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Existence of pyramidal traveling fronts to the buffered bistable systems in $\mathbb{R}^3$

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Abstract. This paper studies the pyramidal calcium concentration waves for buffered bistable systems in $\mathbb{R}^3$. We show the existence of three-dimensional pyramidal traveling fronts by using the fixed point theory and the super-subsolution method combined with the comparison principle. Our result implies that multiple immobile buffers (where all buffers do not diffuse) do not affect the existence of pyramidal calcium concentration waves.

1. Introduction

For a long time, many researchers have sought to understand the traveling fronts of the reaction-diffusion equation [4, 5, 29] and they have observed the phenomenon of wave propagation in various fields, such as biology and chemistry with regards to the FitzHugh-Nagumo model [22] and the Belousov-Zhabotinskii reaction [7, 12]. Among them, the study on the propagation of calcium concentration waves between cells and within them has received widespread attention [1, 3, 9, 17]. In general, the mechanism of calcium wave generation is based on reaction-diffusion, which can be documented by the equation

$$u_t(\mathbf{x}, t) = D\Delta u(\mathbf{x}, t) + f(u(\mathbf{x}, t)), \quad \mathbf{x} \in \mathbb{R}^N, \ t > 0, \tag{1}$$

where u represents the concentration of free cytosolic calcium, Δ is the Laplace operator, $D > 0$ represents diffusion coefficient of free cytosolic calcium in the cytoplasm, N denotes the spatial dimension of cells and the bistable nonlinear reaction term $f(u)$ not only maintains stable self-sustaining waves [4], but also is considered to be critical for the fertilization of calcium waves in mature Xenopus oocytes [6, 33].

Due to the presence of calcium buffers in cells [15], the study of calcium waves is slightly different from other excitable systems. Calcium buffer is a kind of protein in the cytoplasm that can bind to free calcium, thereby limiting the diffusion of free calcium and controlling the release and uptake of calcium [17, 32]. Consequently, whether calcium buffers have an effect on calcium waves has aroused wide concern. One of

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buffered systems can be written as

$$\begin{cases} u_t(\mathbf{x},t) = D\Delta u + f(u) + \sum_{i=1}^{n}\left[k_-^i b_0^i - \left(k_+^i u + k_-^i\right)v_i\right], \\ v_{i,t}(\mathbf{x},t) = D_i\Delta v_i + k_-^i b_0^i - \left(k_+^i u + k_-^i\right)v_i, \quad i = 1,2,\cdots,n, \end{cases} \tag{2}$$

for all $(\mathbf{x},t) \in \mathbb{R}^N \times (0,+\infty)$, where v_i and $D_i \geq 0$ represent the concentration and diffusion coefficient of the i-th free buffer in the cytoplasm respectively, n is the number of species of the free buffer, $b_0^i > 0$ represents the amount of the total concentration of the i-th buffer, including the concentration of the i-th free buffer and the i-th non-free buffer, and $k_\pm^i$ are reaction rates of calcium ions and the i-th free buffer through the following reaction

$$Ca^{2+} + B_i \underset{k_-^i}{\overset{k_+^i}{\rightleftharpoons}} CaB_i, \quad i = 1,\cdots,n,$$

where B_i represents the i-th free buffer and CaB_i represents the i-th non-free buffer.

As a special solution of the development model based on the unbounded region, the traveling front can well describe the properties of the solution of the reaction-diffusion equation. In high-dimensional space, under the influnce of curvature, the equation has traveling fronts whose level set is not hyperplane. Thus it is very vital to study nonplanar traveling fronts in high-dimensional space. For the Fisher-KPP monostable case, Hamel and Nadirashvili [8] proved the existence of an infinite-dimensional manifold of nonplanar traveling fronts in $\mathbb{R}^N$ $(N \geq 2)$. For the degenerate Fisher-KPP monostable case, Wang and Bu [31] established the existence of pyramidal traveling fronts in $\mathbb{R}^3$ and showed the existence and stability of V-shaped traveling fronts. In [2], they further studied the stability of pyramidal traveling fronts in $\mathbb{R}^3$. When the nonlinear term reaction $f(u)$ is bistable, Ninomiya and Taniguchi [16] obtained the existence of the V-shaped traveling fronts in $\mathbb{R}^2$ by constructing supersolutions and subsolutions and using the comparison principle. Later, Taniguchi [23, 24] used a similar method to prove the existence and stability of the pyramidal traveling fronts in $\mathbb{R}^3$. Wang [30] studied the existence, uniqueness and stability of V-shaped traveling fronts for reaction-diffusion bistable systems in $\mathbb{R}^2$. Wang, Li and Ruan [34] also showed the existence, uniqueness and stability of three-dimensional traveling fronts for monotone bistable systems of reaction-diffusion equations in $\mathbb{R}^3$ by using the super-subsolution method combined with the comparison argument. For more research on higher dimensional space, we can refer to literatures [13, 19–21, 25, 26] and the references therein. With regard to system (2), Tsai and Sneyd [27, 28] showed the existence, local stability and uniqueness of one-dimensional traveling waves for $D_i = 0 (i = 1,2,\cdots,n)$ and the stability and uniqueness of one-dimensional traveling waves for $D_i \geq 0 (i = 1,2,\cdots,n)$. Jia et al. [10, 11] obtained the existence, global stability and uniqueness of V-shaped traveling fronts of the buffered bistable systems in $\mathbb{R}^2$. As mentioned above, reaction-diffusion equations may have traveling curved fronts with different types of level sets in spaces of different dimensions. For system (2) with $D_i = 0, (i = 1,2,\cdots,n)$, there is no relevant conclusion on whether a pyramidal traveling front exists in $\mathbb{R}^3$. In this paper, we will answer that question in the affirmative. It is worth noting that only the first equation has a positive diffusion term, while the other n equations have no positive diffusion term which makes the equation lose the regularization estimation, so that the existence of the solution cannot be verified by the prior estimation. Therefore, when improving the regularity of the solution, the requirements are higher and the difficulty will be increased.

In this paper, assume that the nonlinear term $f(u)$ is a simple form

$$f(u) = u(u-a)(1-u), \quad 0 < a < \frac{1}{2}.$$

Without loss of generality, we can research system (2) with only one free buffer. That is, we study the buffered bistable system

$$\begin{cases} u_t(x,y,z,t) = D\Delta u + f(u) + \left[k_- b_0 - (k_+ u + k_-)v\right], \\ v_t(x,y,z,t) = k_- b_0 - (k_+ u + k_-)v, \end{cases} \tag{3}$$

for all $(x, y, z, t) \in \mathbb{R}^3 \times (0, +\infty)$, where D, k_+, k_- and b_0 are positive constants. Studying the existence of traveling fronts of system (3) is complicated by the fact that the diffusion coefficient of the second equation is 0, which makes the system lose regular estimate. Thus we will apply the Banach's fixed point theory to show the existence of pyramidal traveling fronts of system (3).

Let us now give two symbolic definitions. For any two vectors $\mathbf{a}, \mathbf{a}' \in \mathbb{R}^3$, $\mathbf{a} \le (<)\mathbf{a}'$ means $a_i \le (<) a_i'$ with $i = 1, 2, 3$. The interval $[\mathbf{a}, \mathbf{a}'] = \left\{\mathbf{x} \in \mathbb{R}^3 | \mathbf{a} \le \mathbf{x} \le \mathbf{a}'\right\}$. After defining $\phi_1(x, y, z, t) = u(x, y, z, t)$, $\phi_2(x, y, z, t) = b_0 - v(x, y, z, t)$, $\mathbf{\Phi} = (\phi_1, \phi_2)$ and $\mathbf{D} = \mathrm{diag}(D, 0)$, then system (3) can be simplified as

$$\mathbf{\Phi}_t(x, y, z, t) = \mathbf{D}\Delta\mathbf{\Phi}(x, y, z, t) + \mathbf{F}(\mathbf{\Phi}(x, y, z, t)), \tag{4}$$

where

$$\mathbf{F}(\mathbf{\Phi}) = (f_1(\mathbf{\Phi}), f_2(\mathbf{\Phi})) = (f(\phi_1) + k_-\phi_2 - k_+\phi_1(b_0 - \phi_2), -k_-\phi_2 + k_+\phi_1(b_0 - \phi_2)).$$

For convenience, we always denote $\mathbf{0} = (0, 0)$ and $\mathbf{G} = (1, b_0 - b_1) = \left(1, b_0 - \frac{k_- b_0}{k_- + k_+}\right)$. Obviously $0 < b_1 < b_0$, and $\mathbf{0}$ and $\mathbf{G}$ are two equilibria of system (4). From [27], it follows that the Eq.(4) has a unique positive traveling front $\mathbf{\Psi}(\varsigma) = (\psi_1(\varsigma), \psi_2(\varsigma))$ connecting $\mathbf{0}$ and $\mathbf{G}$ and the wave speed c_* is positive, where $\varsigma = (x, y, z) \cdot \mathbf{e} + c_* t$ and $\mathbf{e} \in \mathbb{S}^2$. That is, $(\psi_1(\varsigma), \psi_2(\varsigma))$ satisfies

$$\begin{cases} D\psi_1'' - c_*\psi_1' + f(\psi_1) + k_-\psi_2 - k_+\psi_1(b_0 - \psi_2) = 0, \\ -c_*\psi_2' - k_-\psi_2 + k_+\psi_1(b_0 - \psi_2) = 0, \\ 0 < \psi_1 < 1,\ 0 < \psi_2 < b_0 - b_1,\ \psi_1' > 0,\ \psi_2' > 0, \\ \psi_1(-\infty) = 0,\ \psi_1(+\infty) = 1, \\ \psi_2(-\infty) = 0,\ \psi_2(+\infty) = b_0 - b_1. \end{cases} \tag{5}$$

In addition, the traveling front $(\psi_1(\varsigma), \psi_2(\varsigma))$ has the following asymptotic behavior.

