Filomat 38:9 (2024), 2965–2982
https://doi.org/10.2298/FIL2409965G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence of classical solutions for a class of several types of equations

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Abstract. We study a class of Hamilton-Jacobi equations and a class of incompressible Navier-Stokes equations. A new topological approach is applied to prove the existence of at least one and at least two nonnegative classical solutions. The arguments are based upon recent theoretical results.

1. Introduction

In this paper we investigate the following Hamilton-Jacobi equation

$$\begin{aligned} u_t + u_x^2 &= 0, \quad t > 0, \quad x \in \mathbb{R}, \\ u(0,x) &= u_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{1}$$

where

(H1) $u_0 \in C^1(\mathbb{R})$, $0 \le u_0 \le B$ on $\mathbb{R}$ for some positive constant B.

It has wide applications in optics, mechanics and semi-classical quantum theory.

In this paper, we will inevstigate it for existence of at least one classical solution and existence of at least two non-negative solutions. Next, we investigate a class of IVP for a class of incompressible Navier-Stokes equations for existence of global classical solutions. More precisely, we will study the following incompressible Navier-Stokes equations

$$\begin{aligned} u_t + uu_x + vu_y + wu_z + \frac{1}{\rho}p_x - \nu u_{xx} - \nu u_{yy} - \nu u_{zz} &= 0 \\ v_t + uv_x + vv_y + wv_z + \frac{1}{\rho}p_y - \nu v_{xx} - \nu v_{yy} - \nu v_{zz} &= 0 \\ w_t + uw_x + vw_y + ww_z + \frac{1}{\rho}p_z - \nu w_{xx} - \nu w_{yy} - \nu w_{zz} &= 0 \\ u_x + v_y + w_z &= 0 \quad \text{in} \quad (0,\infty)\times\mathbb{R}^3, \\ u(0,x,y,z) = u_0(x,y,z), \quad v(0,x,y,z) = v_0(x,y,z), \quad w(0,x,y,z) &= w_0(x,y,z), \quad (x,y,z)\in\mathbb{R}^3, \end{aligned} \tag{2}$$

where

2020 *Mathematics Subject Classification*. Primary 47H10; Secondary 35K70, 4G20.
Keywords. Hamilton-Jacobi equation, Navier-Stokes equations, positive solution, fixed point, sum of operators.
Received: 01 June 2023; Revised: 18 September 2023; Accepted: 25 September 2023
Communicated by Dragan S. Djordjević
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(P1) $p, u, v, w : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}$ are unknown, $u_0, v_0, w_0 \in C^2(\mathbb{R}^3)$ are given functions, $0 \le u_0, v_0, w_0 \le B$ on $\mathbb{R}^3$ for some positive constant B.

This is a system of partial differential equations that governs the flow of a viscous incompressible fluid. Here ρ is the density, u the velocity vector, p is the pressure. The first three equations of (2) are Cauchy's momentum equations where the first term is the accelerating time varying term, the second and third are the convective and the hydrostatic terms respectively. The physical example of the convective term can be described as a river that is converging, the case where the term is increasing and the river diverging the case where the term is decreasing. The hydrostatic term describes flow from high pressure to low pressure. The forth term is the viscosity term with the coefficient ν the kinematical viscosity. This term describes the ability of the fluid to induce motion of neighboring particles. On the right hand side we have the external forces density term. This term can include: gravity, magneto-hydrodynamic force, and so on. The fourth equation of (2) is the nullification of the divergence due to incompressibility condition. Turbulent fluid motions are believed to be well modeled by the incompressible Navier-Stokes equations. In the case of the 3D version of the NS equations the existence problem is an unsolved issue([10]).

We recall that global existence of weak solutions of the incompressible Navier-Stokes equations is known to hold in every space dimension. Uniqueness of weak solutions and global existence of strong solutions is known in dimension two [17]. In dimension three, global existence of strong solutions of the incompressible Navier-Stokes equations in thin three-dimensional domains began with the papers [19] and [20], where is used the methods in [13] and [14].

In this paper we propose new method for investigation of equations (2). The proposed method gives existence of classical solutions for the problem (2).

The paper is organized as follows. In the next section, we give some auxiliary results. In Section 3 we investigate the equation (1). In Section 4, we investigate the equations (2).

2. Preliminary Results

The first continuation theorems applicable to nonlinear problems were due to Leray and Schauder (1934) [22, Theorem 10.3.10]. This result is the most famous and most general result of the continuation theorems (see [22, pages 28,29]). In [21] (1955), Scheafer formulated a special case of Leray-Schauder continuation theorem in the form of an alternative, and proves it as a consequence of Schauder fixed point theorem. In this paper, we will use some nonlinear alternatives, in one hand, to develop a new fixed point theorem and in another hand to study the existence of solutions for Problem (1). In what follows we recall these alternatives.

Proposition 2.1. *(Leray-Schauder nonlinear alternative [3]) Let $C \subset E$ be a convex, closed subset in a Banach space E, $0 \in U \subset C$ where U is an open set. Let $f : \overline{U} \to C$ be a continuous, compact map. Then*

(a) *either f has a fixed point in $\overline{U}$,*

(b) *or there exist $x \in \partial U$, and $\lambda \in (0, 1)$ such that $x = \lambda f(x)$.*

As a consequence, we obtain

Proposition 2.2. *(Schaefer's Theorem or Leray-Schauder alternative, [7], p.124 or [22], p.29) Let E be a Banach space and $f : E \to E$ be completely continuous map. Then,*

(a) *either f has a fixed point in E,*

(b) *or for any $\lambda \in (0, 1)$, the set $\{x \in E : x = \lambda f(x)\}$ is unbounded.*

Another version of Scheafer's Theorem is given by:

Proposition 2.3. *(Scheafer's Theorem [21]) Let E be a Banach space and $f : E \to E$ be completely continuous map. Then*

(a) *either there exists for each* $\lambda \in [0,1]$ *one small* $x \in E$ *such that* $x = \lambda f(x)$,

(b) *or the set* $\{x \in E \,:\, x = \lambda f(x),\ 0 < \lambda < 1\}$ *is bounded in* E.

To prove our existence result we will use the following fixed point theorem.

