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Resistance distance and Kirchhoff index of the splitting-joins of two graphs

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Abstract. Let G be a graph. The splitting graph $SP(G)$ of G is the graph received from G by putting a new vertex w' for each $w \in V_G$ and joining w' to all vertices of G adjacent to w. Let S_G be the set of such new vertices of the splitting graph $SP(G)$. Let G_1 and G_2 be two simple connected graphs, the splitting V-vertex join graph is obtained by taking one copy of $SP(G_1)$ and joining each vertex in V_{G_1} to each vertex in V_{G_2}, denoted by $G_1 \veebar G_2$. The splitting S-vertex join of G_1 and G_2, denoted by $G_1 \barwedge G_2$, is a graph obtained from $SP(G_1)$ and G_2 by joining each vertex in S_{G_1} to each vertex in V_{G_2}. In this paper, we calculate the resistance distance and Kirchhoff index of $G_1 \veebar G_2$ and $G_1 \barwedge G_2$ for regular graphs G_1 and G_2, respectively.

1. Introduction

We deal with finite, simple and undirected graphs, and follow [3] for undefined terms and notations. Let $G = (V_G, E_G)$ be a graph with vertex set $V_G = \{v_1, v_2, \ldots, v_n\}$ and edge set E_G, where $n = |V_G|$ is the *order* of G. The *adjacency matrix* of G, denoted by A_G, is the $n \times n$ matrix whose (i, j)-entry is 1 if v_i and v_j are adjacent in G and 0 otherwise. The degree of v_i in G is denoted by $d_i = d_G(v_i)$. The *Laplacian matrix* of G is the matrix $L_G = D_G - A_G$, where D_G is the diagonal matrix with diagonal entries $d_1, d_2, \ldots, d_n$.

For a square matrix M of order n, the characteristic polynomial $\det(tI_n - M)$ of M is denoted by $f_M(t)$, where I_n is the identity matrix with order n. Particularly, for a graph G, $f_{A_G}(t)$ and $f_{L_G}(t)$ are the adjacency and Laplacian *characteristic polynomial* of G, respectively. And their roots are the adjacency and Laplacian eigenvalues of G, separately. The collection of eigenvalues of A_G and L_G together with their multiplicities referred to the *A-spectrum* and *L-spectrum* of G, respectively. Denote the A-spectrum (respectively, L-spectrum) as $Spec_A(G) = \{\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)\}$ (respectively, $Spec_L(G) = \{\mu_1(G), \mu_2(G), \ldots, \mu_n(G)\}$). Note that if G is r-regular graph, then each eigenvalue μ_i of L_G corresponds to an eigenvalue λ_i of A_G via the relation $\mu_i(G) = r - \lambda_i(G)$.

In 1993, Klein and Randić [8] presented the resistance distance between vertices v_i and v_j in graph G, denoted by $r_{ij}(G)$, defined as the effective resistance between v_i and v_j calculated according to Ohm's law when the unit resistance is distributed on each edge of G. The resistance distance of graph is equal to the

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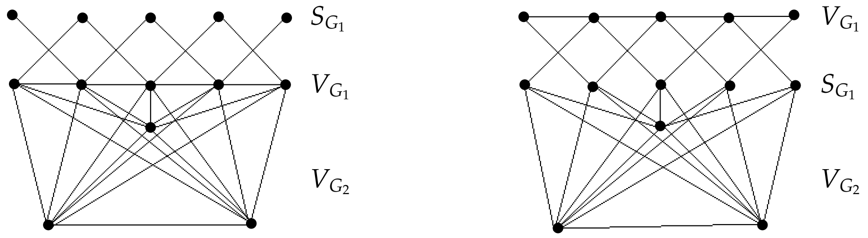
equivalent resistance of electrical network, which is a new metric of graph and has a broad development prospect in chemistry, network analysis, physics and other fields. The Kirchhoff index $Kf(G)$ of G is the sum of the resistance distances between all pairs of vertices of G, i.e., $Kf(G) = \sum_{i<j} r_{ij}$.

The *splitting graph* $SP(G)$ of a graph G is the graph obtained from G by taking a new vertex w' for each $w \in V_G$ and joining w' to all vertices of G adjacent to w. Let S_G be the set of such new vertices of the splitting graph $SP(G)$, i.e., $S_G = V_{SP(G)} \backslash V_G$. Lu et al. [12] introduced two types of graph operations based on the splitting graph as follows.

Definition 1.1. *[12] Let G_i be an n_i-vertex connected graph for $i = 1, 2$. The splitting V-vertex join of G_1 and G_2 is obtained by taking one copy of $SP(G_1)$ and joining each vertex in V_{G_1} to each vertex in V_{G_2}, denoted by $G_1 \veebar G_2$. The splitting S-vertex join of G_1 and G_2 is a graph obtained from $SP(G_1)$ and G_2 by joining each vertex in S_{G_1} to each vertex in V_{G_2}, denoted by $G_1 \barwedge G_2$.*

Let P_n be a path of order n and K_n be complete graph of order n. Figure 1 depicts the splitting V-vertex join and the splitting S-vertex join of P_5 and K_3.



$P_5 \veebar K_3$ $P_5 \barwedge K_3$

Figure 1: The splitting V-vertex join of $P_5 \veebar K_3$ and the splitting S-vertex join of $P_5 \barwedge K_3$.

It is well known that the eigenvalues and eigenvectors of the Laplacian matrix are used to represent the resistance distance of the graph [11]. But this method only works for certain graph classes. According to the components of the generalized inverse of the Laplacian matrix, Babapt [1] introduced the formula for expressing resistance distance and Kirchhoff index. Subsequently, reseachers [6, 7, 9, 16] considered the problems of resistance distance and Kirchhoff index of many graph classes and graph operations, such as the Q-vertex and Q-edge join graphs[13], R-vertex and R-edge join graphs[10], the subdivision-vertex and subdivision-edge join graphs [5], the Q-double join graphs[15] and so on.

Motivated by the above works, in this paper, we utilize the group inverse of matrix to calculate the resistance distances and Kirchhoff indices of the splitting V-vertex join $G_1 \veebar G_2$ and the splitting S-vertex join $G_1 \barwedge G_2$ for regular graphs G_1 and G_2, respectively.

2. Preliminaries

Firstly, we give some definitions and lemmas which are very useful in the proof of the main results.

Let Q be a square matrix. The {1}-*inverse* of Q is a matrix, denoted by $Q^{(1)}$, such that $QQ^{(1)}Q = Q$. Particularly, if Q is singular, then Q has infinitely many 1-inverses [2]. The *group inverse* of Q is the unique matrix, denoted by $Q^{\#}$, satisfying $QQ^{\#}Q = Q$, $Q^{\#}QQ^{\#} = Q^{\#}$, and $QQ^{\#} = Q^{\#}Q$. Ben-Israel et al. [2] and Bu et al. [4], independently, proved that $Q^{\#}$ exists if and only if $rank(Q) = rank(Q^2)$. Specifically, if Q is real symmetric matrix, then $Q^{\#}$ exists and $Q^{\#}$ is a symmetric {1}-inverse of Q.

Let Q_{ij} denote the entry of Q in the i-th row and j-th column and $\mathbf{e}$ be a column vector whose entries are all ones. Let I_n be the identity matrix of size n, and $J_{n\times m}$ denote the $n \times m$ matrix whose all entries are 1.

Let G be a graph. Here we state some lemmas, which indicated that the {1}-inverse and group inverse of L_G can expresses the resistance distance and Kirchhoff index of a graph G. These results play a vital role in demonstrating the main conclusions of this paper.

Lemma 2.1. *[1, 4] Suppose G is a connected graph. If vertices v_i and v_j in V_G, then the resistance distance $r_{ij}(G)$ between them is given as follows:*

$$r_{ij}(G) = (L_G^{(1)})_{ii} + (L_G^{(1)})_{jj} - (L_G^{(1)})_{ij} - (L_G^{(1)})_{ji}$$
$$= (L_G^{\#})_{ii} + (L_G^{\#})_{jj} - 2(L_G^{\#})_{ij}.$$

Lemma 2.2. *[14] Let G be a connected graph on n vertices. Then*

$$Kf(G) = ntr(L_G^{(1)}) - \boldsymbol{e}^T L_G^{(1)} \boldsymbol{e},$$

where $tr(L_G^{(1)})$ is the trace of $L_G^{(1)}$.

Definition 2.3. *[17] For a $n \times n$ matrix A, which can be partitioned as*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square matrices. If A_{11} and A_{22} are nonsingular, then the matrix $A_{22} - A_{21}A_{11}^{-1}A_{12}$ and $A_{11} - A_{12}A_{22}^{-1}A_{21}$ are called the Schur complements of A_{11} and A_{22}, respectively.

Lemma 2.4. *[17] Suppose $W = \begin{pmatrix} S & T \\ P & Q \end{pmatrix}$ is a nonsingular matrix. Let S be nonsingular matrix. Then*

$$W^{-1} = \begin{pmatrix} S^{-1} + S^{-1}TF^{-1}PS^{-1} & -S^{-1}TF^{-1} \\ -F^{-1}PS^{-1} & F^{-1} \end{pmatrix}.$$

where $F = Q - PS^{-1}T$ is the Schur complement of S.

Lemma 2.5. *[5] Let $L_G = \begin{pmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{pmatrix}$ be the Laplacian matrix of a connected graph G. If each column vector of L_2^T is $-\boldsymbol{e}$ or a zero vector, then $H = \begin{pmatrix} L_1^{-1} & 0 \\ 0 & F^{\#} \end{pmatrix}$ is a symmetric $\{1\}$-inverse of L_G, where $F = L_3 - L_2^T L_1^{-1} L_2$ is a Schur complement of L_1.*

Lemma 2.6. *[5] Suppose G is a graph of order n. Then*

$$(L_G + aI_n - \frac{a}{n}J_{n\times n})^{\#} = (L_G + aI_n)^{-1} - \frac{1}{an}J_{n\times n},$$

where a is any positive real number.