Lemma 1.1. *[10, Lemma 1.1] There exist two positive constants C_0 and β_0 such that*

$$\max\{|1 - \psi_1(\varsigma)|, |b_0 - b_1 - \psi_2(\varsigma)|\} + \max\left\{|\psi_1'(\varsigma)|, |\psi_2'(\varsigma)|\right\} + |\psi_1''(\varsigma)| \le C_0 e^{-\beta_0\varsigma}, \quad \varsigma \ge 0,$$

$$\max\{|\psi_1(\varsigma)|, |\psi_2(\varsigma)|\} + \max\left\{|\psi_1'(\varsigma)|, |\psi_2'(\varsigma)|\right\} + |\psi_1''(\varsigma)| \le C_0 e^{-\beta_0|\varsigma|}, \quad \varsigma \le 0.$$

This paper mainly studies the existence of the three-dimensional pyramidal shaped traveling front of Eq.(4). Affected by curvature, we can assume $c > c_*$ and define $m_* = \frac{\sqrt{c^2 - c_*^2}}{c_*}$. Let

$$\mathbf{\Phi}(x, y, z, t) = \mathbf{v}(x_1, x_2, x_3, t),\ (x_1, x_2, x_3) = (x, y, z + ct), \tag{6}$$

which travels in the direction of the z-axis. Substituting $\mathbf{v}$ into Eq.(4), we have

$$\begin{cases} \mathbf{v}_t = \mathbf{D}\Delta\mathbf{v} - c\mathbf{v}_{x_3} + \mathbf{F}(\mathbf{v}), & \mathbf{x} \in \mathbb{R}^3,\ t > 0, \\ \mathbf{v}_0(\mathbf{x}) = \mathbf{v}(\mathbf{x}, 0), & \mathbf{x} \in \mathbb{R}^3, \end{cases} \tag{7}$$

where $\mathbf{x} = (\mathbf{x}', x_3)$ and $\mathbf{x}' = (x_1, x_2)$. The goal of this paper is to find the solution $\mathbf{W}(\mathbf{x}) = (W_1(\mathbf{x}), W_2(\mathbf{x}))$ satisfying the equation

$$\mathcal{L}[\mathbf{W}] := -\mathbf{D}\Delta\mathbf{W} + c\mathbf{W}_{x_3} - \mathbf{F}(\mathbf{W}) = \mathbf{0}, \quad \mathbf{x} \in \mathbb{R}^3. \tag{8}$$

Now we construct a pyramid which comes from [23]. Let $l \in \mathbb{N}$ and $l \ge 3$. Assume that $\left\{\theta_j\right\}_{1 \le j \le l}$ satisfies

$$0 \le \theta_1 < \theta_2 < \cdots < \theta_l < 2\pi \quad \text{and} \quad \max_{1 \le j \le l}\left(\theta_{j+1} - \theta_j\right) < \pi, \tag{9}$$

where $\theta_{l+1} = \theta_1 + 2\pi$. And then $\left(m_* \cos\theta_j, m_* \sin\theta_j, 1\right)$ is the normal vector of surface $\left\{\mathbf{x} \in \mathbb{R}^3 \mid -x_3 = h_j(x_1, x_2)\right\}$, where $h_j(x_1, x_2) = m_* \left(x_1 \cos\theta_j + x_2 \sin\theta_j\right) (1 \le j \le l)$. For any $(x_1, x_2) \in \mathbb{R}^2$, let

$$h(x_1, x_2) = \max_{1 \le j \le l} h_j(x_1, x_2) = m_* \max_{1 \le j \le l} \left(x_1 \cos\theta_j + x_2 \sin\theta_j\right).$$

Then $\left\{\mathbf{x} \in \mathbb{R}^3 \mid -x_3 = h(x_1, x_2)\right\}$ is a pyramid in $\mathbb{R}^3$, and for any $(x_1, x_2) \in \mathbb{R}^2$,

$$h(x_1, x_2) \ge 0, \qquad \lim_{R \to \infty} \inf_{x_1^2 + x_2^2 \ge R^2} h(x_1, x_2) = \infty.$$

Let

$$\Omega_j = \left\{\mathbf{x}' \in \mathbb{R}^2 \mid h(x_1, x_2) = h_j(x_1, x_2)\right\}, \quad j = 1, 2, \cdots, l,$$

then $\mathbb{R}^2 = \bigcup_{j=1}^{l} \Omega_j$. (9) yields that the planes $\Omega_1, \Omega_2, \cdots, \Omega_l$ are arranged in a counterclockwise direction. Let $\partial\Omega_j$ be the boundary of Ω_j and $K = \bigcup_{j=1}^{l} \partial\Omega_j$. Each side of $\left\{\mathbf{x} \in \mathbb{R}^3 \mid -x_3 = h(x_1, x_2)\right\}$ can be represented as

$$S_j = \left\{\mathbf{x} \in \mathbb{R}^3 \mid -x_3 = h_j(x_1, x_2), (x_1, x_2) \in \Omega_j\right\}, \quad j = 1, 2, \cdots, l.$$

Denote each edge of the pyramid $\left\{\mathbf{x} \in \mathbb{R}^3 \mid -x_3 = h(x_1, x_2)\right\}$ as

$$\Gamma_j = \begin{cases} S_j \cap S_{j+1}, & 1 < j < l - 1, \\ S_l \cap S_1, & j = l, \end{cases}$$

then $\bigcup_{j=1}^{l} S_j \subset \mathbb{R}^3$ represents the set of all lateral surfaces of the pyramid, and $\Gamma = \bigcup_{j=1}^{l} \Gamma_j$ represents the set of all edges of the pyramid. For any $\bar{\gamma} \ge 0$, define

$$\mathcal{D}(\bar{\gamma}) = \left\{\mathbf{x} \in \mathbb{R}^3 \mid \operatorname{dist}(\mathbf{x}, \Gamma) \ge \bar{\gamma}\right\}.$$

For any $1 \le j \le l$, it is obvious that $\boldsymbol{\Psi}\left(\frac{c_*}{c}\left(x_3 + h_j(x_1, x_2)\right)\right)$ is the solution of Eq.(8). Define

$$\mathbf{v}^-(\mathbf{x}) = \boldsymbol{\Psi}\left(\frac{c_*}{c}(x_3 + h(x_1, x_2))\right) = \max_{1 \le j \le l} \boldsymbol{\Psi}\left(\frac{c_*}{c}\left(x_3 + h_j(x_1, x_2)\right)\right), \tag{10}$$

then $\mathbf{v}^-(\mathbf{x})$ is a subsolution of Eq.(8), and $\boldsymbol{\Psi}'(\varsigma) > 0$ yields $\mathbf{v}^-_{x_3}(\mathbf{x}) > \mathbf{0}$ for any $\mathbf{x} \in \mathbb{R}^3$.

The main result of this paper is the existence of three-dimensional pyramidal traveling front.

Theorem 1.2. *For any $c > c_*$, the Eq.(4) exists a nonplanar traveling front $\mathbf{W}(\mathbf{x})$ which satisfies Eq.(8), $\mathbf{W}(\mathbf{x}) > \mathbf{v}^-(\mathbf{x})$, $\mathbf{W}_{x_3}(\mathbf{x}) > \mathbf{0}$ for any $\mathbf{x} \in \mathbb{R}^3$, $\mathbf{W}(\mathbf{x}_1', x_3) = \mathbf{W}(\mathbf{x}_2', x_3)$ if $\left|\mathbf{x}_1'\right| = \left|\mathbf{x}_2'\right|$,*

$$\mathbf{W}_{x_1}(0, x_2, x_3) = \mathbf{0}, \quad \mathbf{W}_{x_1}(\mathbf{x}) > \mathbf{0}, \quad \forall (x_1, x_2, x_3) \in (0, +\infty) \times \mathbb{R}^2,$$
$$\mathbf{W}_{x_2}(x_1, 0, x_3) = \mathbf{0}, \quad \mathbf{W}_{x_2}(\mathbf{x}) > \mathbf{0}, \quad \forall (x_1, x_2, x_3) \in \mathbb{R} \times (0, +\infty) \times \mathbb{R},$$

and

$$\lim_{\bar{\gamma} \to \infty} \sup_{\mathbf{x} \in \mathcal{D}(\bar{\gamma})} \left|\mathbf{W}(\mathbf{x}) - \mathbf{v}^-(\mathbf{x})\right| = 0. \tag{11}$$

The rest of this paper is organized as follows. In Section 2, we list some vital and useful notations and preliminaries. In Section 3, we establish the existence of three-dimensional pyramidal traveling front by constructing the supersolution and using comparison principle and fixed point theory. That is, we give the proof of Theorem 1.2.

2. Preliminaries

First of all, we consider the qualitative properties of Jacobian matrix $D\mathbf{F}(\mathbf{\Phi})$.

$$D\mathbf{F}(\mathbf{\Phi}) = \begin{pmatrix} f_{11}(\mathbf{\Phi}) & f_{12}(\mathbf{\Phi}) \\ f_{21}(\mathbf{\Phi}) & f_{22}(\mathbf{\Phi}) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial \phi_1} & \frac{\partial f_1}{\partial \phi_2} \\ \frac{\partial f_2}{\partial \phi_1} & \frac{\partial f_2}{\partial \phi_2} \end{pmatrix} = \begin{pmatrix} -3\phi_1^2 + 2(a+1)\phi_1 + k_+\phi_2 - k_+b_0 - a & k_+\phi_1 + k_- \\ -k_+\phi_2 + k_+b_0 & -k_+\phi_1 - k_- \end{pmatrix}. \tag{12}$$

Similar to the process in [10], we obtain $\lambda^- < 0$ and $\lambda^+ < 0$, where λ^- and λ^+ are the principal eigenvalues of $D\mathbf{F}(\mathbf{0})$ and $D\mathbf{F}(\mathbf{G})$, respectively. In addition, we can get that there exist positive eigenvectors of $D\mathbf{F}(\mathbf{0})$ and $D\mathbf{F}(\mathbf{G})$, which we shall name $\mathbf{P}^- = (P_1^-, P_2^-)$ and $\mathbf{P}^+ = (P_1^+, P_2^+)$ respectively.

