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Induced sequences and weaving of g-frames

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Abstract. In this paper we use the type I induced sequence $\{u_{ik} : i \in I, k \in K_i\}$ of a given g-Bessel sequence $\{\Lambda_i : i \in I\}$ to characterize whether $\{\Lambda_i : i \in I\}$ are g-Riesz frames, near g-Riesz bases and near exact g-frames, and vice versa. We also characterize the precise relationship between the synthesis operators of a given g-Bessel sequence and its type II induced sequence. Finally, we discuss whether the sums $\Lambda + \Delta$ and $\Gamma + \Theta$ are woven, where $\{\Lambda_i : i \in I\}$ and $\{\Gamma_i : i \in I\}$ are woven and Δ, Θ are g-Bessel sequences.

1. Introduction

G-frame, which was proposed by sun [19, 20] in 2006, is a more general frame expressed by bounded linear operators in order to popularize several types of frames such as classical frame, fusion frame, etc. at that time. After that g-frames have been widely studied by many scholars. For more information on g-frames the readers can consult [1, 7–9, 12, 14, 16–21, 25–27] and the papers therein.

In [20], the author introduced an induced sequence $\{u_{ik} : i \in I, k \in K_i\}$ of a g-Bessel sequence $\{\Lambda_i : i \in I\}$ in $\mathcal{U}$ (for more details please see (2.5)), which is called the type I induced sequence in this paper, and investigated the interrelation between $\{u_{ik} : i \in I, k \in K_i\}$ and $\{\Lambda_i : i \in I\}$. In detail, Sun [20] obtained that $\{\Lambda_i : i \in I\}$ is a g-frame (respectively g-Bessel sequence, tight g-frame, g-Riesz basis, g-orthonormal basis) for $\mathcal{U}$ if and only if $\{u_{ik} : i \in I, k \in K_i\}$ is a frame (respectively Bessel sequence, tight frame, Riesz basis, orthonormal basis) for $\mathcal{U}$. Motivated by this, in this paper we will continue to use the type I induced sequence $\{u_{ik} : i \in I, k \in K_i\}$ to characterize whether $\{\Lambda_i : i \in I\}$ is a g-Riesz frame, a near exact g-frame, and a near g-Riesz basis. From the results obtained we know that in general $\{\Lambda_i : i \in I\}$ being a near g-Riesz basis (respectively near exact g-frame), is not equivalent to $\{u_{ik} : i \in I, k \in K_i\}$ being a near Riesz basis (respectively near exact frame).

Let $\{\Lambda_i : i \in I\}$ be a g-Bessel sequence in $\mathcal{U}$ w.r.t. $\{\mathcal{V}_i : i \in I\}$. If the orthonormal basis for $\mathcal{V}_i$ is relaxed to a Riesz basis $\{h_{ik}\}_{k\in K_i}$, by the same way as in [20] we introduce the type II induced sequence $\{v_{ik} : i \in I, k \in K_i\}$ of $\{\Lambda_i : i \in I\}$. Then we characterize the precise relation between the synthesis operators of the g-Bessel sequence $\{\Lambda_i : i \in I\}$ and its type II induced sequence $\{v_{ik} : i \in I, k \in K_i\}$.

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Recall that weaving of frames was first introduced by Bemrose, Casazza, Grochenig, et al. in [2] to simulate a problem in distributed signal processing. Due to the potential applications in wireless sensor networks and signal preprocessing, etc., the weaving of frames has become a hot topic studied by many researchers. Later, the weaving principle has been applied to other frame settings, such as weaving g-frames [6, 13, 15], weaving K-frames [5], weaving Schauder frames [4], etc. For more information on the weaving of frames, the reader can consult [2, 3, 5, 13, 15, 22, 23]. In this paper we continue to investigate whether the sums $\Lambda + \Delta$ and $\Gamma + \Theta$ are woven on a Hilbert space $\mathcal{U}$, where $\Lambda, \Gamma, \Delta, \Theta$ are g-Bessel sequences in $\mathcal{U}$. At the same time, we also consider the case where the sums $\Lambda + \Delta$ and $\Gamma + \Theta$ are woven on $\mathcal{U}$, whether Λ and Γ (or Δ and Θ) are woven on $\mathcal{U}$?

Throughout this paper, we will use such notations. $\mathcal{U}$ and $\mathcal{V}$ are Hilbert spaces, with inner product $\langle \cdot, \cdot \rangle$, and norm $\|\cdot\|$; $L(\mathcal{U}, \mathcal{V})$ is denoted by the collection of all the linear bounded operators from $\mathcal{U}$ to $\mathcal{V}$, if $\mathcal{U} = \mathcal{V}$, then $L(\mathcal{U}, \mathcal{V})$ is abbreviated to $L(\mathcal{U})$; $\{\mathcal{V}_i\}_{i \in I}$ is a sequence of closed subspaces of $\mathcal{V}$, where I is a subset of the integer set $\mathbb{Z}$.

2. Preliminaries of g-frames in Hilbert spaces

Let me first recall the definitions of g-frame, weaving of g-frames, (near) g-Riesz basis, g-Riesz frame and near exact g-frame in Hilbert spaces.

Definition 2.1 *[20] A sequence $\{\Lambda_i \in L(\mathcal{U}, \mathcal{V}_i) : i \in I\}$ is called a g-frame for $\mathcal{U}$ with respect to (w.r.t.) $\{\mathcal{V}_i : i \in I\}$, if there exist $A, B > 0$ such that*

$$A\|f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{U}. \tag{2.1}$$

We call A, B the lower frame bound and upper frame bound of g-frame $\{\Lambda_i : i \in I\}$, respectively. We call $\{\Lambda_i : i \in I\}$ the g-Bessel sequence if the right-hand of (2.1) holds. We call $\{\Lambda_i : i \in I\}$ the tight g-frame if $A = B$, the parseval g-frame if $A = B = 1$.

We call $\{\Lambda_i : i \in I\}$ an exact g-frame for $\mathcal{U}$ w.r.t. $\{\mathcal{V}_i : i \in I\}$ if it ceases to be a g-frame whenever any one of its elements is removed.

Weaving g-frames were first introduced by combining the weaving principle with g-frames by the authors in [6, 13, 15].

Definition 2.2 *[6, 13, 15] Let $\{\Lambda_i : i \in I\}$ and $\{\Gamma_i : i \in I\}$ be g-frames for $\mathcal{U}$ w.r.t. $\{\mathcal{V}_i : i \in I\}$. If for any partition $\{\sigma_j\}_{j=1}^2$ of I, there exist $A, B > 0$ such that $\{\Lambda_i\}_{i \in \sigma_1} \cup \{\Gamma_i\}_{i \in \sigma_2}$ is a g-frame for $\mathcal{U}$ with g-frame bounds A, B, then $\{\Lambda_i : i \in I\}$ and $\{\Gamma_i : i \in I\}$ are said to be woven on $\mathcal{U}$ with universal g-frame bounds A, B, each $\{\Lambda_i\}_{i \in \sigma_1} \cup \{\Gamma_i\}_{i \in \sigma_2}$ is called a weaving.*

Suppose that $\{\Lambda_i : i \in I\}$ is a g-frame for $\mathcal{U}$ w.r.t. $\{\mathcal{V}_i : i \in I\}$. If there exists a g-Bessel sequence $\{\Gamma_i : i \in I\}$ in $\mathcal{U}$ w.r.t. $\{\mathcal{V}_i : i \in I\}$ such that

$$f = \sum_{i \in I} \Gamma_i^* \Lambda_i f = \sum_{i \in I} \Lambda_i^* \Gamma_i f, \quad \forall f \in \mathcal{U}, \tag{2.2}$$

then $\{\Gamma_i : i \in I\}$ is called an alternate dual of $\{\Lambda_i : i \in I\}$. In fact, $\{\Gamma_i : i \in I\}$ satisfying (2.2) is also a g-frame for $\mathcal{U}$.

