &= d_1 - \tilde{\sigma}_t^2\sqrt{T-t}. \end{aligned}$$

*(ii) The put option price formula in a closed form on an underlying stock asset $S$ at exercising time $T$ for the MBSM is given as*

$$\begin{aligned} Put_{MBSM}(S,t) &= \Phi(-d_2)K e^{-r(T-t)} - \Phi(-d_1)S(t) \\ d_1 &= \frac{\ln\left(\frac{S(t)}{K}\right) + \left(r - q + \frac{\tilde{\sigma}_t^2}{2}\right)(T-t)}{\tilde{\sigma}_t\sqrt{T-t}} \\ d_2 &= d_1 - \tilde{\sigma}_t^2\sqrt{T-t}. \end{aligned}$$

**Remark 4.5.**

- *For a non-dividend paying call option, one will set $q = 0$.*
- *The $\Phi(d_1)S(t)$ controls the expected gain from buying a stock completely. The second part $\Phi(d_2)K e^{-r(T-t)}$ estimates the present value of paying the exercise price, $K$, on the maturity date $T$. The two symbols $\Phi(d_1)$ and $\Phi(d_2)$ are standard normal cumulative distribution function.*

## 5. Numerical Results and Analysis

In this section, we give numerical result for European call option. The corresponding European put option prices could easily be obtained via put-call parity having known the call option value.

With constant riskfree interest rate $r = 0.4$, constant volatility $\sigma = 0.5$ for Black-Scholes model (BSM), and with specified values for the variable volatility $\tilde{\sigma}_t$ in each table for the modified Black - Scholes (MBSM), the results in Table 1 and Table 2 were obtained with respect to the models BSM, MBSM and Carr & Madan [12] FFT-algorithm. From the two tables above, it was observed that the stock returns in terms of the

Table 1: **First Simulation result of non-dividend paying European-style call option returns on Nigerian Stock under constant and random (variable) volatility function**

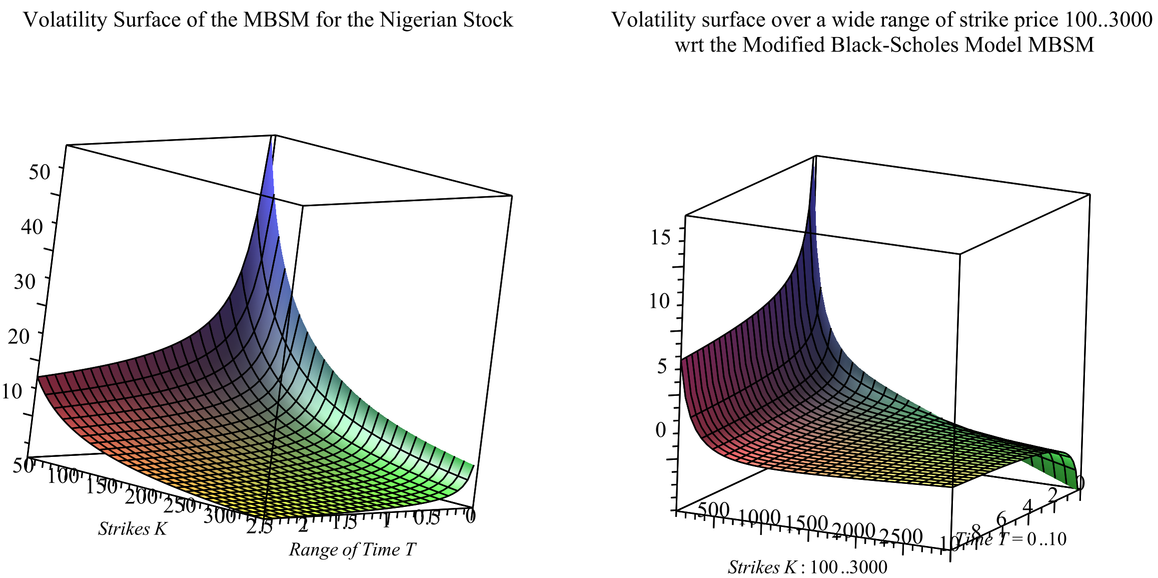
| Asset price **S** | Strike price **K** | Constant Volatility $\sigma$ | Random Volatility $\widetilde{\sigma}$ | Exercise Time (year) | Option Prices BSM | Option Prices MBSM | Option Prices FFT |
|---|---|---|---|---|---|---|---|
| 100.5 | 80 | 0.5 | 0.545 | 0.33 | 31.59417 | 32.01060 | 32.14628 |
| 100.5 | 80 | 0.5 | 0.507 | 0.33 | 31.59417 | 31.65558 | 31.79123 |
| 100.5 | 80 | 0.5 | 0.582 | 0.33 | 31.59417 | 32.38878 | 32.63832 |
| 100.5 | 80 | 0.5 | 0.492 | 0.50 | 36.55854 | 36.47298 | 36.71610 |
| 100.5 | 80 | 0.5 | 0.470 | 0.50 | 36.55854 | 36.24827 | 36.49141 |
| 100.5 | 80 | 0.5 | 0.570 | 0.50 | 36.55854 | 37.38664 | 37.62979 |