Let $\delta_1 > 0$ be a small enough constant such that $\mathbf{P}^+ > \delta_1 \mathbf{P}^-$, and define

$$\mathbf{Q}^- = \delta_1 \mathbf{P}^- = (Q_1^-, Q_2^-), \quad \mathbf{Q}^+ = \mathbf{P}^+ = (Q_1^+, Q_2^+).$$

Obviously, there hold $\mathbf{Q}^+ > \mathbf{Q}^- > \mathbf{0}$. Define

$$\mathbf{H}^- = (H_1^-, H_2^-) = \lambda^- \mathbf{Q}^- = D\mathbf{F}(\mathbf{0})\mathbf{Q}^- < \mathbf{0}, \quad \mathbf{H}^+ = (H_1^+, H_2^+) = \lambda^+ \mathbf{Q}^+ = D\mathbf{F}(\mathbf{G})\mathbf{Q}^+ < \mathbf{0}. \tag{13}$$

Since $D\mathbf{F}(\mathbf{\Phi})$ is continuous, by (13), we can choose $0 < \delta_2 < 1$ small enough such that

$$D\mathbf{F}(\mathbf{\Phi})\mathbf{Q}^- < \frac{1}{2}\mathbf{H}^-, \quad \mathbf{\Phi} \in [\mathbf{G}^-, \delta_2 \mathbf{P}^-], \tag{14}$$

$$D\mathbf{F}(\mathbf{\Phi})\mathbf{Q}^+ < \frac{1}{2}\mathbf{H}^+, \quad \mathbf{\Phi} \in [\mathbf{G} - \delta_2 \mathbf{P}^+, \mathbf{G}^+], \tag{15}$$

$$k_- - \delta_2 k_+ P_1^- > 0, \quad b_1 - \delta_2 P_2^+ > 0, \tag{16}$$

and

$$\begin{aligned} \mathbf{F}(\mathbf{G}^-) &= \mathbf{F}(\mathbf{0}) - \delta_2 D\mathbf{F}(\mathbf{0})\mathbf{P}^- + o(\delta_2 \left|\mathbf{P}^-\right|) = -\delta_2 \lambda^- \mathbf{P}^- + o(\delta_2 \left|\mathbf{P}^-\right|) > \mathbf{0}, \\ \mathbf{F}(\mathbf{G}^+) &= \mathbf{F}(\mathbf{G}) + \delta_2 D\mathbf{F}(\mathbf{G})\mathbf{P}^+ + o(\delta_2 \left|\mathbf{P}^+\right|) = \delta_2 \lambda^+ \mathbf{P}^+ + o(\delta_2 \left|\mathbf{P}^+\right|) < \mathbf{0}, \end{aligned} \tag{17}$$

where $\mathbf{G}^- = (G_1^-, G_2^-) = -\delta_2 \mathbf{P}^-$ and $\mathbf{G}^+ = (G_1^+, G_2^+) = \mathbf{G} + \delta_2 \mathbf{P}^+$. In this paper, we fix the constant δ_2 satisfying (14)–(17) and define

$$\hat{Q} = \max\left\{Q_1^+, Q_2^+\right\}, \ \check{Q} = \min\left\{Q_1^-, Q_2^-\right\}, \ \hat{H} = \max\left\{H_1^+, H_2^+\right\}, \ \check{H} = \max\left\{H_1^-, H_2^-\right\}.$$

Let $\mu_1 = k_+b_0$, $\mu_2 = k_-$, $\bar{f}_1(\mathbf{\Phi}) = f(\phi_1) + k_-\phi_2 + k_+\phi_1\phi_2$ and $\bar{f}_2(\mathbf{\Phi}) = k_+b_0\phi_1 - k_+\phi_1\phi_2$, then Eq.(4) can be rewriten as

$$\begin{cases} \frac{\partial \phi_1}{\partial t} = D\Delta\phi_1 - \mu_1\phi_1 + \bar{f}_1(\mathbf{\Phi}), \\ \frac{\partial \phi_2}{\partial t} = -\mu_2\phi_2 + \bar{f}_2(\mathbf{\Phi}). \end{cases} \tag{18}$$

Following from (17), $\mathbf{G}^-$ and $\mathbf{G}^+$ are the subsolution and supersolution of system (18) respectively. For any $(\phi_1, \phi_2) \in [\mathbf{G}^-, \mathbf{G}^+]$, (16) yields that

$$\frac{\partial \bar{f}_1}{\partial \phi_2}(\phi_1, \phi_2) = k_- + k_+\phi_1 \geq k_- - \delta_2 k_+ P_1^- > 0,$$

$$\frac{\partial \bar{f}_2}{\partial \phi_1}(\phi_1, \phi_2) = k_+(b_0 - \phi_2) \geq k_+(b_1 - \delta_2 P_2^+) > 0,$$

which shows that system (18) is cooperative on $[\mathbf{G}^-, \mathbf{G}^+]$. Let $E = BUC\left(\mathbb{R}^3, \mathbb{R}^2\right)$ be the Banach space that consists of all bounded and uniformly continuous vector-valued functions from $\mathbb{R}^3$ to $\mathbb{R}^2$ and denote $E^+ = \left\{\mathbf{u} \in E | \mathbf{u}(x,y,z) \geq \mathbf{0}, (x,y,z) \in \mathbb{R}^3\right\}$, then E^+ is a closed cone of E. Define a strongly continuous semigroup on E as

$$\mathbf{T}(t) = \mathrm{diag}(T_1(t), T_2(t)), \ t > 0,$$

where for any $(x,y,z) \in \mathbb{R}^3$ and $t > 0$,

$$\begin{aligned} T_1(t)u_1(x,y,z) &= e^{-\mu_1 t} \int_{\mathbb{R}^3} \frac{1}{\left(2\sqrt{\pi D t}\right)^3} e^{-\frac{(x-y_1)^2+(y-y_2)^2+(z-y_3)^2}{4Dt}} u_1(y_1,y_2,y_3)\, dy_1 dy_2 dy_3, \\ T_2(t)u_2(x,y,z) &= e^{-\mu_2 t} u_2(x,y,z). \end{aligned} \tag{19}$$

Now we give the definitions of classical and mild subsolution (supersolution) respectively.

Definition 2.1. *If functions* $\phi_1(x,y,z,t) \in C^{2,1}\left(\mathbb{R}^3 \times (0,+\infty)\right) \cap C\left(\mathbb{R}^3 \times [0,+\infty)\right)$ *and* $\phi_2(x,y,z,t) \in C^{0,1}\left(\mathbb{R}^3 \times (0,+\infty)\right) \cap C\left(\mathbb{R}^3 \times [0,+\infty)\right)$ *satisfy*

$$\begin{cases} \phi_{1,t} - D\Delta\phi_1 + \mu_1\phi_1 - \bar{f_1}(\mathbf{\Phi}) \leq 0 \ (\geq 0), & \forall (x,y,z,t) \in \mathbb{R}^3 \times (0,+\infty), \\ \phi_{2,t} + \mu_2\phi_2 - \bar{f_2}(\mathbf{\Phi}) \leq 0 \ (\geq 0), & \forall (x,y,z,t) \in \mathbb{R}^3 \times (0,+\infty), \end{cases}$$

then the vector-valued function $\mathbf{\Phi}(x,y,z,t)$ *is called a classical subsolution (supersolution) of system* (18).

Definition 2.2. *If the continuous vector-valued function* $\mathbf{\Phi}(x,y,z,t) : \mathbb{R}^3 \times [0,+\infty) \to [\mathbf{G}^-, \mathbf{G}^+]$ *satisfies*

$$\mathbf{\Phi} \leq (\geq) \mathbf{T}(t-s)\mathbf{\Phi}(s) + \int_s^t \mathbf{T}(t-r)\bar{\mathbf{F}}(\mathbf{\Phi}(r))dr, \tag{20}$$

for $0 \leq s < t$, *where* $\bar{\mathbf{F}}(\mathbf{\Phi}) = (\bar{f_1}(\mathbf{\Phi}), \bar{f_2}(\mathbf{\Phi}))$, *then the vector-valued function* $\mathbf{\Phi}(x,y,z,t)$ *is called a mild subsolution (mild supersolution) of system* (18). *Particularly, when* (20) *takes the equal sign, the vector-valued function* $\mathbf{\Phi}(x,y,z,t)$ *can be called a mild solution of system* (18).

By an argument similar to Theorem 2.3 of [10], we can obtain the following comparison principle.

Lemma 2.3. *Suppose that* $\mathbf{\Phi}^-$ *and* $\mathbf{\Phi}^+$ *are mild subsolution and mild supersolution of system* (18) *on* $\mathbb{R}^3 \times [0,+\infty)$ *respectively, and* $\mathbf{\Phi}^-, \mathbf{\Phi}^+ \in [\mathbf{G}^-, \mathbf{G}^+]$ *and* $\mathbf{\Phi}^-(x,y,z,0) \leq \mathbf{\Phi}^+(x,y,z,0)$ *for any* $(x,y,z) \in \mathbb{R}^3$. *Then* $\mathbf{\Phi}^-(x,y,z,t) \leq \mathbf{\Phi}^+(x,y,z,t)$ *for any* $(x,y,z,t) \in \mathbb{R}^3 \times [0,+\infty)$. *Moreover, if the initial value* $\mathbf{\Phi}_0 \in E$ *satisfies*

$$\mathbf{\Phi}^-(x,y,z,0) \leq \mathbf{\Phi}_0(x,y,z) \leq \mathbf{\Phi}^+(x,y,z,0), \quad (x,y,z) \in \mathbb{R}^3,$$

then system (18) *exists a unique mild solution* $\mathbf{\Phi}(x,y,z,t;\mathbf{\Phi}_0)$ *such that*

$$\mathbf{\Phi}^-(x,y,z,t) \leq \mathbf{\Phi}(x,y,z,t;\mathbf{\Phi}_0) \leq \mathbf{\Phi}^+(x,y,z,t), \quad (x,y,z,t) \in \mathbb{R}^3 \times [0,+\infty).$$

Obviously, the above lemma implies that for any $\mathbf{\Phi}_0 \in [\mathbf{G}^-, \mathbf{G}^+]$, one has

$$\mathbf{G}^- \leq \mathbf{\Phi}(x,y,z,t;\mathbf{\Phi}_0) \leq \mathbf{G}^+, \quad \forall (x,y,z,t) \in \mathbb{R}^3 \times [0,+\infty).$$

Similarly, for any $\mathbf{\Phi}_0 \in [\mathbf{0}, \mathbf{G}]$, we can also get

$$\mathbf{0} \leq \mathbf{\Phi}(x,y,z,t;\mathbf{\Phi}_0) \leq \mathbf{G}, \quad \forall (x,y,z,t) \in \mathbb{R}^3 \times [0,+\infty).$$

Next, we are going to mollify the pyramid $\left\{\mathbf{x} \in \mathbb{R}^3 \mid -x_3 = h(x_1,x_2)\right\}$ which will play a vital role in the subsequent proofs, see [23].