Theorem 2.4. *Let* E *be a Banach space,* Y *a closed, convex subset of* E,

$$U = \{x \in Y \,:\, \|x\| < R\},$$

with $R > 0$. *Consider two operators* T *and* S, *where*

$$Tx = \varepsilon\, x,\ x \in \overline{U},$$

for $\varepsilon \in \mathbb{R}$, *and* $S : \overline{U} \to E$ *be such that*

(i) $I - S : \overline{U} \to Y$ *continuous, compact and*

(ii) $\{x \in Y : x = sgn(\varepsilon)\lambda(I - S)x, \quad \|x\| = R\} = \emptyset$, *for any* $\lambda \in \left(0, \frac{1}{|\varepsilon|}\right)$,
where $sgn(\varepsilon)$ *is the signum of* ε.

Then there exists $x^* \in \overline{U}$ *such that*

$$Tx^* + Sx^* = x^*.$$

Proof. We have that the operator $\frac{1}{\varepsilon}(I - S) : \overline{U} \to Y$ is continuous and compact.
Suppose that there exist $x_0 \in \partial U$ and $\mu_0 \in (0,1)$ such that

$$x_0 = \mu_0 \frac{1}{\varepsilon}(I - S)x_0,$$

that is

$$x_0 = sgn(\varepsilon)\frac{\mu_0}{|\varepsilon|}(I - S)x_0.$$

This contradicts the condition (ii). From the Leray-Schauder nonlinear alternative, it follows that there exists $x^* \in \overline{U}$ so that

$$x^* = \frac{1}{\varepsilon}(I - S)x^*$$

or

$$\varepsilon\, x^* + Sx^* = x^*,$$

or

$$Tx^* + Sx^* = x^*.$$

□

Let X be a real Banach space.

Definition 2.5. *A mapping* $K : X \to X$ *is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.*

The concept for l-set contraction is related to that of the Kuratowski measure of noncompactness which we recall for completeness.

Definition 2.6. *Let Ω_X be the class of all bounded sets of X. The Kuratowski measure of noncompactness $\alpha : \Omega_X \to [0, \infty)$ is defined by*

$$\alpha(Y) = \inf\left\{\delta > 0 : Y = \bigcup_{j=1}^{m} Y_j \quad and \quad diam(Y_j) \le \delta, \quad j \in \{1, \dots, m\}\right\},$$

where $diam(Y_j) = \sup\{\|x - y\|_X : x, y \in Y_j\}$ is the diameter of Y_j, $j \in \{1, \dots, m\}$.

For the main properties of the measure of noncompactness we refer the reader to [4].

Definition 2.7. *A mapping $K : X \to X$ is said to be l-set contraction if it is continuous, bounded and there exists a constant $l \ge 0$ such that*

$$\alpha(K(Y)) \le l\alpha(Y),$$

for any bounded set $Y \subset X$. The mapping K is said to be a strict set contraction if $l < 1$.

Obviously, if $K : X \to X$ is a completely continuous mapping, then K is 0-set contraction (see [6]).

Definition 2.8. *Let X and Y be real Banach spaces. A mapping $K : X \to Y$ is said to be expansive if there exists a constant $h > 1$ such that*

$$\|Kx - Ky\|_Y \ge h\|x - y\|_X$$

for any $x, y \in X$.

Definition 2.9. *A closed, convex set $\mathcal{P}$ in X is said to be cone if*

1. *$\alpha x \in \mathcal{P}$ for any $\alpha \ge 0$ and for any $x \in \mathcal{P}$,*
2. *$x, -x \in \mathcal{P}$ implies $x = 0$.*

Denote $\mathcal{P}^* = \mathcal{P}\backslash\{0\}$.
The following result will be used to prove our main result.

Theorem 2.10. *([24]) Let $\mathcal{P}$ be a cone of a Banach space E; Ω a subset of $\mathcal{P}$ and U_1, U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \to \mathcal{P}$ is an expansive mapping with constant $h > 1$, $S : \overline{U}_3 \to E$ is a k-set contraction with $0 \le k < h - 1$ and $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \ne \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \ne \emptyset$, and there exists $u_0 \in \mathcal{P}^*$ such that the following conditions hold:*

(i) *$Sx \ne (I - T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_1 \cap (\Omega + \lambda u_0)$,*

(ii) *there exists $\epsilon \ge 0$ such that $Sx \ne (I - T)(\lambda x)$, for all $\lambda \ge 1 + \epsilon$, $x \in \partial U_2$ and $\lambda x \in \Omega$,*

(iii) *$Sx \ne (I - T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_3 \cap (\Omega + \lambda u_0)$.*

Then $T + S$ has at least two non-zero fixed points $x_1, x_2 \in \mathcal{P}$ such that

$$x_1 \in \partial U_2 \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$$

or

$$x_1 \in (U_2 \setminus U_1) \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

3. Classical Solutions for Hamilton-Jacobi Equations

3.1. Existence of at Least One Classical Solution

The main result in this section is as follows.

Theorem 3.1. *Suppose that* ($H1$) *holds. Then the Cauchy problem* (1) *has at least one classical solutions* $u \in C^1([0,\infty)\times\mathbb{R})$.

Proof. Let $X = C^1([0,\infty)\times\mathbb{R})$ be endowed with the norm

$$\|u\| = \max\left\{\sup_{(t,x)\in[0,\infty)\times\mathbb{R}}|u(t,x)|, \sup_{(t,x)\in[0,\infty)\times\mathbb{R}}|u_t(t,x)|, \sup_{(t,x)\in[0,\infty)\times\mathbb{R}}|u_x(t,x)|\right\},$$

provided it exists. For $u \in X$, define the operator

$$S_1u(t,x) = u(t,x) - u_0(x) + \int_0^t (u_x(t_1,x))^2 dt_1, \quad (t,x)\in[0,\infty)\times\mathbb{R}.$$

Lemma 3.2. *Suppose* ($H1$). *If* $u \in X$ *satisfies the equation*

$$S_1u(t,x) = 0, \quad (t,x)\in[0,\infty)\times\mathbb{R}, \tag{3}$$

then it is a solution of the IVP (1).