Lemma 2.7. *[5] Let Q be a real symmetric matrix. If $Q\boldsymbol{e} = 0$, then we have $Q^{\#}\boldsymbol{e} = 0$ and $\boldsymbol{e}^T Q^{\#} = 0$.*

3. Resistance distance and Kirchhoff index of splitting *V*-vertex join graphs

Now, we calculate the resistance distance and Kirchhoff index of the splitting V-vertex join graph $G_1 \veebar G_2$.

Theorem 3.1. *For $i = 1, 2$, let G_i be an r_i-regular graph of n_i vertices. Assume that $w_k(v_i, v_j) = \left[(A_{G_1} + \frac{1}{r_1}A_{G_1}^2)^k\right]_{ij}$ and $N_{G_1}(v_i) = \{v_j \in V_{G_1} \mid v_iv_j \in E_{G_1 \veebar G_2}\}$. Then we have the following conclusions:*

① *For any $v_i, v_j \in V_{G_1}$, we get*

$$r_{ij}(G_1 \veebar G_2) = \frac{1}{n_2 + 2r_1} \sum_{k=0}^{\infty} \frac{1}{(n_2 + 2r_1)^k} \Big(w_k(v_i, v_i) + w_k(v_j, v_j) - 2w_k(v_i, v_j) \Big);$$

② *For any $v_i, v_j \in V_{G_2}$, we have*

$$r_{ij}(G_1 \veebar G_2) = \Big[(L_{G_2} + n_1 I_{n_2})^{-1}\Big]_{ii} + \Big[(L_{G_2} + n_1 I_{n_2})^{-1}\Big]_{jj} - 2\Big[(L_{G_2} + n_1 I_{n_2})^{-1}\Big]_{ij};$$

③ *For any $v'_i, v'_j \in S_{G_1}$, we know*

$$r_{ij}(G_1 \veebar G_2) = \frac{2}{r_1} + \frac{1}{r_1^2(n_2+2r_1)}\Big(\sum_{\substack{v_s \in N_{G_1}(v'_i)\\ v_t \in N_{G_1}(v'_i)}} \sum_{k=0}^{\infty} \frac{1}{(n_2+2r_1)^k} w_k(v_s, v_t) + \sum_{\substack{v_s \in N_{G_1}(v'_j)\\ v_t \in N_{G_1}(v'_j)}} \sum_{k=0}^{\infty} \frac{1}{(n_2+2r_1)^k} w_k(v_s, v_t) - 2 \sum_{\substack{v_s \in N_{G_1}(v'_i)\\ v_t \in N_{G_1}(v'_j)}} \sum_{k=0}^{\infty} \frac{1}{(n_2+2r_1)^k} w_k(v_s, v_t)\Big);$$

④ *For $v_i \in V_{G_1}, v_j \in V_{G_2}$, we see*

$$r_{ij}(G_1 \veebar G_2) = \frac{1}{n_2+2r_1} \sum_{k=0}^{\infty} \Big(\frac{1}{(n_2+2r_1)^k} w_k(v_i, v_i)\Big) + [(L_{G_2} + n_1 I_{n_2})^{-1}]_{jj} - \frac{1}{n_1 n_2};$$

⑤ *For $v'_i \in S_{G_1}, v_j \in V_{G_2}$, we obtain*

$$r_{ij}(G_1 \veebar G_2) = \frac{1}{r_1} + \frac{1}{r_1^2(n_2+2r_1)} \sum_{\substack{v_s \in N_{G_1}(v'_i)\\ v_t \in N_{G_1}(v'_i)}} \sum_{k=0}^{\infty} \Big(\frac{1}{(n_2+2r_1)^k} w_k(v_s, v_t)\Big) + [(L_{G_2} + n_1 I_{n_2})^{-1}]_{jj} - \frac{1}{n_1 n_2};$$

⑥ *For $v'_i \in S_{G_1}, v_j \in V_{G_1}$, we get*

$$r_{ij}(G_1 \veebar G_2) = \frac{1}{r_1^2(n_2+2r_1)} \sum_{\substack{v_s \in N_{G_1}(v'_i)\\ v_t \in N_{G_1}(v'_i)}} \sum_{k=0}^{\infty} \Big(\frac{1}{(n_2+2r_1)^k} w_k(v_s, v_t)\Big) + \frac{1}{r_1} + \frac{1}{n_2+2r_1} \sum_{k=0}^{\infty} \Big(\frac{1}{(n_2+2r_1)^k} w_k(v_j, v_j)\Big) - \frac{2}{r_1(n_2+2r_1)} \sum_{v_s \in N_{G_1}(v'_i)} \sum_{k=0}^{\infty} \Big(\frac{1}{(n_2+2r_1)^k} w_k(v_s, v_j)\Big).$$

Proof. We mark the vertices of $G_1 \veebar G_2$ as shown in Figure 1, then the Laplacian matrix of $G_1 \veebar G_2$ can be expressed as

$$L(G_1 \veebar G_2) = \begin{array}{c} \\ S_{G_1} \\ V_{G_1} \\ V_{G_2} \end{array} \begin{array}{c} \begin{array}{ccc} S_{G_1} & V_{G_1} & V_{G_2} \end{array} \\ \left(\begin{array}{cc|c} r_1 I_{n_1} & -A_{G_1} & O_{n_1 \times n_2} \\ -A_{G_1}^T & (r_1+n_2) I_{n_1} + L_{G_1} & -J_{n_1 \times n_2} \\ \hline O_{n_2 \times n_1} & -J_{n_2 \times n_1} & n_1 I_{n_2} + L_{G_2} \end{array}\right) \end{array}$$

$$= \left(\begin{array}{c|c} \mathrm{M} & \begin{pmatrix} O_{n_1 \times n_2} \\ -J_{n_1 \times n_2} \end{pmatrix} \\ \hline \begin{pmatrix} O_{n_2 \times n_1} & -J_{n_2 \times n_1} \end{pmatrix} & n_1 I_{n_2} + L_{G_2} \end{array}\right),$$

where $O_{a \times b}$ is the $a \times b$ matrix of all entries equal to zero and $M = \begin{pmatrix} r_1 I_{n_1} & -A_{G_1} \\ -A_{G_1}^T & (r_1+n_2) I_{n_1} + L_{G_1} \end{pmatrix}$.

By Definition 2.3, we know that the Schur complement of $r_1 I_{n_1}$ in M is

$$\begin{aligned} S_M &= (r_1+n_2)I_{n_1} + L_{G_1} - A_{G_1}^T (r_1 I_{n_1})^{-1} A_{G_1} \\ &= (r_1+n_2)I_{n_1} + L_{G_1} - \frac{1}{r_1} A_{G_1}^T A_{G_1} \\ &= (2r_1+n_2)I_{n_1} - A_{G_1} - \frac{1}{r_1} A_{G_1}^T A_{G_1}. \end{aligned} \tag{1}$$

By Lemma 2.4, we have $M^{-1} = \begin{pmatrix} N_1 & N_2 \\ N_3 & S_M^{-1} \end{pmatrix}$, where

$$N_1 = \frac{1}{r_1} I_{n_1} + \frac{1}{r_1^2} A_{G_1} S_M^{-1} A_{G_1}^T, \tag{2}$$

$$N_2 = \frac{1}{r_1} A_{G_1} S_M^{-1}, \tag{3}$$

$$N_3 = \frac{1}{r_1} S_M^{-1} A_{G_1}^T. \tag{4}$$

Let F be the Schur complement of M in $L(G_1 \veebar G_2)$. Then by Definition 2.3, we have

$$\begin{aligned} F =& n_1 I_{n_2} + L_{G_2} - \begin{pmatrix} O_{n_2\times n_1} & -J_{n_2\times n_1} \end{pmatrix} M^{-1} \begin{pmatrix} O_{n_1\times n_2} \\ -J_{n_1\times n_2} \end{pmatrix} \\ =& n_1 I_{n_2} + L_{G_2} - J_{n_2\times n_1} {S_M}^{-1} J_{n_1\times n_2}. \end{aligned} \tag{5}$$

Since

$$\begin{aligned} n_1 J_{n_2\times n_2} &= J_{n_2\times n_1} S_M {S_M}^{-1} J_{n_1\times n_2} \\ &= J_{n_2\times n_1}\Big[(r_1+n_2)I_{n_1} + L_{G_1} - \frac{1}{r_1} A_{G_1}^T A_{G_1}\Big] {S_M}^{-1} J_{n_1\times n_2} \\ &= (r_1+n_2) J_{n_2\times n_1} {S_M}^{-1} J_{n_1\times n_2} - \frac{1}{r_1} J_{n_2\times n_1} A_{G_1}^T A_{G_1} {S_M}^{-1} J_{n_1\times n_2} \\ &= (r_1+n_2) J_{n_2\times n_1} {S_M}^{-1} J_{n_1\times n_2} - r_1 J_{n_2\times n_1} {S_M}^{-1} J_{n_1\times n_2} \\ &= n_2 J_{n_2\times n_1} {S_M}^{-1} J_{n_1\times n_2}, \end{aligned}$$

we get

$$J_{n_2\times n_1} {S_M}^{-1} J_{n_1\times n_2} = \frac{n_1}{n_2} J_{n_2\times n_2}.$$

Hence, from (5), we know

$$F = L_{G_2} + n_1 I_{n_2} - \frac{n_1}{n_2} J_{n_2\times n_2}.$$

From Lemma 2.6, we derive that

$$F^{\#} = (L_{G_2} + n_1 I_{n_2})^{-1} - \frac{1}{n_1 n_2} J_{n_2\times n_2}. \tag{6}$$