Define $\rho(x_1,x_2)=\tilde{\rho}\left(\sqrt{x_1^2+x_2^2}\right)$, where function $\tilde{\rho}(r)\in C^\infty[0,\infty)$ have the properties:
(1) $\tilde{\rho}(r)>0,\ \tilde{\rho}_r(r)\le 0$, for any $r\ge 0$;
(2) $\tilde{\rho}(r)=1$, for any $r>0$ small enough;
(3) $\tilde{\rho}(r)=e^{-r}$, for any $r>R_0>1$ large enough, where R_0 is a constant;
(4) $\int_{\mathbb{R}^2}\tilde{\rho}\left(\sqrt{x_1^2+x_2^2}\right)dx_1dx_2=2\pi\int_0^\infty r\tilde{\rho}(r)dr=1$.
Then $\rho\in C^\infty\left(\mathbb{R}^2\right)$ and $\int_{\mathbb{R}^2}\rho(x_1,x_2)\,dx_1dx_2=1$. For all integers $i_1\ge 0$ and $i_2\ge 0$ with $0\le i_1+i_2\le 3$, one has

$$\left|D_{x_1}^{i_1}D_{x_2}^{i_2}\rho(x_1,x_2)\right|\le M\rho(x_1,x_2),\quad \forall(x_1,x_2)\in\mathbb{R}^2,$$

where $0<M<+\infty$ is a constant, $D_{x_1}^{i_1}=\frac{\partial^{i_1}}{\partial x_1^{i_1}}$ and $D_{x_2}^{i_2}=\frac{\partial^{i_2}}{\partial x_2^{i_2}}$. Define $\varphi(x_1,x_2)=\rho*h$. That is,

$$\begin{aligned}\varphi(x_1,x_2)&=\int_{\mathbb{R}^2}\rho\left(x_1-x_1',x_2-x_2'\right)h\left(x_1',x_2'\right)dx_1'dx_2'\\&=\int_{\mathbb{R}^2}\rho\left(x_1',x_2'\right)h\left(x_1-x_1',x_2-x_2'\right)dx_1'dx_2',\quad\forall(x_1,x_2)\in\mathbb{R}^2.\end{aligned}\tag{21}$$

We call $\left\{\mathbf{x}\in\mathbb{R}^3\mid -x_3=\varphi(x_1,x_2)\right\}$ the mollified pyramid of $\left\{\mathbf{x}\in\mathbb{R}^3\mid -x_3=h(x_1,x_2)\right\}$. Define

$$S(x_1,x_2)=\frac{c}{\sqrt{1+\left|\nabla\varphi(x_1,x_2)\right|^2}}-c_*,\tag{22}$$

where $\nabla\varphi(x_1,x_2)=\left(\varphi_{x_1}(x_1,x_2),\varphi_{x_2}(x_1,x_2)\right)$.

The following two lemmas can be obtained from [23, 24, 31], which show some properties of the functions $\varphi(x_1,x_2)$ and $S(x_1,x_2)$.

Lemma 2.4. *The functions φ and S are defined as* (21) *and* (22) *respectively. Then we have*

$$\begin{aligned}&\sup_{(x_1,x_2)\in\mathbb{R}^2}\left|D_{x_1}^{i_1}D_{x_2}^{i_2}\varphi(x_1,x_2)\right|<\infty,\\&h(x_1,x_2)<\varphi(x_1,x_2)\le h(x_1,x_2)+2\pi m_*\int_0^\infty r^2\tilde{\rho}(r)dr,\\&\left|\nabla\varphi(x_1,x_2)\right|<m_*,\quad 0<S(x_1,x_2)\le c-c_*,\quad\forall(x_1,x_2)\in\mathbb{R}^2,\\&\left|\varphi_{x_1x_1}(x_1,x_2)\right|,\ \left|\varphi_{x_2x_2}(x_1,x_2)\right|\le m_*M,\quad\forall(x_1,x_2)\in\mathbb{R}^2\end{aligned}\tag{23}$$

and

$$\lim_{\lambda\to\infty}\sup\left\{S(x_1,x_2)\mid(x_1,x_2)\in\mathbb{R}^2,\mathrm{dist}\left((x_1,x_2),K\right)\ge\lambda\right\}=0,$$
$$\lim_{\lambda\to\infty}\sup\left\{\varphi(x_1,x_2)-h(x_1,x_2)\mid(x_1,x_2)\in\mathbb{R}^2,\mathrm{dist}\left((x_1,x_2),K\right)\ge\lambda\right\}=0.$$

Lemma 2.5. *There exist two constants β_1 and β_2 such that*

$$0<\beta_1=\inf_{(x_1,x_2)\in\mathbb{R}^2}\frac{\varphi(x_1,x_2)-h(x_1,x_2)}{S(x_1,x_2)}\le\sup_{(x_1,x_2)\in\mathbb{R}^2}\frac{\varphi(x_1,x_2)-h(x_1,x_2)}{S(x_1,x_2)}=\beta_2<\infty.$$

In addition, for any integers $i_1,i_2\ge 0$ satisfying $2\le i_1+i_2\le 3$, there exists a positive constant $\mathcal{H}$ such that

$$\sup_{(x_1,x_2)\in\mathbb{R}^2}\left|\frac{D_{x_1}^{i_1}D_{x_2}^{i_2}\varphi(x_1,x_2)}{S(x_1,x_2)}\right|<\mathcal{H}<+\infty.$$

3. Existence of three-dimensional pyramidal traveling fronts

In this section, we will establish the existence of the three-dimensional pyramidal traveling front $\mathbf{W}(\mathbf{x})$ of Eq.(8). That is, we give the proof of Theorem 1.2.

3.1. Construction of the supersolution

In this subsection, we use the perturbation method to construct the classical supersolution.

For any $\alpha \in (0,1)$, there holds $\frac{1}{\alpha} h(\alpha x_1, \alpha x_2) = h(x_1, x_2)$. Let $z_3 = \alpha x_3$, $\mathbf{z} = (z_1, z_2, z_3) = \alpha \mathbf{x} = (\alpha x_1, \alpha x_2, \alpha x_3)$, $\mathbf{z}' = \alpha \mathbf{x}'$ and

$$\sigma(x_1, x_2) = S(\alpha x_1, \alpha x_2) = S(\mathbf{z}'),$$

$$\varpi(\mathbf{x}) = \frac{c_*}{c}\left(x_3 + \frac{1}{\alpha}\varphi(\alpha x_1, \alpha x_2)\right) = \frac{c_*}{c}\frac{z_3 + \varphi(\mathbf{z}')}{\alpha}, \tag{24}$$

$$\varrho(\mathbf{x}) = \frac{x_3 + \frac{1}{\alpha}\varphi(\alpha x_1, \alpha x_2)}{\sqrt{1 + \left|\nabla\varphi(\alpha x_1, \alpha x_2)\right|^2}} = \frac{z_3 + \varphi(\mathbf{z}')}{\alpha\sqrt{1 + \left|\nabla\varphi(\mathbf{z}')\right|^2}}. \tag{25}$$

Calculate them directly, one has

$$\sigma_{x_i}(x_1, x_2) = \alpha S_{z_i}(\mathbf{z}') \quad \text{and} \quad \sigma_{x_i x_i}(x_1, x_2) = \alpha^2 S_{z_i z_i}(\mathbf{z}'), \quad i = 1, 2,$$

$$\varpi_{x_3} = \frac{c_*}{c}, \quad \varpi_{x_3 x_3} = 0, \quad \varpi_{x_i} = \frac{c_*}{c}\varphi_{z_i}, \quad \varpi_{x_i x_i} = \alpha\frac{c_*}{c}\varphi_{z_i z_i},$$

$$\varrho_{x_3} = \frac{1}{\sqrt{1 + \left|\nabla\varphi(\mathbf{z}')\right|^2}}, \quad \varrho_{x_3 x_3} = 0$$

and for $i = 1, 2$,

$$\varrho_{x_i} = \left(\sqrt{1 + \left|\nabla\varphi(\mathbf{z}')\right|^2}\right)^{-1}\varphi_{z_i} - \alpha\varrho(\mathbf{x})X_i(\mathbf{z}'), \quad \varrho_{x_i x_i} = \alpha Y_i(\mathbf{z}') - \alpha^2\varrho(\mathbf{x})Z_i(\mathbf{z}'),$$

where

$$X_i(\mathbf{z}') = \sqrt{1 + \left|\nabla\varphi(\mathbf{z}')\right|^2}\,\frac{\partial}{\partial z_i}\left(\sqrt{1 + \left|\nabla\varphi(\mathbf{z}')\right|^2}\right)^{-1},$$

$$Y_i(\mathbf{z}') = \frac{\partial}{\partial z_i}\left(\left(\sqrt{1 + \left|\nabla\varphi(\mathbf{z}')\right|^2}\right)^{-1}\varphi_{z_i}\right) - \frac{X_i(\mathbf{z}')}{\sqrt{1 + \left|\nabla\varphi(\mathbf{z}')\right|^2}}\varphi_{z_i},$$

$$Z_i(\mathbf{z}') = \frac{\partial X_i(\mathbf{z}')}{\partial z_i} - X_i^2(\mathbf{z}').$$

Now we define a function $\omega(x) \in C^\infty(\mathbb{R})$ satisfying

$$\begin{cases} \omega(x) = 1, & \text{if } x \geq 1, \\ 0 < \omega(x) < 1,\ 0 < \omega'(x) < 1, & \text{if } -1 < x < 1, \\ \omega(x) = 0, & \text{if } x \leq -1, \end{cases} \tag{26}$$

which plays an important role in later proofs.

