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On invariant submanifolds of normal paracontact metric manifolds on generalized $\mathcal{B}-$curvature tensor

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Abstract. In this article, pseudoparallel submanifolds for normal paracontact metric manifolds are studied. B-curvature tensor in a normal paracontact metric manifold has been considered. For an invariant submanifold of a paracontact metric manifold, $\mathcal{B}-$pseudoparallel, $\mathcal{B}$ 2-pseudoparallel, $\mathcal{B}-$Ricci generalized pseudoparallel, and $\mathcal{B}$ 2− Ricci generalized pseudoparallel has been searched. Also, characterizations of invariant submanifold types are given by means of quasi-conformal, Weyl-conformal, concircular, conharmonic curvature tensors for special cases of generalized $\mathcal{B}-$curvature tensor.

1. Introduction

The study of paracontact geometry was initiated by Kenayuki and Williams [1]. Zamkovoy studied paracontact metric manifolds and their subclasses [2]. Recently Welyczko studied curvature and torsion of Frenet Legendre curves in 3-dimensional nnormal paracontact metric manifolds [3],[4]. In the recent years, contact metric manifolds and their curvature properties have been studied by many authors in [5],[6],[7].

In this article, invariant pseudoparallel submanifolds for a normal paracontact metric manifold are investigated. The normal paracontact metric manifold is considered on the generalized $\mathcal{B}-$curvature tensor. Submanifolds of these manifolds with properties such as $\mathcal{B}-$pseudoparallel, $\mathcal{B}$ 2−pseudoparallel, $\mathcal{B}-$Ricci generalized pseudoparallel, and $\mathcal{B}$ 2−Ricci generalized pseudoparallel has been studied. Also, characterizations of invariant submanifold are given by means of quasi-conformal, Weyl-conformal, concircular, conharmonic curvature tensors for special cases of generalized $\mathcal{B}-$curvature tensor.

For simplicity's sake, the normal paracontact metric manifold expression will be expressed as *NPM*-manifold after this part of the article.

Let's take an $n-$dimensional differentiable W manifold. If it admits a tensor field ϕ of type $(1,1)$, a contravariant vector field ξ and a 1-form η satisfying the following conditions;

$$\phi^2\epsilon_1 = \epsilon_1 - \eta(\epsilon_1)\xi,\ \phi\xi = 0,\ \eta\left(\phi\epsilon_1\right) = 0, \eta(\xi) = 1, \tag{1}$$

and

$$g\left(\phi\epsilon_1, \phi\epsilon_2\right) = -g(\epsilon_1, \epsilon_2) + \eta(\epsilon_1)\eta(\epsilon_2),\ g(\epsilon_1, \xi) = \eta(\epsilon_1), \tag{2}$$

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for all $\epsilon_1, \epsilon_2, \xi \in \chi(W)$, $\left(\phi, \xi, \eta\right)$ is called almost paracontact structure and $\left(W, \phi, \xi, \eta\right)$ is called almost paracontact metric manifold. If the covariant derivative of ϕ satisfies

$$\left(\nabla_{\epsilon_1}\phi\right)\epsilon_2 = -g(\epsilon_1, \epsilon_2)\xi - \eta(\epsilon_2)\epsilon_1 + 2\eta(\epsilon_1)\eta(\epsilon_2)\xi, \tag{3}$$

then, W is called a $NPM-$manifold, where ∇ is Levi-Civita connection, From (3), we can easily to see that

$$\phi\epsilon_1 = \nabla_{\epsilon_1}\xi, \tag{4}$$

for any $\epsilon_1 \in \chi(W)$ [1].

Moreover, if such a manifold has constant sectional curvature equal to c, then its the Riemannian curvature tensor is R given by

$$\begin{aligned} R(\epsilon_1, \epsilon_2)\epsilon_3 &= \tfrac{c+3}{4}\left[g(\epsilon_2, \epsilon_3)\epsilon_1 - g(\epsilon_1, \epsilon_3)\epsilon_2\right] + \tfrac{c-1}{4}\left[\eta(\epsilon_1)\eta(\epsilon_3)\epsilon_2\right. \\ &-\eta(\epsilon_2)\eta(\epsilon_3)\epsilon_1 + g(\epsilon_1, \epsilon_3)\eta(\epsilon_2)\xi - g(\epsilon_2, \epsilon_3)\eta(\epsilon_1)\xi + g\left(\phi\epsilon_2, \epsilon_3\right)\phi\epsilon_1 \\ &\left.-g\left(\phi\epsilon_1, \epsilon_3\right)\phi\epsilon_2 - 2g\left(\phi\epsilon_1, \epsilon_2\right)\phi\epsilon_3\right], \end{aligned} \tag{5}$$

for any vector fields $\epsilon_1, \epsilon_2, \epsilon_3 \in \chi(W)$ [5].

In a normal paracontact metric space form, by direct calculations, we can easily to see that

$$S(\epsilon_1, \epsilon_2) = \frac{c(n-5) + 3n + 1}{4}g(\epsilon_1, \epsilon_2) + \frac{(c-1)(5-n)}{4}\eta(\epsilon_1)\eta(\epsilon_2), \tag{6}$$

which implies that

$$Q\epsilon_1 = \frac{c(n-5) + 4n + 1}{4}\epsilon_1 + \frac{(c-1)(5-n)}{4}\eta(\epsilon_1)\xi, \tag{7}$$

for any $\epsilon_1, \epsilon_2 \in \chi(W)$, where Q is the Ricci operator and S is the Ricci tensor of W.

Lemma 1.1. *Let W be a $\boldsymbol{n}$-dimensional $NPM-$manifold. In this case, the following equations hold.*

$$R(\xi, \epsilon_1)\epsilon_2 = g(\epsilon_1, \epsilon_2)\xi - \eta(\epsilon_2)\epsilon_1, \tag{8}$$

$$R(\epsilon_1, \xi)\epsilon_2 = -g(\epsilon_1, \epsilon_2)\xi + \eta(\epsilon_2)\epsilon_1, \tag{9}$$

$$R(\epsilon_1, \epsilon_2)\xi = \eta(\epsilon_2)\epsilon_1 - \eta(\epsilon_1)\epsilon_2, \tag{10}$$

$$\eta(R(\epsilon_1, \epsilon_2)\epsilon_3) = g(\eta(\epsilon_1)\epsilon_2 - \eta(\epsilon_2)\epsilon_1, \epsilon_3), \tag{11}$$

$$S(\epsilon_1, \xi) = (n-1)\eta(\epsilon_1), \tag{12}$$

$$Q\xi = (n-1)\xi, \tag{13}$$

where R, S and Q are Riemann curvature tensor, Ricci curvature tensor and Ricci operator, respectively.