Therefore, according to Lemma 2.5, we get the expression of $L^{(1)}_{G_1 \veebar G_2}$ as follows

$$L^{(1)}_{G_1 \veebar G_2} \left(\begin{array}{cc:c} N_1 & N_2 & 0 \\ N_3 & S_M^{-1} & 0 \\ \hdashline 0 & 0 & F^{\#} \end{array}\right). \tag{7}$$

① For $v_i, v_j \in V_{G_1}$, combining Lemma 2.1 with (7), we have

$$r_{ij}(G_1 \veebar G_2) = (S_M^{-1})_{ii} + (S_M^{-1})_{jj} - 2(S_M^{-1})_{ij}. \tag{8}$$

In view of (1), we get

$$S_M = (n_2 + 2r_1)\Big[I_{n_1} - \frac{1}{n_2 + 2r_1}(A_{G_1} + \frac{1}{r_1}A_{G_1}^T A_{G_1})\Big].$$

The spectral radius of $\frac{1}{n_2+2r_1}(A_{G_1} + \frac{1}{r_1}A_{G_1}^T A_{G_1})$ is

$$\rho(\frac{1}{n_2 + 2r_1}(A_{G_1} + \frac{1}{r_1}A_{G_1}^T A_{G_1})) = \frac{r_1 + \frac{r_1^2}{r_1}}{n_2 + 2r_1} = \frac{2r_1}{n_2 + 2r_1} < 1,$$

which implies that the power series of $\Big[I_{n_1} - \frac{1}{n_2+2r_1}(A_{G_1} + \frac{1}{r_1}A_{G_1}^T A_{G_1})\Big]^{-1}$ is convergent. Thus, we obtain

$$S_M^{-1} = \frac{1}{n_2 + 2r_1}\sum_{k=0}^{\infty}\Big[\frac{1}{(n_2 + 2r_1)^k}(A_{G_1} + \frac{1}{r_1}A_{G_1}^2)^k\Big]. \tag{9}$$

Suppose that $w_k(v_i, v_j) = \Big[(A_{G_1} + \frac{1}{r_1}A_{G_1}^2)^k\Big]_{ij}$. Then due to (8) and (9), we have

$$r_{ij}(G_1 \veebar G_2) = \frac{1}{n_2 + 2r_1}\sum_{k=0}^{\infty}\frac{1}{(n_2 + 2r_1)^k}\Big(w_k(v_i, v_i) + w_k(v_j, v_j) - 2w_k(v_i, v_j)\Big).$$

② For $v_i, v_j \in V_{G_2}$, by Lemma 2.1 and (7), we have

$$r_{ij}(G_1 \veebar G_2) = (F^{\#})_{ii} + (F^{\#})_{jj} - 2(F^{\#})_{ij}.$$

Based on (6), we obtain

$$r_{ij}(G_1 \veebar G_2) = \Big[(L_{G_2} + n_1 I_{n_2})^{-1}\Big]_{ii} + \Big[(L_{G_2} + n_1 I_{n_2})^{-1}\Big]_{jj} - 2\Big[(L_{G_2} + n_1 I_{n_2})^{-1}\Big]_{ij}.$$

③ For $v_i', v_j' \in S_{G_1}$, according to Lemma 2.1 and (7), we have

$$r_{ij}(G_1 \veebar G_2) = (N_1)_{ii} + (N_1)_{jj} - 2(N_1)_{ij}. \tag{10}$$

Recall that $N_{G_1}(v_i) = \{v_j \in V_{G_1} \mid v_i v_j \in E_{G_1 \veebar G_2}\}$. According to (2) and (9), we can get

$$\begin{aligned}
(N_1)_{ii} &= \frac{1}{r_1} + \frac{1}{r_1^2}(A_{G_1} S_M^{-1} A_{G_1}^T)_{ii} \\
&= \frac{1}{r_1} + \frac{1}{r_1^2}\left(\left(\sum_{v_s \in N_{G_1}(v_i')}(S_M^{-1})_{s1}, \sum_{v_s \in N_{G_1}(v_i')}(S_M^{-1})_{s2}, \ldots, \sum_{v_s \in N_{G_1}(v_i')}(S_M^{-1})_{sn_1}\right)A_{G_1}^T\right)_i \\
&= \frac{1}{r_1} + \frac{1}{r_1^2}\sum_{\substack{v_s \in N_{G_1}(v_i') \\ v_t \in N_{G_1}(v_i')}}(S_M^{-1})_{st} \\
&= \frac{1}{r_1} + \frac{1}{r_1^2(n_2 + 2r_1)}\sum_{\substack{v_s \in N_{G_1}(v_i') \\ v_t \in N_{G_1}(v_i')}}\sum_{k=0}^{\infty}\Big[\frac{1}{(n_2 + 2r_1)^k}(A_{G_1} + \frac{1}{r_1}A_{G_1}^2)^k\Big]_{st}.
\end{aligned} \tag{11}$$

By using a similar analysis as above, we can deduce that

$$(N_1)_{ij} = \frac{1}{r_1^2(n_2+2r_1)} \sum_{\substack{v_s\in N_{G_1}(v_i') \\ v_t\in N_{G_1}(v_j')}} \sum_{k=0}^{\infty} \Big[\frac{1}{(n_2+2r_1)^k}(A_{G_1}+\frac{1}{r_1}A_{G_1}^2)^k\Big]_{st}. \quad (12)$$

Let $w_k(v_s,v_t) = \Big[(A_{G_1}+\frac{1}{r_1}A_{G_1}^2)^k\Big]_{st}$. Then substituting (11) and (12) into (10), we obtain

$$\begin{aligned} r_{ij}(G_1 \veebar G_2) =& \frac{2}{r_1} + \frac{1}{r_1^2(n_2+2r_1)}\Big(\sum_{\substack{v_s\in N_{G_1}(v_i') \\ v_t\in N_{G_1}(v_i')}} \sum_{k=0}^{\infty} \frac{1}{(n_2+2r_1)^k} w_k(v_s,v_t) \\ &+ \sum_{\substack{v_s\in N_{G_1}(v_j') \\ v_t\in N_{G_1}(v_j')}} \sum_{k=0}^{\infty} \frac{1}{(n_2+2r_1)^k} w_k(v_s,v_t) - 2 \sum_{\substack{v_s\in N_{G_1}(v_i') \\ v_t\in N_{G_1}(v_j')}} \sum_{k=0}^{\infty} \frac{1}{(n_2+2r_1)^k} w_k(v_s,v_t)\Big). \end{aligned}$$

④ For $v_i \in V_{G_1}, v_j \in V_{G_2}$, by Lemma 2.1 and (7), we have

$$r_{ij}(G_1 \veebar G_2) = (S_M^{-1})_{ii} + (F^{\#})_{jj}.$$

Combining (9) with (6), we receive

$$r_{ij}(G_1 \veebar G_2) = \frac{1}{n_2+2r_1} \sum_{k=0}^{\infty} \Big(\frac{1}{(n_2+2r_1)^k} w_k(v_i,v_i)\Big) + [(L_{G_2}+n_1 I_{n_2})^{-1}]_{jj} - \frac{1}{n_1 n_2}.$$

⑤ For $v_i' \in S_{G_1}, v_j \in V_{G_2}$, together Lemma 2.1 with (7), we have

$$r_{ij}(G_1 \veebar G_2) = (N_1)_{ii} + (F^{\#})_{jj}.$$

Due to (11) and (6), we have

$$r_{ij}(G_1 \veebar G_2) = \frac{1}{r_1} + \frac{1}{r_1^2(n_2+2r_1)} \sum_{\substack{v_s\in N_{G_1}(v_i') \\ v_t\in N_{G_1}(v_i')}} \sum_{k=0}^{\infty} \Big(\frac{1}{(n_2+2r_1)^k} w_k(v_s,v_t)\Big) + [(L_{G_2}+n_1 I_{n_2})^{-1}]_{jj} - \frac{1}{n_1 n_2}.$$

⑥ For $v_i' \in S_{G_1}, v_j \in V_{G_1}$, by using Lemma 2.1 and

$$r_{ij}(G_1 \veebar G_2) = (N_1)_{ii} + (S_M^{-1})_{jj} - 2(N_2)_{ij}. \quad (13)$$

From (3), we know $N_2 = \frac{1}{r_1} A_{G_1} S_M^{-1}$. Furthermore, using (9), $(N_2)_{ij}$ can be expressed as

$$\begin{aligned} (N_2)_{ij} &= \frac{1}{r_1}(A_{G_1} S_M^{-1})_{ij} \\ &= \frac{1}{r_1}\left(\sum_{v_s\in N_{G_1}(v_i')} (S_M^{-1})_{s1}, \sum_{v_s\in N_{G_1}(v_i')} (S_M^{-1})_{s2}, \ldots, \sum_{v_s\in N_{G_1}(v_i')} (S_M^{-1})_{sn_1} \right)_j \\ &= \frac{1}{r_1} \sum_{v_s\in N_{G_1}(v_i')} (S_M^{-1})_{sj} \\ &= \frac{1}{r_1(n_2+2r_1)} \sum_{v_s\in N_{G_1}(v_i')} \sum_{k=0}^{\infty} \Big(\frac{1}{(n_2+2r_1)^k} w_k(v_s,v_j)\Big). \end{aligned} \quad (14)$$

