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Further norm and numerical radius inequalities for sum of Hilbert space operators

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Abstract. Let $\mathbb{B}(\mathcal{H})$ denote the set of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. In this paper, the authors present some norm inequalities for sum of operators which are a generalization of some recent results. Among other inequalities, it is shown that if $S, T \in \mathbb{B}(\mathcal{H})$ are normal operators, then

$$\|S+T\| \leq \frac{1}{2}(\|S\|+\|T\|)+\frac{1}{2}\sqrt{(\|S\|-\|T\|)^2+4\,\|f_1(|S|)g_1(|T|)\|\,\|f_2(|S|)g_2(|T|)\|},$$

where f_1, f_2, g_1, g_2 are non-negative continuous functions on $[0,\infty)$, in which $f_1(x)f_2(x)=x$ and $g_1(x)g_2(x)=x\,(x\geq 0)$. Moreover, several inequalities for the numerical radius are shown.

1. Introduction

Let $\mathbb{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with an inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. In the case, when $\dim\mathcal{H}=n$, we identify $\mathbb{B}(\mathcal{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n\times n$ matrices with entries in the complex field. For $S\in\mathbb{B}(\mathcal{H})$, let $S=\Re(S)+i\Im(S)$ be the Cartesian decomposition of S, where the Hermitian operators $\Re(S)=\frac{S+S^*}{2}$ and $\Im(S)=\frac{S-S^*}{2i}$ are called the real and imaginary parts of S, respectively. The numerical radius of $S\in\mathbb{B}(\mathcal{H})$ is defined by

$$w(S):=\sup\{|\langle Sx,x\rangle| : x\in\mathcal{H}, \|x\|=1\}.$$

It is well known that $w(\cdot)$ defines a norm on $\mathbb{B}(\mathcal{H})$, which is equivalent to the usual operator norm $\|\cdot\|$. In fact, for any $S\in\mathbb{B}(\mathcal{H})$, $\frac{1}{2}\|S\|\leq w(S)\leq\|S\|$; see [5, 7]. Let $r(\cdot)$ denote the spectral radius. It is well known that for every operator $S\in\mathbb{B}(\mathcal{H})$, we have $r(S)\leq w(S)$. In [13], the authors showed that $w(S)=\sup_{\theta\in\mathbb{R}}\|\Re(e^{i\theta}S)\|=\sup_{\alpha^2+\beta^2=1}\|\alpha\Re(S)+\beta\Im(S)\|$, which is equal to the above definition. For more facts about the numerical radius see [3, 8, 9, 14, 16] and references therein. Let $S,T,X,Y\in\mathbb{B}(\mathcal{H})$. The operator matrices $\begin{bmatrix} S & 0\\ 0 & T\end{bmatrix}$ and $\begin{bmatrix} 0 & X\\ Y & 0\end{bmatrix}$ are called the diagonal and off-diagonal parts of the operator

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matrix $\begin{bmatrix} S & X \\ Y & T \end{bmatrix}$, respectively.
In [12], it has been shown that if S is an operator in $\mathbb{B}(\mathcal{H})$, then

$$w(S) \leq \frac{1}{2}\left(\|S\| + \|S^2\|^{\frac{1}{2}}\right). \tag{1}$$

Several refinements and generalizations of inequality (1) have been given; see [1, 16, 18]. Yamazaki [18] showed that for $S \in \mathbb{B}(\mathcal{H})$ and $t \in [0,1]$, we have

$$w(S) \leq \frac{1}{2}\left(\|S\| + w(\tilde{S}_t)\right), \tag{2}$$

where $S = U|S|$ is the polar decomposition of S and $\tilde{S}_t = |S|^t U |S|^{1-t}$. Horn et al. [10] proved that

$$\|S + T\| \leq \||S| + |T|\|, \tag{3}$$

where $S, T \in \mathbb{B}(\mathcal{H})$ are normal. Davidson and Power [6] proved that if S and T are positive operators in $\mathbb{B}(\mathcal{H})$, then

$$\|S + T\| \leq \max\{\|S\|, \|T\|\} + \|ST\|^{\frac{1}{2}}. \tag{4}$$

Inequality (4) has been generalized in [2, 15]. In [15], the author extended this inequality to the form

