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$*$-Ricci-Yamabe soliton on Kenmotsu manifold with torse forming potential vector field

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Abstract. The goal of the present paper is to deliberate $*$-Ricci-Yamabe soliton, whose potential vector field is torse-forming on the Kenmotsu manifold. Here, we have shown the nature of the soliton and found the scalar curvature when the manifold admitting $*$-Ricci-Yamabe soliton on the Kenmotsu manifold. Next, we have evolved the characterization of the vector field when the manifold satisfies $*$-Ricci-Yamabe soliton. Also, we have embellished some applications of a vector field as torse-forming in terms of $*$-Ricci-Yamabe soliton on the Kenmotsu manifold. We have developed an example of $*$-Ricci-Yamabe soliton on 3-dimensional Kenmotsu manifold to prove our findings.

1. Introduction

In 1972, K. Kenmotsu [20] obtained some tensor equations to characterize the manifolds of the third class. Since then the manifolds of the third class have been called Kenmotsu manifolds. In 1982, R. S. Hamilton [17] introduced the concept of Ricci flow, which is an evolution equation for metrics on a Riemannian manifold. The Ricci flow equation is given by:

$$\frac{\partial g}{\partial t} = -2S, \tag{1.1}$$

on a compact Riemannian manifold M with Riemannian metric g. A self-similar solution to the Ricci flow ([17], [32]) is called a Ricci soliton [18] if it moves only by a one-parameter family of diffeomorphism and scaling. The Ricci soliton equation is given by:

$$\pounds_V g + 2S + 2\Lambda g = 0, \tag{1.2}$$

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where $£_V$ is the Lie derivative in the direction of V, S is Ricci tensor, g is Riemannian metric, V is a vector field and Λ is a scalar. The Ricci soliton is said to be shrinking, steady, and expanding accordingly as Λ is negative zero, and positive respectively. The concept of Yamabe flow was first introduced by Hamilton [18] to construct Yamabe metrics on compact Riemannian manifolds. On a Riemannian or pseudo-Riemannian manifold M, a time-dependent metric $g(\cdot, t)$ is said to evolve by the Yamabe flow if the metric g satisfies the given equation,

$$\frac{\partial}{\partial t}g(t) = -rg(t), \qquad g(0) = g_0, \tag{1.3}$$

where r is the scalar curvature of the manifold M. In 2-dimension the Yamabe flow is equivalent to the Ricci flow [17] (defined by $\frac{\partial}{\partial t}g(t) = -2S(g(t))$, where S denotes the Ricci tensor). But in dimension, > 2 the Yamabe and Ricci flows do not agree, since the Yamabe flow preserves the conformal class of the metric but the Ricci flow does not in general. A Yamabe soliton [2] corresponds to a self-similar solution of the Yamabe flow, and is defined on a Riemannian or pseudo-Riemannian manifold (M, g) as:

$$\frac{1}{2}£_V g = (r - \Lambda)g, \tag{1.4}$$

where $£_V g$ denotes the Lie derivative of the metric g along the vector field V, r is the scalar curvature and Λ is a constant. Moreover, a Yamabe soliton is said to be expanding, steady, and shrinking depending on Λ being positive, zero, and negative respectively. If Λ is a smooth function then (1.4) is called almost Yamabe soliton [2]. Many authors ([11], [12], [22], [26], [27], [30], [28], [10], [13], [14], [3], [7], [25]) have studied Ricci soliton, Yamabe soliton and its generalizations on contact manifolds. Recently in 2019, S. Güler and M. Crasmareanu [15] introduced a new geometric flow which is a scalar combination of Ricci and Yamabe flow under the name Ricci-Yamabe map. This flow is also known as the Ricci-Yamabe flow of the type (α, β_1). Let (M^n, g) be a Riemannian manifold and $T_2^s(M)$ be the linear space of its symmetric tensor fields of (0, 2)-type and $Riem(M) \subsetneq T_2^s(M)$ be the infinite space of its Riemannian metrics. In [15], the authors have stated the following definition:

Definition 1.1:[15] A Riemannian flow on M is a smooth map:

$$g : I \subseteq \mathbb{R} \to Riem(M),$$

where I is a given open interval. We can call it also a time-dependent (or non-stationary) Riemannian metric.

Definition 1.2:[15] The map $RY^{(\alpha_1,\beta_1,g)} : I \to T_2^s(M)$ given by:

$$RY^{(\alpha_1,\beta_1,g)} := \frac{\partial}{\partial t}g(t) + 2\alpha_1 S(t) + \beta_1 r(t)g(t),$$

is called the (α_1, β_1)-Ricci-Yamabe map of the Riemannian flow of (M^n, g), where α, β_1 are some scalars. If $RY^{(\alpha_1,\beta_1,g)} \equiv 0$, then $g(\cdot)$ will be called an (α_1, β_1)-Ricci-Yamabe flow.

Also in [15], the authors characterized that the (α_1, β_1)-Ricci-Yamabe flow is said to be:
- Ricci flow [17] if $\alpha_1 = 1, \beta_1 = 0$.
- Yamabe flow [18] if $\alpha_1 = 0, \beta_1 = 1$.
- Einstein flow ([4], [29]) if $\alpha_1 = 1, \beta_1 = -1$.