Proof. Let $u \in X$ be a solution of the equation (3).

$$0 = u(t,x) - u_0(x) + \int_0^t (u_x(t_1,x))^2 dt_1 \quad (t,x)\in[0,\infty)\times\mathbb{R}. \tag{4}$$

We differentiate (4) with respect to t and we find

$$u_t(t,x) + (u_x(t,x))^2 = 0, \quad (t,x)\in[0,\infty)\times\mathbb{R}.$$

i.e., u satisfies the first equation of (1). Now, we put $t = 0$ in (4) and we arrive at

$$0 = u(0,x) - u_0(x), \quad x\in\mathbb{R}.$$

Therefore u satisfies (1). This completes the proof. □

Let

$$B_1 = \max\left\{2B, B^2\right\}.$$

Lemma 3.3. *Suppose* ($H1$). *For* $u \in X$ *with* $\|u\| \le B$, *we have*

$$|S_1u(t,x)| \le B_1(1+t), \quad (t,x)\in[0,\infty)\times\mathbb{R}.$$

Proof. We have

$$\begin{aligned}|S_1u(t,x)| &= \left|u(t,x) - u_0(x) + \int_0^t (u_x(t_1,x))^2 dt_1\right| \\ &\le |u(t,x)| + |u_0(x)| + \int_0^t (u_x(t_1,x))^2 dt_1 \\ &\le 2B + tB^2 \\ &\le B_1(1+t), \quad (t,x)\in[0,\infty)\times\mathbb{R}.\end{aligned}$$

This completes the proof. □

Suppose that

(H2) there exists a nonnegative function $g \in \mathcal{C}([0,\infty)\times\mathbb{R})$ so that $g(0,x) = g(t,0) = 0$, $(t,x) \in [0,\infty)\times\mathbb{R}$, $g(t,x) > 0$ for $(t,x) \in (0,\infty)\times(\mathbb{R}\backslash\{0\})$, and a positive constant A for which

$$2(1+t+t^2)(1+|x|)\int_0^t \left|\int_0^x g(t_1,x_1)dx_1\right|dt_1 \le A.$$

In the last section, we will give an example for a function g and a constant A that satisfy $(H2)$. For $u \in X$, define the operator

$$S_2u(t,x) = \int_0^t \int_0^x (t-t_1)(x-x_1)g(t_1,x_1)S_1u(t_1,x_1)dx_1dt_1,$$

$(t,x) \in [0,\infty)\times\mathbb{R}$.

Lemma 3.4. *Suppose* $(H1)$ *and* $(H2)$*. For* $u \in X$, $\|u\| \le B$, *we have*

$$\|S_2u\| \le AB_1.$$

Proof. We have

$$\begin{aligned}
|S_2u(t,x)| &= \left|\int_0^t \int_0^x (t-t_1)(x-x_1)g(t_1,x_1)S_1u(t_1,x_1)dx_1dt_1\right| \\
&\le \int_0^t \left|\int_0^x (t-t_1)|x-x_1|g(t_1,x_1)|S_1u(t_1,x_1)|dx_1\right|dt_1 \\
&\le B_1\int_0^t \left|\int_0^x (t-t_1)(1+t_1)|x-x_1|g(t_1,x_1)dx_1\right|dt_1 \\
&\le 2B_1(1+t)t|x|\int_0^t \left|\int_0^x g(t_1,x_1)dx_1\right|dt_1 \\
&\le AB_1, \quad (t,x) \in [0,\infty)\times\mathbb{R},
\end{aligned}$$

and

$$\begin{aligned}
\left|\frac{\partial}{\partial t}S_2u(t,x)\right| &= \left|\int_0^t \int_0^x (x-x_1)g(t_1,x_1)S_1u(t_1,x_1)dx_1dt_1\right| \\
&\le \int_0^t \left|\int_0^x |x-x_1|g(t_1,x_1)|S_1u(t_1,x_1)|dx_1\right|dt_1 \\
&\le B_1\int_0^t \left|\int_0^x (1+t_1)|x-x_1|g(t_1,x_1)dx_1\right|dt_1 \\
&\le 2B_1(1+t)|x|\int_0^t \left|\int_0^x g(t_1,x_1)dx_1\right|dt_1 \\
&\le AB_1, \quad (t,x) \in [0,\infty)\times\mathbb{R},
\end{aligned}$$

and

$$\left|\frac{\partial}{\partial x}S_2u(t,x)\right| = \left|\int_0^t \int_0^x (t-t_1)g(t_1,x_1)S_1u(t_1,x_1)dx_1dt_1\right|$$

$$
\begin{aligned}
&\leq \int_0^t \left| \int_0^x (t-t_1) g(t_1,x_1) |S_1 u(t_1,x_1)| dx_1 \right| dt_1 \\
&\leq B_1 \int_0^t \left| \int_0^x (t-t_1)(1+t_1) g(t_1,x_1) dx_1 \right| dt_1 \\
&\leq B_1(1+t)t \int_0^t \left| \int_0^x g(t_1,x_1) dx_1 \right| dt_1 \\
&\leq AB_1, \quad (t,x) \in [0,\infty) \times \mathbb{R}.
\end{aligned}
$$

Consequently

$$\|S_2 u\| \leq AB_1.$$

This completes the proof. □

Lemma 3.5. *Suppose* (*H*1) *and* (*H*2). *If* $u \in X$ *satisfies the equation*

$$S_2 u(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \tag{5}$$

then u is a solution to the IVP (1).

Proof. We differentiate two times with respect to t and two times with respect to x the equation (5) and we find

$$g(t,x) S_1 u(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

whereupon

$$S_1 u(t,x) = 0, \quad (t,x) \in (0,\infty) \times (\mathbb{R} \backslash \{0\}).$$

Since $S_2 u(\cdot,\cdot)$ is a continuous function on $[0,\infty) \times \mathbb{R}$, we have

$$
\begin{aligned}
0 &= \lim_{t \to 0} S_2 u(t,x) = S_2 u(0,x) = \lim_{x \to 0} S_2 u(t,x) = S_2 u(t,0) \\
&= \lim_{t,x \to 0} S_2 u(t,x) = S_2 u(0,0), \quad (t,x) \in [0,\infty) \times \mathbb{R}.
\end{aligned}
$$

Therefore

$$S_2 u(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Hence and Lemma 3.2, we conclude that u is a solution to the IVP (1). This completes the proof. □

Below, suppose

(H3) $\epsilon \in (0,1)$, A and B satisfy the inequalities $\epsilon B_1(1+A) < 1$ and $AB_1 < 1$.