Hence, plugging (9), (11) and (14) into (13), we get

$$r_{ij}(G_1 \veebar G_2) = \frac{1}{r_1^2(n_2+2r_1)} \sum_{\substack{v_s \in N_{G_1}(v_i') \\ v_t \in N_{G_1}(v_i')}} \sum_{k=0}^{\infty} \Big(\frac{1}{(n_2+2r_1)^k} w_k(v_s,v_t)\Big) + \frac{1}{r_1} + \frac{1}{n_2+2r_1} \sum_{k=0}^{\infty} \Big(\frac{1}{(n_2+2r_1)^k} w_k(v_j,v_j)\Big)$$
$$- \frac{2}{r_1(n_2+2r_1)} \sum_{v_s \in N_{G_1}(v_i')} \sum_{k=0}^{\infty} \Big(\frac{1}{(n_2+2r_1)^k} w_k(v_s,v_j)\Big).$$

□

Theorem 3.2. *Suppose G_i is an r_i-regular graph of n_i vertices. If $\lambda_1(G_i), \lambda_2(G_i), \ldots, \lambda_n(G_i)$ are the eigenvalues of A_{G_i} for $i = 1,2$, then*

$$Kf(G_1 \veebar G_2) = (2n_1+n_2)\Big[\frac{1}{r_1} \sum_{i=1}^{n_1} \frac{\lambda_i^2(G_1)+r_1^2}{r_1(n_2+2r_1-\lambda_i(G_1)) - \lambda_i^2(G_1)} + \sum_{i=1}^{n_2} \frac{1}{(r_2+n_1)-\lambda_i(G_2)}\Big]$$
$$+ \frac{2n_1^2+n_1n_2-n_1}{r_1} - \frac{4n_1^2+2n_1n_2+n_2^2}{n_1n_2}.$$

Proof. By Lemma 2.2, we have

$$Kf(G_1 \veebar G_2) = (2n_1+n_2)tr(L^{(1)}_{G_1 \veebar G_2}) - \mathbf{e}^T L^{(1)}_{G_1 \veebar G_2} \mathbf{e}.$$

Since $L^{(1)}_{G_1 \veebar G_2}$ can be shown from the proof of Theorem 3.1 as in (7), we have

$$tr(L^{(1)}_{G_1 \veebar G_2}) = tr(N_1) + tr(S_M^{-1}) + tr(F^{\#}).$$

From (1), we obtain

$$tr(S_M) = \sum_{i=1}^{n_1} \Big[(2r_1+n_2) - \lambda_i(G_1) - \frac{1}{r_1}\lambda_i^2(G_1)\Big],$$

and so

$$tr(S_M^{-1}) = \sum_{i=1}^{n_1} \frac{1}{(2r_1+n_2) - \lambda_i(G_1) - \frac{1}{r_1}\lambda_i^2(G_1)}.$$

Recall that $N_1 = \frac{1}{r_1} I_{n_1} + \frac{1}{r_1^2} A_{G_1} S_M^{-1} A_{G_1}^T$ from (2). Then we get

$$tr(N_1) = \frac{1}{r_1} tr(I_{n_1}) + \frac{1}{r_1^2} tr(A_{G_1} S_M^{-1} A_{G_1}^T)$$
$$= \frac{n_1}{r_1} + \frac{1}{r_1^2} \sum_{i=1}^{n_1} \frac{\lambda_i^2(G_1)}{(2r_1+n_2) - \lambda_i(G_1) - \frac{1}{r_1}\lambda_i^2(G_1)}.$$

On the other hand, by (6), we gain

$$tr(F^{\#}) = tr\Big[(L_{G_2} + n_1 I_{n_2})^{-1}\Big] - tr\Big(\frac{1}{n_1n_2} J_{n_2 \times n_2}\Big)$$
$$= \sum_{i=1}^{n_2} \frac{1}{(r_2+n_1) - \lambda_i(G_2)} - \frac{1}{n_1}.$$

Therefore, taking the above results together, we have

$$tr(L^{(1)}_{G_1 \veebar G_2}) = \frac{n_1}{r_1} + \frac{1}{r_1^2}\sum_{i=1}^{n_1} \frac{\lambda_i^2(G_1) + r_1^2}{(n_2 + 2r_1) - \lambda_i(G_1) - \frac{1}{r_1}\lambda_i^2(G_1)} + \sum_{i=1}^{n_2} \frac{1}{(r_2 + n_1) - \lambda_i(G_2)} - \frac{1}{n_1}. \tag{15}$$

Moreover, from (7), it is easy to see that

$$\mathbf{e}^T L^{(1)}_{G_1 \veebar G_2}\mathbf{e} = \mathbf{e_1}^T N_1 \mathbf{e_1} + \mathbf{e_1}^T N_2 \mathbf{e_2} + \mathbf{e_2}^T N_3 \mathbf{e_1} + \mathbf{e_2}^T S_M^{-1} \mathbf{e_2} + \mathbf{e_3}^T F^{\#} \mathbf{e_3},$$

where $\mathbf{e_1}$, $\mathbf{e_2}$ and $\mathbf{e_3}$ are the column vectors of size n_1, n_1 and n_2, respectively, whose all entries are 1.

Notice that

$$\begin{aligned} n_1 &= \mathbf{e_2}^T S_M S_M^{-1} \mathbf{e_2} \\ &= \mathbf{e_2}^T \Big((r_1 + n_2) I_{n_1} + L_{G_1} - \frac{1}{r_1} A_{G_1}^T A_{G_1}\Big) S_M{}^{-1} \mathbf{e_2} \\ &= (r_1 + n_2) \mathbf{e_2}^T S_M{}^{-1} \mathbf{e_2} - \frac{1}{r_1} \mathbf{e_2}^T A_{G_1}^T A_{G_1} S_M{}^{-1} \mathbf{e_2} \\ &= (r_1 + n_2) \mathbf{e_2}^T S_M{}^{-1} \mathbf{e_2} - r_1 \mathbf{e_2}^T S_M{}^{-1} \mathbf{e_2} \\ &= n_2 \mathbf{e_2}^T S_M^{-1} \mathbf{e_2}, \end{aligned}$$

which implies that $\mathbf{e_2}^T S_M^{-1} \mathbf{e_2} = \frac{n_1}{n_2}$. Since $N_1 = \frac{1}{r_1} I_{n_1} + \frac{1}{r_1^2} A_{G_1} S_M^{-1} A_{G_1}^T$, $N_2 = \frac{1}{r_1} A_{G_1} S_M^{-1}$ and $N_3 = \frac{1}{r_1} S_M^{-1} A_{G_1}^T$ from (2), (3) and (4), we get

$$\mathbf{e_1}^T N_1 \mathbf{e_1} = \frac{n_1}{r_1} + \frac{1}{r_1^2} \mathbf{e_1}^T A_{G_1} S_M{}^{-1} A_{G_1}^T \mathbf{e_1} = \frac{n_1}{r_1} + \mathbf{e_1}^T S_M{}^{-1} \mathbf{e_1} = \frac{n_1}{r_1} + \frac{n_1}{n_2}.$$

By a similar analysis as above, we can obtain that

$$\mathbf{e_1}^T N_2 \mathbf{e_2} = \mathbf{e_2}^T N_3 \mathbf{e_1} = \frac{1}{r_1} r_1 \mathbf{e_2}^T S_M{}^{-1} \mathbf{e_2} = \frac{n_1}{n_2}.$$

In addition, since $F = L_{G_2} + n_1 I_{n_2} - \frac{n_1}{n_2} J_{n_2 \times n_2}$, by simple calculation, we have F is a real symmetric matrix and $F\mathbf{e_3} = 0$. Hence, from Lemma 2.7, we can obtain $\mathbf{e_3}^T F^{\#} \mathbf{e_3} = 0$.

Finally, Putting the above results together, we get

$$\mathbf{e}^T L_{G_1 \veebar G_2}{}^{(1)} \mathbf{e} = \frac{n_1}{r_1} + 4\frac{n_1}{n_2}. \tag{16}$$

Therefore, combining (15) with (16), we have

$$\begin{aligned} Kf(G_1 \veebar G_2) = &(2n_1 + n_2)\Big[\frac{1}{r_1} \sum_{i=1}^{n_1} \frac{\lambda_i^2(G_1) + r_1^2}{r_1(n_2 + 2r_1 - \lambda_i(G_1)) - \lambda_i^2(G_1)} + \sum_{i=1}^{n_2} \frac{1}{(r_2 + n_1) - \lambda_i(G_2)}\Big] \\ &+ \frac{2n_1^2 + n_1 n_2 - n_1}{r_1} - \frac{4n_1^2 + 2n_1 n_2 + n_2^2}{n_1 n_2}. \end{aligned}$$

□

Now, we provide an example.

Example 3.3 Suppose P_2 denotes a path on 2 vertices. It is easy to get that $Spec_A(P_2) = \{-1, 1\}$. The splitting V-vertex join graph $P_2 \veebar P_2$ of P_2 and P_2 is shown in Figure 2.

Now, applying Theorem 3.1, we have the following conclusions.

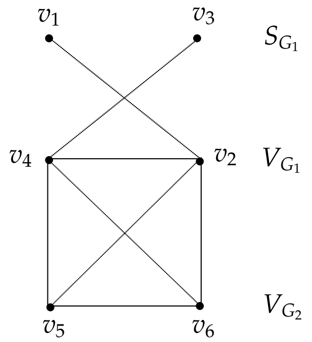


Figure 2: $P_2 \veebar P_2$.