A soliton to the Ricci-Yamabe flow is called Ricci-Yamabe soliton if it moves only by one parameter group of diffeomorphism and scaling. The metric of the Riemannian manifold (M^n, g), $n > 2$ is said to admit (α_1, β_1)-Ricci-Yamabe soliton or simply Ricci-Yamabe soliton (RYS) $(g, V, \Lambda, \alpha_1, \beta_1)$ if it satisfies the equation:

$$£_V g + 2\alpha_1 S + [2\Lambda - \beta_1 r]g = 0, \tag{1.5}$$

where $\pounds_V g$ denotes the Lie derivative of the metric g along the vector field V, S is the Ricci tensor, r is the scalar curvature and $\Lambda, \alpha_1, \beta_1$ are real scalars.

In the above equation if the vector field V is the gradient of a smooth function f (denoted by Df, D denotes the gradient operator) then the equation (1.5) is called gradient Ricci-Yamabe soliton (GRYS) and it is defined as:

$$Hess f + \alpha_1 S + \Big[\Lambda - \frac{1}{2}\beta_1 r\Big] g = 0, \tag{1.6}$$

where $Hess f$ is the Hessian of the smooth function f. Moreover, the Ricci-Yamabe soliton and gradient Ricci-Yamabe soliton are said to be expanding, steady, or shrinking according to Λ is positive, zero, and negative respectively. Also if $\Lambda, \alpha_1, \beta_1$ become smooth functions then (1.5) and (1.6) are called almost Ricci-Yamabe soliton and gradient almost Ricci-Yamabe soliton respectively. The concept of $*$-Ricci tensor on almost Hermitian manifolds and $*$-Ricci tensor of real hypersurfaces in non-flat complex space were introduced by Tachibana [31] and Hamada [16] respectively where the $*$-Ricci tensor is defined by:

$$S^*(V_1, V_2) = \frac{1}{2}(\mathrm{Tr}\{\varphi \circ R(V_1, \varphi V_2)\}), \tag{1.7}$$

for all vector fields V_1, V_2 on M^n, φ is a (1,1)-tensor field and Tr denotes Trace. If $S^*(V_1, V_2) = \lambda g(V_1, V_2) + \nu\eta(V_1)\eta(V_2)$ for all vector fields V_1, V_2 and λ, ν are smooth functions, then the manifold is called $*$-η-Einstein manifold. Further if $\nu = 0$ i.e $S^*(V_1, V_2) = \lambda g(V_1, V_2)$ for all vector fields V_1, V_2 then the manifold becomes $*$-Einstein. In 2014, Kaimakamis and Panagiotidou [19] introduced the notion of $*$-Ricci soliton which can be defined as:

$$\pounds_V g + 2S^* + 2\Lambda g = 0, \tag{1.8}$$

for all vector fields V_1, V_2 on M^n and Λ being a constatnt. In [33], authors have considered $*$-Ricci solitons and gradient almost $*$-Ricci solitons on Kenmotsu manifolds and obtained some beautiful results. Very recently, Ali et al. [23] and Dey et al. [9, 21, 26, 28] have studied $*$-Ricci solitons and their generalizations in the framework of almost contact geometry. Using (1.8) and (1.5), we can introduce $*$-Ricci-Yamabe soliton [8] as:

Definition 1.3: A Riemannian or pseudo-Riemannian manifold (M, g) of dimension n is said to admit $*$-Ricci-Yamabe soliton if

$$\pounds_V g + 2\alpha_1 S^* + [2\Lambda - \beta_1 r^*] g = 0, \tag{1.9}$$

where $\pounds_V g$ denotes the Lie derivative of the metric g along the vector field V, S^* is the $*$-Ricci tensor, $r^* = \mathrm{Tr}(S^*)$ is the $*$- scalar curvature and $\Lambda, \alpha_1, \beta_1$ are real scalars. The $*$-Ricci-Yamabe soliton is said to be expanding, steady, and shrinking depending on Λ being positive, zero, and negative respectively. If the vector field V is of gradient type i.e. $V = grad(f)$, for f is a smooth function on M, then the equation (1.9) is called gradient $*$-Ricci-Yamabe soliton. On the other hand, a nowhere vanishing vector field τ on a Riemannian or pseudo-Riemannian manifold (M, g) is called torse-forming [36] if

$$\nabla_{V_1}\tau = \psi V_1 + \omega(V_1)\tau, \tag{1.10}$$

where ∇ is the Levi-Civita connection of g, ψ is a smooth function and ω is a 1-form. Moreover The vector field τ is called concircular ([5], [35]) if the 1-form ω vanishes identically in the equation (1.10). The vector field τ is called concurrent ([24], [34]) if in (1.10) the 1-form ω vanishes identically and the function $\psi = 1$. The vector field τ is called recurrent if in (1.10) the function $\psi = 0$. Finally if in (1.10) $\psi = \omega = 0$, then the vector field τ is called a parallel vector field. In 2017, Chen [6] introduced a new vector field called a torqued vector field. If the vector field τ satisfies (1.10) with $\omega(\tau) = 0$, then τ is called torqued vector field. Also in this case, ψ is known as the torqued function and the 1-form ω is the torqued form of τ.

The outline of the article goes as follows: In section 2, after a brief introduction, we have discussed some needful results, which will be used in the later section. In section 3, we have contrived $*$-Ricci-Yamabe soliton admitting Kenmotsu manifold and obtained the nature of soliton, Laplacian of the smooth function. We have also proved that the manifold is η-Einstein when the manifold satisfies $*$-Ricci-Yamabe soliton and the vector field is conformal Killing. Next, we have demonstrated some properties of vector fields on $*$-Ricci-Yamabe soliton. Section 5 deals with some geometrical and physical motivation of $*$-Ricci-Yamabe soliton. In section 6, we have constructed an example to illustrate the existence of $*$-Ricci-Yamabe soliton on 3-dimensional Kenmotsu manifold.