Lemma 3.1. *There exist a positive constant $0 < \varepsilon_0^+ < 1$ and a positive function $\alpha_0^+(\varepsilon)$ such that, for any $\varepsilon \in \left(0, \varepsilon_0^+\right)$ and $\alpha \in \left(0, \alpha_0^+(\varepsilon)\right)$, the function*

$$\mathbf{v}^+(\mathbf{x}; \varepsilon, \alpha) = \mathbf{\Psi}(\varrho(\mathbf{x})) + \varepsilon\sigma(\mathbf{x}')\left[\omega(\varpi(\mathbf{x}))\mathbf{Q}^+ + (1 - \omega(\varpi(\mathbf{x})))\mathbf{Q}^-\right]$$

is a classical supersolution of Eq.(8)*. Furthermore, the function* $\mathbf{v}^+(\mathbf{x};\varepsilon,\alpha)$ *satisfies the properties*

$$\lim_{\tilde{\gamma}\to\infty}\sup_{\mathbf{x}\in\mathcal{D}(\tilde{\gamma})}\left|v_j^+(\mathbf{x};\varepsilon,\alpha)-v_j^-(\mathbf{x})\right|\leqslant\varepsilon(c-c_*)\hat{Q},\ j=1,2, \tag{27}$$

$$\mathbf{v}^+(\mathbf{x};\varepsilon,\alpha)>\mathbf{v}^-(\mathbf{x}),\quad\forall\mathbf{x}\in\mathbb{R}^3, \tag{28}$$

$$\mathbf{v}^+_{x_3}(\mathbf{x};\varepsilon,\alpha)>\mathbf{0},\quad\forall\mathbf{x}\in\mathbb{R}^3. \tag{29}$$

Proof. **First of all,** we prove $\mathbf{v}^+(\mathbf{x};\varepsilon,\alpha)$ is a supersolution of Eq.(8). Assume $0<\varepsilon\le\delta_2$, where δ_2 is defined in Section 2. For the sake of convenience, we abbreviate $\mathbf{v}^+(\mathbf{x};\varepsilon,\alpha)$ as $\mathbf{v}^+(\mathbf{x})$, $\varpi(\mathbf{x})$ as ϖ, $\varrho(\mathbf{x})$ as ϱ and $\mathbf{\Psi}(\varrho(\mathbf{x}))$ as $\mathbf{\Psi}(\varrho)$. In order to show $\mathbf{v}^+(\mathbf{x})$ is a supersolution of Eq.(8), we only have to prove that it satisfies

$$\begin{cases}\mathcal{L}_1\left[\mathbf{v}^+\right](\mathbf{x})=-D\left(v^+_{1,x_1x_1}+v^+_{1,x_2x_2}+v^+_{1,x_3x_3}\right)+cv^+_{1,x_3}-f_1(\mathbf{v}^+(\mathbf{x}))\ge0,\quad\forall\mathbf{x}\in\mathbb{R}^3,\\ \mathcal{L}_2\left[\mathbf{v}^+\right](\mathbf{x})=cv^+_{2,x_3}-f_2(\mathbf{v}^+(\mathbf{x}))\ge0,\quad\forall\mathbf{x}\in\mathbb{R}^3.\end{cases} \tag{30}$$

By calculating directly and combining with (5), we can obtain

$$\begin{aligned}\mathcal{L}_1\left[\mathbf{v}^+\right](\mathbf{x})=&D\left(1-\varrho_{x_1}^2-\varrho_{x_2}^2-\varrho_{x_3}^2\right)\psi_1''-D\left(\varrho_{x_1x_1}+\varrho_{x_2x_2}\right)\psi_1'\\&-D\varepsilon\sigma(\mathbf{x}')\omega''\left(\varpi_{x_1}^2+\varpi_{x_2}^2+\varpi_{x_3}^2\right)\left(Q_1^+-Q_1^-\right)\\&-D\varepsilon\sigma(\mathbf{x}')\omega'\left(\varpi_{x_1x_1}+\varpi_{x_2x_2}\right)\left(Q_1^+-Q_1^-\right)\\&-D\varepsilon\left(\sigma_{x_1x_1}+\sigma_{x_2x_2}\right)\left[\omega Q_1^++(1-\omega)Q_1^-\right]\\&-2D\varepsilon\omega'\left(Q_1^+-Q_1^-\right)\left(\sigma_{x_1}\varpi_{x_1}+\sigma_{x_2}\varpi_{x_2}\right)\\&+\left(\frac{c}{\sqrt{1+\left|\nabla\varphi(\mathbf{z}')\right|^2}}-c_*\right)\psi_1'+c\varepsilon\sigma(\mathbf{x}')\omega'\varpi_{x_3}\left(Q_1^+-Q_1^-\right)\\&-\left[f_1\left(\mathbf{v}^+(\mathbf{x})\right)-f_1\left(\mathbf{\Psi}(\varrho)\right)\right]\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_2\left[\mathbf{v}^+\right](\mathbf{x})=&\left(\frac{c}{\sqrt{1+\left|\nabla\varphi(\mathbf{z}')\right|^2}}-c_*\right)\psi_2'+c\varepsilon\sigma(\mathbf{x}')\omega'\varpi_{x_3}\left(Q_2^+-Q_2^-\right)\\&-\left[f_2\left(\mathbf{v}^+(\mathbf{x})\right)-f_2\left(\mathbf{\Psi}(\varrho)\right)\right].\end{aligned}$$

Let

$$A_1=\sup_{\mathbf{z}'\in\mathbb{R}^2}\frac{\sum_{i=1,2}\left|S_{z_iz_i}(\mathbf{z}')\right|}{S(\mathbf{z}')}\quad\text{and}\quad A_2=\sup_{\mathbf{z}'\in\mathbb{R}^2}\frac{\sum_{i=1,2}\left|S_{z_i}(\mathbf{z}')\right|}{S(\mathbf{z}')}.$$

By Lemma 1.1, Lemma 2.4 and Lemma 2.5, there are two positive constants A_3 and A_4 such that

$$\left|D\left(1-\sum_{i=1}^3\varrho_{x_i}^2\right)\psi_1''\right|\le A_3\alpha\sigma(\mathbf{x}')\quad\text{and}\quad\left|D\sum_{i=1,2}\varrho_{x_ix_i}\psi_1'\right|\le A_4\alpha\sigma(\mathbf{x}').$$

Now we divide our proof into three steps.
Step 1: $\varrho<-X'$, where $X'>0$ is a large enough constant.

If $\varrho < 0$, there is $\frac{c}{c_*}\varpi < \varrho < \varpi < 0$. Let's suppose $\varpi \leq -X_1 < -1$, where $X_1 > 0$ is a constant. Then the definition of $\omega(x)$ from (26) yields $\omega \equiv 0$. Thus we obtain

$$\begin{aligned} & - D\varepsilon\left(\sum_{i=1,2}\sigma_{x_ix_i}(\mathbf{x}')\right)\left[\omega Q_1^+ + (1-\omega)\,Q_1^-\right] \\ & = -D\varepsilon\frac{\sum_{i=1,2}\sigma_{x_ix_i}(\mathbf{x}')}{\sigma(\mathbf{x}')}\sigma(\mathbf{x}')Q_1^- = -D\varepsilon\alpha^2\frac{\sum_{i=1,2}S_{z_iz_i}(\mathbf{z}')}{S(\mathbf{z}')}\sigma(\mathbf{x}')Q_1^- \geq -D\varepsilon\alpha^2\sigma(\mathbf{x}')Q_1^- A_1. \end{aligned}$$

For the nonlinear term f_j, we have

$$\begin{aligned} f_j\left(\mathbf{v}^+(\mathbf{x})\right) - f_j\left(\mathbf{\Psi}(\varrho)\right) &= \sum_{i=1,2} f_{ji}\left(\tilde{\theta}_j\mathbf{v}^+(\mathbf{x}) + \left(1-\tilde{\theta}_j\right)\mathbf{\Psi}(\varrho)\right)\varepsilon\sigma(\mathbf{x}')Q_i^- \\ &= \sum_{i=1,2} f_{ji}\left(\mathbf{\Psi}(\varrho) + \varepsilon\tilde{\theta}_j\sigma(\mathbf{x}')\mathbf{Q}^-\right)\varepsilon\sigma(\mathbf{x}')Q_i^-, \end{aligned}$$

where $0 < \tilde{\theta}_j < 1$, $j = 1, 2$. From Lemma 1.1, it follows that

$$\mathbf{\Psi}(\varsigma) \to \mathbf{0} \quad \text{and} \quad \sum_{i=1,2} f_{ji}\left(\mathbf{\Psi}(\varsigma)\right)Q_i^- \to H_j^-, \quad j = 1,2, \quad \text{as } \varsigma \to \infty.$$

And hence, there is a constant $X_2 > 0$ large enough such that for any $\varepsilon \in \left(0, \frac{\delta_2\check{Q}}{\hat{Q}}\right)$,

$$-\delta_2\mathbf{P}^- < \mathbf{\Psi}(\varrho) + \varepsilon\tilde{\theta}\sigma(\mathbf{x}')\mathbf{Q}^- < \delta_2\mathbf{P}^-, \quad \varrho < -X_2,\ \tilde{\theta} \in (0,1).$$

Besides, by (14), it follows that

$$\sum_{i=1,2} f_{ji}\left(\mathbf{\Psi}(\varrho) + \varepsilon\tilde{\theta}_j\sigma\mathbf{Q}^-\right)\varepsilon\sigma(\mathbf{x}')Q_i^- < \frac{1}{2}\varepsilon\sigma(\mathbf{x}')H_j^-, \quad \varrho < -X_2,\ \tilde{\theta}_j \in (0,1),\ j = 1,2.$$

Set $X' = \max\left\{\frac{c}{c_*}X_1, X_2\right\}$ and recall $\frac{c}{c_*}\varpi < \varrho < \varpi < 0$ if $\varrho < 0$. Thus, for $\varrho < -X'$, if

$$0 < \alpha < \min\left\{\frac{-\varepsilon\check{H}}{2\left(A_3 + A_4 + DA_1\hat{Q}\right)}, 1\right\},$$

then one has

$$\begin{aligned} \mathcal{L}_1\left[\mathbf{v}^+\right](\mathbf{x}) &\geq -A_3\alpha\sigma(\mathbf{x}') - A_4\alpha\sigma(\mathbf{x}') - D\varepsilon\alpha^2\sigma(\mathbf{x}')Q_1^- A_1 - \frac{1}{2}\varepsilon\sigma(\mathbf{x}')H_1^- \\ &> \left[-\alpha\left(A_3 + A_4 + D\varepsilon\alpha Q_1^- A_1\right) - \frac{1}{2}\varepsilon H_1^-\right]\sigma(\mathbf{x}') > 0, \end{aligned}$$

and

$$\mathcal{L}_2\left[\mathbf{v}^+\right](\mathbf{x}) \geq -\frac{1}{2}\varepsilon\sigma(\mathbf{x}')H_2^- > 0.$$

Step 2: $\varrho > X''$, where $X'' > 0$ is a large enough constant.