In 2014, Shaik and Kundu in [8], defined and studied a type of tensor field, called generalized $\mathcal{B}-$curvature tensor on a Riemannian manifold. It is a generalization of the quasi-conformal, Weyl-conformal, conharmonic and concircular curvature tensors and is given as

$$\begin{aligned} \mathcal{B}(\epsilon_1, \epsilon_2)\epsilon_3 &= p_0R(\epsilon_1, \epsilon_2)\epsilon_3 + p_1\left[S(\epsilon_2, \epsilon_3)\epsilon_1 - S(\epsilon_1, \epsilon_3)\epsilon_2 + g(\epsilon_2, \epsilon_3)Q\epsilon_1\right. \\ &\left.-g(\epsilon_1, \epsilon_3)Q\epsilon_2\right] + 2p_2r\left[g(\epsilon_2, \epsilon_3)\epsilon_1 - g(\epsilon_1, \epsilon_3)\epsilon_2\right]. \end{aligned} \tag{14}$$

From (14), for n-dimensional $NPM-$manifold, we easily to see that

$$\mathcal{B}(\xi,\epsilon_1)\epsilon_2=\left[p_0+\frac{p_1}{4}\left[c(n-5)+7n-1\right]+2p_2r\right]\left[g(\epsilon_1,\epsilon_2)\xi-\eta(\epsilon_2)\epsilon_1\right], \tag{15}$$

$$\mathcal{B}(\epsilon_1,\xi)\epsilon_2=\left[p_0+\frac{p_1}{4}\left[c(n-5)+7n-1\right]+2p_2r\right]\left[-g(\epsilon_1,\epsilon_2)\xi+\eta(\epsilon_2)\epsilon_1\right], \tag{16}$$

$$\mathcal{B}(\epsilon_1,\epsilon_2)\xi=\left[p_0+\frac{p_1}{4}\left[c(n-5)+7n-1\right]+2p_2r\right]\left[\eta(\epsilon_2)\epsilon_1-\eta(\epsilon_1)\epsilon_2\right]. \tag{17}$$

In particular, the $\mathcal{B}-$curvature tensor is reduced to
i. The Quasi-conformal curvature tensor if

$$p_0=a, p_1=b \text{ and } p_2=-\frac{1}{2n}\left(\frac{a}{n-1}+2b\right). \tag{18}$$

ii. The Weyl-conformal curvature tensor if

$$p_0=1, p_1=-\frac{1}{n-1} \text{ and } p_2=-\frac{1}{2(n-1)(n-2)}. \tag{19}$$

iii. The Concircular curvature tensor if

$$p_0=1, p_1=0 \text{ and } p_2=-\frac{1}{n(n-1)}. \tag{20}$$

iv. The Conharmonic curvature tensor if

$$p_0=1, p_1=-\frac{1}{n-1} \text{ and } p_2=0. \tag{21}$$

Let $\tilde{W}$ be the immersed submanifold of a $NPM-$manifold $W\left(\phi,\xi,\eta,g\right)$. Let the tangent and normal subspaces of $\tilde{W}$ in $W\left(\phi,\xi,\eta,g\right)$ be $\Gamma\left(T\tilde{W}\right)$ and $\Gamma\left(T^{\perp}\tilde{W}\right)$, respectively. Gauss and Weingarten formulas are, respectively, given

$$\nabla_{\epsilon_1}\epsilon_2=\tilde{\nabla}_{\epsilon_1}\epsilon_2+\sigma(\epsilon_1,\epsilon_2), \tag{22}$$

$$\nabla_{\epsilon_1}\epsilon_5=-A_{\epsilon_5}\epsilon_1+\tilde{\nabla}^{\perp}_{\epsilon_1}\epsilon_5, \tag{23}$$

for all $\epsilon_1,\epsilon_2\in\Gamma\left(T\tilde{W}\right)$ and $\epsilon_5\in\Gamma\left(T^{\perp}\tilde{W}\right)$, where $\tilde{\nabla}$ and $\tilde{\nabla}^{\perp}$ are the connections on $\tilde{W}$ and $\Gamma\left(T^{\perp}\tilde{W}\right)$, respectively, σ and A are the second fundamental form and the shape operator of $\tilde{W}$. There is a relation

$$g(A_{\epsilon_5}\epsilon_1,\epsilon_2)=g(\sigma(\epsilon_1,\epsilon_2),\epsilon_5)$$

between second basic form and shape operator. The covariant derivative of the second fundamental form σ is defined as

$$(\nabla_{\epsilon_1}\sigma)(\epsilon_2,\epsilon_3)=\nabla^{\perp}_{\epsilon_1}\sigma(\epsilon_2,\epsilon_3)-\sigma(\tilde{\nabla}_{\epsilon_1}\epsilon_2,\epsilon_3)-\sigma(\epsilon_2,\tilde{\nabla}_{\epsilon_1}\epsilon_3). \tag{24}$$

Specifically, if $\nabla\sigma=0$, second fundamental form is called parallel.
Let $\tilde{R}$ be the Riemann curvature tensor of $\tilde{W}$. In this case, the Gauss equation can be expressed as

$$R(\epsilon_1,\epsilon_2)\epsilon_3=\tilde{R}(\epsilon_1,\epsilon_2)\epsilon_3+A_{\sigma(\epsilon_1,\epsilon_3)}\epsilon_2-A_{\sigma(\epsilon_2,\epsilon_3)}\epsilon_1+(\nabla_{\epsilon_1}\sigma)(\epsilon_2,\epsilon_3)-(\nabla_{\epsilon_2}\sigma)(\epsilon_1,\epsilon_3),$$

for all $\epsilon_1, \epsilon_2, \epsilon_3 \in \Gamma\left(T\tilde{W}\right)$, where if

$$(\nabla_{\epsilon_1}\sigma)(\epsilon_2, \epsilon_3) - (\nabla_{\epsilon_2}\sigma)(\epsilon_1, \epsilon_3) = 0,$$

then submanifold is called curvature-invariant submanifold.

Let $\tilde{W}$ be a Riemannian manifold, T is $(0,k)-$type tensor field and A is $(0,2)-$type tensor field. In this case, Tachibana tensor field $Q(A,T)$ is defined as

$$Q(A,T)(X_1,...,X_k;\epsilon_1,\epsilon_2) = -T((\epsilon_1 \wedge_A \epsilon_2)X_1,...,X_k) - ... - T(X_1,...,X_{k-1},(\epsilon_1 \wedge_A \epsilon_2)X_k), \tag{25}$$

where,

$$(\epsilon_1 \wedge_A \epsilon_2)\epsilon_3 = A(\epsilon_2,\epsilon_3)\epsilon_1 - A(\epsilon_1,\epsilon_3)\epsilon_2, \tag{26}$$

$k \geq 1, X_1, X_2, ..., X_k, \epsilon_1, \epsilon_2 \in \Gamma\left(T\tilde{W}\right)$.

Definition 1.2. *A submanifold of a semi-Riemannian manifold (W,g) is said to be pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel and 2-Ricci generalized pseudoparallel if*

$$R \cdot \sigma \text{ and } Q(g,\sigma)$$

$$R \cdot \nabla\sigma \text{ and } Q(g,\nabla\sigma)$$

$$R \cdot \sigma \text{ and } Q(S,\sigma)$$

$$R \cdot \nabla\sigma \text{ and } Q(S,\nabla\sigma)$$

are linearly dependent, respectively.

For brevity, after this part of the article, 2 pseudoparallel expressions will be shown as $2P$, Ricci generalized pseudoparallel as $RG-$pseudoparallel, and 2-Ricci generalized pseudoparallel as $2-RG$ pseudoparallel.

