Filomat 38:9 (2024), 3243–3259
https://doi.org/10.2298/FIL2409243A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Connectedness and approximative properties of sets in asymmetric spaces

A. R. Alimov[a,*], **I. G. Tsar'kov**[b]

[a] *Lomonosov State University; Steklov Mathematical Institute of Russian Academy of Sciences, Moscow; Moscow Center of Fundamental and Applied Mathematics*
[b] *Lomonosov Moscow State University*

Abstract. In asymmetric normed spaces, we study continuity of the metric projection operator and structural connectedness-type properties of approximating sets. Connectedness of intersections with balls ($\mathring{B}$- and B-connectedness) of approximatively compact sets is examined. The set of points of approximative uniqueness for externally strongly complete subsets uniformly convex spaces that are complete with respect to the symmetrization norm is shown to be dense (in the symmetrization norm). Classical properties of stability of operators of best and near-best approximation and of the distance function in asymmetric spaces are studied. For uniformly convex asymmetric spaces embedded in a complete semilinear space, we also study whether for P_0-connected sets (and, in particular, sets of uniqueness and Chebyshev sets) have connected intersections with open balls.

1. Introduction

The present paper, which continues the studies begun in [8], [9], [30], [37], is concerned with relations between connectedness classes of subsets of asymmetric normed spaces.

By definition, an *asymmetric norm* on a real linear space X is a nonnegative functional $\|\cdot\,|$ such that, for all $x, y \in X$,

(1) $\|x\,| = 0 \Leftrightarrow x = 0$;
(2) $\|\alpha x\,| = \alpha\|x\,|$ for all $\alpha \geq 0$;
(3) $\|x + y\,| \leq \|x\,| + \|y\,|$.

In general, $\|x\,| \neq \|-x\,|$. The functional $\|x\|_{\mathrm{sym}} = \max\{\|x\,|, \|-x\,|\}$, $x \in X$, is known as the *symmetrization norm*. By an *asymmetric* space we mean a real linear space equipped with an asymmetric norm. (Sometimes, where no confusion can arise, an asymmetric norm is simply called a norm.) A space in which the asymmetric norm $\|\cdot\,|$ is equivalent to the symmetrization norm $\|\cdot\|_{\mathrm{sym}}$ (i.e., there exists a number $K \geq 1$ such that

2020 *Mathematics Subject Classification*. Primary 41A65.
Keywords. approximation in asymmetric spaces, $\mathring{B}$-complete set, B-connected set, uniformly convex asymmetric space, approximatively compact set, point of approximative uniqueness, externally strongly complete set.
Received: 05 May 2023; Revised: 25 October 2023; Accepted: 26 October 2023
Communicated by Miodrag Spalević
This research was carried out at Lomonosov Moscow State University with the financial support of the Russian Science Foundation (grant no. 22-11-00129).
* Corresponding author: A. R. Alimov
Email addresses: alexey.alimov@gmail.com (A. R. Alimov), tsar@mech.math.msu.su (I. G. Tsar'kov)

$\frac{1}{K}\|x\|_{\rm sym} \le \|x| \le \|x\|_{\rm sym}$ for all $x \in X$), is called a *symmetrizable* asymmetric space. The topology τ of an asymmetric space is generated by the open balls

$$\mathring{B}(x,r) = \{y \in X \mid \|y - x| < r\}.$$

In general, this topology satisfies only the T_1-separation axiom and may fail to be Hausdorff [18].

On a space with asymmetric norm $(X, \|\cdot|)$, we consider "closed" (right) balls

$$B(x,r) = B^+(x,r) = \{y \mid \|y - x| \leqslant r\},$$

and also the left (mirror) balls

$$B^-(x,r) = \{y \mid \|x - y| \leqslant r\},$$

i.e., the balls defined by the (mirror) asymmetric norm $\|-\cdot|$. Similarly, we define the (right) open balls $\mathring{B}(x,r) = \mathring{B}^+(x,r)$, the (left) open balls $\mathring{B}^-(x,r)$, and the spheres $S^+(x,r)$ and $S^-(x,r)$. For right ("+") objects, the superscript "+" will be as a rule omitted. Note that even in Hausdorff asymmetric spaces, the "closed ball" $B(x,r) = \{y \in X \mid \|y - x| \le r\}$ may fail to be closed (see, for example, [18]). For brevity, we write $B = B(0,1)$.

