Filomat 38:9 (2024), 3121–3135
https://doi.org/10.2298/FIL2409121G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Cubical simplicial algebras and related crossed structures

Özgün Gürmen Alansal[a]

[a] *Kütahya Dumlupınar University, Science and Art Faculty, Department of Mathematics, Turkey*

Abstract. We introduce the concept of Peiffer pairings in the Moore n-complex of an n-dimensional simplicial commutative algebras and, using these pairings operators demonsrate the connection between n-dimensional simplicial commutative algebras and crossed n-cubes. In the content of dimension 3, we provide explicit calculations using Peiffer pairings to establish the close relationship between cubical simplicial algebras, crossed cubes and 3-crossed modules on commutative algebras.

1. Introduction

Crossed modules of groups were introduced by Whitehead in [16]. The notion of crossed square introduced by Guin-Walery and Loday in [9], can be thought as a 2-dimensional version of crossed modules. As crossed modules model 2-types of homotopy connected spaces, crossed squares model 3-types. The general form of crossed squares are crossed n-cubes introduced by Ellis in [8]. These structure is an algebraic model for homotopy $(n+1)$-types. If $\mathbf{G} = \{G_n\}$ is a simplicial group with Moore complex $(N\mathbf{G}, \partial)$, then a 1-truncation of $\mathbf{G}$, $tr_1(\mathbf{G})$ gives a crossed module as $\partial_1 : NG_1 \to NG_0$. Then, a 2-truncation of $\mathbf{G}$, $tr_2(\mathbf{G})$ gives a 2-crossed module $NG_2 \xrightarrow{\partial_2} NG_1 \xrightarrow{\partial_1} NG_0$ introduced by Conduchè in [6]. The connection between n-truncated simplicial groups and crossed n-cubes was proven by Porter in [14]. Using the images of $F_{\alpha,\beta}$ functions introduced by Mutlu and Porter [12], they have proven in theorem 2.2 of [13] that $\overline{\partial_1} : NG_1/(NG_2 \cap D_2) \to NG_0$ is a crossed module, where $\partial_2(NG_2 \cap D_2) = [\ker d_1, \ker d_0]$ is a commutator subgroup generated by the elements $\partial_2(F_{(0),(1)}(x, y)) = s_0d_1(x)ys_0d_1(x^{-1})(xyx^{-1})^{-1}$ for $x, y \in NG_1 = \ker d_0$ in NG_1. The original motivation of this result comes from Brown-Loday lemma given for the equivalence between crossed modules and cat^1-groups (cf. [5]). The connection between 2-crossed modules and simplicial groups with Moore complex of length 2 has been proven by Mutlu and Porter [13] by using the images of $F_{\alpha,\beta}$ functions in the Moore complex. In [10], the general form of $F_{\alpha,\beta}$ functions for the n-complex of a n-dimensional group has been reformulated and using the images of these functions, it was proven that $tr_1(\mathbf{G})$ gives a crossed n-cube over groups for a multisimplicial group G. Then if $\mathbf{G}$ is a bisimplicial group, $tr_1(\mathbf{G})$ gives a crossed square, and if $\mathbf{G}$ is a cubical simplicial group $tr_1(\mathbf{G})$ gives a crossed cube of groups.

In this paper, we will give the commutative algebra version of this result. The commutative algebra version of crossed modules was studied by Porter in [15] and higher dimensional cases has been investigated by Ellis [8]. Arvasi and Porter in [2], by introducing the functions $C_{\alpha,\beta}$ for a simplicial commutative algebra

2020 *Mathematics Subject Classification*. Primary 18G45; Secondary 55U10.
Keywords. Bisimplicial algebras, crossed modules, Moore complex
Received: 05 May 2023; Revised: 25 October 2023; Accepted: 02 November 2023
Communicated by Ljubiša D. R. Kočinac
Email address: ozgun.gurmen@dpu.edu.tr (Özgün Gürmen Alansal)

E, proved that $\overline{\partial_1} : \overline{NE_1} \to NE_0$ is a crossed module of commutative algebras, where $\overline{NE_1} = NE_1/\partial_2(NE_2 \cap D_2)$ and where $\partial_2(NE_2 \cap D_2)$ is an ideal of NE_1 generated by elements of the form $\partial_2(C_{(0),(1)}(x, y)) = xy - xs_0d_1y$ for $x, y \in NE_1$. Similar applications for higher dimensional crossed modules such as crossed squares, 2-crossed modules has been given by these authors in terms of $C_{\alpha,\beta}$ functions in the Moore complex of a simplicial algebra.

In this work, following [10], we will define $C_{\alpha,\beta}$ functions for n-dimensional simplicial algebras or multisimplicial algebras and using these functions, we prove that $tr_1(\mathbf{E})$ gives a crossed n-cube of commutative algebras for multisimplicial algebra **E**. In particular as an explicit application, to see the role of these functions within these structures, we give detailed calculations in dimension 3 and thus we obtain the close relationship among 1-truncated cubical simplicial algebras, crossed cubes, crossed squares and crossed modules. The results and the general methods for n- dimensional simplicial commutative algebras given in section 2 of this work are, of course, inspired by those given for the corresponding group case in [10]. Although, some of the calculations in this work are similar to the case of commutative algebras, to repeat some arguments can be regarded as advisable, for the readers of this work.

2. Multisimplicial algebras

All algebras discussed in this work will be commutative algebras over a fixed commutative ring k. We will denote the category of commutative algebras by $\mathbf{Alg}_k$. A simplicial algebra **E** is a collection of algebras $\{E_n\}$ together with the homomorphisms $d_i^n : E_n \to E_{n-1}, (0 \le i \le n)$ and $s_j^n : E_n \to E_{n+1}, (0 \le j \le n)$ called faces and degeneracies respectively satisfying the usual simplicial identities given in [2]. The Moore complex $(N\mathbf{E}, \partial)$ of a simplicial commutative algebra **E** is a chain complex defined by $NE_n = \bigcap_{i=0}^{n-1} \ker d_i^n$ on each level together with the boundaries $\partial_n : NE_n \to NE_{n-1}$ induced from d_i^n by restriction. The Moore complex is of length k if $NE_n = 0$ for $n \ge k + 1$. A crossed module of algebras is a homomorphism of algebras $\partial : S \to R$ together with an algebra action of R on S given by $s \cdot r$ and $r \cdot s$ on the left and right sides, satisfying the conditions $CM1$. $\partial(s \cdot r) = \partial(s)r$, $\partial(r \cdot s) = r\partial(s)$ and $CM2$. $\partial(s) \cdot s' = ss' = s \cdot \partial(s')$ for all $r \in R$ and $s, s' \in S$. Where the first condition is called the pre crossed module axiom and the second is *Peiffer identity*. Using the action of R on S, we can say that S is an R-module and from condition CM1, ∂ is an R-module morphism.

We know from [12] that for any simplicial algebra **E** and for $x, y \in NE_1$, $C_{(0),(1)}(x, y) = s_1x(s_1y - s_0y) \in NE_2$ and thus, we have $\partial_2(C_{(0),(1)}(x, y)) = xy - xs_0d_1y \in \partial_2(NE_2)$. If I_2 is an ideal generated by elements of the form $C_{(0),(1)}(x, y)$ of NE_2, in [3] it was proven the equality $\partial_2(NE_2) = \partial_2(I_2)$ and thus $\overline{\partial_1} : NE_1/\partial_2(NE_2) \to NE_0$ given by $\overline{\partial_1}(\overline{a}) = \overline{\partial_1}(a + \partial_2(NE_2)) = \partial_1(a)$ is a crossed module of algebras together with action of $x \in NE_0$ on $a \in NE_1$ given by $x \cdot a = s_0(x)a$ and $a \cdot x = as_0(x)$. If the Moore complex is length 1, then we have $NE_2 = \{0\}$ and $\partial_2(NE_2) = \{0\}$ and thus $\partial_1 : NE_1 \to NE_0$ is a crossed module. Thus we can give the following result from Arvasi and Porter [3]:

Proposition 2.1. ([3]) *Let E be a simplicial (commutative) algebra. Then $\overline{\partial_2} : NE_1/\partial_2(NE_2 \cap D_2) \to NE_0$ is a crossed module.*

To give the general form of this result, firstly, we give some definition about multisimplicial algebras and Peiffer pairings on them.