2. Invariant Pseudoparalel Submanifolds of Normal PAracontact Metric Manifolds

Let $\tilde{W}$ be the immersed submanifold of an $n-$dimensional $NPM-$manifold $W\left(\phi,\xi,\eta,g\right)$. If $\phi(T_xW) \subset T_xW$ in every x point, the $\tilde{W}$ manifold is called an invariant submanifold. We note that all of properties of an invariant submanifold inherit the ambient manifold. In the rest of this article, we will assume that $\tilde{W}$ is the invariant submanifold of the $NPM-$manifold $W\left(\phi,\xi,\eta,g\right)$. So, it is clear that

$$\sigma\left(\phi\epsilon_1,\epsilon_2\right) = \sigma\left(\epsilon_1,\phi\epsilon_2\right) = \phi\sigma(\epsilon_1,\epsilon_2) \tag{27}$$

$$\sigma(\epsilon_1,\xi) = 0, \tag{28}$$

for all $\epsilon_1,\epsilon_2 \in \Gamma\left(T\tilde{W}\right)$.

Lemma 2.1. *Let $\tilde{W}$ be an invariant submanifold of the $n-$dimensional normal paracontact manifold $W\left(\phi,\xi,\eta,g\right)$. The second fundamental form σ of $\tilde{W}$ is parallel if and only if $\tilde{W}$ is the total geodesic submanifold.*

Let us now consider a invariant submanifolds of the $NPM-$manifold for the generalized $\mathcal{B}-$curvature tensor.

Definition 2.2. *Let $\tilde{W}$ be an invariant submanifold of the $n-$dimensional $NPM-$manifold $W\left(\phi,\xi,\eta,g\right)$. If $\mathcal{B}\cdot\sigma$ and $Q(g,\sigma)$ are linearly dependent, $\tilde{W}$ is called generalized $\mathcal{B}-$pseudoparallel submanifold.*

In the same sense, it can be said that there is a function $\mathcal{F}_1$ on the set $M_1 = \left\{\epsilon_1 \in \tilde{W} \middle| \sigma(\epsilon_1) \neq g(\epsilon_1)\right\}$ such that

$$\mathcal{B} \cdot \sigma = \mathcal{F}_1 Q(g, \sigma).$$

If $\mathcal{F}_1 = 0$ specifically, $\tilde{W}$ is called a generalized $\mathcal{B}-$ semiparallel submanifold.

Theorem 2.3. *Let $\tilde{W}$ be an invariant submanifold of the $n-$dimensional NPM$-$manifold $W\left(\phi, \xi, \eta, g\right)$. If $\tilde{W}$ is a generalized $\mathcal{B}-$ pseudoparallel submanifold, then $\tilde{W}$ is either a total geodesic or*

$$\mathcal{F}_1 = -\left[p_0 + \frac{p_1}{4}\left[c(n-5) + 7n - 1\right] + 2p_2 r\right].$$

Proof. Let's assume that $\tilde{W}$ is a generalized $\mathcal{B}-$ pseudoparallel submanifold. So, we can write

$$(\mathcal{B}(\epsilon_1, \epsilon_2) \cdot \sigma)(\epsilon_4, \epsilon_5) = \mathcal{F}_1 Q(g, \sigma)(\epsilon_4, \epsilon_5; \epsilon_1, \epsilon_2), \tag{29}$$

for all $\epsilon_1, \epsilon_2, \epsilon_4, \epsilon_5 \in \Gamma\left(T\tilde{W}\right)$. From (29), it is clear that

$$R^{\perp}(\epsilon_1, \epsilon_2)\sigma(\epsilon_4, \epsilon_5) - \sigma(\mathcal{B}(\epsilon_1, \epsilon_2)\epsilon_4, \epsilon_5) - \sigma(\epsilon_4, \mathcal{B}(\epsilon_1, \epsilon_2)\epsilon_5)$$
$$= -\mathcal{F}_1\left\{\sigma\left(\left(\epsilon_1 \wedge_g \epsilon_2\right)\epsilon_4, \epsilon_5\right) + \sigma\left(\epsilon_4, \left(\epsilon_1 \wedge_g \epsilon_2\right)\epsilon_5\right)\right\}.$$

This implies that

$$\begin{aligned} &R^{\perp}(\epsilon_1, \epsilon_2)\sigma(\epsilon_4, \epsilon_5) - \sigma(\mathcal{B}(\epsilon_1, \epsilon_2)\epsilon_4, \epsilon_5) - \sigma(\epsilon_4, \mathcal{B}(\epsilon_1, \epsilon_2)\epsilon_5) = -\mathcal{F}_1\{g(\epsilon_2, \epsilon_4)\sigma(\epsilon_1, \epsilon_5) \\ &-g(\epsilon_1, \epsilon_4)\sigma(\epsilon_2, \epsilon_5) + g(\epsilon_2, \epsilon_5)\sigma(\epsilon_4, \epsilon_1) - g(\epsilon_1, \epsilon_5)\sigma(\epsilon_4, \epsilon_2)\}. \end{aligned} \tag{30}$$

If we choose $\epsilon_1 = \epsilon_5 = \xi$ in (30) and making use of (15), (17), (28), we get

$$\left\{\mathcal{F}_1 + \left[p_0 + \frac{p_1}{4}\left[c(n-5) + 7n - 1\right] + 2p_2 r\right]\right\}\sigma(\epsilon_4, \epsilon_2) = 0.$$

This completes proof of the theorem. □

Thus we have the following corollaries.

Corollary 2.4. *Let $\tilde{W}$ be an invariant submanifold of the $n-$dimensional NPM$-$manifold $W\left(\phi, \xi, \eta, g\right)$. If $\tilde{W}$ is a quasi-conformal pseudoparallel submanifold, then $\tilde{W}$ is either a total geodesic or*

$$\mathcal{F}_1 = \frac{1}{n}\left(\frac{a}{n-1} + 2b\right)r - \left[a + \frac{b}{4}\left[c(n-5) + 7n - 1\right]\right].$$

Corollary 2.5. *Let $\tilde{W}$ be an invariant submanifold of the $n-$dimensional NPM$-$manifold $W\left(\phi, \xi, \eta, g\right)$. If $\tilde{W}$ is a Weyl-conformal pseudoparallel submanifold, then $\tilde{W}$ is either a total geodesic or*

$$\mathcal{F}_1 = \frac{r}{(n-1)(n-2)} + \frac{1}{4(n-1)}\left[c(n-5) + 7n - 1\right] - 1.$$

Corollary 2.6. *Let $\tilde{W}$ be an invariant submanifold of the $n-$dimensional NPM$-$manifold $W\left(\phi, \xi, \eta, g\right)$. If $\tilde{W}$ is a concircular pseudoparallel submanifold, then $\tilde{W}$ is either a total geodesic or*

$$\mathcal{F}_1 = \frac{2r}{n(n-1)} - 1.$$

Corollary 2.7. *Let $\tilde{W}$ be the invariant submanifold of the $n-$dimensional NPM$-$manifold $W\left(\phi,\xi,\eta,g\right)$. If $\tilde{W}$ is a conharmonic pseudoparallel submanifold, then $\tilde{W}$ is either a total geodesic or*

$$\mathcal{F}_1 = \frac{1}{n-1}\left[c\left(n-5\right)+7n-1\right]-1.$$

Definition 2.8. *Let $\tilde{W}$ be an invariant submanifold of the $n-$dimensional NPM-manifold $W\left(\phi,\xi,\eta,g\right)$. If $\mathcal{B}\cdot\nabla\sigma$ and $Q\left(g,\nabla\sigma\right)$ are linearly dependent, then $\tilde{W}$ is called generalized $\mathcal{B}-2P$ submanifold.*

In this case, it can be said that there is a function $\mathcal{F}_2$ on the set $M_2=\left\{\epsilon_1\in\tilde{W}\middle|\nabla\sigma\left(\epsilon_1\right)\neq g\left(\epsilon_1\right)\right\}$ such that

$$\mathcal{B}\cdot\nabla\sigma=\mathcal{F}_2Q\left(g,\nabla\sigma\right).$$

If $\mathcal{F}_2=0$ specifically, $\tilde{W}$ is called a generalized $\mathcal{B}$ 2-semiparallel submanifold.

