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Ricci bi-conformal vector fields on Lorentzian four-dimensional generalized symmetric spaces

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Abstract. In this paper, we completely classify the Ricci bi-conformal vector fields on non-symmetric simply-connected four dimensional pseudo-Riemannian generalized symmetric spaces up to isometry and we show which of them are the Killing vector fields and gradient vector fields.

1. Introduction

Let (M, g) be a smooth n-dimensional pseudo-Riemannian manifold. Geometric vector fields are important in differential geometry and physics. On of the geometric flows is conformal vector field. A vector field X on a Riemannian manifold (M, g) is called conformal vector field if there is a smooth function ψ on M that named a potential function, such that $\mathcal{L}_X g = 2\psi g$. If the potential function $\psi = 0$, then X is called a Killing vector field. Conformal vector fields are completely explained in [10, 16, 28]. Another generalization of Killing vector fields is generalized Kerr-Schild vector field. The generalized Kerr-Schild vector field is defined by

$$\mathcal{L}_X g = \alpha g + \beta l \otimes l, \qquad \mathcal{L}_X l = \gamma l,$$

where α, β, γ are smooth functions over M and l is a null 1-form field on M. When $\beta = 0$ then it is called a Kerr-Schild vector field. A symmetric tensor h on M is called a square root of g if $h_{ik}h^k_j = g_{ij}$. Garcia-Parrado and Senovilla [17] using square root of g defined bi-conformal vector fields. A vector field X is said to be a bi-conformal vector field if it satisfies the following equations:

$$\mathcal{L}_X g = \alpha g + \beta h, \qquad \mathcal{L}_X h = \alpha h + \beta g,$$

where h is a symmetric square root of g and α, β are smooth functions. The functions α and β are called gauges of the symmetry [12, 17] and they play a role analogous to the factor ψ appearing in the definition of the conformal vector fields. After then, De et al. in [13] using the metric tensor g and the Ricci tensor S defined Ricci bi-conformal vector fields as follows.

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Definition 1.1. *A vector field X on a pseudo-Riemannian manifold (M, g) is called Ricci bi-conformal vector field if it satisfies the following equations*

$$(\mathcal{L}_X g)(Y, Z) = \alpha g(Y, Z) + \beta S(Y, Z), \tag{1}$$

and

$$(\mathcal{L}_X S)(Y, Z) = \alpha S(Y, Z) + \beta g(Y, Z), \tag{2}$$

for any vector fields Y, Z and some smooth functions α and β, where S is the Ricci tensor of M with respect to metric g.
Also, Ricci soliton is introduced by Hamilton [19] as follows

$$\mathcal{L}_X g + S = \lambda g, \quad \lambda \in \mathbb{R},$$

which is a natural generalization of Einstein metric. For more details, see [1–4, 7, 23, 24].

Example 1.2. *Consider the manifold $M = \{(x, y) \in \mathbb{R}^2 : \ y < 0\}$ with metric tensor $g = \frac{1}{1+e^{2y}}(dx^2 + dy^2)$. The Ricci tensor of the metric g reprsented by $S = \frac{2e^{2y}}{1+e^{2y}} g$. For an arbitrary vector field $X = X^1(x, y)\frac{\partial}{\partial x} + X^2(x, y)\frac{\partial}{\partial y}$, we have*

$$\mathcal{L}_X g = \begin{pmatrix} 2\frac{(1+e^{2y})\partial_x X^1 - e^{2y} X^2}{(1+e^{2y})^2} & \frac{\partial_x X^2 + \partial_y X^1}{1+e^{2y}} \\ \frac{\partial_x X^2 + \partial_y X^1}{1+e^{2y}} & 2\frac{(1+e^{2y})\partial_y X^2 - e^{2y} X^2}{(1+e^{2y})^2} \end{pmatrix}$$

and

$$\mathcal{L}_Y S = \begin{pmatrix} 4e^{2y}\frac{(1+e^{2y})\partial_x X^1 + (1-e^{2y})X^2}{(1+e^{2y})^3} & \frac{2e^{2y}(\partial_x X^2 + \partial_y X^1)}{(1+e^{2y})^2} \\ \frac{2e^{2y}(\partial_x X^2 + \partial_y X^1)}{(1+e^{2y})^2} & 4e^{2y}\frac{(1+e^{2y})\partial_y X^2 + (1-e^{2y})X^2}{(1+e^{2y})^3} \end{pmatrix}.$$