A multisimplicial algebra or n-simplicial algebra $\mathbf{E}_{\bullet_1\bullet_2\cdots\bullet_n}$ is given by the functor from the product category

$$\Delta^{op} \times \Delta^{op} \times \cdots \times \Delta^{op} = (\Delta^{op})^n$$

to the category of algebras $\mathbf{Alg}_k$, with structural maps denoted by respectively

$$d_{i_j}^{\tau_j} : E_{k_1,\dots,k_j,\dots,k_n} \longrightarrow E_{k_1,\dots,k_j-1,\dots,k_n}, \ (0 \leqslant i_j \leqslant k_j \ , \ 1 \leqslant j \leqslant n,)$$

and

$$s_{i_j}^{\tau_j} : E_{k_1,\dots,k_j,\dots,k_n} \longrightarrow E_{k_1,\dots,k_j+1,\dots,k_n}, \ (0 \leqslant i_j < k_j \ , \ 1 \leqslant j \leqslant n,),$$

where each τ_j indicates the directions of n-simplicial commutative algebra. The Moore multi complex or n-complex (cf. [7]) of an n-simplicial algebra can be given by

$$NE_{k_1,k_2,\dots,k_n} = \bigcap_{(i_1,i_2,\dots,i_n)=(0,0,\dots,0)}^{(k_1-1,k_2-1,\dots,k_n-1)} \mathrm{Ker} d^1_{i_1} \cap \mathrm{Ker} d^2_{i_2} \cap \cdots \cap \mathrm{Ker} d^n_{i_n}$$

with the boundary homomorphisms of algebras

$$\partial^{\tau_j}_{i_j} : NE_{k_1,\dots,k_j,\dots,k_n} \longrightarrow NE_{k_1,\dots,k_j-1,\dots,k_n}$$

induced by $d^{\tau_j}_{i_j}$. We denote the category of n-simplicial commutative algebras by $\mathbf{SimpAlg}^n$.

2.1. Peiffer Pairings in n-simplicial algebras

In this section, we define for multisimplicial algebras the functions $C_{\alpha,\beta}$ given for simplicial algebras in [2]. Recall the following statements about Peiffer pairings in the Moore complex of a simplicial commutative algebra, from [2, 3]. Define the set $P(n)$ consisting of the pairs of elements in the form (α,β) from $S(n)$ with $\alpha \cap \beta = \emptyset$ and $\beta < \alpha$ where $\alpha = (i_r,\dots,i_1)$, $\beta = (j_s,\dots,j_1) \in S(n)$. The k-linear morphisms are,

$$\{C_{\alpha,\beta} : NE_{n-\#\alpha} \otimes NE_{n-\#\beta} \to NE_n | (\alpha,\beta) \in P(n),\ 0 \leq n\}$$

given by composing:

$$\begin{aligned} C_{\alpha,\beta}(x_\alpha \otimes y_\beta) &= p\mu(s_\alpha \otimes s_\beta)(x_\alpha \otimes y_\beta) \\ &= p(s_\alpha(x_\alpha)s_\beta(x_\beta)) \\ &= (1 - s_{n-1}d_{n-1})\dots(1 - s_0d_0)(s_\alpha(x_\alpha)s_\beta(x_\beta)) \end{aligned}$$

where

$$s_\alpha = s_{i_r}\dots s_{i_1} : NE_{n-\#\alpha} \to E_n,\ s_\beta = s_{j_s}\dots s_{j_1} : NE_{n-\#\beta} \to E_n,$$

$p : E_n \to NE_n$ is given as composite projections $p = p_{n-1}\dots p_0$ with

$$p_j = 1 - s_jd_j \text{ for } j = 0,1,\dots,n-1$$

and $\mu : E_n \otimes E_n \to E_n$ denotes multiplication.

Now, we will give this pairings for multisimplicial algebras. For $n,q \in \mathbb{N}$ with $q \leqslant n$ and for $\alpha \in S(n,q)$, the target of α is called $b(\alpha) : q = b(\alpha)$. Recall that the set $S(n)$ is partially ordered by the following relation $\alpha \leqslant \beta$ if, for $i \in [n]$, one has $\alpha(i) \geqslant \beta(i)$ where $[b(\alpha)]$ and $[b(\beta)]$ are considered as subsets of $\mathbb{N}$.

Given $n \neq 0, n \in \mathbb{N}$ and $\mathbf{n} = (k_1,k_2,\dots,k_n) \in \mathbb{N}^n$, let $S(\mathbf{n}) = S(k_1) \times S(k_2) \times \dots \times S(k_n)$ with the product (partial) order.

Let $\alpha,\beta \in S(\mathbf{n})$ and $\alpha = (\alpha_1,\alpha_2,\dots,\alpha_n)$; $\beta = (\beta_1,\beta_2,\dots,\beta_n)$ where $\alpha_i \in S(k_i)$ and $\beta_j \in S\left(k_j\right)$, $1 \leqslant i,j \leqslant n$.

The n-dimensional case of the functions $C_{\alpha,\beta}$ can be given as follows. The pairings that we will need

$$\left\{C_{\alpha,\beta} : NE_{\mathbf{n}-\#\alpha} \otimes NE_{\mathbf{n}-\#\beta} \longrightarrow NE_{\mathbf{n}}\ ;\ \alpha,\beta \in S(\mathbf{n})\right\}$$

are given as composites by the diagram

$$\begin{array}{ccc} NE_{k_1-\#\alpha_1,k_2-\#\alpha_2,\dots,k_n-\#\alpha_n} \otimes NE_{k_1-\#\beta_1,k_2-\#\beta_2,\dots,k_n-\#\beta_n} & \xrightarrow{C_{\alpha,\beta}} & NE_{k_1,k_2,\dots,k_n} \\ \Big\downarrow{\scriptstyle (s_{\alpha_1}s_{\alpha_2}\dots s_{\alpha_n},\, s_{\beta_1}s_{\beta_2}\dots s_{\beta_n})} & & \Big\uparrow{\scriptstyle p} \\ E_{k_1,k_2,\dots,k_n} \otimes E_{k_1,k_2,\dots,k_n} & \xrightarrow[\mu]{} & E_{k_1,k_2,\dots,k_n} \end{array}$$

where $s_\alpha : s_{\alpha_1} s_{\alpha_2} ... s_{\alpha_n}$, for $1 \leqslant i \leqslant n$; $s_{\alpha_i} : s_{i_r} \cdots s_{i_1}$ for $\alpha_i = (i_r, \ldots, i_1) \in S(k_i)$ and similarly s_{β_i}, and p is defined by the composite projection

$$p = (p_{k_1-1} ... p_0)\,(p_{k_2-1} ... p_0) ... (p_{k_n-1} ... p_0)$$

where $p_j(x) = x - s_j d_j(x)$ in each simplicial directions, for any j, and μ is given by the multiplication. Thus the functor $C_{\alpha,\beta}$ is given by $C_{\alpha,\beta}(x_\alpha \otimes y_\beta) = p\mu s_\alpha \otimes s_\beta(x_\alpha \otimes y_\beta)$ where $x_\alpha, y_\beta \in NE_{k_1}$.

2.2. *Crossed n-cubes and n-simplicial algebras*

Crossed n-cubes were defined by Ellis [8] for higher dimensional crossed modules of algebras. The following definition is equivalent to that given in [8]. In this section using the functions $C_{\alpha,\beta}$ for n-simplicial commutative algebras, we will construct a crossed n-cube structure from an n-dimensional simplicial commutative algebra.

Definition 2.2. Let $\langle n \rangle = \{1, 2, ..., n\}$. A crossed n-cube, $\mathbf{K}$, is a family of commutative algebras, $\{K_A : A \subseteq \langle n \rangle\}$, together with homomorphisms, $\eta_i : K_A \to K_{A \setminus \{i\}}$, for $i \in \langle n \rangle$, $A \subseteq \langle n \rangle$, and functions, $h : K_A \times K_B \to K_{A \cup B}$, for all $A, B \subseteq \langle n \rangle$, for $a,\ a' \in K_A,\ b,\ b' \in K_B, c \in K_C,\ k \in \mathbf{k}$, where $\mathbf{k}$ is ring and $i, j \in \langle n \rangle$, the following axioms hold:

1. $\eta a = a$ if $i \notin A$
2. $\eta_i \eta_j a = \eta_j \eta_i a$
3. $\eta_i h(a, b) = h(\eta_i a, \eta_i b)$
4. $h(a, b) = h(\eta_i a, b) = h(a, \eta_i b)$ if $i \in A \cap B$
5. $h(a, a') = aa'$
6. $h(a, b) = h(b, a)$
7. $h(a + a', b) = h(a, b) + h(a', b)$
8. $h(a, b + b') = h(a, b) + h(a, b')$
9. $h(h(a, b), c) = h(a, h(b, c))$
10. $k \cdot h(a, b) = h(k \cdot a, b) = h(a, k \cdot b)$

A morphism of crossed n-cubes

$$\mathbf{f} : \{\mathbf{K}_A\} \longrightarrow \{\mathbf{K}'_A\}$$

is a family of homomorphisms, $\{f_A : K_A \to K'_A \mid A \subseteq \langle n \rangle\}$, which commute with the maps, η_{k_i}, and the h maps.