$$\|S + T\| \leq \max\{\|S\|, \|T\|\} + \frac{1}{2}\left(\||S|^t|T|^{1-t}\| + \||S^*|^{1-t}|T^*|^t\|\right), \tag{5}$$

in which $S, T \in \mathbb{B}(\mathcal{H})$ and $t \in [0,1]$. In [4], the authors showed that a generalization of inequality (5) as follows:

$$\|S + T\| \leq \max\{\|S\|, \|T\|\} + \frac{1}{2}\Big(\|f(|S|)g(|T|)\| + \|f(|S^*|)g(|T^*|)\|\Big),$$

in which $S, T \in \mathbb{B}(\mathcal{H})$ and f, g are two non-negative, non-decreasing continuous functions on $[0, \infty)$ such that $f(x)g(x) = x$ $(x \geq 0)$. Recently, Shi et al. [17] proved the following inequality

$$\|S + T\| \leq \frac{1}{2}(\|S\| + \|T\|) + \frac{1}{2}\sqrt{(\|S\| - \|T\|)^2 + \left\|\frac{1}{t}|S|^r|T|^s + t|S|^{1-r}|T|^{1-s}\right\|^2} \tag{6}$$

for normal operators $S, T \in \mathbb{B}(\mathcal{H})$, $r, s \in [0,1]$ and $t > 0$.

In this study, we consider several norm inequalities for the sum of bounded linear operators. These inequalities refine and generalize inequalities (5) and (6). Moreover, as another application, they show a new numerical radius inequality which is a generalization of [17, Theorem 3.12].

2. main results

Through this section, we give some new inequalities regarding the upper bounds for the sum of two operators. These inequalities are refinements of the previous ones which are indicated in the previous section. To present our results, we need the following lemmas.

Lemma 2.1. *[11] Let $S, T, X, Y \in \mathbb{B}(\mathcal{H})$. Then*

$$\left\|\begin{bmatrix} S & X \\ Y & T \end{bmatrix}\right\| \leq \left\|\begin{bmatrix} \|S\| & \|X\| \\ \|Y\| & \|T\| \end{bmatrix}\right\|.$$

Lemma 2.2. *[20] If $S, T \in \mathbb{B}(\mathcal{H})$ in which ST is selfadjoint, then*

$$\|ST\| \leq \|\Re(TS)\|.$$

In the first result, a generalization of inequality (6) is obtained.

Theorem 2.3. *Let $S, T \in \mathbb{B}(\mathcal{H})$ be normal and f_1, f_2, g_1, g_2 be non-negative continuous functions on $[0, \infty)$, in which $f_1(x)f_2(x) = x$ and $g_1(x)g_2(x) = x$ $(x \geq 0)$. Then*

$$\begin{aligned} &\|S+T\| \\ \leq\ & \frac{1}{2}(\|S\|+\|T\|)+\frac{1}{2}\sqrt{(\|S\|-\|T\|)^2+\left\|\frac{1}{t}f_1(|S|)g_1(|T|)+tf_2(|S|)g_2(|T|)\right\|^2} \end{aligned}$$

for all $t > 0$.

Proof. Assume S and T are positive operators. Then for all $t > 0$, we have

$$\begin{aligned} &\|S+T\| \\ =\ & \left\|\begin{bmatrix} S+T & 0 \\ 0 & 0 \end{bmatrix}\right\| \\ =\ & \left\|\begin{bmatrix} tf_2(S) & g_1(T) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{t}f_1(S) & 0 \\ g_2(T) & 0 \end{bmatrix}\right\| \\ \leq\ & \left\|\Re\left(\begin{bmatrix} \frac{1}{t}f_1(S) & 0 \\ g_2(T) & 0 \end{bmatrix}\begin{bmatrix} tf_2(S) & g_1(T) \\ 0 & 0 \end{bmatrix}\right)\right\| \\ & \qquad \text{(by the Lemma 2.2)} \\ =\ & \left\|\begin{bmatrix} S & \frac{1}{2}(\frac{1}{t}f_1(S)g_1(T)+tf_2(S)g_2(T)) \\ \frac{1}{2}(tf_2(S)g_2(T)+\frac{1}{t}f_1(S)g_1(T)) & T \end{bmatrix}\right\| \\ \leq\ & \left\|\begin{bmatrix} \|S\| & \frac{1}{2}\|\frac{1}{t}f_1(S)g_1(T)+tf_2(S)g_2(T)\| \\ \frac{1}{2}\|tf_2(S)g_2(T)+\frac{1}{t}f_1(S)g_1(T)\| & \|T\| \end{bmatrix}\right\| \\ & \qquad \text{(by Lemma 2.1)} \\ =\ & r\left(\begin{bmatrix} \|S\| & \frac{1}{2}\|\frac{1}{t}f_1(S)g_1(T)+tf_2(S)g_2(T)\| \\ \frac{1}{2}\|\frac{1}{t}f_2(S)g_2(T)+tf_1(S)g_1(T)\| & \|T\| \end{bmatrix}\right) \\ & \qquad \text{(since is Hermitian matrix)} \\ =\ & \frac{1}{2}(\|S\|+\|T\|)+\frac{1}{2}\sqrt{(\|S\|-\|T\|)^2+\left\|\frac{1}{t}f_1(S)g_1(T)+tf_2(S)g_2(T)\right\|^2} \end{aligned}$$