Applying g, S, $\mathcal{L}_X g$, and $\mathcal{L}_X S$ in equations (1) and (2), we obtain

$$\begin{aligned}
&(1 + e^{2y})\partial_x X^1 - e^{2y} X^2 = \frac{1}{2}(1 + e^{2y})\alpha + e^{2y}\beta, \\
&(1 + e^{2y})\partial_y X^2 - e^{2y} X^2 = \frac{1}{2}(1 + e^{2y})\alpha + e^{2y}\beta, \\
&\partial_x X^2 + \partial_y X^1 = 0, \\
&(1 + e^{2y})\partial_x X^1 + (1 - e^{2y})X^1 = \frac{1}{2}(1 + e^{2y})\alpha + \frac{e^{-2y}(1 + e^{2y})^2}{4}\beta, \\
&(1 + e^{2y})\partial_y X^2 + (1 - e^{2y})X^1 = \frac{1}{2}(1 + e^{2y})\alpha + \frac{e^{-2y}(1 + e^{2y})^2}{4}\beta.
\end{aligned}$$

By direct computation, we observe that $X^1 = x, X^2 = y$ and

$$\alpha = \frac{2}{1 + e^{2y}}\left(1 + e^{2y} - xe^{2y} - \frac{4xe^{2y}}{e^{-2y}(1 + e^{2y})^2 - 4e^{2y}}\right), \beta = \frac{4x}{e^{-2y}(1 + e^{2y})^2 - 4e^{2y}}$$

is a solution of the above system. So the manifold M has a non-trivial Ricci bi-conformal vector field. Also, vector field X is a Ricci soliton vector field on manifolds M if and only if $X = a\frac{\partial}{\partial x} + \frac{\partial}{\partial x}$ with $\lambda = 0$ where a is a constant.

Cerny and Kowalski [11] classified pseudo-Riemannian four-dimensional generalized symmetric spaces into four classes, denoted by A, B, C, and D. Except from type C, which is Lorentzian, in the remainder cases associated pseudo-Riemannian metric is of signature $(4, 0), (2, 2)$ and $(0, 4)$. In [8, 15], the Levi-Civita connection, the curvature tensor, and the Ricci tensor of these spaces are computed. Batat and Onda [3] classified, up to isometry, non-symmetric simply-connected four dimensional pseudo-Riemannian generalized symmetric spaces which are algebraic Ricci solitons.

Motivated by [13], we study the Ricci bi-conformal vector fields on Lorentzian four-dimensional generalized symmetric spaces.

The paper is organized as follows. In Section 2, we recall some necessary concepts on non-symmetric simply-connected four dimensional pseudo-Riemannian generalized symmetric spaces which will be used throughout this paper. In the Section 3, we give the main results and their proofs.

2. Preliminaries

Suppose that (M, g) is a connected pseudo-Riemannian manifold and p is a point of M. A symmetry at a point p is an isometry s_p of M having p as isolated fixed point.
A regular s-structure on M is a family of isometries $\{s_p | p \in M\}$ of (M, g) such that

- the mapping $M \times M \to M$, $(p, q) \mapsto s_p(q)$, is smooth,
- p is an isolated fixed point of s_p, $\forall p \in M$,
- $s_p \circ s_q = s_{s_p(q)} \circ s_p$, $\forall p, q \in M$.

The map s_p is called the symmetry centered at p. The order of a regular s-structure is the least integer $k \geq 2$ such that $s_p^k = id_M$ for all $p \in M$. If such an integer does not exist, we say that the regular s-structure has order infinity. A generalized symmetric space is a connected, pseudo-Riemannian manifold (M, g), admitting at least one regular s-structure.

If (M, g) is a generalized symmetric space then the full isometry group $I(M)$ of M acts transitively on it, which means that (M, g) can be identified with $(G/H, g)$, where $G \subset I(M)$ is a subgroup of $I(M)$ acting transitively on M and H is the isotropy group at a fixed point $p \in M$. Moreover it admits at least on structure of reductive homogenous space with an invariant metric [11].

Generalized symmetric spaces have been intensively studied under different points of view [5, 18, 20–22, 26, 27]. Several geometric features of four-dimensional generalized symmetric spaces have been studied: homogeneous geodesic [14], curvature properties [8], harmonicity properties of invariant vector fields [6]. Bouharis and Djebbar [4] studied Ricci solitons on Lorentzian four-dimensional generalized symmetric space of type C.