① For $v_2, v_4 \in V_{G_1}$, we have

$$r_{24}\left(P_2 \veebar P_2\right) = \frac{1}{n_2+2r_1}\sum_{k=0}^{\infty}\frac{1}{(n_2+2r_1)^k}\Big(w_k(v_i,v_i)+w_k(v_j,v_j)-2w_k(v_i,v_j)\Big) = \frac{1}{2}.$$

② For $v_5, v_6 \in V_{G_2}$, we see

$$r_{56}\left(P_2 \veebar P_2\right) = \Big[(L_{G_2}+n_1I_{n_2})^{-1}\Big]_{ii} + \Big[(L_{G_2}+n_1I_{n_2})^{-1}\Big]_{jj} - 2\Big[(L_{G_2}+n_1I_{n_2})^{-1}\Big]_{ij} = \frac{1}{2}.$$

③ For $v_1, v_3 \in S_{G_1}$, we obtain

$$\begin{aligned} r_{13}(P_2 \veebar P_2) =& \frac{2}{r_1} + \frac{1}{r_1^2(n_2+2r_1)}\Big(\sum_{\substack{v_s\in N_{G_1}(v_i')\\ v_t\in N_{G_1}(v_i')}}\sum_{k=0}^{\infty}\frac{1}{(n_2+2r_1)^k}w_k(v_s,v_t) \\ &+ \sum_{\substack{v_s\in N_{G_1}(v_j')\\ v_t\in N_{G_1}(v_j')}}\sum_{k=0}^{\infty}\frac{1}{(n_2+2r_1)^k}w_k(v_s,v_t) - 2\sum_{\substack{v_s\in N_{G_1}(v_i')\\ v_t\in N_{G_1}(v_j')}}\sum_{k=0}^{\infty}\frac{1}{(n_2+2r_1)^k}w_k(v_s,v_t)\Big) = \frac{5}{2}. \end{aligned}$$

④ If $v_i \in V_{G_1}, v_j \in V_{G_2}$, taking v_4 and v_5 as an example, then

$$r_{45}(P_2 \veebar P_2) = \frac{1}{n_2+2r_1}\sum_{k=0}^{\infty}\Big(\frac{1}{(n_2+2r_1)^k}w_k(v_i,v_i)\Big) + [(L_{G_2}+n_1I_{n_2})^{-1}]_{jj} - \frac{1}{n_1n_2} = \frac{1}{2}.$$

⑤ Let $v_i \in S_{G_1}, v_j \in V_{G_2}$, taking v_1 and v_5 as an example. Then

$$r_{15}(P_2 \veebar P_2) = \frac{1}{r_1} + \frac{1}{r_1^2(n_2+2r_1)}\sum_{\substack{v_s\in N_{G_1}(v_i')\\ v_t\in N_{G_1}(v_i')}}\sum_{k=0}^{\infty}\Big(\frac{1}{(n_2+2r_1)^k}w_k(v_s,v_t)\Big) + [(L_{G_2}+n_1I_{n_2})^{-1}]_{jj} - \frac{1}{n_1n_2} = \frac{3}{2}.$$

⑥ Suppose $v_i \in S_{G_1}, v_j \in V_{G_1}$, taking v_1 and v_2 as an example. Then

$$\begin{aligned} r_{ij}(G_1 \veebar G_2) =& \frac{1}{r_1^2(n_2+2r_1)}\sum_{\substack{v_s\in N_{G_1}(v_i')\\ v_t\in N_{G_1}(v_i')}}\sum_{k=0}^{\infty}\Big(\frac{1}{(n_2+2r_1)^k}w_k(v_s,v_t)\Big) + \frac{1}{r_1} + \frac{1}{n_2+2r_1}\sum_{k=0}^{\infty}\Big(\frac{1}{(n_2+2r_1)^k}w_k(v_j,v_j)\Big) \\ &- \frac{2}{r_1(n_2+2r_1)}\sum_{v_s\in N_{G_1}(v_i')}\sum_{k=0}^{\infty}\Big(\frac{1}{(n_2+2r_1)^k}w_k(v_s,v_j)\Big) = 1. \end{aligned}$$

In addition, by Theorem 3.2, we obtain Kirchhoff index of $P_2 \veebar P_2$ as follows:

$$Kf(P_2 \veebar P_2) = (2n_1+n_2)[\frac{1}{r_1}\sum_{i=1}^{n_1}\frac{\lambda_i^2(G_1)+r_1^2}{r_1(n_2+2r_1-\lambda_i(G_1))-\lambda_i^2(G_1)} + \sum_{i=1}^{n_2}\frac{1}{(r_2+n_1)-\lambda_i(G_2)}] + \frac{2n_1^2+n_1n_2-n_1}{r_1} - \frac{4n_1^2+2n_1n_2+n_2^2}{n_1n_2} = \frac{33}{2}.$$

On the other hand, by using Mathematica, we find the resistance distance matrix of $P_2 \veebar P_2$ as shown below:

$$R(P_2 \veebar P_2) = \begin{pmatrix} 0 & 1 & \frac{5}{2} & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\ 1 & 0 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{5}{2} & \frac{3}{2} & 0 & 1 & \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} & 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

This implies that the Theorem 3.1 and Theorem 3.2 are effective ways to compute the resistance distance and the Kirchhoff index.

4. Resistance distance and Kirchhoff index of splitting S-vertex join graphs

In this section, we calculate the resistance distance and Kirchhoff index of the splitting S-vertex join graph $G_1 \overline{\wedge} G_2$.

Theorem 4.1. *Suppose G_i is an r_i-regular graph on n_i vertices for $i = 1, 2$. Let $w_k(v_i, v_j) = \Big[(A_{G_1} + \frac{1}{r_1+n_2}A_{G_1}^2)^k\Big]_{ij}$ and $N_{G_1}(v_i) = \{v_j \in V_{G_1} \mid v_iv_j \in E_{G_1 \overline{\wedge} G_2}\}$. Then we can conclude the following results.*

① *For any $v_i, v_j \in V_{G_1}$, we have*

$$r_{ij}(G_1 \overline{\wedge} G_2) = \frac{1}{2r_1}\sum_{k=0}^{\infty}\frac{1}{(2r_1)^k}\Big(w_k(v_i,v_i) + w_k(v_j,v_j) - 2w_k(v_i,v_j)\Big);$$

② *For any $v_i, v_j \in V_{G_2}$, we get*

$$r_{ij}(G_1 \overline{\wedge} G_2) = \Big[(L_{G_2}+n_1I_{n_2})^{-1}\Big]_{ii} + \Big[(L_{G_2}+n_1I_{n_2})^{-1}\Big]_{jj} - 2\Big[(L_{G_2}+n_1I_{n_2})^{-1}\Big]_{ij};$$

③ *For any $v_i', v_j' \in S_{G_1}$, we obtain*

$$r_{ij}(G_1 \overline{\wedge} G_2) = \frac{2}{r_1+n_2} + \frac{1}{2r_1(r_1+n_2)^2}\Big[\sum_{\substack{v_s\in N_{G_1}(v_i') \\ v_t\in N_{G_1}(v_i')}}\sum_{k=0}^{\infty}\frac{1}{(2r_1)^k}w_k(v_s,v_t) + \sum_{\substack{v_s\in N_{G_1}(v_j') \\ v_t\in N_{G_1}(v_j')}}\sum_{k=0}^{\infty}\frac{1}{(2r_1)^k}w_k(v_s,v_t) - 2\sum_{\substack{v_s\in N_{G_1}(v_i') \\ v_t\in N_{G_1}(v_j')}}\sum_{k=0}^{\infty}\frac{1}{(2r_1)^k}w_k(v_s,v_t)\Big];$$

④ *For $v_i \in V_{G_1}$, $v_j \in V_{G_2}$, we see*

$$r_{ij}(G_1 \overline{\wedge} G_2) = \frac{1}{2r_1}\sum_{k=0}^{\infty}\Big[\frac{1}{(2r_1)^k}w_k(v_i,v_i)\Big] + \Big[(L_{G_2}+n_1I_{n_2})^{-1}\Big]_{jj} - \frac{1}{n_1n_2};$$

⑤ *For $v_i' \in S_{G_1}$, $v_j \in V_{G_2}$, we know*

$$r_{ij}\left(G_1 \overline{\wedge} G_2\right) = \frac{1}{r_1+n_2} + \frac{1}{2r_1(r_1+n_2)^2} \sum_{\substack{v_s \in N_{G_1}(v_i') \\ v_t \in N_{G_1}(v_i')}} \sum_{k=0}^{\infty} \left[\frac{1}{(2r_1)^k} w_k(v_s, v_t)\right] + \left[\left(L_{G_2} + n_1 I_{n_2}\right)^{-1}\right]_{jj} - \frac{1}{n_1 n_2};$$

⑥ *For $v_i' \in S_{G_1}$, $v_j \in V_{G_1}$, we have*

$$\begin{aligned} r_{ij}\left(G_1 \overline{\wedge} G_2\right) = &\frac{1}{2r_1\left(r_1+n_2\right)^2} \sum_{\substack{v_s \in N_{G_1}(v_i') \\ v_t \in N_{G_1}(v_i')}} \sum_{k=0}^{\infty} \left[\frac{1}{(2r_1)^k} w_k(v_s, v_t)\right] + \frac{1}{2r_1} \sum_{k=0}^{\infty} \left[\frac{1}{(2r_1)^k} w_k(v_j, v_j)\right] \\ &- \frac{2}{2r_1(r_1+n_2)} \sum_{v_s \in N_{G_1}(v_i')} \sum_{k=0}^{\infty} \left[\frac{1}{(2r_1)^k} w_k(v_s, v_j)\right] + \frac{1}{r_1+n_2}. \end{aligned}$$

Proof. Marking the vertices of $G_1 \overline{\wedge} G_2$ as shown in Figure 1, we have the Laplacian matrix of $G_1 \overline{\wedge} G_2$ below

$$L_{G_1 \overline{\wedge} G_2} = \begin{array}{c} \\ S_{G_1} \\ V_{G_1} \\ V_{G_2} \end{array} \begin{array}{c} \begin{array}{ccc} S_{G_1} & V_{G_1} & V_{G_2} \end{array} \\ \left(\begin{array}{cc|c} (r_1+n_2)I_{n_1} & -A_{G_1} & -J_{n_1 \times n_2} \\ -A_{G_1}^T & r_1 I_{n_1} + L_{G_1} & O_{n_1 \times n_2} \\ \hline -J_{n_2 \times n_1} & O_{n_2 \times n_1} & n_1 I_{n_2} + L_{G_2} \end{array}\right) \end{array}$$

$$= \left(\begin{array}{c|c} \mathrm{M} & \begin{pmatrix} -J_{n_1 \times n_2} \\ O_{n_1 \times n_2} \end{pmatrix} \\ \hline \begin{pmatrix} -J_{n_2 \times n_1} & O_{n_2 \times n_1} \end{pmatrix} & n_1 I_{n_2} + L_{G_2} \end{array}\right),$$

where $M = \begin{pmatrix} (r_1+n_2)I_{n_1} & -A_{G_1} \\ -A_{G_1}^T & r_1 I_{n_1} + L_{G_1} \end{pmatrix}$ and $O_{a \times b}$ is the $a \times b$ matrix with all entries equal to zero.