Let $\widetilde{Y}$ denote the set of all equi-continuous families in X with respect to the norm $\|\cdot\|$. Let also, $Y = \overline{\widetilde{Y}}$ be the closure of $\widetilde{Y}$,

$$U = \{u \in Y : \|u\| < B\}.$$

For $u \in \overline{U}$ and $\epsilon > 0$, define the operators

$$T(u)(t,x) = \epsilon u(t,x),$$

$$S(u)(t,x) = u(t,x) - \epsilon u(t,x) - \epsilon S_2(u)(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

For $u \in \overline{U}$, we have

$$\begin{aligned} \|(I-S)(u)\| &= \|\epsilon u + \epsilon S_2(u)\| \\ &\leq \epsilon\|u\| + \epsilon\|S_2(u)\| \\ &\leq \epsilon B + \epsilon A B_1. \end{aligned}$$

Thus, $S : \overline{U} \to X$ is continuous and $(I-S)(\overline{U})$ resides in a compact subset of Y. Now, suppose that there is a $u \in Y$ so that $\|u\| = B$ and

$$u = \lambda(I-S)(u)$$

or

$$u = \lambda\epsilon\,(I + S_2)(u), \tag{6}$$

for some $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Note that $(Y, \|\cdot\|)$ is a Banach space. Assume that the set

$$\mathcal{A} = \{u \in Y : u = \mu(I+S_2)(u), \quad 0 < \mu < 1\}$$

is bounded. By (9), it follows that the set $\mathcal{A}$ is not empty set. Then, by Schaefer's Theorem, it follows that there is a $u^* \in Y$ such that

$$u^* = (I + S_2)(u^*), \tag{7}$$

or

$$S_2(u^*) = 0,$$

i.e., u^* is a solution to the problem (1). Assume that the set $\mathcal{A}$ is unbounded. Then, by Schaefer's Theorem, it follows that the equation

$$u = \mu(I+S_2)(u), \quad u \in Y,$$

has at least one small solution $u^* \in Y$ for any $\mu \in [0,1]$. In particular, for $\mu = 1$, there is a $u^* \in Y$ such that (10) holds and then it is a solution to the problem (1). Let now,

$$\{u \in Y : u = \lambda_1(I-S)(u),\ \|u\| = B\} = \emptyset$$

for any $\lambda_1 \in \left(0, \frac{1}{\epsilon}\right)$. Then, from Theorem 2.4, it follows that the operator $T+S$ has a fixed point $u^* \in Y$. Therefore

$$\begin{aligned} u^*(t,x) &= T(u^*)(t,x) + S(u^*)(t,x) \\ &= \epsilon u^*(t,x) + u^*(t,x) \\ &\quad -\epsilon u^*(t,x) - \epsilon S_2(u^*)(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}, \end{aligned}$$

whereupon

$$S_2(u^*)(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

From here, u^* is a solution to the problem (1). From here and from Lemma 3.5, it follows that u is a solution to the IVP (1). This completes the proof. □

3.2. Existence of at Least Two Classical Solutions

The main result in this section is as follows.

Theorem 3.6. *Suppose that* $(H1)$ *holds. Then the Cauchy problem* (1) *has at least two non-negative classical solutions* $u_1, u_2 \in C^1([0,\infty)\times\mathbb{R})$.

Proof. Let X be the space used in the previous section. Suppose

(H4) Let $m > 0$ be large enough and A, B, r, L, R_1 be positive constants that satisfy the following conditions

$$r < L < R_1, \quad \epsilon > 0, \quad R_1 > \left(\frac{2}{5m} + 1\right)L,$$

$$AB_1 < \frac{L}{5}.$$

Let

$$\widetilde{P} = \{u \in X : u \geq 0 \quad \text{on} \quad [0,\infty)\times\mathbb{R}\}.$$

With $\mathcal{P}$ we will denote the set of all equi-continuous families in $\widetilde{P}$. For $v \in X$, define the operators

$$\begin{aligned} T_1 v(t) &= (1+m\epsilon)v(t) - \epsilon\frac{L}{10}, \\ S_3 v(t) &= -\epsilon S_2 v(t) - m\epsilon v(t) - \epsilon\frac{L}{10}, \end{aligned}$$

$t \in [0,\infty)$. Note that any fixed point $v \in X$ of the operator $T_1 + S_3$ is a solution to the IVP (1). Define

$$\begin{aligned} U_1 &= \mathcal{P}_r = \{v \in \mathcal{P} : \|v\| < r\}, \\ U_2 &= \mathcal{P}_L = \{v \in \mathcal{P} : \|v\| < L\}, \\ U_3 &= \mathcal{P}_{R_1} = \{v \in \mathcal{P} : \|v\| < R_1\}, \\ R_2 &= R_1 + \frac{A}{m}B_1 + \frac{L}{5m}, \\ \Omega &= \overline{\mathcal{P}_{R_2}} = \{v \in \mathcal{P} : \|v\| \leq R_2\}. \end{aligned}$$

1. For $v_1, v_2 \in \Omega$, we have

   $$\|T_1 v_1 - T_1 v_2\| = (1+m\varepsilon)\|v_1 - v_2\|,$$

   whereupon $T_1 : \Omega \to X$ is an expansive operator with a constant $h = 1 + m\varepsilon > 1$.
2. For $v \in \overline{\mathcal{P}}_{R_1}$, we get

   $$\begin{aligned} \|S_3 v\| &\leq \varepsilon\|S_2 v\| + m\varepsilon\|v\| + \varepsilon\frac{L}{10} \\ &\leq \varepsilon\left(AB_1 + mR_1 + \frac{L}{10}\right). \end{aligned}$$

   Therefore $S_3(\overline{\mathcal{P}}_{R_1})$ is uniformly bounded. Since $S_3 : \overline{\mathcal{P}}_{R_1} \to X$ is continuous, we have that $S_3(\overline{\mathcal{P}}_{R_1})$ is equi-continuous. Consequently $S_3 : \overline{\mathcal{P}}_{R_1} \to X$ is a 0-set contraction.

3. Let $v_1 \in \overline{\mathcal{P}}_{R_1}$. Set

$$v_2 = v_1 + \frac{1}{m} S_2 v_1 + \frac{L}{5m}.$$

Note that $S_2 v_1 + \frac{L}{5} \geq 0$ on $[t_0, \infty)$. We have $v_2 \geq 0$ on $[t_0, \infty)$ and

$$\begin{aligned} \|v_2\| &\leq \|v_1\| + \frac{1}{m}\|S_2 v_1\| + \frac{L}{5m} \\ &\leq R_1 + \frac{A}{m} B_1 + \frac{L}{5m} \\ &= R_2. \end{aligned}$$

Therefore $v_2 \in \Omega$ and

$$-\varepsilon m v_2 = -\varepsilon m v_1 - \varepsilon S_2 v_1 - \varepsilon \frac{L}{10} - \varepsilon \frac{L}{10}$$

or

$$\begin{aligned} (I - T_1) v_2 &= -\varepsilon m v_2 + \varepsilon \frac{L}{10} \\ &= S_3 v_1. \end{aligned}$$

Consequently $S_3(\overline{\mathcal{P}}_{R_1}) \subset (I - T_1)(\Omega)$.