Definition 2.3 *[20] A sequence $\{\Lambda_i \in L(\mathcal{U}, \mathcal{V}_i) : i \in I\}$ is called a g-Riesz basis for $\mathcal{U}$ w.r.t. $\{\mathcal{V}_i : i \in I\}$, if the following two conditions hold:*

(*i*) *$\{\Lambda_i : i \in I\}$ is g-complete, namely $\{f : \Lambda_i f = 0, i \in I\} = \{0\}$;*

(*ii*) *There exist two positive constants A, B such that for any $J \subset I$, and $g_i \in \mathcal{V}_i, i \in J$,*

$$A \sum_{i \in J} \|g_i\|^2 \leq \left\| \sum_{i \in J} \Lambda_i^* g_i \right\|^2 \leq B \sum_{i \in J} \|g_i\|^2.$$

Definition 2.4 *[1, 17] A sequence $\{\Lambda_i \in L(\mathcal{U}, \mathcal{V}_i) : i \in I\}$ is called a g-Riesz frame for $\mathcal{U}$ w.r.t. $\{\mathcal{V}_i : i \in I\}$, if for any subset $J \subset I$, $\{\Lambda_i : i \in J\}$ is a g-frame for $\mathcal{U}_J$ w.r.t. $\{\mathcal{V}_i : i \in J\}$ with uniform g-frame bounds A and B, where*

$$\mathcal{U}_J = \overline{\left\{ \sum_{i \in J} \Lambda_i^* g_i : \forall g_i \in \mathcal{V}_i, i \in J \right\}}. \tag{2.3}$$

Definition 2.5 *[11] Let $f_i \in \mathcal{U}, \forall i \in I$. If there exists a finite subset $\sigma \subset I$ such that $\{f_i : i \in I \backslash \sigma\}$ is a Riesz basis for $\mathcal{U}$, then $\{f_i : i \in I\}$ is called a $\sigma-$ near Riesz basis for $\mathcal{U}$.*

Since a Riesz basis is also an exact frame, Definition 2.5 can be expressed in another way.

Definition 2.6 *Let $f_i \in \mathcal{U}, \forall i \in I$. If there exists a finite subset $\sigma \subset I$ such that $\{f_i : i \in I \backslash \sigma\}$ is an exact frame for $\mathcal{U}$, then $\{f_i : i \in I\}$ is called a $\sigma-$ near exact frame for $\mathcal{U}$.*

Now we recall the definition of near g-Riesz basis.

Definition 2.7 *[1] Let $\Lambda_i \in L(\mathcal{U}, \mathcal{V}_i), \forall i \in I$. If there exists a finite subset $\sigma \subset I$ such that $\{\Lambda_i : i \in I \backslash \sigma\}$ is a g-Riesz basis for $\mathcal{U}$, then $\{\Lambda_i : i \in I\}$ is called a $\sigma-$ near g-Riesz basis for $\mathcal{U}$ w.r.t. $\{\mathcal{V}_i : i \in I\}$.*

Since an exact g-frame is not a g-Riesz basis in general, it's necessary to introduce the definition of near exact g-frame.

Definition 2.8 *Let $\Lambda_i \in L(\mathcal{U}, \mathcal{V}_i), \forall i \in I$. If there exists a finite subset $\sigma \subset I$ such that $\{\Lambda_i : i \in I \backslash \sigma\}$ is an exact g-frame for $\mathcal{U}$, then $\{\Lambda_i : i \in I\}$ is called a $\sigma-$ near exact g-frame for $\mathcal{U}$ w.r.t. $\{\mathcal{V}_i : i \in I\}$.*

Since a g-Riesz basis is an exact g-frame, a near g-Riesz basis must be a near exact g-frame, but the converse is not true in general.

Remark 2.9 *Note that for a near g-Riesz basis (resp. near exact g-frame, near Riesz basis, near exact frame), we mean that we can only delete finite elements from $\{\Lambda_i : i \in I\}$ such that the left is a g-Riesz basis (resp. an exact g-frame, a Riesz basis, an exact frame).*

Let $\{\Lambda_i : i \in I\}$ be a g-Bessel sequence in $\mathcal{U}$ w.r.t. $\{\mathcal{V}_i : i \in I\}$. The synthesis operator T_Λ of $\{\Lambda_i : i \in I\}$ is defined as follows

$$T_\Lambda : l^2(\{\mathcal{V}_i\}_{i \in I}) \to \mathcal{U}, \quad T(\{g_i\}_{i \in I}) = \sum_{i \in I} \Lambda_i^* g_i, \tag{2.4}$$

where $l^2(\{\mathcal{V}_i\}_{i \in I})$ is a Hilbert space, and is defined as follows:

$$l^2(\{\mathcal{V}_i\}_{i \in I}) = \left\{ \{g_i\}_{i \in I} : g_i \in \mathcal{V}_i, i \in I \text{ and } \sum_{i \in I} \|g_i\|^2 < +\infty \right\},$$

with the inner product $\langle \{f_i\}_{i \in I}, \{g_i\}_{i \in I} \rangle = \sum_{i \in I} \langle f_i, g_i \rangle$.

Let $\{\Lambda_i : i \in I\}$ be a g-Bessel sequence in $\mathcal{U}$ w.r.t. $\{\mathcal{V}_i : i \in I\}$ and for any $i \in I$, let $\{e_{ik}\}_{k \in K_i}$ be an orthonormal basis for $\mathcal{V}_i$, and $\{h_{ik}\}_{k \in K_i}$ be a Riesz basis for $\mathcal{V}_i$ with Riesz bounds C_i, D_i, where $0 < C = \inf_{i \in I}\{C_i\}$, $D = \sup_{i \in I}\{D_i\} < \infty$, and K_i is a subset of $\mathbb{Z}$. In [20] Sun introduced a sequence $\{u_{ik} : i \in I, k \in K_i\}$ corresponding to $\{\Lambda_i : i \in I\}$ with $\{e_{ik}\}_{k \in K_i}, \forall i \in I$ in the following

$$u_{ik} = \Lambda_i^* e_{ik}, \quad \forall i \in I, k \in K_i. \tag{2.5}$$

By the same way we define $\{v_{ik} : i \in I, k \in K_i\}$ corresponding to $\{\Lambda_i : i \in I\}$ and $\{h_{ik}\}_{k\in K_i}, \forall i \in I$ as follows

$$v_{ik} = \Lambda_i^* h_{ik}, \;\; \forall i \in I, k \in K_i. \tag{2.6}$$

Obviously $\{u_{ik} : i \in I, k \in K_i\}$ is a special case of $\{v_{ik} : i \in I, k \in K_i\}$. In the rest of this paper $\{u_{ik} : i \in I, k \in K_i\}$ and $\{v_{ik} : i \in I, k \in K_i\}$ are respectively called **type I** and **type II induced sequences** of $\{\Lambda_i : i \in I\}$.

At the end of this section we recall several results obtained by Sun, Zhu.

Lemma 2.10 *[20] Let $\{u_{ik}\}_{i\in I,k\in K_i}$ be defined as in (2.5). Then $\{\Lambda_i : i \in I\}$ is a g-frame (resp. g-Riesz basis) for $\mathcal{U}$ w.r.t. $\{\mathcal{V}_i : i \in I\}$ with g-frame bounds A and B, if and only if its type I induced sequence $\{u_{ik} : i \in I, k \in K_i\}$ is a frame (resp. Riesz basis) for $\mathcal{U}$ with frame bounds A and B.*

Lemma 2.11 *[27] $\{\Lambda_i : i \in I\}$ is a g-frame for $\mathcal{U}$ w.r.t $\{\mathcal{V}_i : i \in I\}$, if and only if the corresponding synthesis operator T_Λ defined as in (2.4) is bounded and surjective on $\mathcal{U}$.*

3. Characterizations of kinds of g-frames by type I and type II induced sequences

Let $\{\Lambda_i : i \in I\}$ be a g-Bessel sequence in $\mathcal{U}$ w.r.t. $\{\mathcal{V}_i : i \in I\}$, with type I induced sequence $\{u_{ik} : i \in I, k \in K_i\}$. In [20] the author studied the relationship between $\{\Lambda_i : i \in I\}$ and its type I induced sequence $\{u_{ik} : i \in I, k \in K_i\}$, and obtained some important results (see Lemma 2.10). Motivated by sun [20] in this paper we continue to investigate such problems: If $\{\Lambda_i : i \in I\}$ are near g-Riesz bases (resp. near exact g-frames, g-Riesz frames) for $\mathcal{U}$, can we deduce that its type I induced sequence $\{u_{ik} : i \in I, k \in K_i\}$ are near Riesz bases (resp. near exact frames, Riesz frames) for $\mathcal{U}$, and vice versa? In fact, if $\{\Lambda_i : i \in I\}$ is a near g-Riesz basis for $\mathcal{U}$, then $\{u_{ik} : i \in I, k \in K_i\}$ is not a near Riesz basis for $\mathcal{U}$ in general. The reader can check the following counterexample.