By Definition 2.3, we have the Schur complement of $(r_1+n_2)I_{n_1}$ in M is

$$S_M = r_1 I_{n_1} + L_{G_1} - \frac{1}{r_1+n_2}\left(A_{G_1}^T A_{G_1}\right). \tag{17}$$

By Lemma 2.4, we have

$$M^{-1} = \begin{pmatrix} M_1 & M_2 \\ M_3 & S_M^{-1} \end{pmatrix}, \tag{18}$$

where

$$M_1 = \frac{1}{r_1+n_2} I_{n_1} + \frac{1}{(r_1+n_2)^2} A_{G_1} S_M^{-1} A_{G_1}^T, \tag{19}$$

$$M_2 = \frac{1}{r_1+n_2} A_{G_1} S_M^{-1}, \tag{20}$$

$$M_3 = \frac{1}{r_1+n_2} S_M^{-1} A_{G_1}^T. \tag{21}$$

Suppose F is the Schur complement of M in $L(G_1 \overline{\wedge} G_2)$. Then from Definition 2.3 and (18), we get

$$\begin{aligned} F &= n_1 I_{n_2} + L_{G_2} - \begin{pmatrix} -J_{n_2 \times n_1} & O_{n_2 \times n_1} \end{pmatrix} M^{-1} \begin{pmatrix} -J_{n_1 \times n_2} \\ O_{n_1 \times n_2} \end{pmatrix} \\ &= n_1 I_{n_2} + L_{G_2} - \frac{n_1}{r_1+n_2} J_{n_2 \times n_2} - \frac{{r_1}^2}{(r_1+n_2)^2} J_{n_2 \times n_1} S_M^{-1} J_{n_1 \times n_2}. \end{aligned} \tag{22}$$

Since

$$\begin{aligned} n_1 J_{n_2\times n_1} &= J_{n_2\times n_1} S_M S_M^{-1} J_{n_1\times n_2} \\ &= J_{n_2\times n_1}\Big[r_1 I_{n_1} + L_{G_1} - \frac{1}{r_1+n_2}(A_{G_1}^T A_{G_1})\Big] S_M^{-1} J_{n_1\times n_2} \\ &= (r_1 - \frac{r_1^2}{r_1+n_2}) J_{n_2\times n_1} S_M^{-1} J_{n_1\times n_2}, \end{aligned}$$

we get

$$J_{n_2\times n_1} S_M^{-1} J_{n_1\times n_2} = \frac{n_1(r_1+n_2)}{r_1 n_2} J_{n_2\times n_2}. \tag{23}$$

Substituting (23) into (22), we obtain

$$F = n_1 I_{n_2} + L_{G_2} - \frac{n_1}{n_2} J_{n_2\times n_2}.$$

Furthermore, according to Lemma 2.6, we get the expression

$$F^{\#} = (L_{G_2} + n_1 I_{n_2})^{-1} - \frac{1}{n_1 n_2} J_{n_2\times n_2}. \tag{24}$$

Therefore, we get the expression of $L^{(1)}_{G_1 \overline{\wedge} G_2}$ from Lemma 2.5 as follows

$$L^{(1)}_{G_1 \overline{\wedge} G_2} = \left(\begin{array}{cc|c} M_1 & M_2 & 0 \\ M_3 & S_M^{-1} & 0 \\ \hline 0 & 0 & F^{\#} \end{array}\right). \tag{25}$$

① For $v_i, v_j \in V_{G_1}$, combining Lemma 2.1 with (25), we have

$$r_{ij}(G_1 \overline{\wedge} G_2) = (S_M^{-1})_{ii} + (S_M)^{-1}_{jj} - 2(S_M^{-1})_{ij}.$$

In view of (17), we get

$$S_M = 2r_1\Big[I_{n_1} - \frac{1}{2r_1}(A_{G_1} + \frac{1}{r_1+n_2} A_{G_1}^2)\Big].$$

The spectral radius of $\frac{1}{2r_1}(A_{G_1} + \frac{1}{r_1+n_2} A_{G_1}^2)$ is

$$\rho(\frac{1}{2r_1}(A_{G_1} + \frac{1}{r_1+n_2} A_{G_1}^2)) = \frac{2r_1+n_2}{2r_1+2n_2} < 1,$$

which implies that the power series of $\Big[I_{n_1} - \frac{1}{2r_1}(A_{G_1} + \frac{1}{r_1+n_2} A_{G_1}^2)\Big]^{-1}$ is convergent. Thus, we gain

$$S_M^{-1} = \frac{1}{2r_1} \sum_{k=0}^{\infty} \Big[\frac{1}{(2r_1)^k}(A_{G_1} + \frac{1}{r_1+n_2} A_{G_1}^2)^k\Big]. \tag{26}$$

Let $w_k(v_i, v_j) = \Big[(A_{G_1} + \frac{1}{r_1+n_2} A_{G_1}^2)^k\Big]_{ij}$. Then we have

$$r_{ij}(G_1 \overline{\wedge} G_2) = \frac{1}{2r_1} \sum_{k=0}^{\infty} \Big[\frac{1}{(2r_1)^k}\Big(w_k(v_i,v_i) + w_k(v_j,v_j) - 2w_k(v_i,v_j)\Big)\Big].$$

② For $v_i, v_j \in V_{G_2}$, according to Lemma 2.1 and (25), we have

$$r_{ij}\left(G_1 \overline{\wedge} G_2\right) = (F^{\#})_{ii} + (F^{\#})_{jj} - 2(F^{\#})_{ij}.$$

Based on (24), we obtain

$$r_{ij}\left(G_1 \overline{\wedge} G_2\right) = \left[\left(L_{G_2} + n_1 I_{n_2}\right)^{-1}\right]_{ii} + \left[\left(L_{G_2} + n_1 I_{n_2}\right)^{-1}\right]_{jj} - 2\left[\left(L_{G_2} + n_1 I_{n_2}\right)^{-1}\right]_{ij}.$$

③ For $v_i', v_j' \in S_{G_1}$, according to Lemma 2.1 and (25), we have

$$r_{ij}\left(G_1 \overline{\wedge} G_2\right) = (M_1)_{ii} + (M_1)_{jj} - 2\,(M_1)_{ij}\,. \tag{27}$$

Note that $N_{G_1}(v_i) = \{v_j \in V_{G_1} \mid v_i v_j \in E_{G_1 \overline{\wedge} G_2}\}$. According to (19), we can get

$$\begin{aligned}
(M_1)_{ii} &= \frac{1}{r_1 + n_2} + \frac{1}{(r_1 + n_2)^2}\left(A_{G_1} S_M^{-1} A_{G_1}^T\right)_{ii} \\
&= \frac{1}{r_1 + n_2} + \frac{1}{(r_1 + n_2)^2}\left(\left(\sum_{v_s \in N_{G_1}(v_i')} \left(S_M^{-1}\right)_{s1}, \sum_{v_s \in N_{G_1}(v_i')} (S_M^{-1})_{s2}, \ldots, \sum_{v_s \in N_{G_1}(v_i')} (S_M^{-1})_{sn_1}\right) A_{G_1}^T\right)_i \\
&= \frac{1}{r_1 + n_2} + \frac{1}{2r_1(r_1 + n_2)^2} \sum_{\substack{v_s \in N_{G_1}(v_i') \\ v_t \in N_{G_1}(v_i')}} \sum_{k=0}^{\infty} \left(\frac{1}{(2r_1)^k} w_k(v_s, v_t)\right).
\end{aligned} \tag{28}$$

By using a similar method as above, we obtain

$$(M_1)_{ij} = \frac{1}{2r_1(r_1 + n_2)^2} \sum_{\substack{v_s \in N_{G_1}(v_i') \\ v_t \in N_{G_1}(v_j')}} \sum_{k=0}^{\infty} \left(\frac{1}{(2r_1)^k} w_k(v_s, v_t)\right). \tag{29}$$

Therefore, substituting (28) and (29) into (27), we have

$$\begin{aligned}
r_{ij}\left(G_1 \overline{\wedge} G_2\right) = &\frac{1}{2r_1\left(r_1 + n_2\right)^2}\Big[\sum_{\substack{v_s \in N_{G_1}(v_i') \\ v_t \in N_{G_1}(v_i')}} \sum_{k=0}^{\infty} \frac{1}{(2r_1)^k} w_k(v_s, v_t) + \sum_{\substack{v_s \in N_{G_1}(v_j') \\ v_t \in N_{G_1}(v_j')}} \sum_{k=0}^{\infty} \frac{1}{(2r_1)^k} w_k(v_s, v_t) \\
&- 2 \sum_{\substack{v_s \in N_{G_1}(v_i') \\ v_t \in N_{G_1}(v_j')}} \sum_{k=0}^{\infty} \frac{1}{(2r_1)^k} w_k(v_s, v_t)\Big] + \frac{2}{r_1 + n_2}.
\end{aligned}$$