Generalized symmetric spaces of low dimension have been completely classified. Non-symmetric simply-connected four-dimensional pseudo-Riemannian generalized symmetric spaces were classified in four types A, B, C, D, by Cerny and Kowalski [11] which is only type C in Lorentzian form and is as follows: the underlying homogeneous space G/H is the matrix group

$$G = \begin{pmatrix} e^{-t} & 0 & 0 & x \\ 0 & e^{t} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Manifold (M, g) is the space $\mathbb{R}^4(x, y, u, v)$ with the pseudo-Riemannian metric

$$g = \pm(e^{2t}dx^2 + e^{-2t}dy^2) + dzdt. \tag{3}$$

The order is $k = 3$ and the possible signatures are $(1, 3), (3, 1)$.

3. The main results and their proofs

Suppose that (M, g) is a four-dimensional generalized pseudo-Riemannian symmetric space. By ∇, S, and R we denote respectively the Levi-Civita connection, the scalar curvature, and the Riemannian curvature tensor of the manifold M. The Riemannian curvature tensor R is defined by

$$R(X, Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z.$$

The Ricci tensor S of (M, g) is defined by

$$S(X, Y) = \sum_{k=1}^{4} \epsilon_k g(R(X, e_k)Y, e_k),$$

with respect to the pseudo-orthonormal frame field $\{e_1, e_2, e_3, e_4\}$, with $g(e_k, e_k) = \epsilon_k = \pm 1$.

Now, assume that $(M = G/H, g)$ is a non-symmetric simply-connected four-dimensional generalized symmetric space of type C. From [4], the Levi-Civita connection ∇ of M with respect to the coordinates vector fields $\{\partial_1 = \frac{\partial}{\partial x}, \partial_2 = \frac{\partial}{\partial y}, \partial_3 = \frac{\partial}{\partial z}, \partial_4 = \frac{\partial}{\partial w}\}$ is described by

$$\nabla_{\partial_i}\partial_j = \begin{pmatrix} 2\epsilon e^{-2w}\partial_3 & 0 & 0 & -\partial_1 \\ 0 & -2\epsilon e^{2w}\partial_3 & 0 & \partial_2 \\ 0 & 0 & 0 & 0 \\ -\partial_1 & \partial_2 & 0 & 0 \end{pmatrix}, \tag{4}$$

and the Ricci tensor is represented by

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}. \tag{5}$$

Let $X = X_1\partial_1 + X_2\partial_2 + X_3\partial_3 + X_4\partial_4$ be an vector field on (M, g) where $X_i, i = 1, 2, 3, 4$ are smooth functions of the variables x, y, z, w. Therefore the Lie-derivative of g along the vector field $X = X_i\partial_i$ is given by

$$\begin{cases} (\mathcal{L}_X g)_{11} = 2\epsilon e^{-2w}(\partial_1 X_1 - X_4), & (\mathcal{L}_X g)_{12} = \epsilon(e^{2w}\partial_1 X_2 + e^{-2w}\partial_2 X_1), \\ (\mathcal{L}_X g)_{13} = \frac{1}{2}\partial_1 X_4 + \epsilon e^{-2w}\partial_3 X_1, & (\mathcal{L}_X g)_{14} = \frac{1}{2}\partial_1 X_3 + \epsilon e^{-2w}\partial_4 X_1, \\ (\mathcal{L}_X g)_{22} = 2\epsilon e^{2w}(X_4 + \partial_2 X_2), & (\mathcal{L}_X g)_{23} = \frac{1}{2}\partial_2 X_4 + \epsilon e^{2w}\partial_3 X_2, \\ (\mathcal{L}_X g)_{24} = \frac{1}{2}\partial_2 X_3 + \epsilon e^{2w}\partial_4 X_2, & (\mathcal{L}_X g)_{33} = \partial_3 X_4, \\ (\mathcal{L}_X g)_{34} = \frac{1}{2}(\partial_3 X_3 + \partial_4 X_4), & (\mathcal{L}_X g)_{44} = \partial_4 X_3. \end{cases} \tag{6}$$

The Lie-derivative of S along the vector field $X = X_i\partial_i$ is represented by

$$\mathcal{L}_X S = \begin{pmatrix} 0 & 0 & 0 & -2\partial_1 X_4 \\ 0 & 0 & 0 & -2\partial_2 X_4 \\ 0 & 0 & 0 & -2\partial_3 X_4 \\ -2\partial_1 X_4 & -2\partial_2 X_4 & -2\partial_3 X_4 & -4\partial_4 X_4 \end{pmatrix}. \tag{7}$$