If $\varrho > 0$, there is $\frac{c}{c_*}\varpi > \varrho > \varpi > 0$. Without loss of generality, we assume $\varpi \geq X_3 > 1$, where X_3 is a constant. Then the definition of $\omega(x)$ from (26) yields $\omega \equiv 1$. Thus one has

$$\begin{aligned} & - D\varepsilon\left(\sum_{i=1,2}\sigma_{x_ix_i}(\mathbf{x}')\right)\left[\omega Q_1^+ + (1-\omega)\,Q_1^-\right] \\ & = -D\varepsilon\frac{\sum_{i=1,2}\sigma_{x_ix_i}(\mathbf{x}')}{\sigma(\mathbf{x}')}\sigma(\mathbf{x}')Q_1^+ = -D\varepsilon\alpha^2\frac{\sum_{i=1,2}S_{z_iz_i}(\mathbf{z}')}{S(\mathbf{z}')}\sigma(\mathbf{x}')Q_1^+ \geq -D\varepsilon\alpha^2\sigma(\mathbf{x}')Q_1^+ A_1. \end{aligned}$$

For the nonlinear term f_j, we have

$$f_j\left(\mathbf{v}^+(\mathbf{x})\right) - f_j\left(\mathbf{\Psi}(\varrho)\right) = \sum_{i=1,2} f_{ji}\left(\tilde{\theta}_j \mathbf{v}^+(\mathbf{x}) + \left(1-\tilde{\theta}_j\right)\mathbf{\Psi}(\varrho)\right)\varepsilon\sigma(\mathbf{x}')Q_i^+$$
$$= \sum_{i=1,2} f_{ji}\left(\mathbf{\Psi}(\varrho) + \varepsilon\tilde{\theta}_j\sigma(\mathbf{x}')\mathbf{Q}^+\right)\varepsilon\sigma(\mathbf{x}')Q_i^+,$$

where $0<\tilde{\theta}_j<1$, $j=1,2$. From Lemma 1.1, it follows that

$$\mathbf{\Psi}(\varsigma)\to\mathbf{G} \quad\text{and}\quad \sum_{i=1,2} f_{ji}\left(\mathbf{\Psi}(\varsigma)\right)Q_i^+ \to H_j^+,\quad j=1,2,\quad \text{as } \varsigma\to+\infty.$$

Therefore, there is a constant $X_4>0$ large enough such that for any $\varepsilon\in\left(0,\frac{\delta_2\check{Q}}{\hat{Q}}\right)$,

$$\mathbf{G}-\delta_2\mathbf{P}^- < \mathbf{\Psi}(\varrho)+\varepsilon\tilde{\theta}\sigma(\mathbf{x}')\mathbf{Q}^+ < \mathbf{G}+\delta_2\mathbf{P}^-,\quad \varrho>X_4,\ \tilde{\theta}\in(0,1).$$

And hence, by (15), we can get

$$\sum_{i=1,2} f_{ji}\left(\mathbf{\Psi}(\varrho)+\varepsilon\tilde{\theta}_j\sigma\mathbf{Q}^+\right)\varepsilon\sigma(\mathbf{x}')Q_i^+ < \frac{1}{2}\varepsilon\sigma(\mathbf{x}')H_j^+,\quad \varrho>X_4,\ \tilde{\theta}_j\in(0,1),\ j=1,2.$$

Take $X''=\max\left\{\frac{c}{c_*}X_3, X_4\right\}$ and review $\frac{c}{c_*}\varpi>\varrho>\varpi>0$ if $\varrho>0$. Thus, for $\varrho>X''$, if

$$0<\alpha<\min\left\{\frac{-\varepsilon\hat{H}}{2\left(A_3+A_4+DA_1\hat{Q}\right)},1\right\},$$

then we have

$$\mathcal{L}_1\left[\mathbf{v}^+\right](\mathbf{x}) \ge -A_3\alpha\sigma(\mathbf{x}') - A_4\alpha\sigma(\mathbf{x}') - D\varepsilon\alpha^2\sigma(\mathbf{x}')Q_1^+A_1 - \frac{1}{2}\varepsilon\sigma(\mathbf{x}')H_1^+$$
$$> \left[-\alpha\left(A_3+A_4+D\varepsilon\alpha Q_1^+A_1\right) - \frac{1}{2}\varepsilon H_1^+\right]\sigma(\mathbf{x}')>0,$$

and

$$\mathcal{L}_2\left[\mathbf{v}^+\right](\mathbf{x}) \ge -\frac{1}{2}\varepsilon\sigma(\mathbf{x}')H_2^+>0.$$

Step 3: $-X'\le\varrho\le X''$.
Define

$$A_5=\sup_{-X'\le x\le X''}\omega''(x)\left(1+m_*^2\right),\quad N_*=\max_{j\in\{1,2\}}\left(Q_j^+-Q_j^-\right),\quad p_*=\min_{-X'\le x\le X'',j\in\{1,2\}}\psi_j'(x)$$

and for any $i,j\in\{1,2\}$,

$$M_{ji}=\sup_{\mathbf{\Phi}\in[\mathbf{G}^-,\mathbf{G}^+]}\left|f_{ji}(\mathbf{\Phi})\right|,\quad \mathbf{M}_j=\left(M_{j1},M_{j2}\right)\quad\text{and}\quad C_j=\mathbf{M}_j\mathbf{Q}^+,$$

where f_{ji} is defined by (12). By direct calculations, we can get

$$\begin{aligned}
&-D\varepsilon\sigma(\mathbf{x}')\left[\omega''\left(\varpi_{x_1}^2+\varpi_{x_2}^2+\varpi_{x_3}^2\right)+\omega'\left(\varpi_{x_1x_1}+\varpi_{x_2x_2}\right)\right]\left(Q_1^+-Q_1^-\right)\\
&\quad=-D\varepsilon\sigma(\mathbf{x}')\left[\omega''\frac{c_*^2}{c^2}\left(1+\varphi_{z_1}^2+\varphi_{z_2}^2\right)\left(Q_1^+-Q_1^-\right)+\omega'\alpha\frac{c_*}{c}\left(\varphi_{z_1z_1}+\varphi_{z_2z_2}\right)\left(Q_1^+-Q_1^-\right)\right]\\
&\quad\geq -D\varepsilon\sigma(\mathbf{x}')\frac{c_*}{c}\left(Q_1^+-Q_1^-\right)\left[\omega''\frac{c_*}{c}\left(1+m_*^2\right)+2\omega'\alpha m_*M\right]\\
&\quad\geq -D\varepsilon\sigma(\mathbf{x}')N_*\left(A_5+2\alpha m_*M\right),\\
&-D\varepsilon\left(\sigma_{x_1x_1}+\sigma_{x_2x_2}\right)\left[\omega Q_1^++(1-\omega)Q_1^-\right]\\
&\quad=-D\varepsilon\alpha^2\frac{\sum_{i=1,2}S_{z_iz_i}(\mathbf{z}')}{S(\mathbf{z}')}\sigma(\mathbf{x}')\left[\omega Q_1^++(1-\omega)Q_1^-\right]\geq -D\varepsilon\alpha^2\sigma(\mathbf{x}')A_1\hat{Q},\\
&-2D\varepsilon\omega'\left(Q_1^+-Q_1^-\right)\left(\sigma_{x_1}\varpi_{x_1}+\sigma_{x_2}\varpi_{x_2}\right)\\
&\quad=-2D\alpha\varepsilon\omega'\left(Q_1^+-Q_1^-\right)\left(\frac{c_*}{c}S_{z_1}\varphi_{z_1}+\frac{c_*}{c}S_{z_2}\varphi_{z_2}\right)\\
&\quad\geq -2D\alpha\varepsilon\omega'\left(Q_1^+-Q_1^-\right)\frac{c_*}{c}A_2\sigma(\mathbf{x})m_*\geq -2D\alpha\varepsilon N_*m_*A_2\sigma(\mathbf{x}'),
\end{aligned}$$

$$\left(\frac{c}{\sqrt{1+\left|\nabla\varphi\left(\mathbf{z}'\right)\right|^2}}-c_*\right)\psi_j'\geq\sigma(\mathbf{x}')p_*$$

and

$$f_j\left(\mathbf{v}^+(\mathbf{x})\right)-f_j\left(\mathbf{\Psi}(\varrho)\right)\leq C_j\varepsilon\sigma(\mathbf{x}'),\quad j=1,2.$$

Thus if $\alpha<\min\left\{\frac{p_*}{2\left(A_3+A_4+D\hat{Q}A_1\right)},1\right\}$ and $\varepsilon<\min\left\{\frac{p_*}{2[DN_*(A_5+2m_*M)+2DN_*m_*A_2+C_1]},1\right\}$, one has

$$\begin{aligned}
\mathcal{L}_1\left[\mathbf{v}^+\right](\mathbf{x})&\geq -A_3\alpha\sigma(\mathbf{x}')-A_4\alpha\sigma(\mathbf{x}')-D\varepsilon\sigma(\mathbf{x}')N_*\left(A_5+2\alpha m_*M\right)-D\varepsilon\alpha^2\sigma(\mathbf{x}')\hat{Q}A_1\\
&\qquad-2D\alpha\varepsilon N_*m_*A_2\sigma(\mathbf{x}')+\sigma(\mathbf{x})p_*-C_1\varepsilon\sigma(\mathbf{x}')\\
&\geq\sigma\left[-\left(A_3+A_4+D\hat{Q}A_1\right)\alpha+p_*-\left(DN_*\left(A_5+2m_*M\right)+2DN_*m_*A_2+C_1\right)\varepsilon\right]\\
&>0
\end{aligned}$$

and if $\varepsilon<\frac{p_*}{C_2}$, we can also get

$$\mathcal{L}_2\left[\mathbf{v}^+\right](\mathbf{x})\geq p_*\sigma(\mathbf{x})-C_2\varepsilon\sigma(\mathbf{x})>0.$$

Combining with the above three steps, $\mathbf{v}^+(\mathbf{x})$ is proved to be the supersolution of Eq.(8) if

$$0<\varepsilon<\min\left\{1,\frac{\delta_2\check{Q}}{\hat{Q}},\frac{p_*}{C_2},\frac{p_*}{2[DN_*(A_5+2m_*M)+2DN_*m_*A_2+C_1]}\right\}$$

and

$$0<\alpha<\min\left\{1,\frac{-\varepsilon\check{H}}{2\left(A_3+A_4+DA_1\hat{Q}\right)},\frac{-\varepsilon\hat{H}}{2\left(A_3+A_4+DA_1\hat{Q}\right)},\frac{p_*}{2\left(A_3+A_4+D\hat{Q}A_1\right)}\right\}.$$

Secondly, we can prove (27) and (28) by similar discussions of [31, ineqality (2.6)] and [31, ineqality (2.7)] respectively, hence we omit the details.