Example 2.3. (a) For $n = 1$, a crossed 1-cube is the same as a crossed module $K_1 \to K_\emptyset$.

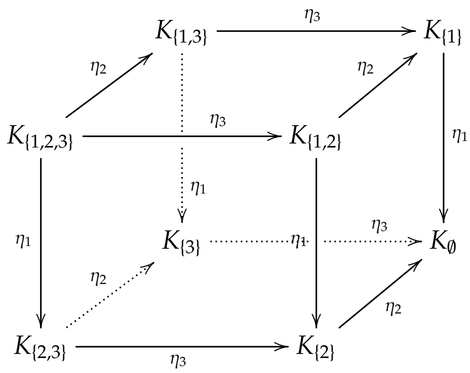
(b) For $n = 2$, one has a crossed square defined by Ellis in [8]

$$\begin{array}{ccc} K_{\{1,2\}} & \xrightarrow{\eta_2} & K_1 \\ {\scriptstyle \eta_1}\downarrow & & \downarrow{\scriptstyle \eta_1} \\ K_2 & \xrightarrow[\eta_2]{} & K_\emptyset, \end{array}$$

where each η_i is a crossed module. The h-maps give actions and a pairing

$$h : K_1 \times K_2 \to K_{\{1,2\}}.$$

(c) For $n = 3$, one has a crossed 3-cube



where each η_i is a crossed module for $i = 1, 2, 3$. The h-maps give actions and the following pairings

$$\begin{array}{ll}
h : K_1 \times K_2 \to K_{\{1,2\}} & , \quad h : K_1 \times K_3 \to K_{\{1,3\}} \\
h : K_2 \times K_3 \to K_{\{2,3\}} & , \quad h : K_{\{1,2\}} \times K_3 \to K_{\{1,2,3\}} \\
h : K_1 \times K_{\{2,3\}} \to K_{\{1,2,3\}} & , \quad h : K_{\{1,3\}} \times K_2 \to K_{\{1,2,3\}} \\
h : K_{\{2,3\}} \times K_{\{1,2\}} \to K_{\{1,2,3\}} & , \quad h : K_{\{1,2\}} \times K_{\{1,3\}} \to K_{\{1,2,3\}} \\
h : K_{\{2,3\}} \times K_{\{1,3\}} \to K_{\{1,2,3\}} & .
\end{array}$$

We can give the main result of this section.

Theorem 2.4. *Let $\mathbf{E}_{\bullet_1\bullet_2\cdots\bullet_n}$ be an n-simplicial algebra with Moore n-complex $\mathbf{NE}_{\bullet_1\bullet_2\cdots\bullet_n}$, such that $NE_{\bullet_1\bullet_2\cdots\bullet_n} = \{1\}$ for any $\bullet_j \geqslant 2$, $(1 \leqslant j \leqslant n)$. Then this Moore n-complex has a crossed n-cube structure over algebras.*

Proof. We will use $C_{\alpha,\beta}$ functions in the proof. First, we define K_A for any subset $A \subset \langle n \rangle = \{1, 2, \ldots, n\}$ by

$$K_A = NE_{\underline{\sigma}}$$

where $\underline{\sigma} = (\sigma_i | 1 \leq i \leq n)$ with $\sigma_i = 1$ if $i \in A$ and 0 otherwise.

The map

$$\eta_i : K_A \longrightarrow K_{A-\{i\}}$$

is given by the face operator $d_1^{\tau_i} : NE_{\underline{\sigma}(:\sigma_i=1)} \longrightarrow NE_{\underline{\sigma}(:\sigma_i=0)}$, where τ_i indicates the simplicial directions. For the subsets $B \subseteq A \subseteq < n >$, the structure morphism $\eta : K_A \to K_B$ is given by the simplicial structure, namely the operator $\prod\limits_{i \in A \setminus B} d_1^i$.

For $A = \{i, i+1, \ldots j\}$ and $B = \{l, l+1, \ldots m\}$ where $1 \leq i, j, l, m \leq n$, we have

$$K_A = NE_{\underline{\sigma}}$$

where $\underline{\sigma} = (\sigma_k : 1 \leq k \leq n)$ and for $i \leq k \leq j$, $\sigma_k = 1$ and 0 otherwise and

$$K_B = NE_{\underline{\sigma}}$$

where $\underline{\sigma} = (\sigma_k : 1 \leq k \leq n)$ and for $l \leq k \leq m$, $\sigma_k = 1$ and 0 otherwise.

Let

$$\chi = \begin{cases}
(\underline{\sigma} = (\sigma_k : 1 \leq k \leq n)), \text{ for } i \leq k \leq m, \sigma_k = 1, \text{ otherwise } 0, & if \quad i \leqslant l,\ j \leqslant m \\
(\underline{\sigma} = (\sigma_k : 1 \leq k \leq n)), \text{ for } i \leq k \leq j, \sigma_k = 1, \text{ otherwise } 0, & if \quad i \leqslant l,\ j \geqslant m \\
(\underline{\sigma} = (\sigma_k : 1 \leq k \leq n)), \text{ for } l \leq k \leq m, \sigma_k = 1, \text{ otherwise } 0, & if \quad i \geqslant l,\ j \leqslant m \\
(\underline{\sigma} = (\sigma_k : 1 \leq k \leq n)), \text{ for } l \leq k \leq j, \sigma_k = 1, \text{ otherwise } 0, & if \quad i \geqslant l,\ j \geqslant m.
\end{cases}$$

The h maps $h : K_A \times K_B \to K_{A\cup B}$ are obtained from the commutative diagram

$$\begin{array}{ccc} NE_{\underline{\sigma}:(\sigma_k=1:i\leq k\leq j)} \otimes NE_{\underline{\sigma}:(\sigma_k=1:l\leq k\leq m)} & \xrightarrow{C_{\alpha,\beta}} & NE_{\chi} \\ \downarrow{\scriptstyle (s_\alpha, s_\beta)} & & \uparrow{\scriptstyle p} \\ E_\chi \otimes E_\chi & \xrightarrow[\mu]{} & E_\chi \end{array}$$

by composing of the maps $p, \mu, (s_\alpha, s_\beta)$, for $K_A,\ K_B$ as follows:

$$\begin{aligned} C_{\alpha,\beta}(x\otimes y) &= p\mu(s_\alpha, s_\beta)(x\otimes y) \\ &= p(s_\alpha(x)s_\beta(y)) \\ &= (1 - s_0^{\tau_i} d_0^{\tau_i})(1 - s_0^{\tau_{i+1}} d_0^{\tau_{i+1}}) \cdots (1 - s_0^{\tau_m} d_0^{\tau_m})(s_\alpha(x)s_\beta(y)) \\ &= s_0^{\tau_i} s_0^{\tau_{i+1}} \cdots s_0^{\tau_j}(x) s_0^{\tau_l} s_0^{\tau_{l+1}} \cdots s_0^{\tau_m}(y) \end{aligned}$$

where τ_i and x_l indicate the simplicial directions and

$$\begin{aligned} \alpha &= (\underbrace{\emptyset, \emptyset, \ldots, \emptyset}_{(i-1)-times}, \underbrace{(0), (0), \ldots, (0)}_{(j-i)-times}, \underbrace{\emptyset, \emptyset, \ldots, \emptyset}_{(n-j)-times}) \\ \beta &= (\underbrace{\emptyset, \emptyset, \ldots, \emptyset}_{(l-1)-times}, \underbrace{(0), (0), \ldots, (0)}_{(m-l)-times}, \underbrace{\emptyset, \emptyset, \ldots, \emptyset}_{(n-l)-times}). \end{aligned}$$

For any subsets $A,\ B \subseteq < n >= \{1, 2, \ldots, n\}$ and $K_A = N_{\underline{\sigma}}$ where $\underline{\sigma} = (\sigma_i | 1 \leq i \leq n)$ with $\sigma_i = 1$ if $i \in A$ and $\sigma_i = 0$ otherwise, and $K_B = N_{\underline{\underline{\sigma}}}$ where $\underline{\underline{\sigma}} = (\sigma_j | 1 \leq j \leq n)$ with $\sigma_j = 1$ if $j \in A$ and $\sigma_j = 0$ otherwise.