2. Preliminaries

Let M be a $(2n+1)$ dimensional connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g) where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is the compatible Riemannian metric such that

$$\phi^2(V_1) = -V_1 + \eta(V_1)\xi, \eta(\xi) = 1, \eta \circ \phi = 0, \phi\xi = 0, \tag{2.1}$$

$$g(\phi V_1, \phi V_2) = g(V_1, V_2) - \eta(V_1)\eta(V_2), \tag{2.2}$$

$$g(V_1, \phi V_2) = -g(\phi V_1, V_2), \tag{2.3}$$

$$g(V_1, \xi) = \eta(V_1) \tag{2.4}$$

for all vector fields $V_1, V_2 \in \chi(M)$.
An almost contact metric manifold is said to be a Kenmotsu manifold [20] if

$$(\nabla_{V_1}\phi)V_2 = -g(V_1, \phi V_2)\xi - \eta(V_2)\phi V_1, \tag{2.5}$$

$$\nabla_{V_1}\xi = V_1 - \eta(V_1)\xi, \tag{2.6}$$

where ∇ denotes the Riemannian connection of g. In a Kenmotsu manifold the following relations hold ([1], [30]):

$$\eta(R(V_1, V_2)V_3) = g(V_1, V_3)\eta(V_2) - g(V_2, V_3)\eta(V_1), \tag{2.7}$$

$$R(V_1, V_2)\xi = \eta(V_1)V_2 - \eta(V_2)V_1, \tag{2.8}$$

$$R(V_1, \xi)V_2 = g(V_1, V_2)\xi - \eta(V_2)V_1, \tag{2.9}$$

where R is the Riemannian curvature tensor.

$$S(V_1, \xi) = -2n\eta(V_1), \tag{2.10}$$

$$S(\phi V_1, \phi V_2) = S(V_1, V_2) + 2n\eta(V_1)\eta(V_2), \tag{2.11}$$

$$(\nabla_{V_1}\eta)V_2 = g(V_1, V_2) - \eta(V_1)\eta(V_2), \tag{2.12}$$

for all vector fields $V_1, V_2, V_3 \in \chi(M)$. Now we know

$$(\pounds_\xi g)(V_1, V_2) = g(\nabla_{V_1}\xi, V_2) + g(V_1, \nabla_{V_2}\xi), \tag{2.13}$$

for all vector fields $V_1, V_2, \in \chi(M)$. Then using (2.6) and (2.13), we get

$$(\pounds_\xi g)(V_1, V_2) = 2[g(V_1, V_2) - \eta(V_1)\eta(V_2)]. \tag{2.14}$$

Proposition 2.1. *[33] On a $(2n+1)$- dimensional Kenmotsu manifold, the $*$-Ricci tensor is given by,*

$$S^*(V_1, V_2) = S(V_1, V_2) + (2n-1)g(V_1, V_2) + \eta(V_1)\eta(V_2). \tag{2.15}$$

Also we take $V_1 = e_i$, $V_2 = e_i$ in the above equation, where e_i's are a local orthonormal frame and summing over $i = 1, 2, ..., (2n+1)$, to infer

$$r^* = r + 4n^2, \tag{2.16}$$

where r^* is the $*$- scalar curvature of M.

3. Main Results

Let M be a $(2n+1)$ dimensional Kenmotsu manifold. Now, we take $V = \xi$ into the identity (1.9) on M to yield

$$(\pounds_\xi g)(V_1, V_2) + 2\alpha_1 S^*(V_1, V_2) + [2\Lambda - \beta_1 r^*]g(V_1, V_2) = 0 \tag{3.1}$$

for all vector fields $V_1, V_2, \in \chi(M)$. We utilize the identities (2.14) and (2.15) into the above equation to yield

$$\alpha_1 S(V_1, V_2) + [\Lambda + \alpha_1(2n-1) + 1 - \frac{\beta_1 r^*}{2}]g(V_1, V_2) + [\alpha_1 - 1]\eta(V_1)\eta(V_2) = 0. \tag{3.2}$$

We set $V_2 = \xi$ in the above equation and making the use of (2.1), (2.10) to obtain

$$[\Lambda - \frac{\beta_1 r^*}{2}]\eta(V_1) = 0. \tag{3.3}$$

Since $\eta(V_1) \neq 0$, the previous equation takes the form

$$\Lambda = \frac{\beta_1 r^*}{2}. \tag{3.4}$$

Now with the help of (2.16), we acquire

$$\Lambda = \frac{\beta_1(r + 4n^2)}{2}. \tag{3.5}$$

This leads to the following:

Theorem 3.1. *If the metric g of a (2n+1) dimensional Kenmotsu manifold satisfies the $*$-Ricci-Yamabe soliton $(g, \xi, \Lambda, \alpha_1, \beta_1)$, where ξ is the Reeb vector field, then the soliton is expanding, steady, shrinking according as $\beta_1(r + 4n^2) \gtreqless 0$.*

Also we have, if the manifold M becomes flat i.e $r = 0$ then (3.5) becomes, $\Lambda = 2\beta_1 n^2$. So we can state

Corollary 3.2. *If the metric g of a (2n+1) dimensional Kenmotsu manifold, which is flat, satisfies the $*$-Ricci-Yamabe soliton $(g, \xi, \Lambda, \alpha_1, \beta_1)$, where ξ is the Reeb vector field, then the soliton is expanding, steady, shrinking according as $\beta_1 \gtreqless 0$.*