④ For $v_i \in V_{G_1}, v_j \in V_{G_2}$, combining Lemma 2.1 with (25), we have

$$r_{ij}\left(G_1 \overline{\wedge} G_2\right) = (S_M^{-1})_{ii} + (F^{\#})_{jj}.$$

Further, according to (26) and (24), we know

$$r_{ij}\left(G_1 \overline{\wedge} G_2\right) = \frac{1}{2r_1} \sum_{k=0}^{\infty} \left(\frac{1}{(2r_1)^k} w_k(v_i, v_i)\right) + [(L_{G_2} + n_1 I_{n_2})^{-1}]_{jj} - \frac{1}{n_1 n_2}.$$

⑤ For $v_i' \in S_{G_1}, v_j \in V_{G_2}$, combining Lemma 2.1 with (25), we get

$$r_{ij}\left(G_1 \overline{\wedge} G_2\right) = (M_1)_{ii} + (F^{\#})_{jj}.$$

Similarly, due to (24) and (28), we have

$$r_{ij}(G_1 \barwedge G_2) = \frac{1}{r_1+n_2} + \frac{1}{2r_1(r_1+n_2)^2} \sum_{\substack{v_s \in N_{G_1}(v_i') \\ v_t \in N_{G_1}(v_i')}} \sum_{k=0}^{\infty} \Big(\frac{1}{(2r_1)^k} w_k(v_s, v_t)\Big) + [(L_{G_2} + n_1 I_{n_2})^{-1}]_{jj} - \frac{1}{n_1 n_2}.$$

⑥ For $v_i' \in S_{G_1}$, $v_j \in V_{G_1}$, based on Lemma 2.1 and (25), we have

$$r_{ij}(G_1 \barwedge G_2) = (M_1)_{ii} + (S_M)^{-1}_{jj} - 2(M_2)_{ij}. \tag{30}$$

Since $M_2 = \frac{1}{r_1+n_2} A_{G_1} S_M^{-1}$ from (20), according to (26), we see

$$\begin{aligned}(M_2)_{ij} &= \frac{1}{r_1+n_2}(A_{G_1} S_M^{-1})_{ij} \\ &= \frac{1}{r_1+n_2}\Big(\sum_{v_s \in N_{G_1}(v_i')} \big(S_M^{-1}\big)_{s1}, \sum_{v_s \in N_{G_1}(v_i')} (S_M^{-1})_{s2}, \ldots, \sum_{v_s \in N_{G_1}(v_i')} (S_M^{-1})_{sn_1}\Big)_j \\ &= \frac{1}{r_1+n_2} \sum_{v_s \in N_{G_1}(v_i')} (S_M^{-1})_{sj} \\ &= \frac{1}{2r_1(r_1+n_2)} \sum_{v_s \in N_{G_1}(v_i')} \sum_{k=0}^{\infty} \Big[\frac{1}{(2r_1)^k} w_k(v_s, v_j)\Big].\end{aligned} \tag{31}$$

Hence, plugging (26), (28) and (31) into (30), we get

$$\begin{aligned} r_{ij}(G_1 \barwedge G_2) = & \frac{1}{2r_1(r_1+n_2)^2} \sum_{\substack{v_s \in N_{G_1}(v_i') \\ v_t \in N_{G_1}(v_i')}} \sum_{k=0}^{\infty} \Big[\frac{1}{(2r_1)^k} w_k(v_s, v_t)\Big] + \frac{1}{2r_1} \sum_{k=0}^{\infty} \Big[\frac{1}{(2r_1)^k} w_k(v_j, v_j)\Big] \\ & - \frac{2}{2r_1(r_1+n_2)} \sum_{v_s \in N_{G_1}(v_i')} \sum_{k=0}^{\infty} \Big[\frac{1}{(2r_1)^k} w_k(v_s, v_j)\Big] + \frac{1}{r_1+n_2}. \end{aligned}$$

□

Theorem 4.2. *Assume G_i is an r_i-regular graph with n_i vertices. If $\lambda_1(G_i), \lambda_2(G_i), \ldots, \lambda_n(G_i)$ are the eigenvalues of A_{G_i} for $i = 1, 2$, then*

$$\begin{aligned} Kf(G_1 \barwedge G_2) = & (2n_1+n_2)\Big[\frac{1}{r_1+n_2} \sum_{i=1}^{n_1} \frac{(r_1+n_2)^2 + \lambda_i^2(G_1)}{(2r_1 - \lambda_i(G_1))(r_1+n_2) - \lambda_i^2(G_1)} + \sum_{i=1}^{n_2} \frac{1}{n_1 + r_2 - \lambda_i(G_2)}\Big] \\ & + \frac{2n_1^2 + n_1 n_2}{n_2 + r_1} - \frac{(4r_1+n_2)n_1^2 + 2r_1 n_1 n_2 + r_1 n_2^2}{r_1 n_1 n_2}. \end{aligned}$$

Proof. By Lemma 2.2, we have

$$Kf(G_1 \barwedge G_2) = (2n_1+n_2) tr(L^{(1)}_{G_1 \barwedge G_2}) - \mathbf{e}^T L^{(1)}_{G_1 \barwedge G_2} \mathbf{e}.$$

Since the expression of $L^{(1)}_{G_1 \barwedge G_2}$ from (25) is shown as follows

$$L^{(1)}_{G_1 \barwedge G_2} = \left(\begin{array}{cc|c} M_1 & M_2 & 0 \\ M_3 & S_M^{-1} & 0 \\ \hline 0 & 0 & F^{\#} \end{array}\right),$$

we have

$$tr(L^{(1)}_{G_1 \barwedge G_2}) = tr(M_1) + tr(S_M^{-1}) + tr(F^{\#}).$$

According to (17) , we obtain

$$tr(S_M) = \sum_{i=1}^{n_1} \Big(2r_1 - \lambda_i(G_1) - \frac{1}{r_1 + n_2}\lambda_i^2(G_1)\Big),$$

which implies that

$$tr(S_M^{-1}) = \sum_{i=1}^{n_1} \frac{1}{2r_1 - \lambda_i(G_1) - \frac{1}{r_1+n_2}\lambda_i^2(G_1)}.$$

Meanwhile, from (19), we get

$$\begin{aligned} tr(M_1) &= tr(\frac{1}{r_1 + n_2} I_{n_1}) + tr\Big(\frac{1}{(r_1 + n_2)^2} A_{G_1} S_M^{-1} A_{G_1}^T\Big) \\ &= \frac{n_1}{r_1 + n_2} + \frac{1}{r_1 + n_2} \sum_{i=1}^{n_1} \frac{\lambda_i^2(G_1)}{(2r_1 - \lambda_i(G_1))(r_1 + n_2) - \lambda_i^2(G_1)}. \end{aligned}$$

On the other hand, by (24), we obtain

$$\begin{aligned} tr(F^{\#}) &= tr\Big((L_{G_2} + n_1 I_{n_2})^{-1}\Big) - \frac{1}{n_1 n_2} tr(J_{n_2 \times n_2}) \\ &= \sum_{i=1}^{n_2} \frac{1}{n_1 + r_2 - \lambda_i(G_2)} - \frac{1}{n_1}. \end{aligned}$$

Therefore, taking the above results together, we have

$$\begin{aligned} tr(L^{(1)}_{G_1 \barwedge G_2}) =& \frac{1}{r_1 + n_2} \sum_{i=1}^{n_1} \frac{(r_1 + n_2)^2 + \lambda_i^2(G_1)}{(2r_1 - \lambda_i(G_1))(r_1 + n_2) - \lambda_i^2(G_1)} + \frac{n_1}{r_1 + n_2} \\ &+ \sum_{i=1}^{n_2} \frac{1}{n_1 + r_2 - \lambda_i(G_2)} - \frac{1}{n_1}. \end{aligned} \tag{32}$$

Moreover, from (25), it is easy to verify that

$$\mathbf{e}^T L^{(1)}_{G_1 \barwedge G_2} \mathbf{e} = \mathbf{e_1}^T M_1 \mathbf{e_1} + \mathbf{e_1}^T M_2 \mathbf{e_2} + \mathbf{e_2}^T M_3 \mathbf{e_1} + \mathbf{e_2}^T S_M^{-1} \mathbf{e_2} + \mathbf{e_3}^T F^{\#} \mathbf{e_3},$$

where $\mathbf{e_1}$, $\mathbf{e_2}$ and $\mathbf{e_3}$ are the column vectors of size n_1, n_1 and n_2, respectively, whose all entries are 1.