Theorem 2.9. *Let $\tilde{W}$ be an invariant submanifold of the $n-$dimensional NPM$-$manifold $W\left(\phi,\xi,\eta,g\right)$. If $\tilde{W}$ is a generalized $\mathcal{B}-$ $2P$ submanifold, then $\tilde{W}$ is either a total geodesic or*

$$\mathcal{F}_2=p_0+\frac{p_1}{4}\left[c\left(n-5\right)+7n-1\right]+2p_2r.$$

Proof. Let's assume that $\tilde{W}$ is a generalized $\mathcal{B}$–$2P$ submanifold. So, we can write

$$\left(\mathcal{B}\left(\epsilon_1,\epsilon_2\right)\cdot\nabla\sigma\right)\left(\epsilon_4,\epsilon_5,\epsilon_3\right)=\mathcal{F}_2Q\left(g,\nabla\sigma\right)\left(\epsilon_4,\epsilon_5,\epsilon_3;\epsilon_1,\epsilon_2\right),\tag{31}$$

for all $\epsilon_1,\epsilon_2,\epsilon_4,\epsilon_5,\epsilon_3\in\Gamma\left(T\tilde{W}\right)$. If we choose $\epsilon_1=\epsilon_3=\xi$ in (31), we can write

$$\begin{aligned}&R^{\perp}\left(\xi,\epsilon_2\right)\left(\nabla_{\epsilon_4}\sigma\right)\left(\epsilon_5,\xi\right)-\left(\nabla_{\mathcal{B}(\xi,\epsilon_2)\epsilon_4}\sigma\right)\left(\epsilon_5,\xi\right)\\&-\left(\nabla_{\epsilon_4}\sigma\right)\left(\mathcal{B}\left(\xi,\epsilon_2\right)\epsilon_5,\xi\right)-\left(\nabla_{\epsilon_4}\sigma\right)\left(\epsilon_5,\mathcal{B}\left(\xi,\epsilon_2\right)\xi\right)\\&=-\mathcal{F}_2\left\{\left(\nabla_{\left(\xi\wedge_g\epsilon_2\right)\epsilon_4}\sigma\right)\left(\epsilon_5,\xi\right)+\left(\nabla_{\epsilon_4}\sigma\right)\left(\left(\xi\wedge_g\epsilon_2\right)\epsilon_5,\xi\right)\right.\\&\left.+\left(\nabla_{\epsilon_4}\sigma\right)\left(\epsilon_5,\left(\xi\wedge_g\epsilon_2\right)\xi\right)\right\}.\end{aligned}\tag{32}$$

Let's calculate all the expressions in (32). So, we can write

$$\begin{aligned}&R^{\perp}\left(\xi,\epsilon_2\right)\left(\nabla_{\epsilon_4}\sigma\right)\left(\epsilon_5,\xi\right)=R^{\perp}\left(\xi,\epsilon_2\right)\left\{\tilde{\nabla}^{\perp}_{\epsilon_4}\sigma\left(\epsilon_5,\xi\right)-\sigma\left(\tilde{\nabla}_{\epsilon_4}\epsilon_5,\xi\right)-\sigma\left(\epsilon_5,\tilde{\nabla}_{\epsilon_4}\xi\right)\right\}\\&=-R^{\perp}\left(\xi,\epsilon_2\right)\phi\sigma\left(\epsilon_5,\epsilon_4\right),\end{aligned}\tag{33}$$

$$\begin{aligned}&\left(\nabla_{\mathcal{B}(\xi,\epsilon_2)\epsilon_4}\sigma\right)\left(\epsilon_5,\xi\right)=\tilde{\nabla}^{\perp}_{\mathcal{B}(\xi,\epsilon_2)\epsilon_4}\sigma\left(\epsilon_5,\xi\right)\\&-\sigma\left(\tilde{\nabla}_{\mathcal{B}(\xi,\epsilon_2)\epsilon_4}\epsilon_5,\xi\right)-\sigma\left(\epsilon_5,\tilde{\nabla}_{\mathcal{B}(\xi,\epsilon_2)\epsilon_4}\xi\right)\\&=-\sigma\left(\epsilon_5,\phi\mathcal{B}\left(\xi,\epsilon_2\right)\epsilon_4\right)\\&=-A\eta\left(\epsilon_4\right)\phi\sigma\left(\epsilon_5,\epsilon_2\right),\end{aligned}\tag{34}$$

for the sake of brevity , we put $A = p_0 + \frac{p_1}{4}\left[c\left(n-5\right)+7n-1\right]+2p_2 r$.

$$\begin{aligned} &\left(\nabla_{\epsilon_4}\sigma\right)\left(\mathcal{B}\left(\xi,\epsilon_2\right)\epsilon_5,\xi\right) = \tilde{\nabla}^{\perp}_{\epsilon_4}\sigma\left(\mathcal{B}\left(\xi,\epsilon_2\right)\epsilon_5,\xi\right) \\ &-\sigma\left(\tilde{\nabla}_{\epsilon_4}\mathcal{B}\left(\xi,\epsilon_2\right)\epsilon_5,\xi\right)-\sigma\left(\mathcal{B}\left(\xi,\epsilon_2\right)\epsilon_5,\tilde{\nabla}_{\epsilon_4}\xi\right) \\ &= -\sigma\left(A\left[g\left(\epsilon_2,\epsilon_5\right)\xi-\eta\left(\epsilon_5\right)\epsilon_2\right],\phi\epsilon_4\right) \\ &= A\eta\left(\epsilon_5\right)\phi\sigma\left(\epsilon_2,\epsilon_4\right), \end{aligned} \tag{35}$$

$$\begin{aligned} &\left(\nabla_{\epsilon_4}\sigma\right)\left(\epsilon_5,\mathcal{B}\left(\xi,\epsilon_2\right)\xi\right) = \left(\nabla_{\epsilon_4}\sigma\right)\left(\epsilon_5,A\left[\eta\left(\epsilon_2\right)\xi-\epsilon_2\right]\right) \\ &= -A\left(\nabla_{\epsilon_4}\sigma\right)\left(\epsilon_5,\epsilon_2\right)+A\left(\nabla_{\epsilon_4}\sigma\right)\left(\epsilon_5,\eta\left(\epsilon_2\right)\xi\right) \\ &= A\left[-\left(\nabla_{\epsilon_4}\sigma\right)\left(\epsilon_5,\epsilon_2\right)-\eta\left(\epsilon_2\right)\phi\sigma\left(\epsilon_5,\epsilon_4\right)\right], \end{aligned} \tag{36}$$