The structure morphism $h : K_A \times K_B \to K_{A\cup B}$ is induced by the multiplication on $E_{A\cup B}$ via the homomorphisms of algebras

$$s_{B\backslash(A\cap B)} := \prod_{i\in B\backslash(A\cap B)} s_0^i : E_A \to E_{A\cup B},\ s_{A\backslash(A\cap B)} := \prod_{j\in A\backslash(A\cap B)} s_0^j : E_B \to E_{A\cup B}.$$

Thus for $x \in K_A,\ y \in K_B$ the h-map is induced by the multiplication

$$s_{B\backslash(A\cap B)}(x)s_{A\backslash(A\cap B)}(y) \in E_{A\cup B}.$$

Using the projection map $p : E_\chi \to NE_\chi$ given above, we obtain the h-map as follows: for $x \in K_A,\ y \in K_B$

$$h(x, y) = p_0^{\tau_k} \ldots p_0^{\tau_i}(s_0^{\tau_i} \ldots s_0^{\tau_k})(x) p_0^{\tau_m} \ldots p_0^{\tau_j}(s_0^{\tau_j} \ldots s_0^{\tau_m})(y) \in K_{A\cup B}$$

where for any $j, p_0^{\tau_j}(a) = as_0^{\tau_j} d_0^{\tau_j}(a)^{-1}$ for all $1 \leq i \leq k \leq n;\ i, \ldots, k \in A\backslash(A\cap B), 1 \leq j \leq m \leq n;\ j, \ldots, m \in B\backslash(A\cap B)$ and where $\tau_i, \ldots \tau_k, \tau_j \ldots x_n$ indicate the simplicial directions.

The action of $a \in K_A$ and $b \in K_B$ for $A \subseteq B \subseteq < n >$, can be given by

$$a \cdot b = (s_0^{\tau_i} \ldots s_0^{\tau_k})(a)b$$

where $i, \ldots, k \in A\backslash B$.

From the definition of $\eta : K_A \to K_B$ given by the operator $\prod\limits_{i\in A\backslash B} d_1^i$, the axioms (1),(2) are immediate.

We show for this h-map the following equalities.

If $i \notin A,\ a \in K_A$ then $\eta_i = d_1^{\tau_i} s_0^{\tau_i}$. We obtain $\eta_i(a) = d_1^{\tau_i} s_0^{\tau_i}(a) = id(a) = a$ from the simplicial identities.

By the commutativity of the face and degeneracy maps in the simplicial directions, we obtain $\eta_i\eta_j = \eta_j\eta_i$.

For $K_A = NE_{\underline{\sigma}}$ where $\underline{\sigma} := (\sigma_i|1 \leq i \leq n)$, $\sigma_i = 1$ if $i \in A$ and 0 otherwise, we obtain for the simplicial directions τ_i, and for $\alpha = (\emptyset, \emptyset, \ldots, (0)_i, \emptyset, \ldots, \emptyset)$ and $\beta = (\emptyset, \emptyset, \ldots, (1)_i, \emptyset, \ldots, \emptyset)$

$$C_{\alpha,\beta}(x \otimes y) = s_0^{\tau_i}(x)s_1^{\tau_i}(y) - s_1^{\tau_i}(x)s_1^{\tau_i}(y) \in NE_{\underline{\underline{\sigma}}}$$

where $\underline{\underline{\sigma}} := (\sigma_i|\sigma_i = 2,\ \text{for}\ i \neq j,\ \sigma_j = 1$ if $j \in A$ and 0 otherwise) and since $NE_{\bullet_1\bullet_2\ldots\bullet_n} = \{1\}$ for $j \geq 2$, we obtain

$$d_2^{\tau_i}(C_{\alpha,\beta}(x \otimes y)) = s_0^{\tau_i}d_1^{\tau_i}(x)y - xy = 0$$

and then

$$h(\eta_i(x), y) = s_0^{\tau_i}d_1^{\tau_i}(x)y = xy = h(x, y).$$

Let $\alpha = (\emptyset, \emptyset, \ldots, \emptyset)$, $\beta = (\emptyset, \emptyset, \ldots, \emptyset)$ and for $x, x' \in K_A$, we have $h(x, x') : K_A \times K_A \to K_A$,

$$h(x, x') = xx'.$$

For $x \in K_A,\ y \in K_B$, we have

$$\begin{aligned} h(x,y) &= s_0^{\tau_i}s_0^{\tau_{i+1}}\cdots s_0^{\tau_j}(x)s_0^{\tau_l}s_0^{\tau_{l+1}}\cdots s_0^{\tau_m}(y) \\ &= s_0^{\tau_l}s_0^{\tau_{l+1}}\cdots s_0^{\tau_m}(y)s_0^{\tau_i}s_0^{\tau_{i+1}}\cdots s_0^{\tau_j}(x) \\ &= h(y,x). \end{aligned}$$

Furthermore we have for $x,\ x' \in K_A,\ y,\ y' \in K_B$

$$\begin{aligned} h(x+x',y) &= s_0^{\tau_i}s_0^{\tau_{i+1}}\cdots s_0^{\tau_j}(x+x')s_0^{\tau_l}s_0^{\tau_{l+1}}\cdots s_0^{\tau_m}(y) \\ &= [s_0^{\tau_i}s_0^{\tau_{i+1}}\cdots s_0^{\tau_j}(x) + s_0^{\tau_i}s_0^{\tau_{i+1}}\cdots s_0^{\tau_j}(x')]s_0^{\tau_l}s_0^{\tau_{l+1}}\cdots s_0^{\tau_m}(y) \\ &= s_0^{\tau_i}s_0^{\tau_{i+1}}\cdots s_0^{\tau_j}(x)s_0^{\tau_l}s_0^{\tau_{l+1}}\cdots s_0^{\tau_m}(y) + s_0^{\tau_i}s_0^{\tau_{i+1}}\cdots s_0^{\tau_j}(x')s_0^{\tau_l}s_0^{\tau_{l+1}}\cdots s_0^{\tau_m}(y) \\ &= h(x,y) + h(x',y) \end{aligned}$$

$$\begin{aligned} h(x,y+y') &= s_0^{\tau_i}s_0^{\tau_{i+1}}\cdots s_0^{\tau_j}(x)s_0^{\tau_l}s_0^{\tau_{l+1}}\cdots s_0^{\tau_m}(y+y') \\ &= s_0^{\tau_i}s_0^{\tau_{i+1}}\cdots s_0^{\tau_j}(x)[s_0^{\tau_l}s_0^{\tau_{l+1}}\cdots s_0^{\tau_m}(y) + s_0^{\tau_l}s_0^{\tau_{l+1}}\cdots s_0^{\tau_m}(y')] \\ &= s_0^{\tau_i}s_0^{\tau_{i+1}}\cdots s_0^{\tau_j}(x)s_0^{\tau_l}s_0^{\tau_{l+1}}\cdots s_0^{\tau_m}(y) + s_0^{\tau_i}s_0^{\tau_{i+1}}\cdots s_0^{\tau_j}(x)s_0^{\tau_l}s_0^{\tau_{l+1}}\cdots s_0^{\tau_m}(y') \\ &= h(x,y) + h(x,y'). \end{aligned}$$

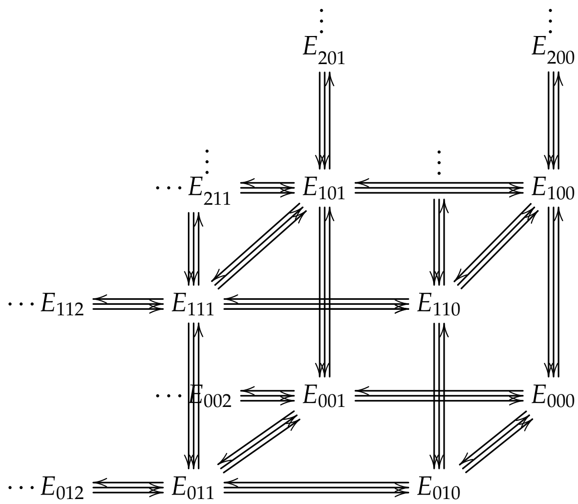
The remaining axioms can be shown similarly. □