With a proof similar to Theorem 3.2, we have

$$\begin{aligned} n_1 &= \mathbf{e_2}^T S_M S_M^{-1} \mathbf{e_2} \\ &= \mathbf{e_2}^T (r_1 I_{n_1} + L_{G_1} - \frac{1}{r_1 + n_2}(A_{G_1}^T A_{G_1})) S_M^{-1} \mathbf{e_2} \\ &= (r_1 - \frac{r_1^2}{r_1 + n_2}) \mathbf{e_2}^T S_M^{-1} \mathbf{e_2}. \end{aligned}$$

Thus, we can obtain ${\mathbf{e_2}}^T S_M^{-1} \mathbf{e_2} = \frac{n_1(r_1+n_2)}{r_1 n_2}$. Further, according to (19), we get

$$\begin{aligned}{\mathbf{e_1}}^T M_1 \mathbf{e_1} &= {\mathbf{e_1}}^T\Big(\frac{1}{r_1+n_2} I_{n_1} + \frac{1}{(r_1+n_2)^2} A_{G_1} S_M^{-1} A_{G_1}^T\Big)\mathbf{e_1}\\ &= \frac{n_1}{r_1+n_2} + \frac{r_1^2}{(r_1+n_2)^2}{\mathbf{e_1}}^T S_M^{-1}\mathbf{e_1}\\ &= \frac{n_1}{n_2}.\end{aligned}$$

By using a similar method as above, we get

$${\mathbf{e_1}}^T M_2 \mathbf{e_2} = {\mathbf{e_2}}^T M_3 \mathbf{e_1} = \frac{r_1}{r_1+n_2}{\mathbf{e_2}}^T S_M^{-1}\mathbf{e_2} = \frac{n_1}{n_2}.$$

Moreover, since $F = n_1 I_{n_2} + L_{G_2} - \frac{n_1}{n_2} J_{n_2\times n_2}$, we have F is a real symmetric matrix and $F\mathbf{e_3} = 0$. So, according to Lemma 2.7, we have $\mathbf{e}^T F^{\#} = 0$ and $\mathbf{e}^T F^{\#}\mathbf{e} = 0$. Hence, we obtain

$$\mathbf{e}^T L^{(1)}_{G_1 \overline{\wedge} G_2}\mathbf{e} = 3\frac{n_1}{n_2} + \frac{n_1(r_1+n_2)}{r_1 n_2}. \tag{33}$$

Finally, combining (32) with (33), we have

$$\begin{aligned}Kf(G_1 \overline{\wedge} G_2) =&(2n_1+n_2)\Big[\frac{1}{r_1+n_2}\sum_{i=1}^{n_1}\frac{(r_1+n_2)^2+\lambda_i^2(G_1)}{(2r_1-\lambda_i(G_1))(r_1+n_2)-\lambda_i^2(G_1)} + \sum_{i=1}^{n_2}\frac{1}{n_1+r_2-\lambda_i(G_2)}\Big]\\ &+\frac{2n_1^2+n_1 n_2}{n_2+r_1} - \frac{(4r_1+n_2)n_1^2+2r_1 n_1 n_2 + r_1 n_2^2}{r_1 n_1 n_2}.\end{aligned}$$

□

At last, we get an example as follows.

Example 4.3

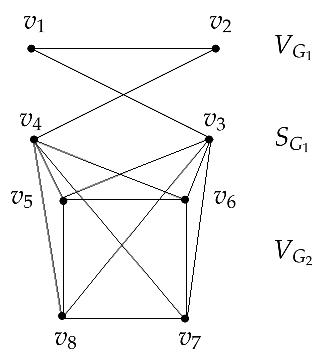


Figure 3: $P_2 \overline{\wedge} C_4$.

Note that $Spec_A(P_2) = \{1, -1\}$ and $Spec_A(C_4) = \{2, 0^2, -2\}$. The splitting S-vertex join $P_2 \overline{\wedge} C_4$ of P_2 and C_4 is shown in Figure 3. According to Theorem 4.1, for any two vertices in $P_2 \overline{\wedge} C_4$, we first calculate the resistance distance.

① For any $v_1, v_2 \in V_{G_1}$, we have

$$r_{12}\left(P_2 \overline{\wedge} C_4\right) = \frac{1}{2r_1}\sum_{k=0}^{\infty}\frac{1}{(2r_1)^k}\Big(w_k(v_i,v_i) + w_k(v_j,v_j) - 2w_k(v_i,v_j)\Big) = \frac{5}{7}.$$

② Let $v_i, v_j \in V_{G_2}$, taking v_5 and v_8 as an example. Then

$$r_{58}\left(P_2 \overline{\wedge} C_4\right) = \left[\left(L_{G_2} + n_1 I_{n_2}\right)^{-1}\right]_{ii} + \left[\left(L_{G_2} + n_1 I_{n_2}\right)^{-1}\right]_{jj} - 2\left[\left(L_{G_2} + n_1 I_{n_2}\right)^{-1}\right]_{ij} = \frac{5}{12}.$$

③ For $v_3, v_4 \in S_{G_1}$, we obtain

$$r_{34}\left(P_2 \overline{\wedge} C_4\right) = \frac{1}{2r_1\left(r_1+n_2\right)^2}\Big[\sum_{\substack{v_s\in N_{G_1}\left(v_i'\right)\\ v_t\in N_{G_1}\left(v_i'\right)}}\sum_{k=0}^{\infty}\frac{1}{(2r_1)^k}w_k(v_s,v_t) + \sum_{\substack{v_s\in N_{G_1}\left(v_j'\right)\\ v_t\in N_{G_1}\left(v_j'\right)}}\sum_{k=0}^{\infty}\frac{1}{(2r_1)^k}w_k(v_s,v_t)$$

$$-2\sum_{\substack{v_s\in N_{G_1}\left(v_i'\right)\\ v_t\in N_{G_1}\left(v_j'\right)}}\sum_{k=0}^{\infty}\frac{1}{(2r_1)^k}w_k(v_s,v_t)\Big] + \frac{2}{r_1+n_2} = \frac{3}{7}.$$

④ Suppose $v_i \in V_{G_1}$, $v_j \in V_{G_2}$, taking v_1 and v_5 as an example. Then

$$r_{15}\left(P_2 \overline{\wedge} C_4\right) = \frac{1}{2r_1}\sum_{k=0}^{\infty}\Big[\frac{1}{(2r_1)^k}w_k(v_i,v_i)\Big] + \left[\left(L_{G_2} + n_1 I_{n_2}\right)^{-1}\right]_{jj} - \frac{1}{n_1 n_2} = \frac{163}{168}.$$

⑤ Let $v_i \in S_{G_1}$, $v_j \in V_{G_2}$, taking v_3 and v_5 as an example. Then

$$r_{35}\left(P_2 \overline{\wedge} C_4\right) = \frac{1}{r_1+n_2} + \frac{1}{2r_1(r_1+n_2)^2}\sum_{\substack{v_s\in N_{G_1}\left(v_i'\right)\\ v_t\in N_{G_1}\left(v_i'\right)}}\sum_{k=0}^{\infty}\Big[\frac{1}{(2r_1)^k}w_k(v_s,v_t)\Big]$$

$$+\left[\left(L_{G_2} + n_1 I_{n_2}\right)^{-1}\right]_{jj} - \frac{1}{n_1 n_2} = \frac{67}{168}.$$

⑥ Assume $v_i \in S_{G_1}$, $v_j \in V_{G_2}$, taking v_1 and v_3 as an example. Then

$$r_{13}\left(P_2 \overline{\wedge} C_4\right) = \frac{1}{2r_1\left(r_1+n_2\right)^2}\sum_{\substack{v_s\in N_{G_1}\left(v_i'\right)\\ v_t\in N_{G_1}\left(v_i'\right)}}\sum_{k=0}^{\infty}\Big[\frac{1}{(2r_1)^k}w_k(v_s,v_t)\Big] + \frac{1}{2r_1}\sum_{k=0}^{\infty}\Big[\frac{1}{(2r_1)^k}w_k(v_j,v_j)\Big]$$

$$-\frac{2}{2r_1(r_1+n_2)}\sum_{v_s\in N_{G_1}(v_i')}\sum_{k=0}^{\infty}\Big[\frac{1}{(2r_1)^k}w_k(v_s,v_j)\Big] + \frac{1}{r_1+n_2} = \frac{5}{7}.$$

Meanwhile, using Theorem 4.2, we can compute Kirchhoff index of $P_2 \overline{\wedge} C_4$ as follows:

$$Kf(P_2 \overline{\wedge} C_4) = (2n_1+n_2)\Big[\frac{1}{r_1+n_2}\sum_{i=1}^{n_1}\frac{(r_1+n_2)^2+\lambda_i^2(G_1)}{(2r_1-\lambda_i(G_1))(r_1+n_2)-\lambda_i^2(G_1)} + \sum_{i=1}^{n_2}\frac{1}{n_1+r_2-\lambda_i(G_2)}\Big]$$

$$+\frac{2n_1^2+n_1n_2}{n_2+r_1} - \frac{(4r_1+n_2)n_1^2+2r_1n_1n_2+r_1n_2^2}{r_1n_1n_2} = \frac{376}{21}.$$

Similarly, by using Mathematica, we obtain the resistance distance matrix of $P_2 \overline{\wedge} C_4$ as shown below:

$$R(P_2 \overline{\wedge} C_4) = \begin{pmatrix} 0 & \frac{5}{7} & \frac{5}{7} & \frac{6}{7} & \frac{163}{168} & \frac{163}{168} & \frac{163}{168} & \frac{163}{168} \\ \frac{5}{7} & 0 & \frac{6}{7} & \frac{5}{7} & \frac{163}{168} & \frac{163}{168} & \frac{163}{168} & \frac{163}{168} \\ \frac{5}{7} & \frac{6}{7} & 0 & \frac{3}{7} & \frac{67}{168} & \frac{67}{168} & \frac{67}{168} & \frac{67}{168} \\ \frac{6}{7} & \frac{5}{7} & \frac{3}{7} & 0 & \frac{67}{168} & \frac{67}{168} & \frac{67}{168} & \frac{67}{168} \\ \frac{163}{168} & \frac{163}{168} & \frac{67}{168} & \frac{67}{168} & 0 & \frac{5}{12} & \frac{1}{2} & \frac{5}{12} \\ \frac{163}{168} & \frac{163}{168} & \frac{67}{168} & \frac{67}{168} & \frac{5}{12} & 0 & \frac{5}{12} & \frac{1}{2} \\ \frac{163}{168} & \frac{163}{168} & \frac{67}{168} & \frac{67}{168} & \frac{1}{2} & \frac{5}{12} & 0 & \frac{5}{12} \\ \frac{163}{168} & \frac{163}{168} & \frac{67}{168} & \frac{67}{168} & \frac{5}{12} & \frac{1}{2} & \frac{5}{12} & 0 \end{pmatrix}.$$

Since our results coincides with the true value of the resistance distance and the Kirchhoff index which could be measured, the Theorem 4.1 and Theorem 4.2 are very useful.

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Conflicts of Interest

The authors declare no conflict of interest.

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