Now, we consider a $*$-Ricci-Yamabe soliton $(g, V, \Lambda, \alpha_1, \beta_1)$ on M as:

$$(\pounds_V g)(V_1, V_2) + 2\alpha_1 S^*(V_1, V_2) + [2\Lambda - \beta_1 r^*]g(V_1, V_2) = 0 \tag{3.6}$$

for all vector fields $V_1, V_2, \in \chi(M)$. We plug $V_1 = e_i$, $V_2 = e_i$ in the equation (3.6), where e_i's are a local orthonormal frame and summing over $i = 1, 2, ..., (2n+1)$ and using (2.16) to arrive

$$divV + (r + 4n^2)\Big[\alpha_1 - \frac{\beta_1(2n+1)}{2}\Big] + \Lambda(2n+1) = 0. \tag{3.7}$$

If we take the vector field V is of gradient type i.e $V = grad(f)$, for f is a smooth function on M, then the equation (3.7) becomes

$$\Delta(f) = -(r + 4n^2)\Big[\alpha_1 - \frac{\beta_1(2n+1)}{2}\Big] - \Lambda(2n+1), \tag{3.8}$$

where $\Delta(f)$ is the Laplacian equation satisfied by f. So, we can state the following theorem:

Theorem 3.3. *If the metric g of a (2n+1) dimensional Kenmotsu manifold satisfies the $*$-Ricci-Yamabe soliton $(g, V, \Lambda, \alpha_1, \beta_1)$, where V is the gradient of a smooth function f, then the Laplacian equation satisfied by f is,*

$$\Delta(f) = -(r + 4n^2)\Big[\alpha_1 - \frac{\beta_1(2n+1)}{2}\Big] - \Lambda(2n+1).$$

Now if $\alpha_1 = 1, \beta_1 = 0$, (1.9) reduces to $*$-Ricci soliton and (3.8) takes the form, $\Delta(f) = -(r+4n^2)-\Lambda(2n+1)$. If $\alpha_1 = 0, \beta_1 = 2$, (1.9) reduces to $*$-Yamabe soliton and (3.8) takes the form, $\Delta(f) = [r+4n^2-\Lambda](2n+1)$. Moreover if $\alpha_1 = \beta_1 = 1$, (1.9) reduces to $*$-Einstein soliton and (3.8) takes the form $\Delta(f) = -(r+4n^2)\big[1-\frac{(2n+1)}{2}\big]-\Lambda(2n+1)$. Then we have

Remark 3.4. **Case-I:** If the metric g of a $(2n+1)$ dimensional Kenmotsu manifold satisfies the $*$-Ricci soliton (g, V, Λ), where V is the gradient of a smooth function f, then the Laplacian equation satisfied by f is

$$\Delta(f) = -(r + 4n^2) - \Lambda(2n+1).$$

Case-II: If the metric g of a $(2n+1)$ dimensional Kenmotsu manifold satisfies the $*$-Yamabe soliton (g, V, Λ), where V is the gradient of a smooth function f, then the Laplacian equation satisfied by f is

$$\Delta(f) = [r + 4n^2 - \Lambda](2n+1).$$

Case-III: If the metric g of a $(2n+1)$ dimensional Kenmotsu manifold satisfies $*$-Einstein soliton (g, V, Λ), where V is the gradient of a smooth function f, then the Laplacian equation satisfied by f is

$$\Delta(f) = -(r + 4n^2)\Big[1 - \frac{(2n+1)}{2}\Big] - \Lambda(2n+1).$$

Also if we consider the vector field V as solenoidal i.e., $divV = 0$, then (3.7) reads

$$r = -\frac{\Lambda(2n+1)}{\Big[\alpha_1 - \frac{\beta_1(2n+1)}{2}\Big]} - 4n^2, \tag{3.9}$$

provided $\Big[\alpha_1 - \frac{\beta_1(2n+1)}{2}\Big] \neq 0$. Again if r takes the form of (3.9), then from (3.7), we obtain $divV = 0$. This leads to the following:

Theorem 3.5. *Let the metric g of a (2n+1) dimensional Kenmotsu manifold admits the $*$-Ricci-Yamabe soliton $(g, V, \Lambda, \alpha_1, \beta_1)$. Then the vector field V is solenoidal iff the scalar curvature takes the form $-\frac{\Lambda(2n+1)}{\big[\alpha_1 - \frac{\beta_1(2n+1)}{2}\big]} - 4n^2$, provided $\Big[\alpha_1 - \frac{\beta_1(2n+1)}{2}\Big] \neq 0$.*

A vector field V is said to be a conformal Killing vector field if the following relation holds:

$$(\pounds_V g)(V_1, V_2) = 2\Omega g(V_1, V_2), \tag{3.10}$$

where Ω is some function of the co-ordinates(conformal scalar). Moreover, if Ω is not constant the conformal Killing vector field V is said to be proper. Also when Ω is constant, V is called a homothetic vector field and when the constant Ω becomes non-zero, V is said to be a proper homothetic vector field. If $\Omega = 0$ in the above equation, then V is called a Killing vector field. Let $(g, V, \Lambda, \alpha_1, \beta_1)$ be a $*$-Ricci-Yamabe soliton on a $(2n+1)$ dimensional Kenmotsu manifold M, where V is a conformal Killing vector field. Then from (1.9), (2.15) and (3.10), we have,

$$\alpha_1 S(V_1, V_2) = -\Big[\alpha_1(2n-1) + \Lambda + \Omega - \frac{\beta_1 r^*}{2}\Big] g(V_1, V_2) - \alpha_1 \eta(V_1)\eta(V_2), \tag{3.11}$$

which leads to the fact that the manifold is η-Einstein, provided $\alpha_1 \neq 0$.
This leads to the following:

Theorem 3.6. *If the metric g of a (2n+1) dimensional Kenmotsu manifold endows the $*$-Ricci-Yamabe soliton $(g, V, \Lambda, \alpha_1, \beta_1)$, where V is a conformal Killing vector field, then the manifold becomes η-Einstein, provided $\alpha_1 \neq 0$.*

We take $V_2 = \xi$ into identity (3.11) and using (2.1), (2.10) to achieve

$$\Big[2\alpha_1 n - \alpha_1(2n-1) - \Lambda - \Omega + \frac{\beta_1 r^*}{2} - \alpha_1\Big]\eta(V_1) = 0. \tag{3.12}$$

Since $\eta(V_1) \neq 0$, we obtain

$$\Omega = \frac{\beta_1 r^*}{2} - \Lambda. \tag{3.13}$$

Then making the use of (2.16), the above equation becomes

$$\Omega = \frac{\beta_1(r + 4n^2)}{2} - \Lambda. \tag{3.14}$$

Hence we can state

Theorem 3.7. *Let the metric g of a (2n+1) dimensional Kenmotsu manifold satisfy the $*$-Ricci-Yamabe soliton $(g, V, \Lambda, \alpha_1, \beta_1)$, where V is a conformal Killing vector field. Then V is*
(i)proper vector field if $\frac{\beta_1(r+4n^2)}{2} - \Lambda$ is not constant.

(ii)homothetic vector field if $\frac{\beta_1(r+4n^2)}{2} - \Lambda$ is constant.

(iii) proper homothetic vector field if $\frac{\beta_1(r+4n^2)}{2} - \Lambda$ is non-zero constant.

(iv) Killing vector field if $\Lambda = \frac{\beta_1(r+4n^2)}{2}$.

Using the property of Lie derivative we can write

$$(£_V g)(V_1, V_2) = g(\nabla_{V_1} V, V_2) + g(\nabla_{V_2} V, V_1) \tag{3.15}$$

for any vector fields V_1, V_2.
Then from the identities (2.10), (2.15), (2.16) and (3.15), (1.9) takes the form

$$g(\nabla_{V_1} V, V_2) + g(\nabla_{V_2} V, V_1) + 2\alpha_1[-2ng(V_1, V_2) + (2n-1)g(V_1, V_2) + \eta(V_1)\eta(V_2)] + [2\Lambda - \beta_1(r + 4n^2)]g(V_1, V_2) = 0, \tag{3.16}$$

which leads to

$$g(\nabla_{V_1} V, V_2) + g(\nabla_{V_2} V, V_1) + \Big[2\Lambda - \beta_1(r + 4n^2) - 2\alpha_1\Big] g(V_1, V_2) + 2\alpha_1 \eta(V_1)\eta(V_2) = 0. \tag{3.17}$$

Suppose θ is a 1-form, which is metrically equivalent to V and is given by $\theta(V_1) = g(V_1, V)$ for an arbitrary vector field V_1. Then the exterior derivative $d\theta$ of θ can be written as:

$$2(d\theta)(V_1, V_2) = g(\nabla_{V_1} V, V_2) - g(\nabla_{V_2} V, V_1). \tag{3.18}$$

As $d\theta$ is skew-symmetric, so if we define a tensor field F of type (1,1) by,

$$(d\theta)(V_1, V_2) = g(V_1, FY), \tag{3.19}$$

then F is skew self-adjoint i.e. $g(V_1, FV_2) = -g(FV_1, V_2)$. So (3.19) can be written as:

$$(d\theta)(V_1, V_2) = -g(FV_1, V_2) \tag{3.20}$$

We feed the equation (3.18) into (3.20) to arrive

$$g(\nabla_{V_1} V, V_2) - g(\nabla_{V_2} V, V_1) = -2g(FV_1, V_2). \tag{3.21}$$

Now, we add the equations (3.21) and (3.17) side by side and factoring out V_2 to infer

$$\nabla_{V_1} V = -FV_1 - \Big[\Lambda - \frac{\beta_1(r + 4n^2)}{2} - \alpha_1\Big]V_1 - \alpha_1\eta(V_1)\xi. \tag{3.22}$$

Substituting the above equation in $R(V_1, V_2)V = \nabla_{V_1}\nabla_{V_2}V - \nabla_{V_2}\nabla_{V_1}V - \nabla_{[V_1,V_2]}V$, we have

$$\begin{aligned} R(V_1, V_2)V &= (\nabla_{V_2}F)V_1 - (\nabla_{V_1}F)V_2 + \beta_1\frac{V_2}{2}(V_1 r) - \beta_1\frac{V_1}{2}(Yr) \\ &+ \eta(V_1)V_2 - \eta(V_2)V_1. \end{aligned} \tag{3.23}$$

Noting that $d\theta$ is closed, we obtain

$$g(V_1, (\nabla_{V_3}F)V_2) + g(V_2, (\nabla_{V_1}F)V_3) + g(V_3, (\nabla_{V_2}F)V_1) = 0. \tag{3.24}$$