4. Assume that for any $u_0 \in \mathcal{P}^*$ there exist $\lambda \geq 0$ and $x \in \partial \mathcal{P}_r \cap (\Omega + \lambda u_0)$ or $x \in \partial \mathcal{P}_{R_1} \cap (\Omega + \lambda u_0)$ such that

$$S_3 x = (I - T_1)(x - \lambda u_0).$$

Then

$$-\epsilon S_2 x - m\epsilon x - \epsilon \frac{L}{10} = -m\epsilon(x - \lambda u_0) + \epsilon \frac{L}{10}$$

or

$$-S_2 x = \lambda m u_0 + \frac{L}{5}.$$

Hence,

$$\|S_2 x\| = \left\| \lambda m u_0 + \frac{L}{5} \right\| > \frac{L}{5}.$$

This is a contradiction.

5. Suppose that for any $\epsilon_1 \geq 0$ small enough there exist a $x_1 \in \partial \mathcal{P}_L$ and $\lambda_1 \geq 1 + \epsilon_1$ such that $\lambda_1 x_1 \in \overline{\mathcal{P}}_{R_1}$ and

$$S_3 x_1 = (I - T_1)(\lambda_1 x_1). \tag{8}$$

In particular, for $\epsilon_1 > \frac{2}{5m}$, we have $x_1 \in \partial \mathcal{P}_L$, $\lambda_1 x_1 \in \overline{\mathcal{P}}_{R_1}$, $\lambda_1 \geq 1 + \epsilon_1$ and (8) holds. Since $x_1 \in \partial \mathcal{P}_L$ and $\lambda_1 x_1 \in \overline{\mathcal{P}}_{R_1}$, it follows that

$$\left(\frac{2}{5m} + 1 \right) L < \lambda_1 L = \lambda_1 \|x_1\| \leq R_1.$$

Moreover,

$$-\epsilon S_2 x_1 - m\epsilon x_1 - \epsilon \frac{L}{10} = -\lambda_1 m\epsilon x_1 + \epsilon \frac{L}{10},$$

or

$$S_2 x_1 + \frac{L}{5} = (\lambda_1 - 1) m x_1.$$

From here,

$$2\frac{L}{5} \geq \left\| S_2 x_1 + \frac{L}{5} \right\| = (\lambda_1 - 1) m \|x_1\| = (\lambda_1 - 1) mL$$

and

$$\frac{2}{5m} + 1 \geq \lambda_1,$$

which is a contradiction.

Therefore all conditions of Theorem 2.10 hold. Hence, the IVP (1) has at least two solutions u_1 and u_2 so that

$$\|u_1\| = L < \|u_2\| < R_1$$

or

$$r < \|u_1\| < L < \|u_2\| < R_1.$$

□

3.3. An Example

Below, we will illustrate our main results. Let

$$R_1 = B = 10, \quad L = 5, \quad r = 4, \quad m = 10^{50}, \quad A = \frac{1}{10 B_1}, \quad \epsilon = \frac{1}{5 B_1 (1 + A)}.$$

Then

$$B_1 = 10^2$$

and

$$A B_1 = \frac{1}{10} < B, \quad \epsilon B_1 (1 + A) < 1,$$

i.e., $(H3)$ holds. Next,

$$r < L < R_1, \quad \epsilon > 0, \quad R_1 > \left(\frac{2}{5m} + 1 \right) L, \quad A B_1 < \frac{L}{5}.$$

i.e., $(H4)$ holds. Take

$$h(s) = \log \frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}}, \quad l(s) = \arctan \frac{s^{11}\sqrt{2}}{1 - s^{22}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1.$$

Then

$$h'(s) = \frac{22\sqrt{2} s^{10} (1 - s^{22})}{(1 - s^{11}\sqrt{2} + s^{22})(1 + s^{11}\sqrt{2} + s^{22})},$$

$$l'(s) = \frac{11\sqrt{2}s^{10}(1+s^{20})}{1+s^{40}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1.$$

Therefore

$$-\infty < \lim_{s\to\pm\infty}(1+s+s^2)h(s) < \infty,$$

$$-\infty < \lim_{s\to\pm\infty}(1+s+s^2)l(s) < \infty.$$

Hence, there exists a positive constant C_1 so that

$$(1+s+s^2)^3\left(\frac{1}{44\sqrt{2}}\log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}}+\frac{1}{22\sqrt{2}}\arctan\frac{s^{11}\sqrt{2}}{1-s^{22}}\right) \leq C_1,$$

$s \in \mathbb{R}$. Note that $\lim_{s\to\pm 1} l(s) = \frac{\pi}{2}$ and by [18] (pp. 707, Integral 79), we have

$$\int \frac{dz}{1+z^4} = \frac{1}{4\sqrt{2}}\log\frac{1+z\sqrt{2}+z^2}{1-z\sqrt{2}+z^2}+\frac{1}{2\sqrt{2}}\arctan\frac{z\sqrt{2}}{1-z^2}.$$

Let

$$Q(s) = \frac{s^{10}}{(1+s^{44})(1+s+s^2)^2}, \quad s \in \mathbb{R},$$

and

$$g_1(t,x) = Q(t)Q(x), \quad t \in [0,\infty), \quad x \in \mathbb{R}.$$

Then there exists a constant $C > 0$ such that

$$2\left(1+t+t^2\right)(1+|x|)\int_0^t \left|\int_0^x g_1(t_1,x_1)dx_1\right|dt_1 \leq C, \quad (t,x) \in [0,\infty)\times\mathbb{R}.$$

Let

$$g(t,x) = \frac{A}{C}g_1(t,x), \quad (t,x) \in [0,\infty)\times\mathbb{R}.$$

Then

$$2\left(1+t+t^2\right)(1+|x|)\int_0^t \left|\int_0^x g(t_1,x_1)dx_1\right|dt_1 \leq A, \quad (t,x) \in [0,\infty)\times\mathbb{R},$$

i.e., $(H2)$ holds. Therefore for the IVP

$$u_t + u_x^2 = 0, \quad t > 0, \quad x \in \mathbb{R},$$

$$u(0,x) = \frac{1}{(1+x^2)^8}, \quad x \in \mathbb{R},$$

are fulfilled all conditions of Theorem 3.1 and Theorem 3.6.

4. Classical Solutions for Incompressible Navier-Stokes Equations

4.1. Existence of at Least One Classical Solution

Without loss of generality, suppose that $\rho = \nu = 1$. Our result in this section is as follows.