$$\begin{aligned} &\left(\nabla_{\left(\xi\wedge_g\epsilon_2\right)\epsilon_4}\sigma\right)\left(\epsilon_5,\xi\right) = \tilde{\nabla}^{\perp}_{\left(\xi\wedge_g\epsilon_2\right)\epsilon_4}\sigma\left(\epsilon_5,\xi\right) \\ &-\sigma\left(\tilde{\nabla}_{\left(\xi\wedge_g\epsilon_2\right)\epsilon_4}\epsilon_5,\xi\right)-\sigma\left(\epsilon_5,\tilde{\nabla}_{\left(\xi\wedge_g\epsilon_2\right)\epsilon_4}\xi\right) \\ &= \eta\left(\epsilon_4\right)\phi\sigma\left(\epsilon_5,\epsilon_2\right), \end{aligned} \tag{37}$$

$$\begin{aligned} &\left(\nabla_{\epsilon_4}\sigma\right)\left(\left(\xi\wedge_g\epsilon_2\right)\epsilon_5,\xi\right) = \tilde{\nabla}^{\perp}_{\epsilon_4}\sigma\left(\left(\xi\wedge_g\epsilon_2\right)\epsilon_5,\xi\right) \\ &-\sigma\left(\tilde{\nabla}_{\epsilon_4}\left(\xi\wedge_g\epsilon_2\right)\epsilon_5,\xi\right)-\sigma\left(\left(\xi\wedge_g\epsilon_2\right)\epsilon_5,\tilde{\nabla}_{\epsilon_4}\xi\right) \\ &= -\sigma\left(g\left(\epsilon_2,\epsilon_5\right)\xi-g\left(\xi,\epsilon_5\right)\epsilon_2,\phi\epsilon_4\right) \\ &= \eta\left(\epsilon_5\right)\phi\sigma\left(\epsilon_2,\epsilon_4\right), \end{aligned} \tag{38}$$

$$\begin{aligned} &\left(\nabla_{\epsilon_4}\sigma\right)\left(\epsilon_5,\left(\xi\wedge_g\epsilon_2\right)\xi\right) = \left(\nabla_{\epsilon_4}\sigma\right)\left(\epsilon_5,\eta\left(\epsilon_2\right)\xi-\epsilon_2\right) \\ &= \left(\nabla_{\epsilon_4}\sigma\right)\left(\epsilon_5,\eta\left(\epsilon_2\right)\xi\right)-\left(\nabla_{\epsilon_4}\sigma\right)\left(\epsilon_5,\epsilon_2\right) \\ &= -\eta\left(\epsilon_2\right)\phi\sigma\left(\epsilon_5,\epsilon_4\right)-\left(\nabla_{\epsilon_4}\sigma\right)\left(\epsilon_5,\epsilon_2\right). \end{aligned} \tag{39}$$

Hence, we substitute (33) , (34) , (35) , (36) , (37) , (38) , (39) in (32) , we obtain

$$\begin{aligned} &-R^{\perp}\left(\xi,\epsilon_2\right)\phi\sigma\left(\epsilon_5,\epsilon_4\right)-A\eta\left(\epsilon_4\right)\phi\sigma\left(\epsilon_5,\epsilon_2\right) \\ &-A\eta\left(\epsilon_5\right)\phi\sigma\left(\epsilon_2,\epsilon_4\right)+A\eta\left(\epsilon_2\right)\phi\sigma\left(\epsilon_5,\epsilon_4\right) \\ &+A\left(\nabla_{\epsilon_4}\sigma\right)\left(\epsilon_5,\epsilon_2\right) = -\mathcal{F}_2\left\{\eta\left(\epsilon_5\right)\phi\sigma\left(\epsilon_4,\epsilon_2\right)\right. \\ &+\eta\left(\epsilon_4\right)\phi\sigma\left(\epsilon_2,\epsilon_5\right)-\eta\left(\epsilon_2\right)\phi\sigma\left(\epsilon_5,\epsilon_4\right) \\ &\left.-\left(\nabla_{\epsilon_4}\sigma\right)\left(\epsilon_5,\epsilon_2\right)\right\}. \end{aligned} \tag{40}$$

If we choose $\epsilon_5 = \xi$ in (40) and use,

$$(\nabla_{\epsilon_4}\sigma)(\xi,\epsilon_2) = -\phi\sigma(\epsilon_4,\epsilon_2),$$

we get

$$[\mathcal{F}_2 - A]\,\phi\sigma(\epsilon_2,\epsilon_4) = 0.$$

This completes proof of the theorem. □

Taking account of (14) and Theorem 2, the following corollaries can be given.

Corollary 2.10. *Let $\tilde{W}$ be an invariant submanifold of the $n-$dimensional $NPM-$manifold $W\left(\phi,\xi,\eta,g\right)$. If $\tilde{W}$ is a quasi-conformal $2P$ submanifold, then $\tilde{W}$ is either a total geodesic or*

$$\mathcal{F}_2 = a + \frac{b}{4}\left[c(n-5) + 7n - 1\right] - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right).$$

Corollary 2.11. *Let $\tilde{W}$ be an invariant submanifold of the $n-$dimensional $NPM-$manifold $W\left(\phi,\xi,\eta,g\right)$. If $\tilde{W}$ is a Weyl-conformal $2P$ submanifold, then $\tilde{W}$ is either a total geodesic or*

$$\mathcal{F}_2 = 1 - \frac{1}{4(n-1)}\left[c(n-5) + 7n - 1\right] - \frac{r}{(n-1)(n-2)}.$$

Corollary 2.12. *Let $\tilde{W}$ be an invariant submanifold of the $n-$dimensional $NPM-$manifold $W\left(\phi,\xi,\eta,g\right)$. If $\tilde{W}$ is a concircular $2P$ submanifold, then $\tilde{W}$ is either a total geodesic or*

$$\mathcal{F}_2 = 1 - \frac{2r}{n(n-1)}.$$

Corollary 2.13. *Let $\tilde{W}$ be an invariant submanifold of the $n-$dimensional $NPM-$manifold $W\left(\phi,\xi,\eta,g\right)$. If $\tilde{W}$ is a conharmonic $2P$ submanifold, then $\tilde{W}$ is either a total geodesic or*

$$\mathcal{F}_2 = 1 - \frac{1}{4(n-1)}\left[c(n-5) + 7n - 1\right].$$

Definition 2.14. *Let $\tilde{W}$ be an invariant submanifold of the $n-$dimensional $NPM-$manifold $W\left(\phi,\xi,\eta,g\right)$. If $\mathcal{B}\cdot\sigma$ and $Q(S,\sigma)$ are linearly dependent, $\tilde{W}$ is called generalized $\mathcal{B}-$ $RG-$pseudoparallel submanifold.*

In this case, there is a function $\mathcal{F}_3$ on the set $M_3 = \left\{\epsilon_1 \in \tilde{W} \middle| \sigma(\epsilon_1) \neq S(\epsilon_1)\right\}$ such that

$$\mathcal{B}\cdot\sigma = \mathcal{F}_3 Q(S,\sigma).$$

If specifically $\mathcal{F}_3 = 0$, $\tilde{W}$ is called a $\mathcal{B}-$Ricci generalized semiparallel submanifold.