3. Cubical simplicial algebras and applications

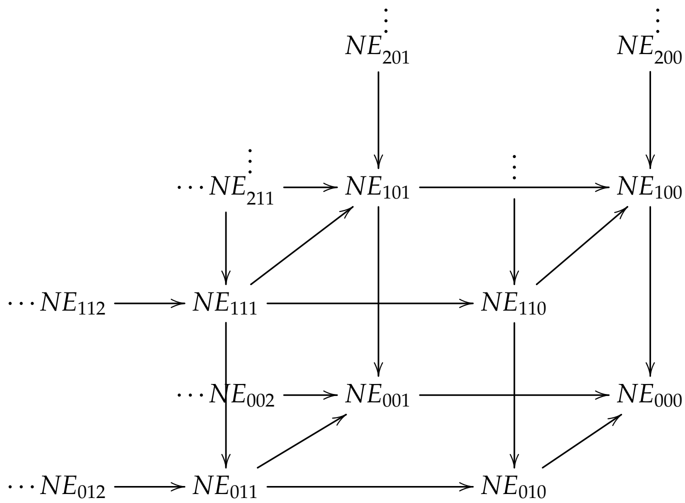
In this section, for dimension 3, using the functions $C_{\alpha,\beta}$ for a cubical simplicial algebras, we will give the relations among cubical simplicial algebra, crossed modules, crossed squares, 2-crossed modules, crossed 3-cubes and 3-crossed modules of algebras.

A cubical simplicial algebra $\mathbf{E}_{\bullet_1\bullet_2\bullet_3}$ is a collection of algebras $\{E_{ijk}\}$ with $i, j, k \geq 0,\ i, j, k \in \mathbb{N}$ together the face operators $d_i^n : \{E_{ijk}\} \to \{E_{i-1jk}\}$, $d_j^n : \{E_{ijk}\} \to \{E_{ij-1k}\}$, $d_k^n : \{E_{ijk}\} \to \{E_{ijk-1}\}$ and $s_i^n : \{E_{ijk}\} \to \{E_{i+1jk}\}$, $s_j^n : \{E_{ijk}\} \to \{E_{ij+1k}\}$, $s_k^n : \{E_{ijk}\} \to \{E_{ijk+1}\}$ satisfying the usual simplicial identities. A cubical simplicial algebra

$\mathbf{E}_{\bullet_1\bullet_2\bullet_3}$ can be represented by the following diagram



The Moore 3-complex of a cubical simplicial algebra can be given by the following diagram



In particular, for example, the Moore complex components given in this diagram can be explained as

$$NE_{000} = E_{000},\ NE_{100} = \ker d_0^{\tau_1}, NE_{201} = \ker d_0^{\tau_1} \cap \ker d_1^{\tau_1} \cap \ker d_0^{\tau_3}.$$

3.1. Crossed modules from cubical simplicial algebras

Let $\mathbf{E}_{\bullet_1\bullet_2\bullet_3}$ be a cubical simplicial algebra with Moore complex $\mathbf{NE}_{\bullet_1\bullet_2\bullet_3}$ such that $NE_{\bullet_1\bullet_2\bullet_3} = \{0\}$ for any $\bullet_j \geqslant 2,\ (1 \leqslant j \leqslant 3)$. Then this Moore 3-complex has twelve crossed modules as follows:

$$\begin{array}{ll}
d_1{}^{\tau_2} : NE_{111} \to NE_{101}, & d_1{}^{\tau_1} : NE_{111} \to NE_{011} \\
d_1{}^{\tau_3} : NE_{111} \to NE_{110}, & d_1{}^{\tau_1} : NE_{101} \to NE_{001} \\
d_1{}^{\tau_3} : NE_{101} \to NE_{100}, & d_1{}^{\tau_2} : NE_{110} \to NE_{100} \\
d_1{}^{\tau_1} : NE_{110} \to NE_{010}, & d_1{}^{\tau_1} : NE_{100} \to NE_{000} \\
d_1{}^{\tau_2} : NE_{011} \to NE_{001}, & d_1{}^{\tau_3} : NE_{011} \to NE_{010} \\
d_1{}^{\tau_2} : NE_{010} \to NE_{000}, & d_1{}^{\tau_3} : NE_{001} \to NE_{000}
\end{array}$$

For example NE_{011} acts on NE_{111} via $s_0^{\tau_1}$. An action of $x \in NE_{011}$ on $a \in NE_{111}$ is given by

$$x \cdot a = s_0^{\tau_1}(x)a$$

From the Peiffer pairings we know that for $x, y \in NE_{111}$

$$s_0^{\tau_1}(x)s_1^{\tau_1}(y) - s_1^{\tau_1}(x)s_1^{\tau_1}(y) \in NE_{211}$$

Let us explain now how we are using the pairings within this structure.

Since $NE_{\bullet_1\bullet_2\bullet_3} = \{0\}$ for $\bullet_j \geqslant 2$, the Moore 3-complex of the cubical simplicial algebra $\mathbf{E}_{\bullet_1\bullet_2\bullet_3}$ is of length 1, we have $NE_{211} \cap D_{211} = \{0\}$ and then we obtain $\partial_2(NE_{211} \cap D_{211}) = \{0\}$ and thus

$$d_2^{\tau_1}(s_0^{\tau_1}(x)s_1^{\tau_1}(y) - s_1^{\tau_1}(x)s_1^{\tau_1}(y)) = s_0^{\tau_1}d_1^{\tau_1}(x)y - xy = 0$$

and then we obtain $\partial_1^{\tau_1}(x) \cdot y = xy$. Similarly we have for $x \in NE_{011}$

$$d_1^{\tau_1}(x \cdot a) = d_1^{\tau_1}(s_0^{\tau_1}(x)a) = xd_1^{\tau_1}(a).$$

Thus $d_1^{\tau_1} : NE_{111} \to NE_{011}$ is a crossed module of algebras. Same method can be used for other homomorphisms, given above.

3.2. *Crossed squares from cubical simplicial algebras*

In this section, we will obtain six different crossed squares from a cubical simplicial algebra with Moore 3-complex of length 1. We suppose that $\mathbf{E}_{\bullet_1\bullet_2\bullet_3}$ is a cubical simplicial algebra with Moore complex $\mathbf{NE}_{\bullet_1\bullet_2\bullet_3} = \{0\}$ for $\bullet_j \geqslant 2$, $(1 \leqslant j \leqslant 3)$. Then we have following crossed squares

$$\begin{array}{ccc} NE_{111} & \xrightarrow{d_1^{\tau_3}} & NE_{110} \\ {\scriptstyle d_1^{\tau_1}}\downarrow & & \downarrow{\scriptstyle d_1^{\tau_1}} \\ NE_{011} & \xrightarrow[d_1^{\tau_3}]{} & NE_{010} \end{array} \qquad \begin{array}{ccc} NE_{101} & \xrightarrow{d_1^{\tau_3}} & NE_{100} \\ {\scriptstyle d_1^{\tau_1}}\downarrow & & \downarrow{\scriptstyle d_1^{\tau_1}} \\ NE_{001} & \xrightarrow[d_1^{\tau_3}]{} & NE_{000} \end{array}$$

$$\begin{array}{ccc} NE_{111} & \xrightarrow{d_1^{\tau_3}} & NE_{110} \\ {\scriptstyle d_1^{\tau_2}}\downarrow & & \downarrow{\scriptstyle d_1^{\tau_2}} \\ NE_{101} & \xrightarrow[d_1^{\tau_3}]{} & NE_{100} \end{array} \qquad \begin{array}{ccc} NE_{011} & \xrightarrow{d_1^{\tau_3}} & NE_{010} \\ {\scriptstyle d_1^{\tau_2}}\downarrow & & \downarrow{\scriptstyle d_1^{\tau_2}} \\ NE_{001} & \xrightarrow[d_1^{\tau_3}]{} & NE_{000} \end{array}$$

$$\begin{array}{ccc} NE_{111} & \xrightarrow{d_1^{\tau_2}} & NE_{101} \\ {\scriptstyle d_1^{\tau_1}}\downarrow & & \downarrow{\scriptstyle d_1^{\tau_1}} \\ NE_{011} & \xrightarrow[d_1^{\tau_2}]{} & NE_{001} \end{array} \qquad \begin{array}{ccc} NE_{110} & \xrightarrow{d_1^{\tau_2}} & NE_{100} \\ {\scriptstyle d_1^{\tau_1}}\downarrow & & \downarrow{\scriptstyle d_1^{\tau_2}} \\ NE_{010} & \xrightarrow[d_1^{\tau_2}]{} & NE_{000} \end{array}$$