Making inner product of (3.23) with respect to V_3, we acquire

$$\begin{aligned} g(R(V_1, V_2)V, V_3) &= g((\nabla_{V_2}F)V_1, V_3) - g((\nabla_{V_1}F)V_2, V_3) + \eta(V_1)g(V_2, V_3) \\ &- \eta(V_2)g(V_1, V_3) + \beta_1\frac{Xr}{2}g(V_2, V_3) - \beta_1\frac{V_2 r}{2}g(V_1, V_3). \end{aligned} \tag{3.25}$$

As F is skew self-adjoint, then $\nabla_{V_1}F$ is also skew self-adjoint. Then using (3.24), (3.25) takes the form

$$\begin{aligned} g(R(V_1, V_2)V, V_3) &= g((\nabla_{V_3}F)V_2, V_1) + \eta(V_1)g(V_2, V_3) - \eta(V_2)g(V_1, V_3) \\ &+ \beta_1\frac{g(V_1, Dr)}{2}g(V_2, V_3) - \beta_1\frac{g(V_2, Dr)}{2}g(V_1, V_3). \end{aligned} \tag{3.26}$$

We put $V_1 = V_3 = e_i$ in the above equation, where e_i's are a local orthonormal frame and summing over $i = 1, 2, 3, .., (2n + 1)$ to find

$$S(V_2, V) = -2n\eta(V_2) - (divF)V_2 - \beta_1 n g(V_2, Dr), \tag{3.27}$$

where $divF$ is the divergence of the tensor field F. Using (2.10), the previous equation becomes

$$(divF)V_2 = 2n[g(V_2, V) - \eta(V_2)] - \beta_1 n g(V_2, Dr). \tag{3.28}$$

Now we compute the covariant derivative of the squared g-norm of V using (3.22) as follows:

$$\begin{aligned} \nabla_{V_1} \mid V \mid^2 &= 2g(\nabla_{V_1} V, V) \\ &= -2g(FV_1, V) - \Big[2\Lambda - \beta_1(r + 4n^2) - 2\alpha_1\Big]g(V_1, V) \\ &- 2\alpha_1\eta(V_1)\eta(V). \end{aligned} \tag{3.29}$$

Again making the use of (3.15), (3.17) provides

$$(\pounds_V g)(V_1, V_2) = -\Big[2\Lambda - \beta_1(r + 4n^2) - 2\alpha_1\Big]g(V_1, V_2) - 2\alpha_1\eta(V_1)\eta(V_2). \tag{3.30}$$

Then we fetch the identity (3.29) into (3.30) to yield

$$\nabla_{V_1} \mid V \mid^2 + 2g(FV_1, V) - (\pounds_V g)(V_1, V) = 0. \tag{3.31}$$

So we can state

Theorem 3.8. *If the metric g of a (2n+1) dimensional Kenmotsu manifold endows the $*$-Ricci-Yamabe soliton $(g, V, \Lambda, \alpha_1, \beta_1)$ then the vector V and its metric dual 1-form θ satisfies the relation*

$$(divF)V_2 = 2n[g(V_2, V) - \eta(V_2)] - \beta_1 n g(V_2, Dr),$$

and

$$\nabla_{V_1} \mid V \mid^2 + 2g(FV_1, V) - (\pounds_V g)(V_1, V) = 0.$$

4. Application of torse forming vector field on Kenmotsu manifold admitting $*$-Ricci-Yamabe soliton

Let $(g, \tau, \Lambda, \alpha_1, \beta_1)$ be a $*$-Ricci-Yamabe soliton on a $(2n+1)$ dimensional Kenmotsu manifold M, where τ is a torse-forming vector field. Then from (1.9), (2.15) and (2.16), we have,

$$(\pounds_\tau g)(V_1, V_2) + 2\alpha_1[S(V_1, V_2) + (2n-1)g(V_1, V_2) + \eta(V_1)\eta(V_2)] + [2\Lambda - \beta_1(r + 4n^2)]g(V_1, V_2) = 0, \quad (4.1)$$

where $\pounds_\tau g$ denotes the Lie derivative of the metric g along the vector field τ.
Now with the help of the identity (1.10), we obtain

$$\begin{aligned} (\pounds_\tau g)(V_1, V_2) &= g(\nabla_{V_1}\tau, V_2) + g(V_1, \nabla_{V_2}\tau) \\ &= 2\psi g(V_1, V_2) + \omega(V_1)g(\tau, V_2) + \omega(V_2)g(\tau, V_1), \end{aligned} \quad (4.2)$$

for all $V_1, V_2 \in M$. Then making use of (4.2) and (4.1), we get

$$\Big[\frac{\beta_1(r + 4n^2)}{2} - \Lambda - \psi - \alpha_1(2n-1)\Big]g(V_1, V_2) - \alpha_1 S(V_1, V_2) - \alpha_1\eta(V_1)\eta(V_2) = \frac{1}{2}\Big[\omega(V_1)g(\tau, V_2) + \omega(V_2)g(\tau, V_1)\Big]. \quad (4.3)$$

We contract the equation (4.3) over V_1 and V_2 to find

$$\Big[\frac{\beta_1(r + 4n^2)}{2} - \Lambda - \psi - \alpha_1(2n-1)\Big](2n+1) - \alpha_1 r - \alpha_1 = \omega(\tau), \quad (4.4)$$

which leads to

$$\Lambda = \frac{\beta_1(r + 4n^2)}{2} - \psi - \alpha_1(2n-1) - \frac{\alpha_1 r + \alpha_1 + \omega(\tau)}{(2n+1)}. \quad (4.5)$$

So, we can state the following theorem:

Theorem 4.1. *If the metric g of a (2n+1) dimensional Kenmotsu manifold admits the $*$-Ricci-Yamabe soliton $(g, \tau, \Lambda, \alpha_1, \beta_1)$, where τ is a torse-forming vector field, then $\Lambda = \frac{\beta_1(r+4n^2)}{2} - \psi - \alpha_1(2n-1) - \frac{\alpha_1 r + \alpha_1 + \omega(\tau)}{(2n+1)}$ and the soliton is expanding, steady, shrinking according as $\frac{\beta_1(r+4n^2)}{2} - \psi - \alpha_1(2n-1) - \frac{\alpha_1 r + \alpha_1 + \omega(\tau)}{(2n+1)} \gtreqless 0$.*

Now in (4.5), if the 1-form ω vanishes identically then $\Lambda = \frac{\beta_1(r+4n^2)}{2} - \psi - \alpha_1(2n-1) - \frac{\alpha_1 r + \alpha_1}{(2n+1)}$. If the 1-form ω vanishes identically and the function $\psi = 1$ in (4.5), then $\Lambda = \frac{\beta_1(r+4n^2)}{2} - 1 - \alpha_1(2n-1) - \frac{\alpha_1 r + \alpha_1}{(2n+1)}$. In (4.5), if the function $\psi = 0$, then $\Lambda = \frac{\beta_1(r+4n^2)}{2} - \alpha_1(2n-1) - \frac{\alpha_1 r + \alpha_1 + \omega(\tau)}{(2n+1)}$. If $\psi = \omega = 0$ in (4.5), then $\Lambda = \frac{\beta_1(r+4n^2)}{2} - \alpha_1(2n-1) - \frac{\alpha_1 r + \alpha_1}{(2n+1)}$. Finally in (4.5), if $\omega(\tau) = 0$, then $\Lambda = \frac{\beta_1(r+4n^2)}{2} - \psi - \alpha_1(2n-1) - \frac{\alpha_1 r + \alpha_1}{(2n+1)}$. Then we have

Corollary 4.2. *Let the metric g of a (2n+1) dimensional Kenmotsu manifold endows the $*$-Ricci-Yamabe soliton $(g, \tau, \Lambda, \alpha_1, \beta_1)$, where τ is a torse-forming vector field, then if τ is*

(i) con-circular, then $\Lambda = \frac{\beta_1(r+4n^2)}{2} - \psi - \alpha_1(2n-1) - \frac{\alpha_1 r+\alpha_1}{(2n+1)}$ and the soliton is expanding, steady, shrinking according as $\frac{\beta_1(r+4n^2)}{2} - \psi - \alpha_1(2n-1) - \frac{\alpha_1 r+\alpha_1}{(2n+1)} \gtreqless 0$.

(ii) concurrent, then $\Lambda = \frac{\beta_1(r+4n^2)}{2} - 1 - \alpha_1(2n-1) - \frac{\alpha_1 r+\alpha_1}{(2n+1)}$ and the soliton is expanding, steady, shrinking according as $\Lambda = \frac{\beta_1(r+4n^2)}{2} - 1 - \alpha_1(2n-1) - \frac{\alpha_1 r+\alpha_1}{(2n+1)} \gtreqless 0$.

(iii) recurrent, then $\Lambda = \frac{\beta_1(r+4n^2)}{2} - \alpha_1(2n-1) - \frac{\alpha_1 r+\alpha_1+\omega(\tau)}{(2n+1)}$ and the soliton is expanding, steady, shrinking according as $\Lambda = \frac{\beta_1(r+4n^2)}{2} - \alpha_1(2n-1) - \frac{\alpha_1 r+\alpha_1+\omega(\tau)}{(2n+1)} \gtreqless 0$.

(iv) parallel, then $\Lambda = \frac{\beta_1(r+4n^2)}{2} - \alpha_1(2n-1) - \frac{\alpha_1 r+\alpha_1}{(2n+1)}$ and the soliton is expanding, steady, shrinking according as $\Lambda = \frac{\beta_1(r+4n^2)}{2} - \alpha_1(2n-1) - \frac{\alpha_1 r+\alpha_1}{(2n+1)} \gtreqless 0$.

(v) torqued, then $\Lambda = \frac{\beta_1(r+4n^2)}{2} - \psi - \alpha_1(2n-1) - \frac{\alpha_1 r+\alpha_1}{(2n+1)}$ and the soliton is expanding, steady, shrinking according as $\Lambda = \frac{\beta_1(r+4n^2)}{2} - \psi - \alpha_1(2n-1) - \frac{\alpha_1 r+\alpha_1}{(2n+1)} \gtreqless 0$.

5. Geometrical and physical motivation of $*$-Ricci-Yamabe soliton

The notion of $*$-Ricci-Yamabe soliton is replaced by Ricci-Yamabe soliton as a kinematic solution of Ricci-Yamabe flow, whose profile develops a characterization of spaces of constant sectional curvature along with the locally symmetric spaces. Also, a geometric phenomenon of $*$-Ricci-Yamabe solitons can evolve an aqueduct between a sectional curvature inheritance symmetry of space-time and the class of Ricci-Yamabe solitons. As an application to relativity, there are some physical models of perfect fluid Ricci-Yamabe soliton space times which generates a curvature inheritance symmetry. Here, we can find some physical and geometrical models of perfect $*$-Ricci-Yamabe soliton space-time and that will give the physical significance, to the concept of $*$-Ricci-Yamabe soliton. As an application to cosmology and general relativity by investigating the kinetic and potential nature of relativistic space-time, we present a physical model of 3-class namely, shrinking, steady, and expanding of perfect and dust fluid solution of $*$-Ricci-Yamabe soliton space-time. The first case shrinking ($\Lambda < 0$) which exists on a minimal time interval $-\alpha_1 < t < b$ where $b < \alpha_1$, steady ($\Lambda = 0$) that exists for all time or expanding ($\Lambda > 0$) which exists on maximal time interval $a < t < \alpha_1, a > -\alpha_1$. These three classes give an example of ancient, eternal, and immortal solutions.