for positive operators S and T. Now, if S and T are normal operators, then by using inequality (3), we have

$$\begin{aligned} &\|S+T\| \\ \leq\ & \||S|+|T|\| \qquad \text{(by inequality (3))} \\ \leq\ & \frac{1}{2}(\||S|\|+\||T|\|)+\frac{1}{2}\sqrt{(\||S|\|-\||T|\|)^2+\left\|\frac{1}{t}f_1(|S|)g_1(|T|)+tf_2(|S|)g_2(|T|)\right\|^2} \\ =\ & \frac{1}{2}(\|S\|+\|T\|)+\frac{1}{2}\sqrt{(\|S\|-\|T\|)^2+\left\|\frac{1}{t}f_1(|S|)g_1(|T|)+tf_2(|S|)g_2(|T|)\right\|^2}. \end{aligned}$$

□

Lemma 2.4 (Young's inequality). *If a, b are nonnegative real numbers and $p, q > 1$ are real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Equality holds if and only if $a^p = b^q$.

As a consequence of Theorem 2.3, we have the following result.

Corollary 2.5. *Suppose that $S, T \in \mathbb{B}(\mathcal{H})$ are normal and f_1, f_2, g_1, g_2 are non-negative continuous functions on $[0, \infty)$ such that $f_1(x)f_2(x) = x$ and $g_1(x)g_2(x) = x$ $(x \geq 0)$. Then*

$$\|S + T\| \leq \frac{1}{2}(\|S\| + \|T\|) + \frac{1}{2}\sqrt{(\|S\| - \|T\|)^2 + 4\|f_1(|S|)g_1(|T|)\|\,\|f_2(|S|)g_2(|T|)\|}.$$

Proof. If S and T are normal operators, then

$$\begin{aligned}
&\|S + T\| \\
&\leq \frac{1}{2}(\|S\| + \|T\|) + \frac{1}{2}\sqrt{(\|S\| - \|T\|)^2 + \left\|\frac{1}{t}f_1(|S|)g_1(|T|) + tf_2(|S|)g_2(|T|)\right\|^2} \\
&\leq \frac{1}{2}(\|S\| + \|T\|) + \frac{1}{2}\sqrt{(\|S\| - \|T\|)^2 + \left(\frac{1}{t}\|f_1(|S|)g_1(|T|)\| + t\|f_2(|S|)g_2(|T|)\|\right)^2},
\end{aligned}$$

where the above inequality follows by the fact that function $f(x) = x^{\frac{1}{2}}$ is increasing on $[0, \infty)$. Next, taking the infimum to both sides of the above inequality over all positive real number t, we obtain

$$\begin{aligned}
&\|S + T\| \\
&\leq \frac{1}{2}(\|S\| + \|T\|) + \frac{1}{2}\sqrt{(\|S\| - \|T\|)^2 + \min_{t>0}\left(\frac{1}{t}\|f_1(|S|)g_1(|T|)\| + t\|f_2(|S|)g_2(|T|)\|\right)^2} \\
&= \frac{1}{2}(\|S\| + \|T\|) + \frac{1}{2}\sqrt{(\|S\| - \|T\|)^2 + 4\|f_1(|S|)g_1(|T|)\|\,\|f_2(|S|)g_2(|T|)\|}.
\end{aligned}$$

The last equation follows from Lemma 2.4. □

For invertible and normal operators, we get the next result.