Example 3.1 *Suppose that $\{e_i\}_{i=1}^\infty$ is an orthonormal basis for $\mathcal{U}$, and $\mathcal{V}_1 = \mathcal{U}$, $\mathcal{V}_2 = span\{e_1, e_2\}$, $\mathcal{V}_3 = span\{e_3, e_4\}$, $\mathcal{V}_i = span\{e_{i+1}\}, i \geq 4$. Now for any $f \in \mathcal{U}$, define*

$$\Lambda_1 f = \langle f, e_5\rangle e_5, \;\; \Lambda_2 f = 2\sum_{i=1}^{2}\langle f, e_i\rangle e_i,$$

$$\Lambda_3 f = \sum_{i=3}^{4}\langle f, e_i\rangle e_i, \;\; \Lambda_i f = \langle f, e_{i+1}\rangle e_{i+1}, \; i \geq 4.$$

We first show that $\{\Lambda_i\}_{i=2}^\infty$ is a g-Riesz basis for $\mathcal{U}$. For any $f \in \mathcal{U}$, we have

$$\|f\|^2 \leq \sum_{i=2}^{\infty}\|\Lambda_i f\|^2 \leq 4\|f\|^2,$$

hence $\{\Lambda_i\}_{i=2}^\infty$ is a g-frame for $\mathcal{U}$, and consequently $\{\Lambda_i\}_{i=2}^\infty$ is g-complete on $\mathcal{U}$. For any $f \in \mathcal{U}$, $g_2 \in \mathcal{V}_2$, there exist c_1, c_2 such that $g_2 = \sum_{i=1}^2 c_i e_i$, now we have

$$\begin{aligned}\langle \Lambda_2^* g_2, f\rangle &= \langle g_2, \Lambda_2 f\rangle = 2\Big\langle \sum_{i=1}^{2} c_i e_i, \sum_{i=1}^{2}\langle f, e_i\rangle e_i\Big\rangle \\ &= 2\sum_{i=1}^{2} c_i\overline{\langle f, e_i\rangle} = \Big\langle 2\sum_{i=1}^{2} c_i e_i, f\Big\rangle = \langle 2g_2, f\rangle.\end{aligned}$$

Since $f \in \mathcal{U}$ is arbitrary, hence $\Lambda_2^* g_2 = 2g_2$. Similarly we can get $\Lambda_i^* g_i = g_i$, $i \geq 3$. And since $\{g_i\}_{i=2}^\infty$ is orthogonal, for any subset $J \subset I = \{2, 3, \cdots\}$, we have

$$\sum_{i\in J}\|g_i\|^2 \leq \Big\|\sum_{i\in J}\Lambda_i^* g_i\Big\|^2 \leq 4\sum_{i\in J}\|g_i\|^2.$$

Therefore $\{\Lambda_i\}_{i=2}^{\infty}$ is a g-Riesz basis for $\mathcal{U}$, and $\{\Lambda_i\}_{i=1}^{\infty}$ is a near g-Riesz basis for $\mathcal{U}$.

Next we show that the type I induced sequence $\{u_{ik}\}_{i=1,k\in K_i}^{\infty}$ of $\{\Lambda_i\}_{i=1}^{\infty}$ is not a near Riesz basis for $\mathcal{U}$. By direct calculations we get

$$u_{15} = \Lambda_1^* e_5 = e_5,\ u_{1k} = \Lambda_1^* e_k = 0, k \neq 5, u_{2k} = \Lambda_2^* e_k = 2e_k, k = 1,2,$$
$$u_{3k} = \Lambda_3^* e_{k+2} = e_{k+2}, k = 1,2,\ u_{i1} = \Lambda_i^* e_{i+1} = e_{i+1}, i \geq 4.$$

Obviously $\{u_{21}, u_{22}, u_{31}, u_{32}, u_{i1}, i \geq 4\}$ and $\{u_{21}, u_{22}, u_{31}, u_{32}, u_{15}, u_{i1}, i \geq 6\}$ are Riesz bases for $\mathcal{U}$. But both cases we have to erase infinite elements from $\{u_{ik}\}_{i=1,k\in K_i}^{\infty}$, hence $\{u_{ik}\}_{i=1,k\in K_i}^{\infty}$ of $\{\Lambda_i\}_{i=1}^{\infty}$ is not a near Riesz basis for $\mathcal{U}$. □

The following counterexample tells us that if the type I induced sequence $\{u_{ik} : i \in I, k \in K_i\}$ is a near Riesz basis for $\mathcal{U}$, then in general $\{\Lambda_i : i \in I\}$ is not a near g-Riesz basis for $\mathcal{U}$.

Example 3.2 *Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{U}$, and let $\mathcal{V}_i = span\{e_i, e_{i+1}\}, i = 1,2,3$, $\mathcal{V}_i = span\{e_i\}, i \geq 4$. Now for any $f \in \mathcal{U}$, define*

$$\Lambda_1 f = \sum_{i=1}^{2} \langle f, e_i \rangle e_i,\ \ \Lambda_i f = \langle f, e_i \rangle e_i, i \geq 2.$$

By direct calculations we get

$$\Lambda_i^* g_i = c_i e_i, \forall g_i = c_i e_i + c_{i+1} e_{i+1} \in \mathcal{V}_i, i = 2,3,\ \Lambda_i^* g_i = g_i, \forall g_i \in \mathcal{V}_i, i = 1,4,5,\cdots.$$

Now we have

$$u_{i1} = \Lambda_i^* e_i = e_i, i \geq 1,\ u_{12} = e_2,\ u_{i2} = \Lambda_i^* e_{i+1} = 0, i = 2,3.$$

Since we can erase u_{12}, u_{22}, u_{32} from $\{u_{ik} : i \in \mathbf{N}, k \in K_i\}$ such that the left is an orthonormal basis for $\mathcal{U}$, hence $\{u_{ik} : i \in \mathbf{N}, k \in K_i\}$ is a near Riesz basis for $\mathcal{U}$. Next we show that $\{\Lambda_i\}_{i=1}^{\infty}$ is not a near g-Riesz basis for $\mathcal{U}$. For that we divide two cases as follows.

Case I *The subset σ in Definition 2.7 is an empty set. It means that we can delete no elements from $\{\Lambda_i\}_{i=1}^{\infty}$. We show that $\{\Lambda_i\}_{i=1}^{\infty}$ is not a g-Riesz basis for $\mathcal{U}$. If we take $g_2 = e_3 \in \mathcal{V}_2$, $g_3 = e_4 \in \mathcal{V}_3$, otherwise $g_i = 0 \in \mathcal{V}_i$, then we have*

$$\left\| \sum_{i=1}^{\infty} \Lambda_i^* g_i \right\|^2 = \|\Lambda_2^* g_2 + \Lambda_3^* g_3\|^2 = \|\Lambda_2^* e_3 + \Lambda_3^* e_4\|^2 = 0,$$

and $\sum_{i=1}^{\infty} \|g_i\|^2 = \|e_3\|^2 + \|e_4\|^2 = 2$. So the condition (ii) in Definition 2.3 doesn't hold, and $\{\Lambda_i\}_{i=1}^{\infty}$ is not a g-Riesz basis for $\mathcal{U}$.

Case II *The subset σ in Definition 2.7 is not empty. Note that we can only delete Λ_2 such that the left $\{\Lambda_1\} \cup \{\Lambda_i\}_{i=3}^{\infty}$ is a g-frame for $\mathcal{U}$. But $\{\Lambda_1\} \cup \{\Lambda_i\}_{i=3}^{\infty}$ is not a g-Riesz basis for $\mathcal{U}$. In fact, if we take $g_3 = e_4 \in \mathcal{V}_3$, otherwise $g_i = 0 \in \mathcal{V}_i$, then we have*

$$\left\| \sum_{i=1}^{\infty} \Lambda_i^* g_i \right\|^2 = \|\Lambda_3^* g_3\|^2 = \|\Lambda_3^* e_4\|^2 = 0,$$

and $\sum_{i=1}^{\infty} \|g_i\|^2 = \|g_3\|^2 = \|e_4\|^2 = 1$. So the condition (ii) in Definition 2.3 doesn't hold, hence $\{\Lambda_1\} \cup \{\Lambda_i\}_{i=3}^{\infty}$ is not a g-Riesz basis for $\mathcal{U}$.