Theorem 4.1. *Suppose that $(P1)$ holds. Then the equations (2) has at least one solution $(u, v) \in (C^1([0, \infty), C^2(\mathbb{R}^3)))^4$.*

Proof. Let $X^1 = C^1([0, \infty), C^2(\mathbb{R}^3))$ be endowed with the norm

$$\begin{aligned}
\|u\|_{X^1} &= \max\Big\{ \sup_{(t,x,y,z)\in[0,\infty)\times\mathbb{R}^3} |u(t,x,y,z)|, \quad \sup_{(t,x,y,z)\in[0,\infty)\times\mathbb{R}^3} |u_t(t,x,y,z)|, \\
&\qquad \sup_{(t,x,y,z)\in[0,\infty)\times\mathbb{R}^3} |u_x(t,x,y,z)|, \quad \sup_{(t,x,y,z)\in[0,\infty)\times\mathbb{R}} |u_{xx}(t,x,y,z)|, \\
&\qquad \sup_{(t,x,y,z)\in[0,\infty)\times\mathbb{R}^3} |u_y(t,x,y,z)|, \quad \sup_{(t,x,y,z)\in[0,\infty)\times\mathbb{R}} |u_{yy}(t,x,y,z)|, \\
&\qquad \sup_{(t,x,y,z)\in[0,\infty)\times\mathbb{R}^3} |u_z(t,x,y,z)|, \quad \sup_{(t,x,y,z)\in[0,\infty)\times\mathbb{R}} |u_{zz}(t,x,y,z)|\Big\},
\end{aligned}$$

provided it exists. Let $X = X^1 \times X^1 \times X^1 \times X^1$ be endowed with the norm

$$\|(u, v, w, p)\| = \max\{\|u\|_{X^1}, \quad \|v\|_{X^1}, \quad \|w\|_{X^1}, \quad \|p\|_{X^1}\}, \quad (u, v, w, p) \in X,$$

provided it exists. For $(u, v, w, p) \in X$, we will write $(u, v, w, p) \geq 0$ if $u(t, x, y, z) \geq 0$, $v(t, x, y, z) \geq 0$, $w(t, x, y, z) \geq 0$, $p(t, x, y, z) \geq 0$ for any $(t, x, y, z) \in [0, \infty) \times \mathbb{R}^3$. For $(u, v, w, p) \in X$, define the operators

$$\begin{aligned}
S_1^1(u, v, w, p)(t, x, y, z) &= u(t, x, y, z) - u_0(x, y, z) + \int_0^t \Big(u(s, x, y, z)u_x(s, x, y, z) \\
&\quad + v(s, x, y, z)u_y(s, x, y, z) + w(s, x, y, z)u_z(s, x, y, z) + p_x(s, x, y, z) \\
&\quad - u_{xx}(s, x, y, z) - u_{yy}(s, x, y, z) - u_{zz}(s, x, y, z)\Big) ds, \\
S_1^2(u, v, w, p)(t, x, y, z) &= v(t, x, y, z) - v_0(x, y, z) + \int_0^t \Big(u(s, x, y, z)v_x(s, x, y, z) \\
&\quad + v(s, x, y, z)v_y(s, x, y, z) + w(s, x, y, z)v_z(s, x, y, z) + p_y(s, x, y, z) \\
&\quad - v_{xx}(s, x, y, z) - v_{yy}(s, x, y, z) - v_{zz}(s, x, y, z)\Big) ds, \\
S_1^3(u, v, w, p)(t, x, y, z) &= w(t, x, y, z) - w_0(x, y, z) + \int_0^t \Big(u(s, x, y, z)w_x(s, x, y, z) \\
&\quad + v(s, x, y, z)w_y(s, x, y, z) + w(s, x, y, z)w_z(s, x, y, z) + p_z(s, x, y, z) \\
&\quad - w_{xx}(s, x, y, z) - w_{yy}(s, x, y, z) - w_{zz}(s, x, y, z)\Big) ds, \\
S_1^4(u, v, w, p)(t, x, y, z) &= u_x(t, x, y, z) + v_y(t, x, y, z) + w_z(t, x, y, z),
\end{aligned}$$

$$S_1(u,v,w,p)(t,x,y,z) = \Big(S_1^1(u,v,w,p)(t,x,y,z), S_1^2(u,v,w,p)(t,x,y,z),$$
$$S_1^3(u,v,w,p)(t,x,y,z), S_1^4(u,v,w,p)(t,x,y,z)\Big), \quad (t,x,y,z) \in [0,\infty) \times \mathbb{R}^3.$$

As in Section 3, one can prove the following lemmas.

Lemma 4.2. *Suppose (P1). If $(u,v,w,p) \in X$ satisfies the equation*

$$S_1(u,v,w,p)(t,x,y,z) = 0, \quad (t,x,y,z) \in [0,\infty) \times \mathbb{R}^3,$$

then it is a solution of the IVP (2).

Let

$$B_1 = 3B^2 + 4B.$$

Lemma 4.3. *Suppose (P1). For $(u,v,w,p) \in X$ with $\|(u,v,w,p)\| \le B$, we have*

$$|S_1^1(u,v,w,p)(t,x,y,z)| \le B_1(1+t),$$
$$|S_1^2(u,v,w,p)(t,x,y,z)| \le B_1(1+t),$$
$$|S_1^3(u,v,w,p)(t,x,y,z)| \le B_1(1+t),$$
$$|S_1^4(u,v,w,p)(t,x,y,z)| \le B_1(1+t), \quad (t,x,y,z) \in [0,\infty) \times \mathbb{R}^3.$$

Suppose

(P2) $g \in C([0,\infty) \times \mathbb{R}^3)$, $g(t,x,y,z) > 0$ for $(t,x,y,z) \in (0,\infty) \times (\mathbb{R}\backslash\{\{x=0\} \cup \{y=0\} \cup \{z=0\}\})$,

$$g(0,x,y,z) = g(t,0,y,z) = g(t,x,0,z) = g(t,x,y,0) = 0,$$

$(t,x,y,z) \in [0,\infty) \times \mathbb{R}^3$, and

$$8(1+t)^2\left(1+|x|+x^2\right)\left(1+|y|+y^2\right)\left(1+|z|+z^2\right)$$
$$\times \int_0^t \left|\int_0^x \int_0^y \int_0^z g(t_1,x_1,y_1,z_1)dx_1dy_1dz_1\right| dt_1 \le A,$$

$(t,x,y,z) \in [0,\infty) \times \mathbb{R}^3$, for some constant $A > 0$.