Table 2: **Second Simulation result of non-dividend paying European-style call options returns on Nigerian Stock under constant and (variable) volatility function**

| Asset price **S** | Strike prices **K** | Constant Volatility $\sigma$ | Random Volatility $\widetilde{\sigma}$ | Exercise Time (year) | Option Prices BSM | Option Prices MBSM | Option Prices FFT |
|---|---|---|---|---|---|---|---|
| 100.5 | 80 | 0.5 | 0.470 | 0.33 | 31.59417 | 31.34639 | 31.59593 |
| 100.5 | 80 | 0.5 | 0.646 | 0.33 | 31.59417 | 33.10784 | 33.35738 |
| 100.5 | 80 | 0.5 | 0.650 | 0.33 | 31.59417 | 33.15521 | 33.40474 |
| 100.5 | 80 | 0.5 | 0.518 | 0.50 | 36.55854 | 36.75831 | 37.00145 |
| 100.5 | 80 | 0.5 | 0.547 | 0.50 | 36.55854 | 37.09985 | 37.34299 |
| 100.5 | 80 | 0.5 | 0.616 | 0.50 | 36.55854 | 37.99786 | 38.12711 |

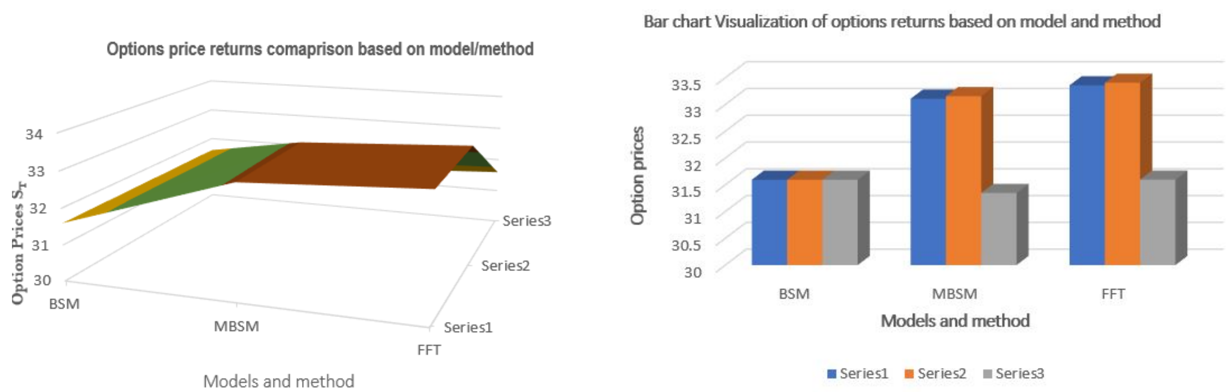
BSM remain constant throughout due to the constant assumption of the volatility term while the MBSM in which the volatility term is random, the option prices varied. At times, the option returns increased and decreased in some cases duet to the randomness of the volatility term. We can inferred from here that the MBSM returns demonstrated the stock market behaviour in the actual sense unlike the BSM. It is hoped this assumption of the MBSM will interest investors rather than the BSM.

### 5.1. Volatility Surface for Nigerian Flourmill Stock using the MBSM




(a) (b)

Figure 2: Option price volatility surfaces: (a) describes the Volatility surface for Nigerian Flourmill stock price during Covid-19 and (b) describes the Volatility surface over a wide range of Strike prices for Nigerian Flourmill stock.




(a) (b)

Figure 3: Option price comparison based on Models & method: (a) describes the surface plot for option price with respect to the BSM, MBSM & FFT; and, (b) describes the Bar chart visualization of option returns in the BSM, MBSM & FFT respectively.