Finally, we can get $\mathbf{v}_{x_3}^+>\mathbf{0}$ from the definition of $\mathbf{v}^+(\mathbf{x};\varepsilon,\alpha)$, which proves (29). In conclusion, let

$$\varepsilon_0^+=\min\left\{1,\frac{\delta_2\check{Q}}{\hat{Q}},\frac{p_*}{C_2},\frac{p_*}{2[DN_*(A_5+2m_*M)+2DN_*m_*A_2+C_1]}\right\}$$

and

$$\alpha_0^+(\varepsilon) = \min\left\{1, \frac{-\varepsilon\check{H}}{2\big(A_3+A_4+DA_1\check{Q}\big)}, \frac{-\varepsilon\hat{H}}{2\big(A_3+A_4+DA_1\hat{Q}\big)}, \frac{p_*}{2\big(A_3+A_4+D\hat{Q}A_1\big)}, \frac{\varepsilon c_*^3 e^2 \beta_0^2 \beta_1 \check{Q}}{4cC_0}\right\}.$$

We completed the proof of this lemma. □

3.2. Existence

In this subsection, we show the existence of pyramidal traveling fronts to Eq.(8). Before this, we give two theorems which play important roles in the proof of Theorem 1.2.

Theorem 3.2. *If the initial value $\mathbf{\Phi}_0(x,y,z) \in BUC\big(\mathbb{R}^3, [\mathbf{G}^-, \mathbf{G}^+]\big)$ is differentiable with respect to the variable z, even on $x, y \in \mathbb{R}$ and non-decreasing in $x, y \in [0,+\infty)$, and $\mathbf{0} \le \mathbf{\Phi}_{0,z}(x,y,z) \in BUC\big(\mathbb{R}^3\big)$, then there is a unique solution $\mathbf{\Phi}(x,y,z,t;\mathbf{\Phi}_0) \in BUC\big(\mathbb{R}^3\times[0,+\infty), [\mathbf{G}^-,\mathbf{G}^+]\big)$ to Eq.(4) such that $\phi_1(x,y,z,t)$ is of C^2 in $(x,y,z) \in \mathbb{R}^3$ and is of C^1 in $t \in (0,+\infty)$, $\phi_2(x,y,z,t)$ is of C^1 in both $z \in \mathbb{R}$ and $t \in (0,+\infty)$, $\mathbf{\Phi}_z(x,y,z,t) \ge \mathbf{0}$ in $(x,y,z,t) \in \mathbb{R}^3\times[0,+\infty)$ and $\mathbf{\Phi}(x,y,z,t;\mathbf{\Phi}_0)$ is even on $x,y \in \mathbb{R}$ and non-decreasing in $x, y \in [0,+\infty)$.*

Proof. Define

$$\nu_1 > \sup_{\mathbf{\Phi}\in[\mathbf{G}^-,\mathbf{G}^+]} \left|\mu_1 - \partial_{\phi_1}\bar{f}_1(\mathbf{\Phi})\right|, \quad \nu_2 > \max\left\{\sup_{\mathbf{\Phi}\in[\mathbf{G}^-,\mathbf{G}^+]} \left|\mu_2 - \partial_{\phi_2}\bar{f}_2(\mathbf{\Phi})\right|, k_+(b_0 - G_2^-), 1\right\}$$

and

$$g_1 = -\mu_1\phi_1 + \nu_1\phi_1 + \bar{f}_1(\mathbf{\Phi}), \quad g_2 = -\mu_2\phi_2 + \nu_2\phi_2 + \bar{f}_2(\mathbf{\Phi}).$$

Then for any $(x,y,z,t) \in \mathbb{R}^3\times[0,+\infty)$, Eq.(4) is equivalent to the integral formula

$$\begin{cases} \phi_1(x,y,z,t) = e^{-\nu_1 t}\displaystyle\int_{\mathbb{R}^3} \frac{1}{\big(2\sqrt{\pi D t}\big)^3} e^{-\frac{(x-y_1)^2+(y-y_2)^2+(z-y_3)^2}{4Dt}} \phi_{01}(y_1,y_2,y_3)\,dy_1dy_2dy_3 \\ \qquad + \displaystyle\int_0^t e^{-\nu_1(t-s)} \int_{\mathbb{R}^3} \frac{1}{\big(2\sqrt{\pi D(t-s)}\big)^3} e^{-\frac{(x-y_1)^2+(y-y_2)^2+(z-y_3)^2}{4D(t-s)}} \\ \qquad\qquad \times g_1(\mathbf{\Phi}(y_1,y_2,y_3,s))\,dy_1dy_2dy_3ds, \\ \phi_2(x,y,z,t) = e^{-\nu_2 t}\phi_{02}(x,y,z) + \displaystyle\int_0^t e^{-\nu_2(t-s)} g_2(\mathbf{\Phi}(x,y,z,s))ds. \end{cases} \tag{31}$$

For any fixed $T \in \left(0, \frac{\ln 2}{\nu_2}\right]$, we construct a set of vector-valued functions

$$S_T = \left\{ \mathbf{\Phi}(x,y,z,t) \;\middle|\; \begin{array}{c} \mathbf{\Phi}(x,y,z,t) \in BUC\big(\mathbb{R}^3\times[0,T], [\mathbf{G}^-,\mathbf{G}^+]\big); \\ \mathbf{0} \le \mathbf{\Phi}_z(x,y,z,t) \in BUC\big(\mathbb{R}^3\times[0,T]\big); \\ \mathbf{\Phi} \text{ is non-decreasing in } x \in [0,+\infty) \text{ and even on } x \in \mathbb{R}; \\ \mathbf{\Phi} \text{ is non-decreasing in } y \in [0,+\infty) \text{ and even on } y \in \mathbb{R}; \\ \sup_{(x,y,z)\in\mathbb{R}^3, t\in[0,T]} \left|\phi_{j,z}(x,y,z,t)\right| \le \hat{C}_j,\ j=1,2. \end{array} \right\},$$

where

$$\hat{C}_1 = \sup_{(x,y,z)\in\mathbb{R}^3} \left|\phi_{01,z}((x,y,z))\right| + \sqrt{\frac{\ln 2}{D\pi}} \|g_1\|_{L^\infty([\mathbf{G}^-,\mathbf{G}^+])} \text{ and } \hat{C}_2 = 2\sup_{(x,y,z)\in\mathbb{R}^3} \left|\phi_{02,z}((x,y,z))\right| + 2\hat{C}_1.$$

Define the norm on S_T by

$$\|\Phi\|_\tau = \sup_{(x,y,z)\in\mathbb{R}^3, t\in[0,T]} \left|\phi_1(x,y,z,t)\right| e^{-\tau t} + \sup_{(x,y,z)\in\mathbb{R}^3, t\in[0,T]} \left|\frac{\partial}{\partial z}\phi_1(x,y,z,t)\right| e^{-\tau t} + \sup_{(x,y,z)\in\mathbb{R}^3, t\in[0,T]} \left|\phi_2(x,y,z,t)\right| e^{-\tau t} + \sup_{(x,y,z)\in\mathbb{R}^3, t\in[0,T]} \left|\frac{\partial}{\partial z}\phi_2(x,y,z,t)\right| e^{-\tau t},$$

where τ is a positive constant. We can know $(S_T, \|\cdot\|_\tau)$ is a Banach space. For any $\Phi \in S_T$ and $(x,y,z,t) \in \mathbb{R}^3 \times [0,T]$, define $\mathcal{A}_T = (\mathcal{A}_{1T}, \mathcal{A}_{2T})$ by

$$\begin{cases} \mathcal{A}_{1T}[\Phi](x,y,z,t) = e^{-\nu_1 t} \int_{\mathbb{R}^3} \frac{1}{\left(2\sqrt{\pi D t}\right)^3} e^{-\frac{y_1^2+y_2^2+y_3^2}{4Dt}} \phi_{01}\left(x-y_1, y-y_2, z-y_3\right) dy_1 dy_2 dy_3 \\ \quad + \int_0^t e^{-\nu_1(t-s)} \int_{\mathbb{R}^3} \frac{1}{\left(2\sqrt{\pi D(t-s)}\right)^3} e^{-\frac{y_1^2+y_2^2+y_3^2}{4D(t-s)}} g_1\left(\Phi\left(x-y_1, y-y_2, z-y_3, s\right)\right) dy_1 dy_2 dy_3 ds, \\ \mathcal{A}_{2T}[\Phi](x,y,z,t) = e^{-\nu_2(t-s)}\phi_{02}(x,y,z) + \int_0^t e^{-\nu_2(t-s)} g_2(\Phi(x,y,z,s)) ds. \end{cases}$$

Applying the Banach's fixed point theory, the rest of the proof process are similar to Theorem 3.2 of [10], we omit the details. □

Similarly to the proof of Theorem 3.3 in [10], and combining with Theorem 5.1.4 in [14], we can obtain the following theorem.

Theorem 3.3. *There is a constant $L > 0$ such that Eq.*(4) *exists a unique solution $\Phi(x,y,z,t;\mathbf{v}^-)$ with the initial value $\mathbf{v}^-$ satisfying*

$$\begin{aligned} &\left|\phi_{j,z}(x,y,z,t)\right| \le L, && (x,y,z,t) \in \mathbb{R}^3 \times [0,+\infty),\ j=1,2, \\ &\left|\phi_j(\bar{x},y,z,t) - \phi_j(\tilde{x},y,z,t)\right| \le L\left|\bar{x}-\tilde{x}\right|, && \bar{x},\tilde{x},y,z \in \mathbb{R}, t \ge 0,\ j=1,2, \\ &\left|\phi_j(x,\bar{y},z,t) - \phi_j(x,\tilde{y},z,t)\right| \le L\left|\bar{y}-\tilde{y}\right|, && \bar{y},\tilde{y},x,z \in \mathbb{R}, t \ge 0,\ j=1,2, \\ &\left|\phi_{j,z}(\bar{x},y,z,t) - \phi_{j,z}(\tilde{x},y,z,t)\right| \le L\left|\bar{x}-\tilde{x}\right|, && \bar{x},\tilde{x},y,z \in \mathbb{R}, t \ge 0,\ j=1,2, \\ &\left|\phi_{j,z}(x,\bar{y},z,t) - \phi_{j,z}(x,\tilde{y},z,t)\right| \le L\left|\bar{y}-\tilde{y}\right|, && \bar{y},\tilde{y},x,z \in \mathbb{R}, t \ge 0,\ j=1,2, \\ &\left|\phi_{1,zz}(x,y,z,t)\right| \le L, && (x,y,z,t) \in \mathbb{R}^3 \times [0,+\infty), \\ &\left|\phi_{2,z}(x,y,\bar{z},t) - \phi_{2,z}(x,y,\tilde{z},t)\right| \le L\left|\bar{z}-\tilde{z}\right|, && \bar{z},\tilde{z},x,y \in \mathbb{R},\ t \ge 0, \\ &\left|\phi_{2,t}(x,y,z,t)\right| \le L, && (x,y,z,t) \in \mathbb{R}^3 \times [0,+\infty). \end{aligned}$$

For any $\theta \in (0,1)$ and $\varepsilon > 0$ small enough, there is a positive constant $J = J(\theta, \varepsilon)$ such that

$$\|\phi_1\|_{C^{2+\theta,1}(\mathbb{R}^3\times[\varepsilon,+\infty))} \le J.$$

Now we prove Theorem 1.2. That is, we show the existence of pyramidal traveling front to Eq.(8).