For $(u,v,w,p) \in X$, define the operators

$$S_2^1(u,v,w,p)(t,x,y,z) = \int_0^t \int_0^x (t-t_1)(x-x_1)^2(y-y_1)^2(z-z_1)^2 g(t_1,x_1,y_1,z_1)$$
$$S_1^1(u,v,w,p)(t_1,x_1,y_1,z_1)dx_1dy_1dz_1dt_1,$$

$$S_2^2(u,v,w,p)(t,x,y,z) = \int_0^t \int_0^x (t-t_1)(x-x_1)^2(y-y_1)^2(z-z_1)^2 g(t_1,x_1,y_1,z_1)$$

$$S_1^2(u,v,w,p)(t_1,x_1,y_1,z_1)dx_1dy_1dz_1dt_1,$$

$$S_2^3(u,v,w,p)(t,x,y,z) = \int_0^t \int_0^x (t-t_1)(x-x_1)^2(y-y_1)^2(z-z_1)^2 g(t_1,x_1,y_1,z_1)$$
$$S_1^3(u,v,w,p)(t_1,x_1,y_1,z_1)dx_1dy_1dz_1dt_1,$$

$$S_2^4(u,v,w,p)(t,x,y,z) = \int_0^t \int_0^x (t-t_1)(x-x_1)^2(y-y_1)^2(z-z_1)^2 g(t_1,x_1,y_1,z_1)$$
$$S_1^4(u,v,w,p)(t_1,x_1,y_1,z_1)dx_1dy_1dz_1dt_1,$$

$$S_2(u,v,w,p)(t,x) = \Big(S_2^1(u,v,w,p)(t,x,y,z), S_2^2(u,v,w,p)(t,x,y,z),$$
$$S_2^3(u,v,w,p)(t,x,y,z), S_2^4(u,v,w,p)(t,x,y,z)\Big), \quad (t,x,y,z) \in [0,\infty)\times\mathbb{R}^3,$$

$(t,x,y,z) \in [0,\infty)\times\mathbb{R}^3$.

Lemma 4.4. *Suppose* (*P*1) *and* (*P*2). *For* $(u,v,w,p) \in X$, $\|(u,v,w,p)\| \leq B$, *we have*

$$\|S_2(u,v,w,p)\| \leq AB_1.$$

Next,

Lemma 4.5. *Suppose* (*P*1) *and* (*P*2). *If* $(u,v,w,p) \in X$ *satisfies the equation*

$$S_2(u,v,w,p)(t,x,y,z) = 0, \quad (t,x,y,z) \in [0,\infty)\times\mathbb{R}^3,$$

then (u,v,w,p) *is a solution to the IVP* (2).

Below, suppose

(P3) $\epsilon \in (0,1)$, A, B and B_1 satisfy the inequalities $\epsilon B_1(1+A) < 1$ and $AB_1 < B$.

Let $\widetilde{\widetilde{\widetilde{Y}}}$ denote the set of all equi-continuous families in X with respect to the norm $\|\cdot\|$. Let also, $\widetilde{\widetilde{Y}} = \overline{\widetilde{\widetilde{\widetilde{Y}}}}$ be the closure of $\widetilde{\widetilde{\widetilde{Y}}}$, $\widetilde{Y} = \widetilde{\widetilde{Y}} \cup \{(u_0,v_0,w_0)\}$,

$$Y = \{(u,v) \in \widetilde{Y} : (u,v,w,p) \geq 0, \quad \|(u,v,w,p)\| \leq B\}.$$

Note that Y is a compact set in X. For $(u,v,w,p) \in X$, define the operators

$$T(u,v,w,p)(t,x,y,z) = -\epsilon(u,v,w,p)(t,x,y,z),$$

$$S(u,v,w,p)(t,x,y,z) = (u,v,w,p)(t,x,y,z) + \epsilon(u,v,w,p)(t,x,y,z)$$
$$+\epsilon S_2(u,v,w,p)(t,x,y,z), \quad (t,x,y,z) \in [0,\infty)\times\mathbb{R}^3.$$

Let $\widetilde{Y}$ denote the set of all equi-continuous families in X with respect to the norm $\|\cdot\|$. Let also, $Y = \overline{\widetilde{Y}}$ be the closure of $\widetilde{Y}$,

$$U = \{(u,v,w,p) \in Y : \|(u,v,w,p)\| < B\}.$$

For $(u, v, w, p) \in \overline{U}$ and $\epsilon > 0$, define the operators

$$T(u,v,w,p)(t,x,y,z) = \epsilon(u,v,w,p)(t,x,y,z),$$

$$S(u,v,w,p)(t,x,y,z) = (u,v,w,p)(t,x,y,z) - \epsilon(u,v,w,p)(t,x,y,z) - \epsilon S_2(u,v,w,p)(t,x,y,z),$$

$(t, x, y, z) \in [0, \infty) \times \mathbb{R}^3$. For $(u, v, w, p) \in \overline{U}$, we have

$$\begin{aligned} \|(I-S)(u,v,w,p)\| &= \|\epsilon(u,v,w,p) + \epsilon S_2(u,v,w,p)\| \\ &\leq \epsilon\|(u,v,w,p)\| + \epsilon\|S_2(u,v,w,p)\| \\ &\leq \epsilon B + \epsilon A B_1. \end{aligned}$$

Thus, $S : \overline{U} \to X$ is continuous and $(I - S)(\overline{U})$ resides in a compact subset of Y. Now, suppose that there is a $(u, v, w, p) \in Y$ so that $\|(u, v, w, p)\| = B$ and

$$(u,v,w,p) = \lambda(I-S)(u,v,w,p)$$

or

$$(u,v,w,p) = \lambda\epsilon\,(I+S_2)(u,v,w,p), \tag{9}$$

for some $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Note that $(Y, \|\cdot\|)$ is a Banach space. Assume that the set

$$\mathcal{A} = \{(u,v,w,p) \in Y : (u,v,w,p) = \mu(I+S_2)(u,v,w,p), \quad 0 < \mu < 1\}$$

is bounded. By (9), it follows that the set $\mathcal{A}$ is not empty set. Then, by Schaefer's Theorem, it follows that there is a $(u^*, v^*, w^*, p^*) \in Y$ such that