where h-maps are given by

$$h: \begin{array}{rcl} NE_{110} \times NE_{011} & \longrightarrow & NE_{111} \\ (x, y) & \longmapsto & s_0^{\tau_3}(x)s_0^{\tau_1}(y) \end{array}, \qquad h: \begin{array}{rcl} NE_{100} \times NE_{001} & \longrightarrow & NE_{101} \\ (x, y) & \longmapsto & s_0^{\tau_3}(x)s_0^{\tau_1}(y) \end{array},$$

$$h: \begin{array}{rcl} NE_{110} \times NE_{101} & \longrightarrow & NE_{111} \\ (x, y) & \longmapsto & s_0^{\tau_3}(x)s_0^{\tau_2}(y) \end{array}, \qquad h: \begin{array}{rcl} NE_{010} \times NE_{001} & \longrightarrow & NE_{011} \\ (x, y) & \longmapsto & s_0^{\tau_3}(x)s_0^{\tau_2}(y) \end{array},$$

$$h: \begin{array}{rcl} NE_{101} \times NE_{011} & \longrightarrow & NE_{111} \\ (x,y) & \longmapsto & s_0^{\tau_2}(x)s_0^{\tau_1}(y) \end{array}, \quad h: \begin{array}{rcl} NE_{100} \times NE_{010} & \longrightarrow & NE_{110} \\ (x,y) & \longmapsto & s_0^{\tau_2}(x)s_0^{\tau_1}(y) \end{array},$$

For example we show that

$$\begin{array}{ccc} NE_{111} & \xrightarrow{d_1^{\tau_3}} & NE_{110} \\ {\scriptstyle d_1^{\tau_1}}\downarrow & & \downarrow{\scriptstyle d_1^{\tau_1}} \\ NE_{011} & \xrightarrow[d_1^{\tau_3}]{} & NE_{010} \end{array}$$

is a crossed square. The h-map $h : NE_{110} \times NE_{011} \to NE_{111}$ can be defined by

$$h(x,y) = s_0^{\tau_3}(x)s_0^{\tau_1}(y)$$

for $x \in NE_{110}$ and $y \in NE_{011}$.

In the following calculations, we will show that the conditions of a crossed square are satisfied

1. $d_1^{\tau_1}$, $d_1^{\tau_3}$ and $d_1^{\tau_1}d_1^{\tau_3} = d_1^{\tau_3}d_1^{\tau_1}$ crossed modules.

2. The maps are $d_1^{\tau_1}$, $d_1^{\tau_3}$ preserve the actions of NE_{010}.

3. $kh(x,y) = k(s_0^{\tau_3}(x)s_0^{\tau_1}(y)) = s_0^{\tau_3}(kx)s_0^{\tau_1}(y) = h(kx,y)$
$kh(x,y) = k(s_0^{\tau_3}(x)s_0^{\tau_1}(y)) = s_0^{\tau_3}(x)s_0^{\tau_1}(ky) = h(x,ky)$

4. For $x,x' \in NE_{110}$ and $y,y' \in NE_{011}$, we have

$$\begin{array}{rcl} h(x+x',y) & = & s_0^{\tau_3}(x+x')s_0^{\tau_1}(y) \\ & = & (s_0^{\tau_3}(x)+s_0^{\tau_3}(x'))s_0^{\tau_1}(y) \\ & = & s_0^{\tau_3}(x)s_0^{\tau_1}(y)+s_0^{\tau_3}(x')s_0^{\tau_1}(y) \\ & = & h(x,y)+h(x',y) \end{array}$$

$$\begin{array}{rcl} h(x,y+y') & = & s_0^{\tau_3}(x)s_0^{\tau_1}(y+y') \\ & = & s_0^{\tau_3}(x)(s_0^{\tau_1}(y)+s_0^{\tau_1}(y')) \\ & = & s_0^{\tau_3}(x)s_0^{\tau_1}(y)+s_0^{\tau_3}(x)s_0^{\tau_1}(y') \\ & = & h(x,y)+h(x,y'). \end{array}$$

5. For $x \in NE_{110}$, $y \in NE_{011}$ and $r \in NE_{010}$, we have

$$\begin{array}{rcl} r \cdot h(x,y) & = & r \cdot (s_0^{\tau_3}(x+x')s_0^{\tau_1}(y)) \\ & = & r \cdot s_0^{\tau_3}(x+x')s_0^{\tau_1}(y) \\ & = & s_0^{\tau_3}(r \cdot x)s_0^{\tau_1}(y) \\ & = & h(r \cdot x,y) \end{array}$$

Similarly $r \cdot h(x,y) = h(x, r \cdot y)$.

The other crossed squares can be proven by a similar way.

3.3. 2-crossed modules from cubical simplicial algebras

For a crossed square

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ {\scriptstyle \lambda'}\downarrow & & \downarrow{\scriptstyle \mu} \\ N & \xrightarrow[\nu]{} & P \end{array}$$

using Loday's mapping cone complex, Conduché in [7] proved that

$$L \xrightarrow{(\lambda,\lambda'^{-1})} M \rtimes N \xrightarrow{(\mu,\nu)} P$$

is a 2-crossed module. The commutative algebra version can be found in Arvasi in [1]. In the previous section, we have obtained six different crossed squares. For each crossed square, we can say that there is a corresponding 2-crossed module. For example from the following diagram

$$\begin{array}{ccc} NE_{111} & \xrightarrow{d_1^{\tau_3}} & NE_{110} \\ {\scriptstyle d_1^{\tau_1}}\downarrow & & \downarrow{\scriptstyle d_1^{\tau_1}} \\ NE_{011} & \xrightarrow[d_1^{\tau_3}]{} & NE_{010} \end{array}$$

we obtain a 2-crossed module.

$$NE_{111} \xrightarrow{\delta_2} NE_{110} \times NE_{011} \xrightarrow{\delta_1} NE_{010}$$

where $\delta_1(x,y) = d_1^{\tau_1}(x) + d_1^{\tau_3}(y)$ and $\delta_2(a) = (d_1^{\tau_3}(a), -d_1^{\tau_1}(a))$ for all $x \in NE_{110},\ y \in NE_{011}$ and $a \in NE_{111}$. The Peiffer lifting map for this 2-crossed module can be given by

$$\{-,-\} : (NE_{110} \times NE_{011}) \times (NE_{110} \times NE_{011}) \to NE_{111}$$

$$\{(x,y),(a,c)\} = h(xa,c) = s_0^{\tau_3}(xa)s_0^{\tau_1}(y).$$

Similarly, we can define other 2-crossed modules which are associated to the crossed squares given in previous section, by

$$NE_{101} \longrightarrow NE_{100} \times NE_{001} \longrightarrow NE_{000}$$

$$NE_{111} \longrightarrow NE_{110} \times NE_{101} \longrightarrow NE_{100}$$

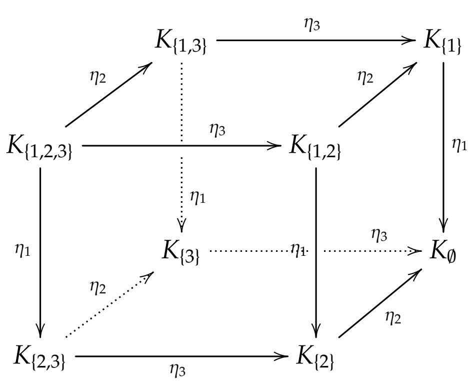
$$NE_{011} \longrightarrow NE_{010} \times NE_{001} \longrightarrow NE_{000}$$

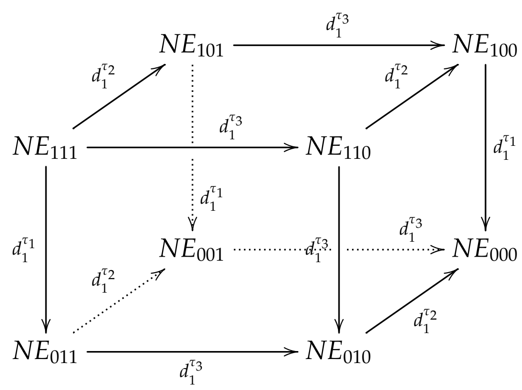
$$NE_{111} \longrightarrow NE_{101} \times NE_{011} \longrightarrow NE_{001}$$

$$NE_{110} \longrightarrow NE_{100} \times NE_{010} \longrightarrow NE_{000}$$

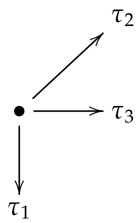
4. Crossed cubes from cubical simplicial algebras