In conclusion there are no g-Riesz bases contained in $\{\Lambda_i\}_{i=1}^{\infty}$, therefore $\{\Lambda_i\}_{i=1}^{\infty}$ is not a near g-Riesz basis for $\mathcal{U}$. □

We first use the type I induced sequence of $\{\Lambda_i : i \in I\}$ to characterize $\{\Lambda_i : i \in I\}$ to be a near g-Riesz basis.

Theorem 3.3 *Let $\Lambda_i \in L(\mathcal{U}, \mathcal{V}_i), i \in I$, and $\{u_{ik} : i \in I, k \in K_i\}$ be the type I induced sequence of $\{\Lambda_i : i \in I\}$. Suppose that for any $i \in I$, $\dim \mathcal{V}_i = 1$. If $\{u_{ik} : i \in I, k \in K_i\}$ is a near Riesz basis for $\mathcal{U}$, then $\{\Lambda_i : i \in I\}$ is a near g-Riesz basis for $\mathcal{U}$.*

Proof. Suppose that $\{u_{ik} : i \in I, k \in K_i\}$ is a near Riesz basis for $\mathcal{U}$. For the trivial case, if $\{u_{ik} : i \in I, k \in K_i\}$ is a Riesz basis for $\mathcal{U}$, by Lemma 2.10 we obtain that $\{\Lambda_i : i \in I\}$ is a g-Riesz basis for $\mathcal{U}$. Next we show the nontrivial case. Assume that there exist $\emptyset \neq \sigma \subset I$, $\emptyset \neq \tau_i \subset K_i, i \in \sigma$ with $\sum_{i\in\sigma} |\tau_i| < \infty$, such that $\{u_{ik} : i \in I\backslash\sigma, k \in K_i\} \cup \{u_{ik} : i \in \sigma, k \in K_i\backslash\tau_i\}$ is a Riesz basis for $\mathcal{U}$. For any $i \in I$, $\dim \mathcal{V}_i = 1$, so $|K_i| = 1, i \in I$. And since $\emptyset \neq \tau_i \subset K_i, i \in \sigma$, $\{u_{ik} : i \in I\backslash\sigma, k \in K_i\} \cup \{u_{ik} : i \in \sigma, k \in K_i\backslash\tau_i\}$ can be rewritten as $\{u_{ik} : i \in I\backslash\sigma, k \in K_i\}$. Hence $\{u_{ik} : i \in I\backslash\sigma, k \in K_i\}$ is a Riesz basis for $\mathcal{U}$. Again by Lemma 2.10 then $\{\Lambda_i : i \in I\backslash\sigma\}$ is a g-Riesz basis for $\mathcal{U}$. Since $\sum_{i\in\sigma} |\tau_i| < \infty$, we have $|\sigma| < \infty$. Therefore $\{\Lambda_i : i \in I\}$ is a near g-Riesz basis for $\mathcal{U}$. □

We also obtain a result as follows.

Theorem 3.4 *Let $\Lambda_i \in L(\mathcal{U}, \mathcal{V}_i), i \in I$, and $\{u_{ik} : i \in I, k \in K_i\}$ be the type I induced sequence of $\{\Lambda_i : i \in I\}$. If $\{u_{ik} : i \in I, k \in K_i\}$ is a $\cup_{i\in\sigma} K_i$-near Riesz basis for $\mathcal{U}$, then $\{\Lambda_i : i \in I\}$ is a σ-near g-Riesz basis for $\mathcal{U}$.*

Proof. $\{u_{ik} : i \in I, k \in K_i\}$ is a $\cup_{i\in\sigma} K_i$-near Riesz basis for $\mathcal{U}$, so $\sum_{i\in\sigma} |K_i| < \infty$ and $\{u_{ik} : i \in I\backslash\sigma, k \in K_i\}$ is a Riesz basis for $\mathcal{U}$. By Lemma 2.10 $\{\Lambda_i : i \in I\backslash\sigma\}$ is a g-Riesz basis for $\mathcal{U}$. Since $\sum_{i\in\sigma} |K_i| < \infty$, we obtain $|\sigma| < \infty$. Hence $\{\Lambda_i : i \in I\backslash\sigma\}$ is a g-Riesz basis for $\mathcal{U}$ by deleting $|\sigma|(< \infty)$ elements from $\{\Lambda_i : i \in I\}$. Therefore $\{\Lambda_i : i \in I\}$ is a σ-near g-Riesz basis for $\mathcal{U}$. □

We then use $\{\Lambda_i : i \in I\}$ to characterize its type I induced sequence to be a near Riesz basis.

Theorem 3.5 *Let $\Lambda_i \in L(\mathcal{U}, \mathcal{V}_i), i \in I$, and $\{u_{ik} : i \in I, k \in K_i\}$ be the type I induced sequence of $\{\Lambda_i : i \in I\}$. If $\{\Lambda_i : i \in I\}$ is a σ-near g-Riesz basis for $\mathcal{U}$, and for any $i \in \sigma$, $\dim \mathcal{V}_i < \infty$, then $\{u_{ik} : i \in I, k \in K_i\}$ is a $\cup_{i\in\sigma} K_i$-near Riesz basis for $\mathcal{U}$.*

Proof. Suppose that $\{\Lambda_i : i \in I\}$ is a σ-near g-Riesz basis for $\mathcal{U}$. Then $\{\Lambda_i : i \in I\backslash\sigma\}$ is a g-Riesz basis for $\mathcal{U}$. By Lemma 2.10 $\{u_{ik} : i \in I\backslash\sigma, k \in K_i\}$ is a Riesz basis for $\mathcal{U}$. Since $|K_i| = \dim \mathcal{V}_i < \infty, i \in \sigma$, and $|\sigma| < \infty$, we have $\sum_{i\in\sigma} |K_i| < \infty$. It means that by deleting $\sum_{i\in\sigma} |K_i|$ elements from $\{u_{ik} : i \in I, k \in K_i\}$ the left $\{u_{ik} : i \in I\backslash\sigma, k \in K_i\}$ is a Riesz basis for $\mathcal{U}$. Therefore $\{u_{ik} : i \in I, k \in K_i\}$ is a near Riesz basis for $\mathcal{U}$. □

The next result is easily followed by Theorem 3.5.

Corollary 3.6 *Let $\Lambda_i \in L(\mathcal{U}, \mathcal{V}_i), i \in I$, and $\{u_{ik} : i \in I, k \in K_i\}$ be the type I induced sequence of $\{\Lambda_i : i \in I\}$. Suppose that for any $i \in I$, $\dim \mathcal{V}_i < \infty$. If $\{\Lambda_i : i \in I\}$ is a near g-Riesz basis for $\mathcal{U}$, then $\{u_{ik} : i \in I, k \in K_i\}$ is a near Riesz basis for $\mathcal{U}$.*

Combing with Theorems 3.3 and 3.5 we can obtain the following corollary.

Corollary 3.7 *Let $\Lambda_i \in L(\mathcal{U}, \mathcal{V}_i), i \in I$, and $\{u_{ik} : i \in I, k \in K_i\}$ be the type I induced sequence of $\{\Lambda_i : i \in I\}$. Suppose that for any $i \in I$, $\dim \mathcal{V}_i = 1$. Then $\{\Lambda_i : i \in I\}$ is a near g-Riesz basis for $\mathcal{U}$, if and only if $\{u_{ik} : i \in I, k \in K_i\}$ is a near Riesz basis for $\mathcal{U}$.*

Next we use the type I induced sequence of $\{\Lambda_i : i \in I\}$ to characterize $\{\Lambda_i : i \in I\}$ to be a near exact g-frame.