$$(u^*,v^*,w^*,p^*) = (I+S_2)(u^*,v^*,w^*,p^*), \tag{10}$$

or

$$S_2(u^*,v^*,w^*,p^*) = 0,$$

i.e., (u^*, v^*, w^*, p^*) is a solution to the problem (1). Assume that the set $\mathcal{A}$ is unbounded. Then, by Schaefer's Theorem, it follows that the equation

$$(u,v,w,p) = \mu(I+S_2)(u,v,w,p), \quad (u,v,w,p) \in Y,$$

has at least one small solution $(u^*, v^*, w^*, p^*) \in Y$ for any $\mu \in [0, 1]$. In particular, for $\mu = 1$, there is a $(u^*, v^*, w^*, p^*) \in Y$ such that (10) holds and then it is a solution to the problem (1). Let now,

$$\{(u,v,w,p) \in Y : (u,v,w,p) = \lambda_1(I-S)(u,v,w,p),\ \|(u,v,w,p)\| = B\} = \emptyset$$

for any $\lambda_1 \in \left(0, \frac{1}{\epsilon}\right)$. Then, from Theorem 2.4, it follows that the operator $T + S$ has a fixed point $(u^*, v^*, w^*, p^*) \in Y$. Therefore

$$\begin{aligned} (u^*,v^*,w^*,p^*)(t,x,y,z) &= T(u^*,v^*,w^*,p^*)(t,x,y,z) + S(u^*,v^*,w^*,p^*)(t,x,y,z) \\ &= \epsilon(u^*,v^*,w^*,p^*)(t,x,y,z) + (u^*,v^*,w^*,p^*)(t,x,y,z) \\ &\quad -\epsilon(u^*,v^*,w^*,p^*)(t,x,y,z) - \epsilon S_2(u^*,v^*,w^*,p^*)(t,x,y,z), \end{aligned}$$

$(t, x, y, z) \in [0, \infty) \times \mathbb{R}^3$, whereupon

$$S_2(u^*,v^*,w^*,p^*)(t,x,y,z) = 0, \quad (t,x,y,z) \in [0,\infty) \times \mathbb{R}^3.$$

From here, (u^*, v^*, w^*, p^*) is a solution to the problem (1). □

4.2. *Existence of at Least Two Non-Negative Solutions*

The main result in this section is as follows.

Theorem 4.6. *Suppose that* $(P1)$ *holds. Then the IVP* (2) *has at least two non-negative solutions in* $(C^1([0,\infty), C^2(\mathbb{R}^3)))^4$.

Proof. Let X be the space used in the previous section. Suppose

(P4) Let $m > 0$ be large enough and A, B, r, L, R_1 be positive constants that satisfy the following conditions

$$r < L < R_1 \le B, \quad \epsilon > 0, \quad R_1 > \left(\frac{2}{5m} + 1\right) L,$$

$$AB_1 < \frac{L}{5}.$$

Let

$$\widetilde{P} = \{(u,v,w,p) \in X : (u,v,w,p) \ge 0 \quad \text{on} \quad [0,\infty) \times \mathbb{R}^3\}.$$

With $\mathcal{P}$ we will denote the set of all equi-continuous families in $\widetilde{P}$. For $(u,v,w,p) \in X$, define the operators

$$T_1(u,v,w,p)(t,x,y,z) = (1 + m\epsilon)(u,v,w,p)(t,x,y,z) - \left(\epsilon\frac{L}{10}, \epsilon\frac{L}{10}\right),$$

$$S_3(u,v,w,p)(t,x,y,z) = -\epsilon S_2(u,v,w,p)(t,x,y,z) - m\epsilon(u,v,w,p)(t,x,y,z) - \left(\epsilon\frac{L}{10}, \epsilon\frac{L}{10}\right),$$

$(t,x,y,z) \in [0,\infty) \times \mathbb{R}^3$. Note that any fixed point $(u,v,w,p) \in X$ of the operator $T_1 + S_3$ is a solution to the IVP (2). Define

$$U_1 = \mathcal{P}_r = \{(u,v,w,p) \in \mathcal{P} : \|(u,v,w,p)\| < r\},$$

$$U_2 = \mathcal{P}_L = \{(u,v,w,p) \in \mathcal{P} : \|(u,v,w,p)\| < L\},$$

$$U_3 = \mathcal{P}_{R_1} = \{(u,v,w,p) \in \mathcal{P} : \|(u,v,w,p)\| < R_1\},$$

$$R_2 = R_1 + \frac{A}{m}B_1 + \frac{L}{5m},$$

$$\Omega = \overline{\mathcal{P}_{R_2}} = \{(u,v) \in \mathcal{P} : \|(u,v,w,p)\| \le R_2\}.$$

Now, the proof repeats the proof of Theorem ??. □

4.3. *An Example*

Let A, B, R_1, L, r, m, A, ϵ be as in Section 3.3. Then

$$B_1 = 340$$

and $(P3)$ and $(P4)$ hold. Take Q as in Section 3.3. Take

$$g_1(t,x,y,z) = Q(t)Q(x)Q(y)Q(z), \quad (t,x,y,z) \in [0,\infty) \times \mathbb{R}^3.$$

Then there exists a constant $C_4 > 0$ such that

$$8(1+t)^2\left(1+|x|+x^2\right)\left(1+|y|+y^2\right)\left(1+|z|+z^2\right)$$

$$\int_0^t \left| \int_0^x \int_0^y \int_0^z g_1(t_1, x_1, y_1, z_1) dx_1 dy_1 dz_1 \right| dt_1 \leq C_4, \quad (t, x, y, z) \in [0, \infty) \times \mathbb{R}^3.$$

Let

$$g(t, x, y, z) = \frac{A}{C_4} g_1(t, x, y, z), \quad (t, x) \in [0, \infty) \times \mathbb{R}^3.$$

Then

$$8(1+t)^2 \left(1 + |x| + x^2\right)\left(1 + |y| + y^2\right)\left(1 + |z| + z^2\right)$$

$$\int_0^t \left| \int_0^x \int_0^y \int_0^z g(t_1, x_1, y_1, z_1) dx_1 dy_1 dz_1 \right| dt_1 \leq C_4, \quad (t, x, y, z) \in [0, \infty) \times \mathbb{R}^3.$$

i.e., $(P2)$ holds. Therefore for the functions

$$u_0(x, y, z) = v_0(x, y, z) = w_0(x, y, z) = \frac{x^2 + y^2 + z^2}{10 + x^4 + y^4 + z^4}, \quad (x, y, z) \in \mathbb{R}^3,$$

the IVP (2) satisfies all conditions of Theorem 4.1 and Theorem 4.6.

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