Applying (6) and (7) in (1) and (2), we have

$$2(\partial_1 X_1 - X_4) = \alpha, \tag{8}$$
$$\beta = 0, \tag{9}$$
$$e^{2w}\partial_1 X_2 + e^{-2w}\partial_2 X_1 = 0, \tag{10}$$
$$\frac{1}{2}\partial_1 X_4 + \epsilon e^{-2w}\partial_3 X_1 = 0, \tag{11}$$
$$\frac{1}{2}\partial_1 X_3 + \epsilon e^{-2w}\partial_4 X_1 = 0, \tag{12}$$
$$\partial_1 X_4 = 0, \tag{13}$$
$$2(X_4 + \partial_2 X_2) = \alpha, \tag{14}$$
$$\frac{1}{2}\partial_2 X_4 + \epsilon e^{2w}\partial_3 X_2 = 0, \tag{15}$$
$$\frac{1}{2}\partial_2 X_3 + \epsilon e^{2w}\partial_4 X_2 = 0, \tag{16}$$
$$\partial_2 X_4 = 0, \tag{17}$$
$$\partial_3 X_4 = 0, \tag{18}$$
$$\partial_3 X_3 + \partial_4 X_4 = \alpha, \tag{19}$$
$$\partial_4 X_3 = 0, \tag{20}$$
$$2\partial_4 X_4 = \alpha. \tag{21}$$

Equations (13), (17), and (18) imply that $X_4 = X_4(w)$. Hence, equation (21) yields $\alpha = \alpha(w)$. Taking derivative of equations (12) and (16) with respect to w, we get

$$\partial_4^2 X_1 - 2\partial_4 X_1 = 0, \tag{22}$$
$$\partial_4^2 X_2 + 2\partial_4 X_2 = 0. \tag{23}$$

Using (11) and (13), we conclude that

$$\partial_3 X_1 = 0. \tag{24}$$

Also, using (15) and (17), we infer

$$\partial_3 X_2 = 0. \tag{25}$$

Solving differential equations (22) and (23), we obtain

$$X_1 = e^{2w}h(x, y) + H(x, y), \tag{26}$$
$$X_2 = e^{-2w}k(x, y) + K(x, y), \tag{27}$$

where h, H, k, and K are smooth functions depending on x and y. From (19) and (21), we find

$$2\partial_3 X_3 = \alpha. \tag{28}$$

By taking derivative with respect to w and using (20) we obtain $\partial_4\alpha = 0$. Then α is a constant. Applying (8) and (14), we arrive at

$$\partial_1 X_1 + \partial_2 X_2 = \alpha. \tag{29}$$

Inserting (26) and (27) into (29), we deduce that

$$e^{2w}\partial_1 h + \partial_1 H + e^{-2w}\partial_2 k + \partial_2 K = \alpha. \tag{30}$$

By taking derivative with respect to w of both sides of (30), we infer

$$e^{2w}\partial_1 h - e^{-2w}\partial_2 k = 0. \tag{31}$$

Since w is arbitrary, (31) gives $\partial_1 h = \partial_2 k = 0$ and so (30) leads to

$$\partial_1 H + \partial_2 K = \alpha. \tag{32}$$

Then h depends only on y and k depends only on x. We replace X_1 and X_2 in equation (10) to find

$$\partial_1 k + e^{2w}\partial_1 K + \partial_2 h + e^{-2w}\partial_2 H = \alpha. \tag{33}$$

Taking differentiation with respect to w of both sides of (33), we get $\partial_1 K = \partial_2 H = 0$ and

$$\partial_1 k + \partial_2 h = 0. \tag{34}$$

Then H depends only on x and K depends only on y. Since x and y are arbitrary, from (34) we conclude that $\partial_1 k = -\partial_2 h = a_1$ for some constant a_1. Thus, $h = -a_1 y + a_2$ and $k = a_1 x + a_3$, where $a_2, a_3 \in \mathbb{R}$. Similarly, $\partial_1 H = \alpha - \partial_2 K = b_1$ for some constant b_1, then $H = b_1 x + b_2$ and $K = (\alpha - b_1)y + b_3$, where $b_2, b_3 \in \mathbb{R}$. Equation (8) leads to $X_4 = \partial_1 H - \frac{\alpha}{2} = b_1 - \frac{\alpha}{2}$. Hence, $\partial_4 X_4 = 0$ and equation (21) implies that $\alpha = 0$ and

$$\begin{cases} X_1 = (-a_1 y + a_2)e^{2w} + b_1 x + b_2, \\ X_2 = (a_1 x + a_3)e^{-2w} - b_1 y + b_3, \\ X_4 = b_1. \end{cases}$$