A crossed 3-cube can be obtained from a 3-simplicial commutative algebra as follows:
For $< n >= \{1,2,3\}$ we have the following diagrams





we show the simplicial directions by



The sets K_A can be given by

$$\begin{array}{lcl} K_{\emptyset} = NE_{000} = E_{000} & , & K_{\{1\}} = NE_{100} = Kerd_0^{\tau_1} \\ K_{\{2\}} = NE_{010} = Kerd_0^{\tau_2} & , & K_{\{3\}} = NE_{001} = Kerd_0^{\tau_3} \\ K_{\{1,2\}} = NE_{110} = Kerd_0^{\tau_1} \cap Kerd_0^{\tau_2} & , & K_{\{2,3\}} = NE_{011} = Kerd_0^{\tau_2} \cap Kerd_0^{\tau_3} \\ K_{\{1,3\}} = NE_{101} = Kerd_0^{\tau_1} \cap Kerd_0^{\tau_3} & , & K_{\{1,2,3\}} = NE_{111} = Kerd_0^{\tau_1} \cap Kerd_0^{\tau_2} \cap Kerd_0^{\tau_3}. \end{array}$$

The maps $\eta_i : K_A \to K_{A-\{i\}}$ are given in the above diagram.

The h-maps can be defined as follows:

$$\begin{array}{llll} h: & NE_{100} \times NE_{010} & \longrightarrow NE_{110} \\ & (x,y) & \longmapsto s_0^{\tau_2}(x)s_0^{\tau_1}(y) \end{array}, \quad \begin{array}{llll} h: & NE_{100} \times NE_{001} & \longrightarrow NE_{101} \\ & (x,y) & \longmapsto s_0^{\tau_3}(x)s_0^{\tau_1}(y) \end{array},$$

$$\begin{array}{llll} h: & NE_{010} \times NE_{001} & \longrightarrow NE_{011} \\ & (x,y) & \longmapsto s_0^{\tau_3}(x)s_0^{\tau_2}(y) \end{array}, \quad \begin{array}{llll} h: & NE_{110} \times NE_{001} & \longrightarrow NE_{111} \\ & (x,y) & \longmapsto s_0^{\tau_3}(x)s_0^{\tau_1}s_0^{\tau_2}(y) \end{array},$$

$$\begin{array}{llll} h: & NE_{100} \times NE_{011} & \longrightarrow NE_{111} \\ & (x,y) & \longmapsto s_0^{\tau_2}s_0^{\tau_3}(x)s_0^{\tau_1}(y) \end{array}, \quad \begin{array}{llll} h: & NE_{101} \times NE_{010} & \longrightarrow NE_{111} \\ & (x,y) & \longmapsto s_0^{\tau_2}(x)s_0^{\tau_1}s_0^{\tau_3}(y) \end{array},$$

$$\begin{array}{llll} h: & NE_{011} \times NE_{101} & \longrightarrow NE_{111} \\ & (x,y) & \longmapsto s_0^{\tau_1}(x)s_0^{\tau_2}(y) \end{array}, \quad \begin{array}{llll} h: & NE_{110} \times NE_{101} & \longrightarrow NE_{111} \\ & (x,y) & \longmapsto s_0^{\tau_3}(x)s_0^{\tau_2}(y) \end{array},$$

$$\begin{array}{llll} h: & NE_{110} \times NE_{011} & \longrightarrow NE_{111} \\ & (a,b) & \longmapsto s_0^{\tau_3}(a)s_0^{\tau_1}(b) \end{array}$$

We can prove the axioms of crossed 3-cubes as follows:

1. Let $A = \{2,3\}$. Then if we have $i = 1 \notin A$, $\eta_i : K_A \to K_A$ is given by

$$\eta_i = \eta_1 = d_1^{\tau_1}s_0^{\tau_1}.$$

From the simplicial identities, we have $d_1^{\tau_1}s_0^{\tau_1} = id$. Therefore, for $i = 1 \notin A = \{2,3\}$ we obtain $\eta_i(a) = a$.

3. For the h-map given by

$$\begin{array}{llll} h: & NE_{110} \times NE_{011} & \longrightarrow NE_{111} \\ & (a,b) & \longmapsto s_0^{\tau_3}(a)s_0^{\tau_1}(b) \end{array}$$

we can write,

$$\begin{array}{rcl} \eta_2 h(a,b) & = & d_1^{\tau_2}(s_0^{\tau_3}(a)s_0^{\tau_1}(b)) \\ & = & s_0^{\tau_3}d_1^{\tau_2}(a)s_0^{\tau_1}d_1^{\tau_2}(b) \ (\because \text{commutativity of simplicial directions}) \\ & = & h(\eta_2 a, \eta_2 b) \end{array}$$

Similarly

$$\begin{aligned}\eta_3 h(a,b) &= d_1^{\tau_3}(s_0^{\tau_3}(a)s_0^{\tau_1}(b))\\ &= s_0^{\tau_3}d_1^{\tau_3}(a)s_0^{\tau_1}d_1^{\tau_3}(b) \text{ } (\because \text{ commutativity of simplicial directions})\\ &= h(\eta_3 a, \eta_3 b).\end{aligned}$$

By using similar calculations, this result for the other η_i maps can be proven.

4. For example, for $x, y \in NE_{110}$, we obtain

$$C_{((0),\emptyset,\emptyset)((1),\emptyset,\emptyset)}(x \otimes y) = s_0^{\tau_1}(x)s_1^{\tau_1}(y) - s_1^{\tau_1}(x)s_1^{\tau_1}(y) \in NE_{110}$$

Since $NE_{210} = \{0\}$, we obtain

$$d_2^{\tau_1}(C_{\alpha,\beta}(x \otimes y)) = s_0^{\tau_1}d_1^{\tau_1}(x)y - xy = 0 \in NE_{110}.$$

Thus we obtain

$$h(\eta_1(x), y) = s_0^{\tau_1}d_1^{\tau_1}(x)y = xy \in NE_{110}.$$

and for $x, y \in NE_{111}$,

$$C_{(\emptyset,\emptyset,(0))(\emptyset,\emptyset,(1))}(x \otimes y) = s_0^{\tau_3}(x)s_1^{\tau_3}(y) - s_1^{\tau_3}(x)s_1^{\tau_3}(y) \in NE_{112}$$

Since $NE_{112} = \{0\}$, we obtain

$$d_2^{\tau_3}(C_{\alpha,\beta}(x \otimes y)) = s_0^{\tau_3}d_1^{\tau_3}(x)y - xy = 0 \in NE_{111}.$$

Thus we obtain

$$h(\eta_1(x), y) = s_0^{\tau_3}d_1^{\tau_3}(x)y = xy = h(x, y)$$

5. Let $\alpha = (\emptyset,\emptyset,\emptyset),\ \beta = (\emptyset,\emptyset,\emptyset)$ and for $a, a' \in K_A$, we have $h : K_A \times K_A \to K_A$,

$$h(a,a') = aa'$$

6. For the map $h : NE_{110} \times NE_{011} \to NE_{111}$, we have

$$h(a,b) = s_0^{\tau_3}(a)s_0^{\tau_1}(b) = s_0^{\tau_1}(b)s_0^{\tau_3}(a) = h(b,a)$$

7. For the map $h : NE_{110} \times NE_{011} \to NE_{111}$, we obtain

$$\begin{aligned}h(a+a',b) &= s_0^{\tau_3}(a+a')s_0^{\tau_1}(b)\\ &= [s_0^{\tau_3}(a) + s_0^{\tau_3}(a')]s_0^{\tau_1}(b)\\ &= s_0^{\tau_3}(a)s_0^{\tau_1}(b) + s_0^{\tau_3}(a')s_0^{\tau_1}(b)\\ &= h(a,b) + h(a',b)\end{aligned}$$

8. For the map $h : NE_{110} \times NE_{011} \to NE_{111}$, we obtain

$$\begin{aligned}h(a,b+b') &= s_0^{\tau_3}(a)s_0^{\tau_1}(b+b')\\ &= s_0^{\tau_3}(a)[s_0^{\tau_1}(b) + s_0^{\tau_1}(b')]\\ &= s_0^{\tau_3}(a)s_0^{\tau_1}(b) + s_0^{\tau_3}(a)s_0^{\tau_1}(b')\\ &= h(a,b) + h(a,b')\end{aligned}$$