Theorem 2.15. *Let $\tilde{W}$ be an invariant submanifold of the $n-$dimensional $NPM-$manifold $W\left(\phi,\xi,\eta,g\right)$. If $\tilde{W}$ is $\mathcal{B}-RG-$pseudoparallel submanifold, then $\tilde{W}$ is either a total geodesic or*

$$\mathcal{F}_3 = -\frac{p_0}{(n-1)} - \frac{p_1}{4(n-1)}\left[c(n-5) + 7n - 1\right] - \frac{2p_2 r}{(n-1)}.$$

Proof. Let's assume that $\tilde{W}$ is a generalized $\mathcal{B}-RG-$pseudoparallel submanifold. So, we can write

$$(\mathcal{B}(\epsilon_1,\epsilon_2)\cdot\sigma)(\epsilon_4,\epsilon_5)=\mathcal{F}_3 Q(S,\sigma)(\epsilon_4,\epsilon_5;\epsilon_1,\epsilon_2),$$

that is,

$$\begin{aligned}&R^{\perp}(\epsilon_1,\epsilon_2)\sigma(\epsilon_4,\epsilon_5)-\sigma(\mathcal{B}_1(\epsilon_1,\epsilon_2)\epsilon_4,\epsilon_5)\\ &-\sigma(\epsilon_4,\mathcal{B}(\epsilon_1,\epsilon_2)\epsilon_5)=-\mathcal{F}_3\{\sigma((\epsilon_1\wedge_S\epsilon_2)\epsilon_4,\epsilon_5)\\ &+\sigma(\epsilon_4,(\epsilon_1\wedge_S\epsilon_2)\epsilon_5)\}.\end{aligned} \tag{41}$$

for all $\epsilon_1,\epsilon_2,\epsilon_4,\epsilon_5\in\Gamma\left(T\tilde{W}\right)$. If we choose $\epsilon_1=\epsilon_5=\xi$ in (41) and in view of (15), (17), (28), we get

$$[(n-1)\mathcal{F}_3+A]\sigma(\epsilon_4,\epsilon_2)=0,$$

where $A=\left[p_0+\frac{p_1}{4}[c(n-5)+7n-1]+2p_2r\right]$. This completes proof of the theorem. □

Corollary 2.16. *Let $\tilde{W}$ be an invariant submanifold of the $n-$dimensional $NPM-$manifold $W\left(\phi,\xi,\eta,g\right)$. If $\tilde{W}$ is quasi-conformal $RG-$pseudoparallel submanifold, then $\tilde{W}$ is either a total geodesic or*

$$\mathcal{F}_3=-\frac{a}{(n-1)}-\frac{b}{4(n-1)}[c(n-5)+7n-1]+\frac{1}{(n-1)}\left[\frac{a}{n-1}+2b\right].$$

Corollary 2.17. *Let $\tilde{W}$ be an invariant submanifold of the $n-$dimensional $NPM-$manifold $W\left(\phi,\xi,\eta,g\right)$. If $\tilde{W}$ is Weyl-conformal $RG-$pseudoparallel submanifold, then $\tilde{W}$ is either a total geodesic or*

$$\mathcal{F}_3=-\frac{1}{(n-1)}-\frac{b}{4(n-1)^2}[c(n-5)+7n-1]-\frac{r}{(n-1)^2(n-2)}.$$

Corollary 2.18. *Let $\tilde{W}$ be an invariant submanifold of the $n-$dimensional $NPM-$manifold $W\left(\phi,\xi,\eta,g\right)$. If $\tilde{W}$ is concircular $RG-$pseudoparallel submanifold, then $\tilde{W}$ is either a total geodesic or*

$$\mathcal{F}_3=-\frac{1}{(n-1)}+\frac{2r}{n(n-1)^2}.$$

Corollary 2.19. *Let $\tilde{W}$ be an invariant submanifold of the $n-$dimensional $NPM-$manifold $W\left(\phi,\xi,\eta,g\right)$. If $\tilde{W}$ is conharmonic $RG-$pseudoparallel submanifold, then $\tilde{W}$ is either a total geodesic or*

$$\mathcal{F}_3=-\frac{1}{(n-1)}+\frac{p_1}{4(n-1)}[c(n-5)+7n-1].$$

Definition 2.20. *Let $\tilde{W}$ be an invariant pseudoparallel submanifold of the $n-$dimensional $NPM-$manifold $W\left(\phi,\xi,\eta,g\right)$. If $\mathcal{B}\cdot\tilde{\nabla}\sigma$ and $Q(S,\tilde{\nabla}\sigma)$ are linearly dependent, $\tilde{W}$ is called $\mathcal{B}-2$ $RG-$pseudoparallel submanifold.*

Then, there is a function $\mathcal{F}_4$ on the set $M_4=\left\{\epsilon_1\in\tilde{W}\middle|\nabla\sigma(\epsilon_1)\neq S(\epsilon_1)\right\}$ such that

$$\mathcal{B}\cdot\nabla\sigma=\mathcal{F}_4 Q(S,\nabla\sigma).$$

If $\mathcal{F}_4=0$ specifically, W is called a $\mathcal{B}$ 2$-$Ricci generalized semiparallel submanifold.

Theorem 2.21. *Let $\tilde{W}$ be an invariant pseudoparallel submanifold of the $n-$dimensional $NPM-$manifold $W\left(\phi,\xi,\eta,g\right)$. If $\tilde{W}$ is a generalized $\mathcal{B}-2$ $RG-$pseudoparallel submanifold, then $\tilde{W}$ is either a total geodesic or*

$$\mathcal{F}_4=\frac{p_0}{(n-1)}+\frac{p_1}{4(n-1)}[c(n-5)+7n-1]+\frac{2p_2r}{(n-1)}.$$

Proof. Let's assume that $\tilde{W}$ is a generalized $\mathcal{B}\,2-RG-$pseudoparallel submanifold. So, we can write

$$\left(\mathcal{B}\left(\epsilon_1,\epsilon_2\right)\cdot\nabla\sigma\right)\left(\epsilon_4,\epsilon_5,\epsilon_3\right)=\mathcal{F}_4Q\left(S,\nabla\sigma\right)\left(\epsilon_4,\epsilon_5,\epsilon_3;\epsilon_1,\epsilon_2\right), \tag{42}$$

for all $\epsilon_1,\epsilon_2,\epsilon_4,\epsilon_5,\epsilon_3\in\Gamma\left(T\tilde{W}\right)$. If we choose $\epsilon_1=\epsilon_5=\xi$ in (42), we can write

$$\begin{aligned}&R^{\perp}\left(\xi,\epsilon_2\right)\left(\nabla_{\epsilon_4}\sigma\right)\left(\xi,\epsilon_3\right)-\left(\nabla_{\mathcal{B}(\xi,\epsilon_2)\epsilon_4}\sigma\right)\left(\xi,\epsilon_3\right)\\&-\left(\nabla_{\epsilon_4}\sigma\right)\left(\mathcal{B}\left(\xi,\epsilon_2\right)\xi,\epsilon_3\right)-\left(\nabla_{\epsilon_4}\sigma\right)\left(\xi,\mathcal{B}\left(\xi,\epsilon_2\right)\epsilon_3\right)\\&=-\mathcal{F}_4\left\{\left(\nabla_{(\xi\wedge_S\epsilon_2)\epsilon_4}\sigma\right)\left(\xi,\epsilon_3\right)+\left(\nabla_{\epsilon_4}\sigma\right)\left(\left(\xi\wedge_S\epsilon_2\right)\xi,\epsilon_3\right)\right.\\&\left.+\left(\tilde{\nabla}_{\epsilon_4}\sigma\right)\left(\xi,\left(\xi\wedge_S\epsilon_2\right)\epsilon_3\right)\right\}.\end{aligned} \tag{43}$$