Corollary 2.6. *Assume $S, T \in \mathbb{B}(\mathcal{H})$ are invertible and normal. If f, g are two positive continuous functions on $[0, \infty)$, then for all $t > 0$,*

$$\begin{aligned}
&\|S + T\| \\
&\leq \frac{1}{2}(\|S\| + \|T\|) + \frac{1}{2}\sqrt{(\|S\| - \|T\|)^2 + \left\|\frac{1}{t}f(|S|)g(|T|) + t|S|(f(|S|))^{-1}|T|(g(|T|))^{-1}\right\|^2}.
\end{aligned}$$

In particular,

$$\|S + T\| \leq \frac{1}{2}(\|S\| + \|T\|) + \frac{1}{2}\sqrt{(\|S\| - \|T\|)^2 + \left\|\frac{1}{t}|S|^r|T|^s + t|S|^{1-r}|T|^{1-s}\right\|^2},$$

where $r, s \in \mathbb{R}$.

Proof. For the first inequality, it is enough to put $f_1(t) = f(t)$, $f_2(t) = \frac{t}{f(t)}$, $g_1(t) = g(t)$ and $g_2(t) = \frac{t}{g(t)}$ $(t > 0)$ in Theorem 2.3. In particular, if $f(t) = t^r$ and $g(t) = t^s$ $(r, s \in \mathbb{R})$, we get the second inequality. □

Corollary 2.7. *Let $S,T \in \mathbb{B}(\mathcal{H})$ be invertible and normal. If f,g are two positive continuous functions on $[0,\infty)$, then*

$$\|S+T\| \leq \max\{\|S\|,\|T\|\} + \|f(|S|)g(|T|)\|^{\frac{1}{2}} \left\||S|(f(|S|))^{-1}|T|(g(|T|))^{-1}\right\|^{\frac{1}{2}}.$$

In particular,

$$\|S+T\| \leq \max\{\|S\|,\|T\|\} + \left\||S|^r|T|^{1-s}\right\|^{\frac{1}{2}} \left\||S|^{1-r}|T|^s\right\|^{\frac{1}{2}}, \tag{7}$$

where $r,s \in \mathbb{R}$.

Proof. Since $S,T \in \mathbb{B}(\mathcal{H})$ are invertible and normal and f,g are two positive continuous functions on $[0,\infty)$, applying Corollary 2.6, implies that

$$\begin{aligned}
&\|S+T\| \\
\leq\ & \frac{1}{2}(\|S\|+\|T\|) + \frac{1}{2}\sqrt{(\|S\|-\|T\|)^2 + \left\|\frac{1}{t}f(|S|)g(|T|) + t|S|(f(|S|))^{-1}|T|(g(|T|))^{-1}\right\|^2} \\
\leq\ & \frac{1}{2}(\|S\|+\|T\|) + \frac{1}{2}|\,\|S\|-\|T\|\,| + \frac{1}{2}\left\|\frac{1}{t}f(|S|)g(|T|) + t|S|(f(|S|))^{-1}|T|(g(|T|))^{-1}\right\| \\
& \qquad\qquad\qquad\qquad\qquad\qquad \text{(by the inequality } \sqrt{x^2+y^2} \leq \sqrt{x^2}+\sqrt{y^2}) \\
=\ & \max\{\|S\|,\|T\|\} + \frac{1}{2}\left\|\frac{1}{t}f(|S|)g(|T|) + t|S|(f(|S|))^{-1}|T|(g(|T|))^{-1}\right\| \\
\leq\ & \max\{\|S\|,\|T\|\} + \frac{1}{2}\left(\frac{1}{t}\|f(|S|)g(|T|)\| + t\left\||S|(f(|S|))^{-1}|T|(g(|T|))^{-1}\right\|\right).
\end{aligned}$$

Now, by taking the infimum on t $(t>0)$, we obtain

$$\begin{aligned}
&\|S+T\| \\
\leq\ & \max\{\|S\|,\|T\|\} + \frac{1}{2}\min_{t>0}\left(\frac{1}{t}\|f(|S|)g(|T|)\| + t\left\||S|(f(|S|))^{-1}|T|(g(|T|))^{-1}\right\|\right) \\
=\ & \max\{\|S\|,\|T\|\} + \|f(|S|)g(|T|)\|^{\frac{1}{2}}\left\||S|(f(|S|))^{-1}|T|(g(|T|))^{-1}\right\|^{\frac{1}{2}}.
\end{aligned}$$