Proof. [Proof of Theorem 1.2] Since $\mathbf{v}^-(x,y,z+ct)$ is a subsolution of Eq.(4), the comparison principle implies that $\mathbf{v}^-(x,y,z+ct) \le \Phi(x,y,z,t;\mathbf{v}^-)$. Let $t=\varepsilon$, we have $\mathbf{v}^-(x,y,z+c\varepsilon) \le \Phi(x,y,z,\varepsilon;\mathbf{v}^-)$. From Lemma 2.3, we have $\Phi(x,y,z+c\varepsilon,t;\mathbf{v}^-) \le \Phi(x,y,z,t+\varepsilon;\mathbf{v}^-)$. Recall that $\mathbf{v}(x,y,z+ct,t;\mathbf{v}^-) = \Phi(x,y,z,t;\mathbf{v}^-)$ in (6), thus $\mathbf{v}(x,y,z+c(t+\varepsilon),t;\mathbf{v}^-) \le \mathbf{v}(x,y,z+c(t+\varepsilon),t+\varepsilon;\mathbf{v}^-)$. Let $(x_1,x_2,x_3) = (x,y,z+c(t+\varepsilon))$, and then we can obtain $\mathbf{v}(x_1,x_2,x_3,t;\mathbf{v}^-) \le \mathbf{v}(x_1,x_2,x_3,t+\varepsilon;\mathbf{v}^-)$ for any $(x_1,x_2,x_3,t) \in \mathbb{R}^3 \times (0,+\infty)$. Therefore, the solution $\mathbf{v}(\mathbf{x},t;\mathbf{v}^-)$ of system (7) with the initial value $\mathbf{v}^-(\mathbf{x})$ is non-decreasing in $t \in [0,+\infty)$ with $\mathbf{x} = (x_1,x_2,x_3)$. And because $\mathbf{0} \le \mathbf{v}(\mathbf{x},t;\mathbf{v}^-) \le \mathbf{G}$ for any $(\mathbf{x},t) \in \mathbb{R}^3 \times [0,+\infty)$ by $\mathbf{v}^-(\mathbf{x}) \in [\mathbf{0},\mathbf{G}]$, the limit function

$$\mathbf{W}(\mathbf{x}) = \lim_{t\to+\infty} \mathbf{v}(\mathbf{x},t;\mathbf{v}^-) \tag{32}$$

is well-defined and independent of α and ε. By the properties of $\mathbf{\Phi}(x, y, z, t; \mathbf{v}^-)$, the definition of (32) and the virtue of $\mathbf{\Phi}(x, y, x_3 - ct, t; \mathbf{v}^-) = \mathbf{v}(x_1, x_2, x_3, t; \mathbf{v}^-)$, we know that $\mathbf{W}(\mathbf{x}, t)$ is global Lipschitz continuous with a positive constant L that is obtained in Theorem 3.3, and is differentiable with respect to x_3. Besides, we have

$$\lim_{t\to\infty} \left\| v_1(\mathbf{x}, t; \mathbf{v}^-) - W_1(\mathbf{x}) \right\|_{C^2_{\mathrm{loc}}(\mathbb{R}^3)} = 0,$$
$$\lim_{t\to\infty} \left\| v_2(\mathbf{x}, t; \mathbf{v}^-) - W_2(\mathbf{x}) \right\|_{C_{\mathrm{loc}}(\mathbb{R}^3)} = 0,$$
$$\lim_{t\to\infty} \left\| v_{2,x_3}(\mathbf{x}, t; \mathbf{v}^-) - W_{2,x_3}(\mathbf{x}) \right\|_{C_{\mathrm{loc}}(\mathbb{R}^3)} = 0.$$

From [18], we know $\mathbf{W}(\mathbf{x})$ satisfies Eq.(8) and there holds $\mathbf{v}^-(\mathbf{x}) \le \mathbf{W}(\mathbf{x}) \le \mathbf{v}^+(\mathbf{x}; \varepsilon, \alpha)$ by the comparison principle. Since the arbitrariness of α and ε, we can get (11) by (27). Since $\mathbf{v}^-(\mathbf{x})$ is even on $x_1, x_2 \in \mathbb{R}$, the definition of $\mathbf{W}(\mathbf{x})$ implies that $\mathbf{W}(x_1, x_2, x_3) = \mathbf{W}(-x_1, x_2, x_3)$ and $\mathbf{W}(x_1, x_2, x_3) = \mathbf{W}(x_1, -x_2, x_3)$ for any $\mathbf{x} \in \mathbb{R}^3$.

By Theorem 3.2 and the monotonicity of $\mathbf{v}^-(\mathbf{x})$ on x_3, it holds $\mathbf{W}_{x_3}(\mathbf{x}) \ge \mathbf{0}$ for any $\mathbf{x} \in \mathbb{R}^3$. Using the strong maximum priciple, and $W_{1,x_3}(\mathbf{x})$ satisfies

$$-D\Delta W_{1,x_3}(\mathbf{x}) + c\partial_{x_3} W_{1,x_3}(\mathbf{x}) - f_{11}(\mathbf{W})\, W_{1,x_3}(\mathbf{x}) \ge 0, \quad \forall \mathbf{x} \in \mathbb{R}^3,$$

one has $W_{1,x_3}(\mathbf{x}) > 0$ for any $\mathbf{x} \in \mathbb{R}^3$. Using proof by contradiction, we can obtain $W_{2,x_3}(\mathbf{x}) > 0$ for any $\mathbf{x} \in \mathbb{R}^3$. In fact, if there is a point $(x_1^*, x_2^*, x_3^*) \in \mathbb{R}^3$ such that $W_{2,x_3}(x_1^*, x_2^*, x_3^*) = 0$, then $W_{2,x_3x_3}(x_1^*, x_2^*, x_3^*) = 0$ by $W_{2,x_3}(\mathbf{x}) \ge 0$ for any $\mathbf{x} \in \mathbb{R}^3$. However, $W_{2,x_3}(\mathbf{x})$ satisfies

$$cW_{2,x_3x_3}(\mathbf{x}) = -\left(k_- + k_+ W_1\right) W_{2,x_3} + k_+ W_{1,x_3}\left(b_0 - W_2\right), \quad \forall \mathbf{x} \in \mathbb{R}^3,$$

which implies

$$W_{2,x_3x_3}(x_1^*, x_2^*, x_3^*) = \frac{k_+}{c} W_{1,x_3}(x_1^*, x_2^*, x_3^*)\left(b_0 - W_2(x_1^*, x_2^*, x_3^*)\right) > 0.$$

This is in contradiction with $W_{2,x_3x_3}(x_1^*, x_2^*, x_3^*) = 0$.

According to Theorem 3.2 and Theorem 3.3, $\mathbf{W}(\mathbf{x})$ is differentiable with respect to x_1, $\mathbf{W}_{x_1}(\mathbf{x}) \ge \mathbf{0}$ for any $\mathbf{x} \in (0, +\infty) \times \mathbb{R}^2$ and $\mathbf{W}_{x_1}(0, x_2, x_3) = \mathbf{0}$ for any $(x_2, x_3) \in \mathbb{R}^2$. And since $W_{1,x_1}(\mathbf{x}) \ge 0$ satisfies

$$-D\Delta W_{1,x_1}(\mathbf{x}) + c\partial_{x_3} W_{1,x_1}(\mathbf{x}) - f_{11}(\mathbf{W})\, W_{1,x_1}(\mathbf{x}) \ge 0, \quad \forall \mathbf{x} \in (0, +\infty) \times \mathbb{R}^2,$$

using the strong maximum priciple, we have $W_{1,x_1}(\mathbf{x}) > 0$ for any $\mathbf{x} \in (0, +\infty) \times \mathbb{R}^2$. Using proof by contradiction again, we can obtain $W_{2,x_1}(\mathbf{x}) > 0$ for any $\mathbf{x} \in (0, +\infty) \times \mathbb{R}^2$. In fact, since $W_{2,x_1}(\mathbf{x}) \ge 0$ for any $\mathbf{x} \in (0, +\infty) \times \mathbb{R}^2$, if there exists a point $(x_1^*, x_2^*, x_3^*) \in (0, +\infty) \times \mathbb{R}^2$ such that $W_{2,x_1}(x_1^*, x_2^*, x_3^*) = 0$, then it is obvious that $W_{2,x_1x_3}(x_1^*, x_2^*, x_3^*) = 0$. Moreover, $W_{2,x_3}(\mathbf{x})$ satisfies

$$cW_{2,x_1x_3}(\mathbf{x}) = -\left(k_- + k_+ W_1\right) W_{2,x_1} + k_+ W_{1,x_1}\left(b_0 - W_2\right), \quad \forall \mathbf{x} \in (0, +\infty) \times \mathbb{R}^2,$$

thus

$$W_{2,x_1x_3}(x_1^*, x_2^*, x_3^*) = \frac{k_+}{c} W_{1,x_1}(x_1^*, x_2^*, x_3^*)\left(b_0 - W_2(x_1^*, x_2^*, x_3^*)\right) > 0,$$

which is in contradiction with $W_{2,x_1}(x_1^*, x_2^*, x_3^*) = 0$.

Similar to prove that $\mathbf{W}_{x_1}(\mathbf{x}) > \mathbf{0}$ for any $\mathbf{x} \in (0, +\infty) \times \mathbb{R}^2$ and $\mathbf{W}_{x_1}(0, x_2, x_3) = \mathbf{0}$ for any $(x_2, x_3) \in \mathbb{R}^2$, we can also prove that $\mathbf{W}_{x_2}(\mathbf{x}) > \mathbf{0}$ for any $\mathbf{x} \in \mathbb{R} \times (0, +\infty) \times \mathbb{R}$ and $\mathbf{W}_{x_2}(x_1, 0, x_3) = \mathbf{0}$ for any $(x_1, x_3) \in \mathbb{R}^2$. We completed the proof. □

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