Let's calculate all the expressions in (43). Firstly, we will calculate

$$\begin{aligned}&R^{\perp}\left(\xi,\epsilon_2\right)\left(\nabla_{\epsilon_4}\sigma\right)\left(\xi,\epsilon_3\right)=R^{\perp}\left(\xi,\epsilon_2\right)\left\{\tilde{\nabla}^{\perp}_{\epsilon_4}\sigma\left(\xi,\epsilon_3\right)\right.\\&\left.-\sigma\left(\tilde{\nabla}_{\epsilon_4}\epsilon_3,\xi\right)-\sigma\left(\epsilon_3,\tilde{\nabla}_{\epsilon_4}\xi\right)\right\}\\&=-R^{\perp}\left(\xi,\epsilon_2\right)\phi\sigma\left(\epsilon_3,\epsilon_4\right),\end{aligned} \tag{44}$$

$$\begin{aligned}&\left(\nabla_{\mathcal{B}(\xi,\epsilon_2)\epsilon_4}\sigma\right)\left(\xi,\epsilon_3\right)=\nabla^{\perp}_{\mathcal{B}(\xi,\epsilon_2)\epsilon_4}\sigma\left(\xi,\epsilon_3\right)\\&-\sigma\left(\tilde{\nabla}_{\mathcal{B}(\xi,\epsilon_2)\epsilon_4}\xi,\epsilon_3\right)-\sigma\left(\xi,\tilde{\nabla}_{\mathcal{B}(\xi,\epsilon_2)\epsilon_4}\epsilon_3\right)\\&=-A\eta\left(\epsilon_4\right)\phi\sigma\left(\epsilon_2,\epsilon_3\right),\end{aligned} \tag{45}$$

$$\begin{aligned}&\left(\nabla_{\epsilon_4}\sigma\right)\left(\mathcal{B}\left(\xi,\epsilon_2\right)\xi,\epsilon_3\right)=\left(\nabla_{\epsilon_4}\sigma\right)\left(A\left[\eta\left(\epsilon_2\right)\xi-\epsilon_2\right],\epsilon_3\right)\\&=-A\left[\left(\nabla_{\epsilon_4}\sigma\right)\left(\epsilon_2,\epsilon_3\right)+\eta\left(\epsilon_2\right)\phi\sigma\left(\epsilon_4,\epsilon_3\right)\right],\end{aligned} \tag{46}$$

$$\begin{aligned}&\left(\nabla_{\epsilon_4}\sigma\right)\left(\xi,\mathcal{B}\left(\xi,\epsilon_2\right)\epsilon_3\right)=\tilde{\nabla}^{\perp}_{\epsilon_4}\sigma\left(\xi,\mathcal{B}\left(\xi,\epsilon_2\right)\epsilon_3\right)\\&-\sigma\left(\tilde{\nabla}_{\epsilon_4}\xi,\mathcal{B}\left(\xi,\epsilon_2\right)\epsilon_3\right)-\sigma\left(\xi,\tilde{\nabla}_{\epsilon_4}\mathcal{B}\left(\xi,\epsilon_2\right)\epsilon_3\right)\\&=A\eta\left(\epsilon_3\right)\phi\sigma\left(\epsilon_4,\epsilon_2\right)\end{aligned} \tag{47}$$

$$\begin{aligned}&\left(\nabla_{(\xi\wedge_S\epsilon_2)\epsilon_4}\sigma\right)\left(\xi,\epsilon_3\right)=\tilde{\nabla}^{\perp}_{(\xi\wedge_S\epsilon_2)\epsilon_4}\sigma\left(\xi,\epsilon_3\right)\\&-\sigma\left(\tilde{\nabla}_{(\xi\wedge_S\epsilon_2)\epsilon_4}\xi,\epsilon_3\right)-\sigma\left(\xi,\tilde{\nabla}_{(\xi\wedge_S\epsilon_2)\epsilon_4}\epsilon_3\right)\\&=(n-1)\,\eta\left(\epsilon_4\right)\phi\sigma\left(\epsilon_2,\epsilon_3\right),\end{aligned} \tag{48}$$

$$\begin{aligned}
&(\nabla_{\epsilon_4}\sigma)((\xi\wedge_S \epsilon_2)\xi,\epsilon_3) = (\nabla_{\epsilon_4}\sigma)(S(\epsilon_2,\xi)\xi - S(\xi,\xi)\epsilon_2,\epsilon_3)\\
&= (\nabla_{\epsilon_4}\sigma)((n-1)\eta(\epsilon_2)\xi - (n-1)\epsilon_2,\epsilon_3)\\
&= (n-1)\left\{\tilde{\nabla}^{\perp}_{\epsilon_4}\sigma(\eta(\epsilon_2)\xi,\epsilon_3) - \sigma(\tilde{\nabla}_{\epsilon_4}\eta(\epsilon_2)\xi,\epsilon_3)\right.\\
&-\sigma(\eta(\epsilon_2)\xi,\tilde{\nabla}_{\epsilon_4}\epsilon_3) + 2n(\nabla_{\epsilon_4}\sigma)(\epsilon_2,\epsilon_3)\}\\
&= -(n-1)\left[(\nabla_{\epsilon_4}\sigma)(\epsilon_2,\epsilon_3) + \eta(\epsilon_2)\phi\sigma(\epsilon_4,\epsilon_3)\right],
\end{aligned} \tag{49}$$

$$\begin{aligned}
&(\nabla_{\epsilon_4}\sigma)(\xi,(\xi\wedge_S \epsilon_2)\epsilon_3) = (\nabla_{\epsilon_4}\sigma)(\xi,S(\epsilon_2,\epsilon_3)\xi - S(\xi,\epsilon_3)\epsilon_2)\\
&= (\nabla_{\epsilon_4}\sigma)(\xi,S(\epsilon_2,\epsilon_3)\xi) - (n-1)(\tilde{\nabla}_{\epsilon_4}\sigma)(\xi,\eta(\epsilon_3)\epsilon_2)\\
&= (n-1)\eta(\epsilon_3)\phi\sigma(\epsilon_4,\epsilon_2).
\end{aligned} \tag{50}$$