For the second inequality, put $f(t)=t^r$ and $g(t)=t^{1-s}$ $(r,s\in\mathbb{R})$ in the first inequality. □

Remark 2.8. *If $S,T \in \mathbb{B}(\mathcal{H})$ are normal operators, then inequality (7) is a refinement and generalization of inequality (5) for normal operators. In fact, in this case*

$$\begin{aligned}
\|S+T\| &\leq \max\{\|S\|,\|T\|\} + \left\||S|^r|T|^{1-s}\right\|^{\frac{1}{2}}\left\||S|^{1-r}|T|^s\right\|^{\frac{1}{2}} \\
&\leq \max\{\|S\|,\|T\|\} + \frac{1}{2}\left(\left\||S|^r|T|^{1-s}\right\| + \left\||S|^{1-r}|T|^s\right\|\right), \\
&\qquad\qquad (\textit{by the arithmetic-geometric mean inequality})
\end{aligned}$$

where $r,s \in [0,1]$.

In the next result, the authors show a generalization of inequality (6) for arbitrary operators $S,T \in \mathbb{B}(\mathcal{H})$.

Theorem 2.9. *Suppose that $S,T \in \mathbb{B}(\mathcal{H})$ and f_1,f_2,g_1,g_2 are non-negative continuous functions on $[0,\infty)$, in which $f_1(x)f_2(x)=x$ and $g_1(x)g_2(x)=x$ $(x\geq 0)$. Then*

$$\|S+T\| \leq \frac{1}{2}(\|S\|+\|T\|) + \frac{1}{2}\sqrt{(\|S\|-\|T\|)^2 + \max\{\alpha,\beta\}},$$

where $\alpha = \left\|\frac{1}{t}f_1(|S^*|)g_1(|T^*|) + tf_2(|S^*|)g_2(|T^*|)\right\|^2$ and $\beta = \left\|\frac{1}{t}f_1(|S|)g_1(|T|) + tf_2(|S|)g_2(|T|)\right\|^2$.

Proof. Let $\tilde{S} = \begin{bmatrix} 0 & S \\ S^* & 0 \end{bmatrix}$ and $\tilde{T} = \begin{bmatrix} 0 & T \\ T^* & 0 \end{bmatrix}$. Then $\tilde{S}$ and $\tilde{T}$ are normal operators and

$$\|\tilde{S} + \tilde{T}\| = \left\| \begin{bmatrix} 0 & S+T \\ S^*+T^* & 0 \end{bmatrix} \right\| = \|S+T\|.$$

Hence, applying Theorem 2.3, we get

$$\begin{aligned} \|S+T\| &= \|\tilde{S}+\tilde{T}\| \\ &\leq \frac{1}{2}(\|\tilde{S}\| + \|\tilde{T}\|) + \frac{1}{2}\sqrt{(\|\tilde{S}\| - \|\tilde{T}\|)^2 + \left\| \frac{1}{t} f_1(|\tilde{S}|) g_1(|\tilde{T}|) + t f_2(|\tilde{S}|) g_2(|\tilde{T}|) \right\|^2}. \end{aligned} \tag{8}$$

Moreover, it follows from $\|\tilde{S}\| = \|S\|$, $\|\tilde{T}\| = \|T\|$ and

$$f(|\tilde{S}|) = \begin{bmatrix} f(|S^*|) & 0 \\ 0 & f(|S|) \end{bmatrix}, \quad f(|\tilde{T}|) = \begin{bmatrix} f(|T^*|) & 0 \\ 0 & f(|T|) \end{bmatrix}$$

for any non-negative continuous functions f on $[0, \infty)$, that

$$\begin{aligned} & \left\| \frac{1}{t} f_1(|\tilde{S}|) g_1(|\tilde{T}|) + t f_2(|\tilde{S}|) g_2(|\tilde{T}|) \right\| \\ = & \left\| \begin{bmatrix} \frac{1}{t} f_1(|S^*|) g_1(|T^*|) + t f_2(|S^*|) g_2(|T^*|) & 0 \\ 0 & \frac{1}{t} f_1(|S|) g_1(|T|) + t f_2(|S|) g_2(|T|) \end{bmatrix} \right\| \\ = & \max \left\{ \left\| \frac{1}{t} f_1(|S^*|) g_1(|T^*|) + t f_2(|S^*|) g_2(|T^*|) \right\|, \left\| \frac{1}{t} f_1(|S|) g_1(|T|) + t f_2(|S|) g_2(|T|) \right\| \right\}. \end{aligned}$$