Theorem 3.8 *Let $\Lambda_i \in L(\mathcal{U}, \mathcal{V}_i), i \in I$, and $\{u_{ik} : i \in I, k \in K_i\}$ be the type I induced sequence of $\{\Lambda_i : i \in I\}$. If $\{u_{ik} : i \in I, k \in K_i\}$ is a near exact frame for $\mathcal{U}$, then $\{\Lambda_i : i \in I\}$ is a near exact g-frame for $\mathcal{U}$.*

Proof. Suppose that $\{u_{ik} : i \in I, k \in K_i\}$ is a near exact frame for $\mathcal{U}$. So $\{u_{ik} : i \in I, k \in K_i\}$ is also a frame for $\mathcal{U}$, by Lemma 2.10 we obtain that $\{\Lambda_i : i \in I\}$ is a g-frame for $\mathcal{U}$. By contradiction we assume that $\{\Lambda_i : i \in I\}$ is not a near exact g-frame for $\mathcal{U}$. Then there exists a subset $\sigma \subset I$ with $|\sigma| = \infty$ such that $\{\Lambda_i : i \in I\backslash\sigma\}$ is a g-frame for $\mathcal{U}$. Again by Lemma 2.10 $\{u_{ik} : i \in I\backslash\sigma, k \in K_i\}$ is a frame for $\mathcal{U}$. Since $|\sigma| = \infty$, so $\sum_{j\in\sigma} |K_j| = \infty$. $\{u_{ik} : i \in I\backslash\sigma, k \in K_i\}$ being a frame for $\mathcal{U}$, means that we can delete infinite elements from $\{u_{ik} : i \in I, k \in K_i\}$ such that the left is a frame for $\mathcal{U}$. We can also delete infinite elements from $\{u_{ik} : i \in I, k \in K_i\}$ such that the left is an exact frame for $\mathcal{U}$. By Remark 2.9 $\{u_{ik} : i \in I, k \in K_i\}$ is not a near exact frame for $\mathcal{U}$. Hence $\{\Lambda_i : i \in I\}$ is indeed a near exact g-frame for $\mathcal{U}$. □

An exact frame is also a Riesz basis, so a near exact frame is a near Riesz basis. Suppose that $\{\Lambda_i : i \in I\}$ is a near exact g-frame for $\mathcal{U}$, Example 3.1 also implies that $\{u_{ik} : i \in I, k \in K_i\}$ is not a near exact frame for $\mathcal{U}$. But if we make some restrictions on $\dim V_i$, $i \in I$, $\{\Lambda_i : i \in I\}$ is a near exact g-frame for $\mathcal{U}$ can deduce that $\{u_{ik} : i \in I, k \in K_i\}$ is a near exact frame for $\mathcal{U}$.

Theorem 3.9 *Let $\Lambda_i \in L(\mathcal{U}, \mathcal{V}_i), i \in I$, and $\{u_{ik} : i \in I, k \in K_i\}$ be the type I induced sequence of $\{\Lambda_i : i \in I\}$. Suppose that for any $i \in I$, $\dim \mathcal{V}_i = 1$. If $\{\Lambda_i : i \in I\}$ is a near exact g-frame for $\mathcal{U}$, then $\{u_{ik} : i \in I, k \in K_i\}$ is a near exact frame for $\mathcal{U}$.*

Proof. Assume that $\{\Lambda_i : i \in I\}$ is a near exact g-frame for $\mathcal{U}$. Then there exists a subset $\sigma \subset I$ with $|\sigma| < \infty$ such that $\{\Lambda_i : i \in I\backslash\sigma\}$ is an exact g-frame for $\mathcal{U}$. By Lemma 2.10 $\{u_{ik} : i \in I\backslash\sigma, k \in K_i\}$ is a frame for $\mathcal{U}$. Next we show that $\{u_{ik} : i \in I\backslash\sigma, k \in K_i\}$ is an exact frame for $\mathcal{U}$. By contradiction we assume that $\{u_{ik} : i \in I\backslash\sigma, k \in K_i\}$ is not exact. Then there exist $\emptyset \neq \tau \subset I\backslash\sigma$, $\emptyset \neq \kappa_i \subset K_i$, $i \in \tau$, such that $\{u_{ik} : i \in I\backslash\sigma\backslash\tau, k \in K_i\} \cup \{u_{ik} : i \in \tau, k \in K_i\backslash\kappa_i\}$ is a frame for $\mathcal{U}$. Since $|K_i| = \dim \mathcal{V}_i = 1$, $i \in I$, and $\emptyset \neq \kappa_i \subset K_i$, $i \in \tau$, so $K_i\backslash\kappa_i = \emptyset$ for any $i \in \tau$. Hence $\{u_{ik} : i \in I\backslash\sigma\backslash\tau, k \in K_i\}$ is a frame for $\mathcal{U}$. Again by Lemma 2.10 $\{\Lambda_i : i \in I\backslash\sigma\backslash\tau\}$ is a g-frame for $\mathcal{U}$. It contradicts that $\{\Lambda_i : i \in I\backslash\sigma\}$ is an exact g-frame for $\mathcal{U}$. Therefore $\{u_{ik} : i \in I\backslash\sigma, k \in K_i\}$ is an exact frame for $\mathcal{U}$. It implies that $\{u_{ik} : i \in I, k \in K_i\}$ is a near exact frame for $\mathcal{U}$ since $\sum_{i\in\sigma} |K_i| = |\sigma| < \infty$. □

The following result can be obtained by combining the Theorems 3.8 and 3.9.

Corollary 3.10 *Let $\Lambda_i \in L(\mathcal{U}, \mathcal{V}_i), i \in I$, and $\{u_{ik} : i \in I, k \in K_i\}$ be the type I induced sequence of $\{\Lambda_i : i \in I\}$. Suppose that for any $i \in I$, $\dim \mathcal{V}_i = 1$. Then $\{u_{ik} : i \in I, k \in K_i\}$ is a near exact frame for $\mathcal{U}$, if and only if $\{\Lambda_i : i \in I\}$ is a near exact g-frame for $\mathcal{U}$.*

The following result tells us that the type I induced sequence of $\{\Lambda_i : i \in I\}$, which is a Riesz frame, can infer that $\{\Lambda_i : i \in I\}$ is a g-Riesz frame.

Theorem 3.11 *Let $\{\Lambda_i : i \in I\}$ be a g-Bessel sequence in $\mathcal{U}$ w.r.t. $\{\mathcal{V}_i : i \in I\}$, with the type I induced sequence $\{u_{ik} : i \in I, k \in K_i\}$. If $\{u_{ik} : i \in I, k \in K_i\}$ is a Riesz frame for $\mathcal{U}$, then $\{\Lambda_i : i \in I\}$ is a g-Riesz frame for $\mathcal{U}$ w.r.t. $\{\mathcal{V}_i\}_{i\in I}$.*

Proof. Suppose that $\{u_{ik} : i \in I, k \in K_i\}$ is a Riesz frame for $\mathcal{U}$ with uniform frame bounds A and B. Then for any subset $J \subset I$, $\{u_{ik} : i \in J, k \in K_i\}$ is a frame for $\mathcal{W}_J$ with frame bounds A and B, where

$$\mathcal{W}_J = \overline{\left\{ \sum_{i\in J} \sum_{k\in K_i} c_{ik} u_{ik} : \forall i \in J, k \in K_i \right\}}.$$

By Lemma 2.10 $\{\Lambda_i : i \in J\}$ is a g-frame for $\mathcal{W}_J$ with g-frame bounds A and B. It follows that $\mathcal{U}_J = R(T_J) = \mathcal{W}_J$ by Lemma 2.11, where T_J is the synthesis operator of $\{\Lambda_i : i \in J\}$, $\mathcal{U}_J$ is defined by (2.3). Hence we obtain that for any $J \subset I$, $\{\Lambda_i : i \in J\}$ is a g-frame for $\mathcal{U}_J$ with uniform g-frame bounds A and B. Hence $\{\Lambda_i : i \in I\}$ is a g-Riesz frame for $\mathcal{U}$. □

Let $\{\Lambda_i : i \in I\}$ be a g-Bessel sequence in $\mathcal{U}$ w.r.t. $\{\mathcal{V}_i : i \in I\}$, with the type I induced sequence $\{u_{ik} : i \in I, k \in K_i\}$. At the moment we can't answer, if $\{\Lambda_i : i \in I\}$ is a g-Riesz frame for $\mathcal{U}$, whether $\{u_{ik} : i \in I, k \in K_i\}$ is a Riesz frame for $\mathcal{U}$. We can only get such a result under the condition $\dim \mathcal{V}_i = 1, \forall i \in I$.