6. Example of a 3-dimensional Kenmotsu manifold admitting $*$-Ricci-Yamabe soliton

We consider the three-dimensional manifold $M = \{(x_1, y_1, z_1) \in \mathbb{R}^3, (x_1, y_1, z_1) \neq (0,0,0)\}$, where (x_1, y_1, z_1) are standard coordinates in $\mathbb{R}^3$. The vector fields

$$e_1 = z_1 \frac{\partial}{\partial x_1}, \quad e_2 = z_1 \frac{\partial}{\partial y_1}, \quad e_3 = -z_1 \frac{\partial}{\partial z_1}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0,$$

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by $\eta(V_3) = g(V_3, e_3)$, for any $V_3 \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on M and ϕ be the (1, 1)-tensor field defined by

$$\phi e_1 = -e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0.$$

Then using the linearity of ϕ and g, we have,

$$\eta(e_3) = 1, \quad \phi^2 V_3 = -V_3 + \eta(V_3)e_3, \quad g(\phi V_3, \phi W) = g(V_3, W) - \eta(V_3)\eta(W),$$

for any $V_3, W \in \chi(M)$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M. Let ∇ be the Levi-Civita connection with respect to the Riemannian metric g. Then we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2.$$

The connection ∇ of the metric g is given by,

$$\begin{aligned} 2g(\nabla_{V_1} V_2, V_3) &= V_1 g(V_2, V_3) + V_2 g(V_3, V_1) - V_3 g(V_1, V_2) \\ &- g(V_1, [V_2, V_3]) - g(V_2, [V_1, V_3]) + g(V_3, [V_1, V_2]), \end{aligned}$$

which is known as Koszul's formula. Using Koszul's formula, we can easily calculate,

$$\nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = e_1,$$

$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_3 = e_2,$$

$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

From the above it follows that the manifold satisfies $\nabla_{V_1}\xi = V_1 - \eta(V_1)\xi$, for $\xi = e_3$. Hence the manifold is a Kenmotsu Manifold. Also, the Riemannian curvature tensor R is given by

$$R(V_1, V_2)V_3 = \nabla_{V_1}\nabla_{V_2} V_3 - \nabla_{V_2}\nabla_{V_1} V_3 - \nabla_{[V_1, V_2]} V_3.$$

Hence,

$$R(e_1, e_2)e_2 = -e_1, \quad R(e_1, e_3)e_3 = -e_1, \quad R(e_2, e_1)e_1 = -e_2,$$

$$R(e_2, e_3)e_3 = -e_2, \quad R(e_3, e_1)e_1 = -e_3, \quad R(e_3, e_2)e_2 = -e_3,$$

$$R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_1 = 0, \quad R(e_3, e_1)e_2 = 0.$$

Then, the Ricci tensor S is given by

$$S(e_1, e_1) = -2, \quad S(e_2, e_2) = -2, \quad S(e_3, e_3) = -2. \tag{6.1}$$

Also the scalar curvature becomes

$$r = \sum_{i=1}^{3} S(e_i, e_i) = -6. \tag{6.2}$$

Using (2.15) and (6.1), we have

$$S^*(e_1, e_1) = -1, \quad S^*(e_2, e_2) = -1, \quad S^*(e_3, e_3) = 0. \tag{6.3}$$

Hence

$$r^* = \mathrm{Tr}(S^*) = -2. \tag{6.4}$$

Let us take the the potential vector field as $V = 2x_1 \frac{\partial}{\partial x_1} + 2y_1 \frac{\partial}{\partial y_1} + z_1 \frac{\partial}{\partial z_1}$. Then $(\pounds_V g)(e_1, e_1) = -2g(\pounds_V e_1, e_1) = 2$.

Similarly, $(\pounds_V g)(e_2, e_2) = 2, \quad (\pounds_V g)(e_3, e_3) = 0$. Hence we have,

$$\sum_{i=1}^{3} (\pounds_V g)(e_i, e_i) = 4. \tag{6.5}$$

Now putting $V_1 = V_2 = e_i$ in the (1.9), summing over $i = 1, 2, 3$ and using (6.4) and (6.5), we obtain

$$\Lambda = \frac{2\alpha_1 - 3\beta_1 - 2}{3}. \tag{6.6}$$

As this Λ, defined as above satisfies (3.7), so g defines a $*$-Ricci-Yamabe soliton on the 3-dimensional Kenmotsu manifold M. Also we can state

Remark 6.1. **Case-I:** When $\alpha_1 = 1, \beta_1 = 0$, (6.6) gives $\Lambda = 0$ and hence (g, V, Λ) is a $*$-Ricci soliton which is steady.

Case-II: When $\alpha_1 = 0, \beta_1 = 2$, (6.6) gives $\Lambda = -\frac{8}{3}$ and hence (g, V, Λ) is a $*$-Yamabe soliton which is shrinking.

Case-III: When $\alpha_1 = 1, \beta_1 = 1$, (6.6) gives $\Lambda = -1$ and hence (g, V, Λ) is a $*$-Einstein soliton which is also shrinking.

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