Using the recent equality and inequality (8), we reach the desired result. □

Applying Theorem 2.9 and the same argument in the proof of Corollary 2.6, we get the next result.

Corollary 2.10. *[11, Corollary 2.22] Suppose that $S, T \in \mathbb{B}(\mathcal{H})$. Then*

$$\begin{aligned} & \|S+T\| \\ \leq & \max\{\|S\|, \|T\|\} + \max\left\{ \||S^*|^r |T^*|^{1-s}\|^{\frac{1}{2}} \||S^*|^{1-r} |T^*|^s\|^{\frac{1}{2}}, \||S|^r |T|^{1-s}\|^{\frac{1}{2}} \||S|^{1-r} |T|^s\|^{\frac{1}{2}} \right\}, \end{aligned}$$

where $r, s \in [0, 1]$. In particular,

$$\|S+T\| \leq \max\{\|S\|, \|T\|\} + \max\left\{ \||S^*|^{\frac{1}{2}} |T^*|^{\frac{1}{2}}\|, \||S|^{\frac{1}{2}} |T|^{\frac{1}{2}}\| \right\}. \tag{9}$$

3. Some results for the numerical radius

In this section, as an application of the norm inequalities for sum of operators, we present some inequalities for the numerical radius.

Theorem 3.1. *Assume $S, T \in \mathbb{B}(\mathcal{H})$. Then*

$$\begin{aligned} w(S-T) \geq\ & 2\max\{w(S), w(T)\} - \max\{\|S\|, \|T\|\} \\ & - \max\left\{ \||S^*|^r |T^*|^{1-s}\|^{\frac{1}{2}} \||S^*|^{1-r} |T^*|^s\|^{\frac{1}{2}}, \||S|^r |T|^{1-s}\|^{\frac{1}{2}} \||S|^{1-r} |T|^s\|^{\frac{1}{2}} \right\}, \end{aligned}$$

where $r, s \in [0, 1]$.

Proof. If $S, T \in \mathbb{B}(\mathcal{H})$, then

$$\begin{aligned} 2\max\{w(S), w(T)\} &= 2w\left(\begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix}\right) \\ &= w\left(\begin{bmatrix} S+T & 0 \\ 0 & T+S \end{bmatrix} + \begin{bmatrix} S-T & 0 \\ 0 & T-S \end{bmatrix}\right) \\ &\leq w\left(\begin{bmatrix} S+T & 0 \\ 0 & T+S \end{bmatrix}\right) + w\left(\begin{bmatrix} S-T & 0 \\ 0 & T-S \end{bmatrix}\right) \\ &= w(S+T) + w(S-T), \end{aligned}$$

and so

$$w(S-T) \geq 2\max\{w(S), w(T)\} - w(S+T).$$

It follows from $w(S+T) \leq \|S+T\|$ and Corollary 2.10 that

$$\begin{aligned} w(S-T) &\geq 2\max\{w(S), w(T)\} - w(S+T) \\ &\geq 2\max\{w(S), w(T)\} - \|S+T\| \\ &\geq 2\max\{w(S), w(T)\} - \max\{\|S\|, \|T\|\} \\ &\quad - \max\left\{\||S^*|^r|T^*|^{1-s}\|^{\frac{1}{2}}\||S^*|^{1-r}|T^*|^s\|^{\frac{1}{2}}, \||S|^r|T|^{1-s}\|^{\frac{1}{2}}\||S|^{1-r}|T|^s\|^{\frac{1}{2}}\right\}, \\ &\qquad \text{(by Corollary 2.10)} \end{aligned}$$

where $r, s \in [0,1]$. □

Remark 3.2. *If $S, T \in \mathbb{B}(\mathcal{H})$ are normal operators, then $|S| = |S^*|$, $|T| = |T^*|$, $w(S) = \|S\|$ and $w(T) = \|T\|$. These conclude that Theorem 3.4 appear as*

$$\|S-T\| \geq w(S-T) \geq \max\{\|S\|, \|T\|\} - \||S|^r|T|^{1-s}\|^{\frac{1}{2}}\||S|^{1-r}|T|^s\|^{\frac{1}{2}}.$$