Theorem 3.12 *Let $\{\Lambda_i : i \in I\}$ be a g-Bessel sequence in $\mathcal{U}$ w.r.t. $\{\mathcal{V}_i : i \in I\}$, with the type I induced sequence $\{u_{ik} : i \in I, k \in K_i\}$. Suppose that for any $i \in I$, $\dim \mathcal{V}_i = 1$. If $\{\Lambda_i : i \in I\}$ is a g-Riesz frame for $\mathcal{U}$ w.r.t. $\{\mathcal{V}_i\}_{i\in I}$, then $\{u_{ik} : i \in I, k \in K_i\}$ is a Riesz frame for $\mathcal{U}$.*

Proof. Assume that $\{\Lambda_i : i \in I\}$ is a g-Riesz frame for $\mathcal{U}$ with uniform g-frame bounds A and B. For any $\emptyset \neq \sigma \subset I$, $\emptyset \neq \tau_i \subset K_i$, $i \in \sigma$, we need to show that $\{u_{ik}\}_{i\in\sigma,k\in\tau_i}$ is a frame for $\mathcal{W}_\sigma$ with uniform frame bounds, where

$$\mathcal{W}_\sigma = \overline{\Big\{ \sum_{i\in\sigma} \sum_{k\in\tau_i} c_{ik} u_{ik} : \forall i \in \sigma, k \in \tau_i \Big\}}.$$

Since for any $i \in I$, $\dim \mathcal{V}_i = 1$, so $|K_i| = 1$, $i \in I$. And since $\emptyset \neq \tau_i \subset K_i$, $i \in \sigma$, hence $\tau_i = K_i$, $i \in \sigma$. Therefore $\{u_{ik}\}_{i\in\sigma,k\in\tau_i}$ can be rewritten as $\{u_{ik}\}_{i\in\sigma,k\in K_i}$, and $\mathcal{W}_\sigma$ can be rewritten as $\overline{\{\sum_{i\in\sigma} \sum_{k\in\tau_i} c_{ik} u_{ik} : \forall i \in \sigma, k \in K_i\}}$. Since $\{\Lambda_i : i \in I\}$ is a g-Riesz frame for $\mathcal{U}$ with uniform g-frame bounds A and B, so $\{\Lambda_i : i \in \sigma\}$ is a g-frame for $\mathcal{U}_\sigma = \overline{\{\sum_{i\in\sigma} \Lambda_i^* g_i : \forall g_i \in \mathcal{V}_i, i \in \sigma\}}$ with g-frame bounds A and B, by Lemma 2.10 $\{u_{ik}\}_{i\in\sigma,k\in K_i}$ is a frame for $\mathcal{U}_\sigma$ with frame bounds A and B. We can also have $\mathcal{W}_\sigma = R(T_\sigma) = \mathcal{U}_\sigma$, where T_σ is the synthesis operator of $\{u_{ik}\}_{i\in\sigma,k\in K_i}$. Hence $\{u_{ik}\}_{i\in\sigma,k\in\tau_i}$ is a frame for $\mathcal{W}_\sigma$ with uniform frame bounds A and B. And $\sigma \subset I$, $\tau_i \subset K_i$, $i \in \sigma$ are arbitrary, therefore $\{u_{ik} : i \in I, k \in K_i\}$ is a Riesz frame for $\mathcal{U}$. □

Combining with Theorems 3.11 and 3.12 we can obtain the following result.

Corollary 3.13 *Let $\{\Lambda_i : i \in I\}$ be a g-Bessel sequence in $\mathcal{U}$ w.r.t. $\{\mathcal{V}_i : i \in I\}$, with the type I induced sequence $\{u_{ik} : i \in I, k \in K_i\}$. Suppose that for any $i \in I$, $\dim \mathcal{V}_i = 1$. Then $\{\Lambda_i : i \in I\}$ is a g-Riesz frame for $\mathcal{U}$, if and only if $\{u_{ik} : i \in I, k \in K_i\}$ is a Riesz frame for $\mathcal{U}$.*

At the end of this section, we give the exact relationship between the synthesis operators of $\{\Lambda_i : i \in I\}$ and its type II induced sequence.

Theorem 3.14 *Let $\{\Lambda_i : i \in I\}$ be a g-Bessel sequence in $\mathcal{U}$ w.r.t. $\{\mathcal{V}_i : i \in I\}$ and for any $i \in I$, $\{h_{ik}\}_{k\in K_i}$ be a Riesz basis for $\mathcal{V}_i$ with Riesz bounds C_i, D_i, where $0 < C = \inf_{i\in I}\{C_i\}$, $D = \sup_{i\in I}\{D_i\} < \infty$. Let $\{v_{ik} : i \in I, k \in K_i\}$ be the type II induced sequence of $\{\Lambda_i : i \in I\}$. Then there exists an invertible operator $Q \in L(l^2(\{\mathcal{V}_i\}_{i\in I}), l^2)$, such that $T_\Lambda = T_v Q$, where T_Λ and T_v are respectively the synthesis operators of $\{\Lambda_i : i \in I\}$ and $\{v_{ik} : i \in I, k \in K_i\}$.*

Proof. Define $Q \in L(l^2(\{\mathcal{V}_i\}_{i\in I}), l^2)$ as follows

$$Q(\{g_i\}_{i\in I}) = \{\langle g_i, S_i^{-1} h_{ik}\rangle\}_{i\in I,k\in K_i}, \tag{3.1}$$

where S_i is the frame operator of $\{h_{ik}\}_{k\in K_i}$, $i \in I$.

We first show that Q is a bounded operator on $l^2(\{\mathcal{V}_i\}_{i\in I})$. For any $i \in I$, $\{h_{ik}\}_{k\in K_i}$ is a Riesz basis for $\mathcal{V}_i$ with Riesz bounds C_i, D_i, so $\{S_i^{-1} h_{ik}\}_{k\in K_i}$ is also a frame for $\mathcal{V}_i$ with frame bounds $\frac{1}{D_i}, \frac{1}{C_i}$. Now for any $\{g_i\}_{i\in I} \in l^2(\{\mathcal{V}_i\}_{i\in I})$, we have

$$\begin{aligned}
\|Q(\{g_i\}_{i\in I})\|^2 &= \|\{\langle g_i, S_i^{-1} h_{ik}\rangle\}_{i\in I,k\in K_i}\|^2 \\
&= \sum_{i\in I} \sum_{k\in K_i} |\langle g_i, S_i^{-1} h_{ik}\rangle|^2 \\
&\leq \sum_{i\in I} \frac{1}{C_i}\|g_i\|^2 \leq \sum_{i\in I} \frac{1}{C}\|g_i\|^2 = \frac{1}{C}\|\{g_i\}_{i\in I}\|^2.
\end{aligned}$$

Hence $Q \in L(l^2(\{\mathcal{V}_i\}_{i\in I}), l^2)$.

We then calculate Q^*. For any $\{g_i\}_{i\in I} \in l^2(\{\mathcal{V}_i\}_{i\in I})$, $\{c_{ik}\}_{i\in I,k\in K_i} \in l^2$, we obtain

$$\begin{aligned}
\langle \{g_i\}_{i\in I}, Q^*(\{c_{ik}\}_{i\in I,k\in K_i})\rangle &= \langle Q(\{g_i\}_{i\in I}), \{c_{ik}\}_{i\in I,k\in K_i}\rangle \\
&= \langle \{\langle g_i, S_i^{-1}h_{ik}\rangle\}_{i\in I,k\in K_i}, \{c_{ik}\}_{i\in I,k\in K_i}\rangle \\
&= \sum_{i\in I}\sum_{k\in K_i}\langle g_i, S_i^{-1}h_{ik}\rangle\overline{c_{ik}} \\
&= \sum_{i\in I}\sum_{k\in K_i}\langle g_i, c_{ik}S_i^{-1}h_{ik}\rangle \\
&= \sum_{i\in I}\Big\langle g_i, \sum_{k\in K_i} c_{ik}S_i^{-1}h_{ik}\Big\rangle \\
&= \Big\langle \{g_i\}_{i\in I}, \Big\{\sum_{k\in K_i} c_{ik}S_i^{-1}h_{ik}\Big\}_{i\in I}\Big\rangle.
\end{aligned}$$

It follows that $Q^*(\{c_{ik}\}_{i\in I,k\in K_i}) = \{\sum_{k\in K_i} c_{ik}S_i^{-1}h_{ik}\}_{i\in I}$ since $\{g_i\}_{i\in I} \in l^2(\{\mathcal{V}_i\}_{i\in I})$ is arbitrary.