A natural example of an asymmetric norm is the Minkowski functional of a convex (not necessarily symmetric) set containing the origin in its kernel and bounded along any ray emanating from the origin. The name "asymmetric norm" was given by M. I. Krein in 1938 (see [23]) in relation to the moment problem.

The theory of asymmetric spaces and their applications is in active development at present: for example, problems related to functional analysis and topology are considered in [17], [14], [18], [25], [31], optimal placement problems (location problems) with asymmetric norms are studied, for example, in [19], [26], [27] (an important role in problems of this kind is played by Chebyshev centers and networks with respect to asymmetric norms), problems pertaining to the statistical principal component method (one of the most popular methods of compact data representation) are studied in [29] (this list is far from being complete). For other applications, see also [17]. In geometric approximation theory, asymmetric norms appear naturally in various problems (see, for example, [1], [5], [7], [20], [24], [31], [35], [38]). Asymmetric distances are also natural in the theory of approximation of functions, where they play the role of a "bridge" between best approximations and best one-sided approximations. Among recent studies on approximative and geometric properties of sets in asymmetric spaces, we mention [3], [5], [7], [8], [9], [24], [22], [35], [38]. For a survey of some results on general theory of asymmetric normed spaces and the problem of characterization of best approximants by convex sets, see [14], [17] and [2].

The principal difficulty in dealing with T_1-asymmetric spaces comes from the lack of metrizability of such spaces (or of their natural topology τ), which leads to discontinuity of the distance function $\rho(\cdot, M)$ to a set M (see [32], [33]). In addition, the most familiar laws and results that hold in symmetric spaces become unclear or even incorrect in the asymmetric case. Problems of existence, uniqueness, and stability of best approximation play a central role in approximation theory (both in symmetric and in asymmetric cases).

Given a nonempty set $M \subset X$, the *right* (*left*) *distance function*, or the right (left) distance from a point $x \in X$ to a set $M \subset X$ is defined as follows:

$$\rho(x,M) := \inf_{y\in M} \|y - x|, \qquad \rho^-(x,M) := \inf_{y\in M} \|x - y| \tag{1.1}$$

(the distance to a set is defined also similarly in semilinear spaces (cones)). In the first case, the distance is measured "from a point to a set", and in the second case, "from a set to a point". On symmetrizable asymmetric spaces, the distance function $\rho(\cdot, M)$ is continuous; however, for arbitrary asymmetric T_1-spaces it is only lower semicontinuous (see [32, p. 146]). The set of all right (left) *nearest points* from a point M to a set $x \in X$ is denoted by $P_M x$ ($P^-_M x$), i.e.,

$$P_M x := \Big\{y \in M \mid \rho(x,M) = \|y - x|\Big\}, \qquad P^-_M x := \Big\{y \in M \mid \rho^-(x,M) = \|x - y|\Big\}.$$

A point x with nonempty $P_M x \neq \varnothing$ is called a point of existence. Below:

– $\mathrm{E}(M)$ is the set points of existence for M (with respect to the asymmetric norm $\|\cdot|$);

– $\mathrm{E}_{\rm sym}(M)$ is the set points of existence for M with respect to the symmetrization norm $\|\cdot\|_{\rm sym}$.

Definition 1.1. A set M is said to be $\mathring{B}$-*complete*[1)] (see [8], [9]) if, for all $x \in X$ and $r > 0$,

$$\text{the condition } M_0 := (\mathring{B}(x,r) \cap M) \neq \varnothing \text{ implies that } \overline{M}_0 \supset (M \cap B(x,r)). \tag{1.2}$$

Property (1.2) means that any point from the intersection of M with the sphere $S(x,r)$ can be "approached" from the intersection of the open ball $\mathring{B}(x,r)$ with the set M under the condition that this intersection is nonempty. It is known (see [9], [7]) that in terms of $\mathring{B}$-complete sets one can characterize the unimodal sets (or LG-sets), and, in a number of spaces, the strict protosuns. $\mathring{B}$-completeness can be looked upon as a new connectedness-type property — for example, it is known (see [8]) that if M is $\mathring{B}$-complete and if its intersection with any open ball is connected, then M has connected intersections also with "closed" balls $B(x,r)$. For applications of $\mathring{B}$-completeness in the study of approximative properties of concrete and abstract sets, see [7], [9], [11]. If Q denotes some property (for example, connectedness), then we say that a set M has the property:

P-Q if, for each $x \in X$, the set $P_M x$ is nonempty and has the property Q;

P_0-Q if $P_M x$ has the property Q for all $x \in X$;

B-Q if $M \cap B(x,r)$ has the property Q for all $x \in X$, $r > 0$;

$\mathring{B}$-Q if $M \cap \mathring{B}(x,r)$ has the property Q for all $x \in X$, $r > 0$.