In particular, if S and T are positive, then for $r = s = \frac{1}{2}$, we have [19, Theorem 4]

$$\|S-T\| \geq \max\{\|S\|, \|T\|\} - \|S^{\frac{1}{2}}T^{\frac{1}{2}}\|.$$

In the next result, we obtain an upper bound for the numerical radius.

Theorem 3.3. *Let $S \in \mathbb{B}(\mathcal{H})$ and f_1, f_2, g_1, g_2 be non-negative continuous functions on $[0,\infty)$, in which $f_1(x)f_2(x) = x$ and $g_1(x)g_2(x) = x$ $(x \geq 0)$. Then*

$$w(S) \leq \frac{1}{2}\|S\| + \frac{1}{4}\max\{\alpha, \beta\},$$

in which $\alpha = \|f_1(|S^*|)g_1(|S|) + f_2(|S^*|)g_2(|S|)\|$ and $\beta = \|f_1(|S|)g_1(|S^*|) + f_2(|S|)g_2(|S^*|)\|$. In particular,

$$w(S) \leq \frac{1}{2}\|S\| + \frac{1}{4}\|f_1(|S^*|)f_2(|S|) + f_2(|S^*|)f_1(|S|)\|.$$

Proof. Using $w(\cdot)$ and Theorem 2.9 for $t = 1$, we have

$$\begin{aligned} w(S) &= \sup_{\theta\in\mathbb{R}}\|\Re(e^{i\theta}S)\| \\ &= \frac{1}{2}\sup_{\theta\in\mathbb{R}}\|e^{i\theta}S + e^{-i\theta}S^*\| \\ &\leq \frac{1}{4}(\|S\| + \|S^*\|) + \frac{1}{4}\max\{\alpha, \beta\} \qquad \text{(by the Theorem 2.9)} \\ &= \frac{1}{2}\|S\| + \frac{1}{4}\max\{\alpha, \beta\}, \end{aligned}$$

in which $\alpha = \|f_1(|S^*|)g_1(|S|) + f_2(|S^*|)g_2(|S|)\|$ and $\beta = \|f_1(|S|)g_1(|S^*|) + f_2(|S|)g_2(|S^*|)\|$. In the special case for $g_1 = f_2$ and $g_2 = f_1$, we get the second inequality. □

Theorem 3.4. *Suppose that* $S \in \mathbb{B}(\mathcal{H})$. *Then*

$$w(S) \leq \max\{\|\Re(S)\|, \|\Im(S)\|\} + \frac{\sqrt{2}}{2}\||\Re(S)|^{\frac{1}{2}}|\Im(S)|^{\frac{1}{2}}\|.$$

Proof. Using inequality (9) and the definition $w(\cdot)$, we have

$$\begin{aligned} w(S) &= \sup_{\alpha^2+\beta^2=1} \|\alpha\Re(S) + \beta\Im(S)\| \\ &\leq \sup_{\alpha^2+\beta^2=1} \left(\max\{\|\alpha\Re(S)\|, \|\beta\Im(S)\|\} + \max\left\{\||\alpha\Re(S)|^{\frac{1}{2}}|\beta\Im(S)|^{\frac{1}{2}}\|, \||\alpha\Re(S)|^{\frac{1}{2}}|\beta\Im(S)|^{\frac{1}{2}}\|\right\}\right) \\ &\qquad \text{(by inequality (9))} \\ &\leq \max\{\|\Re(S)\|, \|\Im(S)\|\} + \sup_{\alpha^2+\beta^2=1}\left(\sqrt{|\alpha\beta|}\||\Re(S)|^{\frac{1}{2}}|\Im(S)|^{\frac{1}{2}}\|\right) \\ &\leq \max\{\|\Re(S)\|, \|\Im(S)\|\} + \frac{\sqrt{2}}{2}\||\Re(S)|^{\frac{1}{2}}|\Im(S)|^{\frac{1}{2}}\|. \end{aligned}$$

□

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