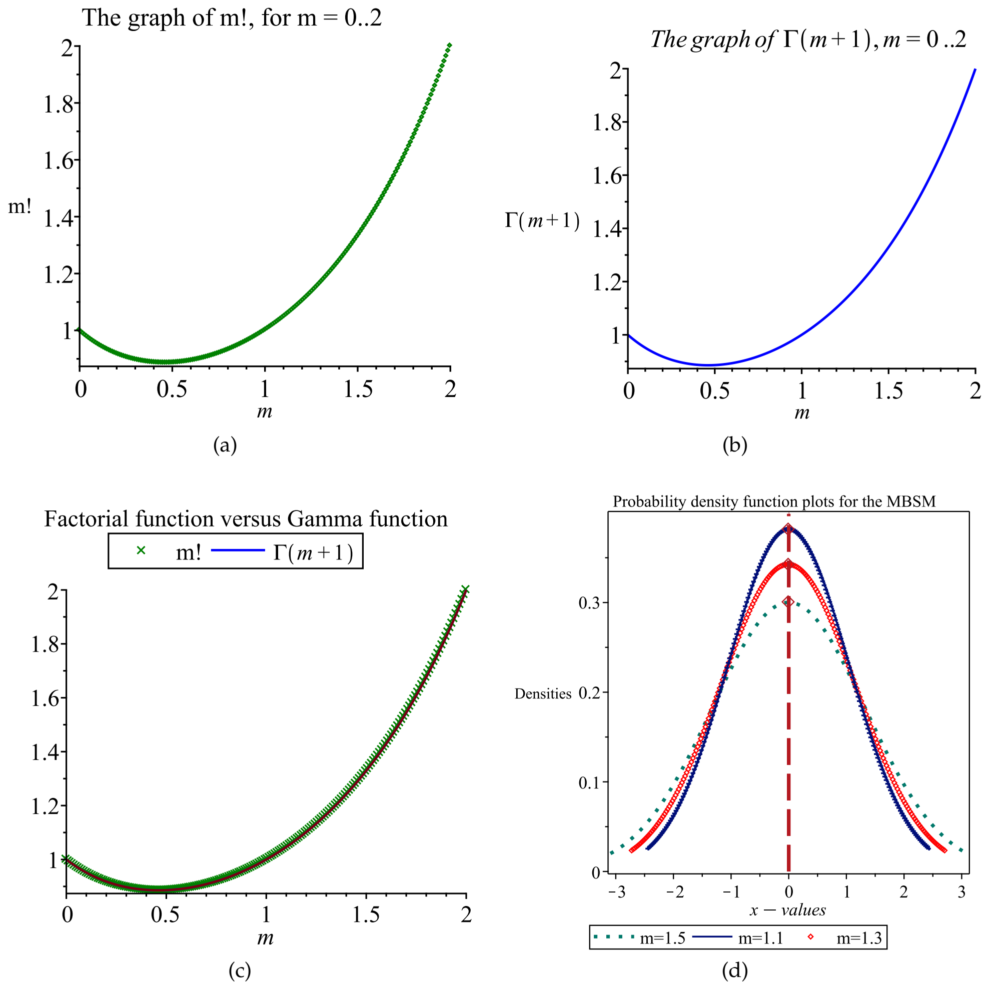



Figure 4: (a) Factorial function plot, (b) Gamma function plot, (c) Factorial & Gamma function plot,(d) The graph of probability density function for the MBSM at varied values of m!.

The figure 4 verified the similarity of factorial function and gamma function in equation (16), while 4(d) shows the probability density function graph at varied values of $m!$ in the MBSM. It is observed that an increased in the value of $m$ increases the fat tail of the density but decrease in the amplitude of the density. Studies such as [17], [18] shows that stock return distributions exhibit fat tail during financial crisis than expected probability of return in financial markets especially risky assets. The crisis of covid-19 is not left behind among other sources of stock asset return uncertainty. A nonlinear fitting of Nigerian Flourmill stock data to the MBSM-model gave the volatility surfaces in Figure 2(a) and Figure 2(b). The visualizations truly formed curved surface which accounts for an improvement on the classical Black-Scholes model of 1973. The major observation is that the classical BSM is common to have flat surface owing to constant volatility structure. In figure 3(b), we see that in the case of BSM, the option price did not change for three consecutive simulations owing to the constant volatility syndrome unlike that of MBSM and its FFT output.

## 6. Conclusion

Options valuation in a volatile economy under Covid-19 pandemic induced uncertainties was studied. We relaxed the assumption of constant volatility term structure in Black & Scholes options pricing model.

Fourier transform methodology was adopted to solve the modified Black-Scholes model (MBSM) partial differential equation. The interval for the random variable volatility was set following the historical volatility of the option to forecast future time volatility. In real life, volatility term is never constant, hence it demands using stochastic form of volatility term in which we have subjected the popular Black - Scholes volatility term to a form of random variable volatility forming accumulation points around the constant volatility term which in turn varies the option prices. Numerical results were generated and compared with the Black - Scholes Model. From the tables of result, the MBSM-prices and FFT-prices showed varied option returns for replicated MBSM simulation which are more realistic unlike the BSM that assumed constant volatility term. The BSM has been criticized by various studies in the past. Hence, our findings is in agreement with [13], [22] which encouraged stochastic form of volatilities for option valuation. The options prices on the underlying Nigerian stock obtained in the Table 1 and Table 2 show the payoff uncertainties from the Covid-19. Therefore, a form of random volatility function is better used in option pricing to account for uncertainties in the stock output under uncertain events like Covid-19. Hence, it is recommended to relax constant volatility term structure of Black - Scholes and apply stochastic form of volatility in which we used random variable volatility.

### Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publishing of this paper.


### Acknowledgement

The authors would like to thank the editor and the reviewers for their germane suggestions and remarks which helped to improve this study.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