If we substitute (44), (45), (46), (47), (48), (49), (50) in (43), we obtain

$$\begin{aligned}
&-R^{\perp}(\xi,\epsilon_2)\phi\sigma(\epsilon_3,\epsilon_4) + A\eta(\epsilon_4)\phi\sigma(\epsilon_2,\epsilon_3)\\
&+A\eta(\epsilon_2)\phi\sigma(\epsilon_4,\epsilon_3) - A\eta(\epsilon_3)\phi\sigma(\epsilon_4,\epsilon_2)\\
&+A(\nabla_{\epsilon_4}\sigma)(\epsilon_2,\epsilon_3) = -\mathcal{F}_4\left\{(n-1)\eta(\epsilon_4)\phi\sigma(\epsilon_2,\epsilon_3)\right.\\
&-(n-1)\eta(\epsilon_2)\phi\sigma(\epsilon_4,\epsilon_3) + (n-1)\eta(\epsilon_3)\phi\sigma(\epsilon_4,\epsilon_2)\\
&-(n-1)(\nabla_{\epsilon_4}\sigma)(\epsilon_2,\epsilon_3)\}.
\end{aligned} \tag{51}$$

If we choose $\epsilon_3 = \xi$ in (51), and one can easily to see

$$(\nabla_{\epsilon_4}\sigma)(\epsilon_2,\xi) = -\phi\sigma(\epsilon_4,\epsilon_2).$$

Thus we have

$$[A - (n-1)\mathcal{F}_4]\,\sigma(\epsilon_4,\epsilon_2) = 0,$$

where $A = p_0 + \frac{p_1}{4}[c(n-5) + 7n - 1] + 2p_2 r$. This completes the proof of the theorem. □

Corollary 2.22. *Let $\tilde{W}$ be an invariant pseudoparallel submanifold of the $n-$dimensional $NPM-$manifold $W(\phi,\xi,\eta,g)$. If $\tilde{W}$ is a quasi-conformal 2 $RG-$pseudoparallel submanifold, then $\tilde{W}$ is either a total geodesic or*

$$\mathcal{F}_4 = \frac{a}{(n-1)} + \frac{b}{4(n-1)}[c(n-5) + 7n - 1] - \frac{r}{n(n-1)}\left(\frac{a}{n-1} + 2b\right).$$

Corollary 2.23. *Let $\tilde{W}$ be an invariant pseudoparallel submanifold of the $n-$dimensional $NPM-$manifold $W(\phi,\xi,\eta,g)$. If $\tilde{W}$ is a weyl-conformal 2 $RG-$pseudoparallel submanifold, then $\tilde{W}$ is either a total geodesic or*

$$\mathcal{F}_4 = \frac{1}{(n-1)} - \frac{1}{4(n-1)^2}[c(n-5) + 7n - 1] - \frac{r}{(n-1)^2(n-2)}.$$

Corollary 2.24. *Let $\tilde{W}$ be an invariant pseudoparallel submanifold of the $n-$dimensional $NPM-$manifold $W(\phi,\xi,\eta,g)$. If $\tilde{W}$ is a concircular 2 $RG-$pseudoparallel submanifold, then $\tilde{W}$ is either a total geodesic or*

$$\mathcal{F}_4 = \frac{1}{(n-1)} - \frac{2r}{n(n-1)^2}.$$

Corollary 2.25. *Let $\tilde{W}$ be an invariant pseudoparallel submanifold of the $n-$dimensional NPM$-$manifold $W\left(\phi,\xi,\eta,g\right)$. If $\tilde{W}$ is a conharmonic 2 RG$-$pseudoparallel submanifold, then $\tilde{W}$ is either a total geodesic or*

$$\mathcal{F}_4=\frac{1}{(n-1)}-\frac{1}{4\left(n-1\right)^2}\left[c\left(n-5\right)+7n-1\right].$$

Example 2.26. *Let us the 5-dimensional manifold*

$$W^5=\{(x_1,x_2,x_3,x_4,x_5):x_i\in R,\}$$

where (x_i) denote the cartesian coordinate in $\mathbb{R}^5$ for $1\leq i\leq 5$. Then the vector fields

$$e_1=\frac{\partial}{\partial x_1},e_2=\frac{\partial}{\partial x_2},e_3=2x_2\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_3},e_4=2x_3\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_4},e_5=-2x_4\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_4}$$

are linearly independent at each point of W^5. By g, we denote the semi-Riemannian metric tensor such that

$$g(e_i,e_j)=\begin{cases}1; & i=j=1,3,4\\ -1; & i=j=2,5\\ 0; & i\neq j\end{cases}$$

Let η be the 1-form defined by $\eta(X)=g(X,e_1)$ for all $X\in\Gamma(TW)$. Now, we definite the paracontact metric structure φ such that

$$\varphi e_1=0,\ \ \varphi e_2=-e_3,\ \ \varphi e_3=-e_2,\ \ \varphi e_4=-e_5,\ \ \varphi e_5=-e_4.$$

Then we can easily see that

$$\eta(e_1)=1,\ \ \varphi^2X=X-\eta(X)\xi,\ \ e_1=\xi$$

and

$$g(\varphi X,\varphi Y)=-g(X,Y)+\eta(X)\eta(Y)$$

for all $X,Y\in\Gamma(W)$. Thus $W^5(\varphi,\eta,\xi,g)$ defines an almost paracontact metric manifold. By ∇, we denote the Levi-Civita connection on W^5. Then by direct calculations, we have non-zero the components

$$[e_2,e_3]=2e_1,\ \ [e_3,e_4]=2e_1,\ \ [e_4,e_5]=-2e_1.$$

Let ∇ be the Levi-Civita connection on W^5. Using the properties of paracontact metric structure and Kozsul formulae, we can observe the non-zero components

$$\nabla e_2e_1=-e_3=\varphi e_2,\ \ \nabla e_3e_1=-e_2=\varphi e_3,\ \ \nabla e_4e_1=-e_5=\varphi e_4,\ \ \nabla e_5e_1=-e_4=\varphi e_5$$

Thus one can easily verified

$$\nabla_Xe_1=\varphi X,$$

for all $X\in\Gamma(TW)$ This tells us that $W^5(\varphi,\eta,\xi,g)$ is a normal paracontact metric manifold with paracontact metric structure (φ,η,ξ,g). By straightforward calculations, we can easily see that non-zero components of the Riemannian curvature tensor R,

$$R(e_i,e_1)e_1=-e_i,\ \ 2\leq i\leq 5.$$

This tell us that

$$R(X,Y)Z = g(X,Z)Y - g(Y,Z)X,$$

for all $X, Y, Z \in \Gamma(TW)$, *that is,* $W^5(\varphi,\eta,\xi,g)$ *is real space form with constant sectional curvature* 1.
Now, we define a submanifold of a normal paracontact metric manifold $W^5(\varphi,\eta,\xi,g)$ *by*

$$\begin{cases} E_1 = \frac{\partial}{\partial x_1}, \\ E_2 = 2x_3\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}, \\ E_3 = 2\left(x_3 - x_4\right)\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}. \end{cases}$$

One can easily to see that

$$\varphi E_1 = 0, \varphi E_2 = -E_3, \varphi E_3 = -E_2,$$

that is, distribution which is spanned by $\{E_1, E_2, E_3\}$ *is an invariant. By direct calculations, we observe*

$$\nabla_{E_2}E_3 = 0, \nabla_{E_2}E_1 = -E_3 \text{ and } \nabla_{E_3}E_1 = -E_2.$$

Thus, the distribution defined by $\{E_1, E_2, E_3\}$ *is integrallenebility and its inegral manifold invariant and totally geodesic submanifold of* $W^5(\varphi,\eta,\xi,g)$. *Consequently, Our example shows that our works are clearly valid.*

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