Equations (12) and (16) give

$$\partial_1 X_3 = -4\epsilon(-a_1 y + a_2), \tag{35}$$

$$\partial_2 X_3 = 4\epsilon(a_1 x + a_3). \tag{36}$$

Also, equation (9) leads to

$$\partial_3 X_3 = 0. \tag{37}$$

Using (20), (35), (36) and (37), we obtain

$$X_3 = -4\epsilon(-a_1 y + a_2)x + 4\epsilon a_3 y + a_4. \tag{38}$$

Therefore, we have the following result:

Theorem 3.1. *A four-dimensional pseudo-Riemannian generalized symmetric space of type C has Ricci bi-conformal vector field $X = X_i\partial_i$ if and only if $\alpha = \beta = 0$ and*

$$\begin{cases} X_1 = (-a_1 y + a_2)e^{2w} + b_1 x + b_2, \\ X_2 = (a_1 x + a_3)e^{-2w} - b_1 y + b_3, \\ X_3 = -4\epsilon(-a_1 y + a_2)x + 4\epsilon a_3 y + a_4, \\ X_4 = b_1, \end{cases} \tag{39}$$

where $a_1, a_2, a_3, a_4, b_1, b_2, b_3 \in \mathbb{R}$.

Now, we consider the vector fields in the form of $X = \nabla f$ for some smooth function f which are Ricci bi-conformal vector fields on a four-dimensional pseudo-Riemannian generalized symmetric space of type C. On a four-dimensional Lorentzian generalized symmetric space, we have

$$\nabla f = \epsilon e^{2w}(\partial_1 f)e_1 + \epsilon e^{-2w}(\partial_2 f)e_2 + 2(\partial_4 f)e_3 + 2(\partial_3 f)e_4. \tag{40}$$

From (39) and (40), we have

$$\partial_1 f = \epsilon(-a_1 y + a_2) + \epsilon(b_1 x + b_2)e^{-2w}, \tag{41}$$

$$\partial_2 f = \epsilon(a_1 x + a_3) + \epsilon(-b_1 y + b_3)e^{2w}, \tag{42}$$

$$\partial_3 f = \frac{b_1}{2}, \tag{43}$$

$$\partial_4 f = -2\epsilon(-a_1 y + a_2)x + 2\epsilon a_3 y + \frac{a_4}{2}. \tag{44}$$

By taking derivative of the equation (41) with respect to w and taking derivative of the equation (44) with respect to x, we infer

$$(b_1 x + b_2)e^{-2w} = -a_1 y + a_2. \tag{45}$$

Hence, $a_1 = a_2 = b_1 = b_2 = 0$. Also, taking derivative of the equation (42) with respect to w and taking derivative of the equation (44) with respect to y, we get

$$(-b_1 y + b_3)e^{2w} = a_1 x + a_3, \tag{46}$$

and $a_3 = b_3 = 0$. Thus

$$\partial_1 f = \partial_2 f = \partial_3 f = 0, \text{ and } \partial_4 f = \frac{a_4}{2}, \tag{47}$$

and

$$f = \frac{a_4}{2} w + a_5, \tag{48}$$

where $a_5 \in \mathbb{R}$. Therefore, we have the following corollary:

Corollary 3.2. *A four-dimensional pseudo-Riemannian generalized symmetric space of type C has Ricci bi-conformal vector field as $X = \nabla f$ if and only if $f = \frac{a}{2}w + b$, where $a, b \in \mathbb{R}$.*

Remark 3.3. *A vector field X on (M, g) is called a Killing vector field if*

$$\mathcal{L}_X g = 0.$$

Then, from Theorem 3.1 we conclude that all Ricci bi-conformal vector fields on four-dimensional Lorentzian generalized symmetric spaces are Killing vector fields. From [25], Ricci bi-conformal vector fields are infinitesimal harmonic transformations, becuase $\mathcal{L}_X g = 0$ implies that $\mathcal{L}_X \nabla = 0$ and $\mathrm{trac}_g(\mathcal{L}_X \nabla) = 0$. Also, (M, g) is said to be Yamabe soliton if it admits a vector field X such that

$$\mathcal{L}_X g = (r - \Lambda)g,$$

where r denotes the scalar curvature of (M, g) and Λ is a real number. Moreover, we say that the Yamabe soliton is a gradient Yamabe soliton if $X = \nabla f$ for some potential function f. Thus, by Theorem 3.1 we deduce that all Ricci bi-conformal vector fields on four-dimensional Lorentzian generalized symmetric spaces admit in Yamabe soliton equation with $\Lambda = r$.

Remark 3.4. *A vector field X is called a Ricci collineation vector field [9] whenever $\mathcal{L}_X S = 0$. Using Theorem 3.1, Ricci bi-conformal vector fields on four-dimensional Lorentzian generalized symmetric spaces become Ricci collineation vector field.*

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