9. We must show that

$$h(h(a,b),c) = h(a,h(b,c))$$

We calculate that for $a \in NE_{100},\ b \in NE_{010},\ c \in NE_{001}$,

$$\begin{aligned}h(h(a,b),c) &= h(s_0^{\tau_2}(a)s_0^{\tau_1}(b), c)\\ &= s_0^{\tau_3}s_0^{\tau_2}(a)s_0^{\tau_3}s_0^{\tau_1}(b)s_0^{\tau_1}s_0^{\tau_2}(c)\\ &= s_0^{\tau_2}s_0^{\tau_3}(a)s_0^{\tau_1}s_0^{\tau_3}(b)s_0^{\tau_1}s_0^{\tau_2}(c) \text{ } (\because \text{ commutativity of simplicial directions})\\ &= h(a, s_0^{\tau_3}(b)s_0^{\tau_2}(c))\\ &= h(a, h(b,c))\end{aligned}$$

10. Finally, we show that

$$k\cdot h(a,b)=h(k\cdot a,b)=h(a,k\cdot b)$$

$$\begin{aligned} k\cdot h(a,b) &= k\cdot (s_0^{\tau_3}(a)s_0^{\tau_1}(b)) \\ &= k\cdot s_0^{\tau_3}(a)s_0^{\tau_1}(b) \\ &= h(k\cdot a,b). \end{aligned}$$

4.1. 3-crossed modules from cubical simplicial algebras

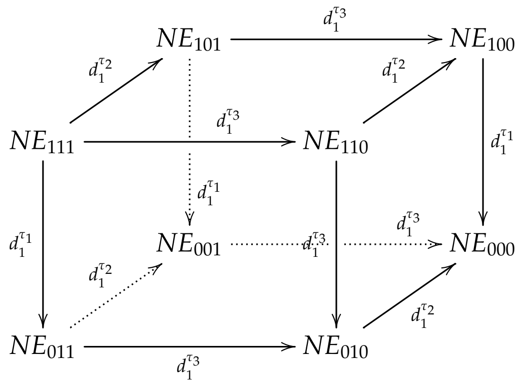
As an algebraic model for homotopy (connected) 4-types, the notion of a 3-crossed module has been introduced in [4]. The connection between simplicial groups with Moore complex of length 4 and 3-crossed modules has been proven in [4], in terms of hypercrossed complex pairings in the Moore complex of a simplicial group. The commutative algebra version this equivalence has been studied in [11]. In this section, using the Loday's mapping cone complex, we will give a 3-crossed module which is associated to the crossed cube obtained from a cubical simplicial algebra in previous section.

Recall from [11] that a 3-crossed module of algebras is a complex of algebras

$$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

together with $\partial_3,\partial_2,\partial_1$, which are $C_0,\ C_1-$algebra morphisms, an action of C_0 on $C_3,\ C_2,\ C_1$, an action of C_1 on $C_3,\ C_2$, and an action of C_2 on C_3 further $C_0,\ C_1-$bilinear maps satisfying the conditions 3CM1-3CM16 given in [11]

Now consider the crossed cube



obtained from cubical simplicial algebra. Its mapping cone complex C is given by

$$NE_{111} \xrightarrow{\partial_3} NE_{101}\rtimes NE_{011}\rtimes NE_{110} \xrightarrow{\partial_2} (NE_{100}\rtimes NE_{001})\rtimes (NE_{001}\rtimes NE_{010})\rtimes (NE_{010}\rtimes NE_{100}) \xrightarrow{\partial_1} NE_{000}$$

together with the homomorphisms

$$\begin{aligned} \partial_3(\gamma) &= (d_1^{\tau_2}(\gamma),d_1^{\tau_1}(\gamma),d_1^{\tau_3}(\gamma)) \\ \partial_2(\beta) &= ((d_1^{\tau_3}(x),-d_1^{\tau_1}(x)),(d_1^{\tau_2}(y),-d_1^{\tau_3}(y)),(d_1^{\tau_1}(z),-d_1^{\tau_2}(z))) \\ \partial_1(\alpha) &= (d_1^{\tau_1}(a)+d_1^{\tau_3}(a'))+(d_1^{\tau_3}(b)+d_1^{\tau_2}(b'))+(d_1^{\tau_2}(c)+d_1^{\tau_1}(c')) \end{aligned}$$

for $\gamma\in NE_{111},\ \beta=(x,y,z)\in NE_{101}\rtimes NE_{011}\rtimes NE_{110}$ and $\alpha=((a,a'),(b,b'),(c,c'))\in (NE_{100}\rtimes NE_{001})\rtimes (NE_{001}\rtimes NE_{010})\rtimes (NE_{010}\rtimes NE_{100})$.

$$\begin{aligned} \partial_2\partial_3(\gamma) &= ((d_1^{\tau_3}d_1^{\tau_2}(\gamma),-d_1^{\tau_1}d_1^{\tau_2}(\gamma)),(d_1^{\tau_2}d_1^{\tau_1}(\gamma),-d_1^{\tau_3}d_1^{\tau_1}(\gamma)),(d_1^{\tau_1}d_1^{\tau_3}(\gamma),-d_1^{\tau_2}d_1^{\tau_3}(\gamma))) \\ &= ((0,0),(0,0),(0,0)) \\ \partial_1\partial_2(\beta) &= d_1^{\tau_1}d_1^{\tau_3}(x)-d_1^{\tau_3}d_1^{\tau_1}(x)+d_1^{\tau_3}d_1^{\tau_2}(y)-d_1^{\tau_2}d_1^{\tau_3}(y)+d_1^{\tau_2}d_1^{\tau_1}(z)-d_1^{\tau_1}d_1^{\tau_2}(z) \\ &= 0 \end{aligned}$$

Using the mapping cone complex and Conduché's result for crossed squares and 2-crossed modules, the bilinear maps for 3-crossed module can be obtained, similarly. Thus, we can say that this mapping cone has a 3-crossed modules structure.

References

[1] Z. Arvasi, *Crossed squares and 2-crossed modules of commutative algebras*, Theory Appl. Categories **3** (1997), 160–181.
[2] Z. Arvasi, T. Porter, *Higher dimensional Peiffer elements in simplicial commutative algebras*, Theory Appl. Categories **3** (1997), 1–23.
[3] Z. Arvasi, T. Porter, *Freeness conditions for 2-crossed module of commutative algebras*, Appl. Categorical Struct. **6** (1998), 455–477.
[4] Z. Arvasi, T. S. Kuzpinari, E. O. Uslu, *Three-crossed modules*, Homology, Homotopy Appl. **11** (2009), 161–187.
[5] R. Brown, JL. Loday, *Van kampen theorems for diagrams of spaces*, Topology **26** (1987), 311–335.
[6] D. Conduché, *Modules croisés généraliés de longueur 2*, J. Pure Appl. Algebra **34** (1984), 155–178.
[7] D. Conduché, *Simplicial crossed modules and mapping cones*, Georgian Math. J. **10** (2003), 623–636.
[8] G. J. Ellis, *Higher dimensional crossed modules of algebras*, J. Pure Appl. Algebra **52** (1988), 277–282.
[9] D. Guin-Walery, J. L. Loday, *Obstructions à l'excision en K-théorie algèbrique, in evanston conference on algebraic K-theory*, Springer, Lecture Notes Math. **854** (1981), 179–216.
[10] Ö. Gürmen Alansal, E. Ulualan, *Peiffer pairings in multisimplicial groups and crossed n-cubes and applications for bisimplicial groups*, Turkish J. Math. **45** (2021), 360–386.
[11] T. S. Kuzpinari, A. Odabaş, E. O. Uslu, *3-crossed modules of commutative algebras*, arXiv:1003.0985v2, 2010.
[12] A. Mutlu, T. Porter, *Applications of Peiffer pairings in the Moore complex of a simplicial group*, Theory Appl. Categories **4** (1998), 148–173.
[13] A. Mutlu, T. Porter, *Freeness conditions for 2-crossed modules and complexes*, Theory Appl. Categories **4** (1998), 174–194.
[14] T. Porter, *N-types of simplicial groups and crossed N-cubes*, Topology **32** (1993), 5–24.
[15] T. Porter, *Homology of commutative algebras and an invariant of Simis and Vasconceles*, J. Algebra **99** (1986), 458–465.
[16] J. H. C. Whitehead, *Combinatorial homotopy II*, Bull. A,er. Math. Soc. **55** (1949), 453–496.