Next we prove that Q is invertible on $l^2(\{\mathcal{V}_i\}_{i\in I})$. Suppose that there exists some $g = \{g_i\}_{i\in I} \in l^2(\{\mathcal{V}_i\}_{i\in I})$ such that $0 = Qg = Q(\{g_i\}_{i\in I}) = \{\langle g_i, S_i^{-1}h_{ik}\rangle\}_{i\in I,k\in K_i}$. Then $\langle g_i, S_i^{-1}h_{ik}\rangle = 0, \forall i \in I, k \in K_i$. Since for any $i \in I$, $\{S_i^{-1}h_{ik}\}_{k\in K_i}$ is a frame for $\mathcal{V}_i$, it follows that $g_i = 0, \forall i \in I$ and $g = 0$. Hence Q is injective. Suppose that there exists $c = \{c_{ik}\}_{i\in I,k\in K_i} \in l^2$ such that $0 = Q^*c = Q^*(\{c_{ik}\}_{i\in I,k\in K_i}) = \{\sum_{k\in K_i} c_{ik}S_i^{-1}h_{ik}\}_{i\in I}$. It follows that for any $i \in I$, $0 = \sum_{k\in K_i} c_{ik}S_i^{-1}h_{ik} = S_i^{-1}(\sum_{k\in K_i} c_{ik}h_{ik})$. Since S_i^{-1} is invertible on $\mathcal{V}_i$, we get $\sum_{k\in K_i} c_{ik}h_{ik} = 0$. It follows that $c_{ik} = 0, \forall i \in I, k \in K_i$ since $\{h_{ik}\}_{k\in K_i}$ is a Riesz basis for $\mathcal{V}_i$, $i \in I$. Hence Q^* is injective on l^2 and consequently Q is surjective on $l^2(\{\mathcal{V}_i\}_{i\in I})$. Therefore Q is invertible on $l^2(\{\mathcal{V}_i\}_{i\in I})$.

It suffices to show that $T_\Lambda = T_v Q$. In fact, for any $\{g_i\}_{i\in I} \in l^2(\{\mathcal{V}_i\}_{i\in I})$, we obtain

$$\begin{aligned}
T_vQ(\{g_i\}_{i\in I}) &= T_v(\{\langle g_i, S_i^{-1}h_{ik}\rangle\}_{i\in I,k\in K_i}) \\
&= \sum_{i\in I}\sum_{k\in K_i}\langle g_i, S_i^{-1}h_{ik}\rangle v_{ik} \\
&= \sum_{i\in I}\sum_{k\in K_i}\langle g_i, S_i^{-1}h_{ik}\rangle \Lambda_i^* h_{ik} \\
&= \sum_{i\in I}\Lambda_i^*\Big(\sum_{k\in K_i}\langle g_i, S_i^{-1}h_{ik}\rangle h_{ik}\Big) \\
&= \sum_{i\in I}\Lambda_i^* g_i = T_\Lambda(\{g_i\}_{i\in I}).
\end{aligned}$$

It follows that $T_\Lambda = T_vQ$ since $\{g_i\}_{i\in I} \in l^2(\{\mathcal{V}_i\}_{i\in I})$ is arbitrary. □

4. Weaving of g-frames in Hilbert spaces

In this section we mainly discuss the weaving of the sums $\{\Lambda_i + \Delta_i\}_{i\in I}$ and $\{\Gamma_i + \Theta_i\}_{i\in I}$ whether are woven on $\mathcal{U}$, where $\mathcal{U}$ is a Hilbert space and $\{\Lambda_i\}_{i\in I}$, $\{\Gamma_i\}_{i\in I}$, $\{\Delta_i\}_{i\in I}$, $\{\Theta_i\}_{i\in I}$ are g-Bessel sequences in $\mathcal{U}$.

Theorem 4.1 *Suppose that $\{\Lambda_i : i \in I\}$ and $\{\Gamma_i : i \in I\}$ are woven on $\mathcal{U}$ with universal g-frame bounds A, B. Let $T_1, T_2 \in L(\mathcal{U})$ and $\{\Delta_i : i \in I\}$, $\{\Theta_i : i \in I\}$ be g-Bessel sequences in $\mathcal{U}$ with g-Bessel bounds B_Δ, B_Θ, respectively. If $A > 2(B_\Delta\|T_1\|^2 + B_\Theta\|T_2\|^2)$, then $\{\Lambda_i + \Delta_iT_1^* : i \in I\}$ and $\{\Gamma_i + \Theta_iT_2^* : i \in I\}$ are woven on $\mathcal{U}$ with universal g-frame bounds*

$$\frac{1}{2}[A - 2(B_\Delta\|T_1\|^2 + B_\Theta\|T_2\|^2)],\ \ 2(B + B_\Delta\|T_1\|^2 + B_\Theta\|T_2\|^2).$$

Proof. For any partition $\{\sigma_j\}_{j=1}^2$ of I, and any $f \in \mathcal{U}$, we have

$$\begin{aligned}
\sum_{i\in\sigma_1}\|\Lambda_i f\|^2 &= \sum_{i\in\sigma_1}\|(\Lambda_i + \Delta_i T_1^*)f - \Delta_i T_1^* f\|^2 \\
&\le 2\sum_{i\in\sigma_1}\|(\Lambda_i + \Delta_i T_1^*)f\|^2 + 2\sum_{i\in\sigma_1}\|\Delta_i T_1^* f\|^2 \qquad (4.1)\\
&\le 2\sum_{i\in\sigma_1}\|(\Lambda_i + \Delta_i T_1^*)f\|^2 + 2\sum_{i\in I}\|\Delta_i T_1^* f\|^2 \\
&\le 2\sum_{i\in\sigma_1}\|(\Lambda_i + \Delta_i T_1^*)f\|^2 + 2B_\Delta\|T_1^* f\|^2 \\
&\le 2\sum_{i\in\sigma_1}\|(\Lambda_i + \Delta_i T_1^*)f\|^2 + 2B_\Delta\|T_1\|^2\cdot\|f\|^2. \qquad (4.2)
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
\sum_{i\in\sigma_2}\|\Gamma_i f\|^2 &= \sum_{i\in\sigma_2}\|(\Gamma_i + \Theta_i T_2^*)f - \Theta_i T_2^* f\|^2 \\
&\le 2\sum_{i\in\sigma_2}\|(\Gamma_i + \Theta_i T_2^*)f\|^2 + 2B_\Theta\|T_2\|^2\cdot\|f\|^2. \qquad (4.3)
\end{aligned}$$

For any partition $\{\sigma_j\}_{j=1}^2$ of I and any $f \in \mathcal{U}$, combing with (4.2) and (4.3) we have

$$\begin{aligned}
&\sum_{i\in\sigma_1}\|(\Lambda_i + \Delta_i T_1^*)f\|^2 + \sum_{i\in\sigma_2}\|(\Gamma_i + \Theta_i T_2^*)f\|^2 \\
\ge\ & \frac{1}{2}\Big(\sum_{i\in\sigma_1}\|\Lambda_i f\|^2 + \sum_{i\in\sigma_2}\|\Gamma_i f\|^2\Big) - (B_\Delta\|T_1\|^2 + B_\Theta\|T_2\|^2)\|f\|^2 \\
\ge\ & \frac{A}{2}\|f\|^2 - (B_\Delta\|T_1\|^2 + B_\Theta\|T_2\|^2)\|f\|^2 \\
=\ & \frac{1}{2}[A - 2(B_\Delta\|T_1\|^2 + B_\Theta\|T_2\|^2)]\|f\|^2,
\end{aligned}$$

where the second inequality is deduced by that $\{\Lambda_i : i \in I\}$ and $\{\Gamma_i : i \in I\}$ are woven on $\mathcal{U}$.

On the other hand, we have

$$\begin{aligned}
&\sum_{i\in\sigma_1}\|(\Lambda_i + \Delta_i T_1^*)f\|^2 + \sum_{i\in\sigma_2}\|(\Gamma_i + \Theta_i T_2^*)f\|^2 \\
\le\ & 2\Big(\sum_{i\in\sigma_1}\|\Lambda_i f\|^2 + \sum_{i\in\sigma_2}\|\Gamma_i f\|^2\Big) + 2\sum_{i\in\sigma_1}\|\Delta_i T_1^* f\|^2 + 2\sum_{i\in\sigma_2}\|\Theta_i T_2^* f\|^2 \\
\le\ & 2(B + B_\Delta\|T_1\|^2 + B_\Theta\|T_2\|^2)\|f\|^2.
\end{aligned}$$

Therefore $\{\Lambda_i + \Delta_i T_1^* : i \in I\}$ and $\{\Gamma_i + \Theta_i T_2^* : i \in I\}$ are woven on $\mathcal{U}$. □

If $T_1 = T_2 = I_{\mathcal{U}}$ in Theorem 4.1, the following corollary is followed by Theorem 4.1.

Corollary 4.2 *Suppose that $\{\Lambda_i : i \in I\}$ and $\{\Gamma_i : i \in I\}$ are woven on $\mathcal{U}$ with universal g-frame bounds A, B. Let $\{\Delta_i : i \in I\}$, $\{\Theta_i : i \in I\}$ be g-Bessel sequences in $\mathcal{U}$ with g-Bessel bounds B_Δ, B_Θ, respectively. If $A > 2(B_\Delta + B_\Theta)$, then $\{\Lambda_i + \Delta_i : i \in I\}$ and $\{\Gamma_i + \Theta_i : i \in I\}$ are woven on $\mathcal{U}$ with universal g-frame bounds $\frac{1}{2}[A - 2(B_\Delta + B_\Theta)]$, $2(B + B_\Delta + B_\Theta)$.*

Moreover, if $\{\Delta_i : i \in I\}$ and $\{\Theta_i : i \in I\}$ are also woven on $\mathcal{U}$, from the proof of Theorem 4.1 we can obtain another corollary as follows.

Corollary 4.3 *Suppose that $\{\Lambda_i : i \in I\}$ and $\{\Gamma_i : i \in I\}$, $\{\Delta_i : i \in I\}$ and $\{\Theta_i : i \in I\}$ are woven on $\mathcal{U}$ with universal g-frame bounds A, B and C, D, respectively. If $A > 2D$, then $\{\Lambda_i + \Delta_i : i \in I\}$ and $\{\Gamma_i + \Theta_i : i \in I\}$ are woven on $\mathcal{U}$ with universal g-frame bounds $\frac{A}{2} - D$, $2(B + D)$.*

Proof. For any partition $\{\sigma_j\}_{j=1}^2$ of I and any $f \in \mathcal{U}$, similar to (4.1) we have

$$\begin{aligned}\sum_{i\in\sigma_2} \|\Gamma_i f\|^2 &= \sum_{i\in\sigma_2} \|(\Gamma_i + \Theta_i) f - \Theta_i f\|^2 \\ &\leq 2\sum_{i\in\sigma_2} \|(\Gamma_i + \Theta_i) f\|^2 + 2\sum_{i\in\sigma_2} \|\Theta_i f\|^2. \end{aligned} \tag{4.4}$$

Combing with (4.1) and (4.4) we obtain

$$\begin{aligned}&\sum_{i\in\sigma_1} \|(\Lambda_i + \Delta_i) f\|^2 + \sum_{i\in\sigma_2} \|(\Gamma_i + \Theta_i) f\|^2 \\ \geq\ & \frac{1}{2}\Big(\sum_{i\in\sigma_1} \|\Lambda_i f\|^2 + \sum_{i\in\sigma_2} \|\Gamma_i f\|^2\Big) - \Big(\sum_{i\in\sigma_1} \|\Delta_i f\|^2 + \sum_{i\in\sigma_2} \|\Theta_i f\|^2\Big) \\ \geq\ & (\frac{A}{2} - D)\|f\|^2.\end{aligned}$$

The upper bound of each weaving is trivial. Hence $\{\Lambda_i + \Delta_i : i \in I\}$ and $\{\Gamma_i + \Theta_i : i \in I\}$ are woven on $\mathcal{U}$. □

Next we consider the converse of the Corollary 4.3. That is, if $\{\Lambda_i + \Delta_i : i \in I\}$ and $\{\Gamma_i + \Theta_i : i \in I\}$, $\{\Lambda_i : i \in I\}$ and $\{\Gamma_i : i \in I\}$ are woven on $\mathcal{U}$, can we deduce that the g-Bessel sequences $\{\Delta_i : i \in I\}$ and $\{\Theta_i : i \in I\}$ are whether woven on $\mathcal{U}$? We give a sufficient condition for this question as follows.

Theorem 4.4 *Suppose that $\{\Lambda_i : i \in I\}$, $\{\Gamma_i : i \in I\}$, $\{\Delta_i : i \in I\}$, and $\{\Theta_i : i \in I\}$ are g-Bessel sequences in $\mathcal{U}$. If $\{\Lambda_i : i \in I\}$ and $\{\Gamma_i : i \in I\}$, $\{\Lambda_i + \Delta_i : i \in I\}$ and $\{\Gamma_i + \Theta_i : i \in I\}$ are woven on $\mathcal{U}$ with universal g-frame bounds A, B and C, D, respectively, and $C > B$, then $\{\Delta_i : i \in I\}$ and $\{\Theta_i : i \in I\}$ are woven on $\mathcal{U}$ with universal g-frame bounds $(\sqrt{C} - \sqrt{B})^2, (\sqrt{B} + \sqrt{D})^2$.*

Proof. For any partition $\{\sigma_j\}_{j=1}^2$ of I and any $f \in \mathcal{U}$, we obtain

$$\begin{aligned}\Big(\sum_{i\in\sigma_1} \|\Delta_i f\|^2 + \sum_{i\in\sigma_2} \|\Theta_i f\|^2\Big)^{\frac{1}{2}} &= \|\{\Delta_i f\}_{i\in\sigma_1} + \{\Theta_i f\}_{i\in\sigma_2}\|_{l^2(\{\mathcal{V}_i\}_{i\in I})} \\ &= \|\{\Delta_i f + \Lambda_i f\}_{i\in\sigma_1} + \{\Theta_i f + \Gamma_i f\}_{i\in\sigma_2} \\ &\quad - (\{\Lambda_i f\}_{i\in\sigma_1} + \{\Gamma_i f\}_{i\in\sigma_2})\|_{l^2(\{\mathcal{V}_i\}_{i\in I})} \\ &\geq \|\{\Delta_i f + \Lambda_i f\}_{i\in\sigma_1} + \{\Theta_i f + \Gamma_i f\}_{i\in\sigma_2}\|_{l^2(\{\mathcal{V}_i\}_{i\in I})} \\ &\quad - \|(\{\Lambda_i f\}_{i\in\sigma_1} + \{\Gamma_i f\}_{i\in\sigma_2})\|_{l^2(\{\mathcal{V}_i\}_{i\in I})} \\ &= \Big(\sum_{i\in\sigma_1} \|(\Delta_i + \Lambda_i) f\|^2 + \sum_{i\in\sigma_2} \|(\Theta_i + \Gamma_i) f\|^2\Big)^{\frac{1}{2}} \\ &\quad - \Big(\sum_{i\in\sigma_1} \|\Lambda_i f\|^2 + \sum_{i\in\sigma_2} \|\Gamma_i f\|^2\Big)^{\frac{1}{2}} \\ &\geq (\sqrt{C} - \sqrt{B})\|f\|,\end{aligned} \tag{4.5}$$

where the last inequality is deduced by that $\{\Lambda_i : i \in I\}$ and $\{\Gamma_i : i \in I\}$, $\{\Lambda_i + \Delta_i : i \in I\}$ and $\{\Gamma_i + \Theta_i : i \in I\}$ are woven on $\mathcal{U}$. It follows that

$$\sum_{i\in\sigma_1} \|\Delta_i f\|^2 + \sum_{i\in\sigma_2} \|\Theta_i f\|^2 \geq (\sqrt{C} - \sqrt{B})^2\|f\|^2.$$

On the other hand, from (4.5) we have

$$\begin{aligned}
&\Big(\sum_{i\in\sigma_1}\|\Delta_i f\|^2+\sum_{i\in\sigma_2}\|\Theta_i f\|^2\Big)^{\frac{1}{2}}\\
\le\ & \|\{\Delta_i f+\Lambda_i f\}_{i\in\sigma_1}+\{\Theta_i f+\Gamma_i f\}_{i\in\sigma_2}\|_{l^2(\{\mathcal{V}_i\}_{i\in I})}+\|(\{\Lambda_i f\}_{i\in\sigma_1}+\{\Gamma_i f\}_{i\in\sigma_2})\|_{l^2(\{\mathcal{V}_i\}_{i\in I})}\\
=\ & \Big(\sum_{i\in\sigma_1}\|(\Delta_i+\Lambda_i)f\|^2+\sum_{i\in\sigma_2}\|(\Theta_i+\Gamma_i)f\|^2\Big)^{\frac{1}{2}}+\Big(\sum_{i\in\sigma_1}\|\Lambda_i f\|^2+\sum_{i\in\sigma_2}\|\Gamma_i f\|^2\Big)^{\frac{1}{2}}\\
\le\ & (\sqrt{B}+\sqrt{D})\|f\|.
\end{aligned}$$

It follows that

$$\sum_{i\in\sigma_1}\|\Delta_i f\|^2+\sum_{i\in\sigma_2}\|\Theta_i f\|^2\le(\sqrt{B}+\sqrt{D})^2\|f\|^2.$$

Therefore $\{\Delta_i : i \in I\}$ and $\{\Theta_i : i \in I\}$ are woven on $\mathcal{U}$. □

5. Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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