For example, a closed nonempty subset of a finite-dimensional space is P-nonempty, i.e., is an existence set.

The paper is structured as follows. In § 2, we study ϱ-continuity of the metric projection and structural connectedness-type properties of approximating sets (Theorem 2.5, Proposition 2.8). We also study $\mathring{B}$- and B-connectedness of approximatively compact and regularly approximatively compact sets (Theorems 2.15, 2.17). The results obtained partially extend several known results for normed spaces and symmetrizable asymmetric normed spaces to the case of arbitrary asymmetric spaces (satisfying the T_1-separation axiom). In § 3, we establish results on density (with respect to the symmetrization norm) of the set of points of approximative uniqueness for externally strongly complete subsets of uniformly convex spaces that are complete with respect to the symmetrization norm (Theorem 3.8). In § 4, we study classical stability properties of operators of best and near-best approximation and of the distance function for arbitrary asymmetric spaces (Theorems 4.1–4.6). In § 5, for uniformly convex asymmetric spaces lying in a complete semilinear space, we study $\mathring{B}$-connectedness of P_0-connected sets, sets of uniqueness, and Chebyshev sets (Theorem 5.2, Corollary 5.4).

We will need the following classes of sets:

($\mathscr{F}$) is the class of nonempty closed sets;

($\mathring{B}$) is the class of $\mathring{B}$-connected sets; (B) is the class of B-connected sets;

(P) is the class of P-connected sets ($P_M x$ is nonempty and connected for each $M \in (P)$).

It is easily checked that, in any asymmetric space (cf. [39]),

$$(B) \subset (\mathring{B}). \tag{1.3}$$

Note that in each infinite-dimensional Banach space there exists a P_0-connected but not P-connected closed set; each separable infinite-dimensional Banach space contains a $\mathring{B}$-connected but not B-connected closed set (see [30]).

We mainly follow the definitions of [4], [6]. Below, $X = (X, \|\cdot|)$ is a real asymmetric normed space.

[1)] In this definition, "$\mathring{B}$" denotes the open unit ball.

2. Connectedness and approximative properties of sets

In this section, we study structural properties of sets with good approximative properties. More precisely, we will study various forms of connectedness for approximatively compact sets in asymmetric spaces.

Definition 2.1. Consider the homogeneous symmetric nonnegative functional measuring the length of an interval $[x, y]$, where $x, y \in X$:

$$\varrho(x, y) := \min\{\|x - y\,|, \|y - x\,|\}.$$

In asymmetric case, such functional was studied in [21]. In general, the functional $\varrho(x, y)$, which we call the *distance* between x and y, is not a metric. Let us also define the following distance between sets:[2)]

$$\varrho(A, C) := \inf\Big\{\varrho(a, c) \mid a \in A,\ c \in C\Big\}.$$

Definition 2.2. A sequence $(x_n) \subset X$ is called a *Cauchy sequence* [17] if, for each $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\|x_m - x_n| < \varepsilon$ for all $m \geqslant n \geqslant N$.

An asymmetric space $X = (X, \|\cdot|)$ is called a *right-* (*left-*) *complete* [17] if, for each Cauchy sequence $(x_n) \subset X$, there exists a point $x \in X$ such that $\|x - x_n| \to 0$ (respectively, $|x_n - x| \to 0$) as $n \to \infty$. For brevity, a right-complete space will be simply called a *complete space*.

It is known that left completeness of a space ensures its nice topological properties, while right completeness is suitable for obtaining extensions of quasi-metric completeness to function spaces and hyperspaces as well as for delivering theorems like the Ekeland variational principle and the Caristi fixed point theorem (see, for example, [28], [15]).

An analogue of the following result for Banach spaces was in fact proved by Vlasov in the proof of his Theorem 1 in [39].

Lemma 2.3. *Let X be a left-complete asymmetric normed space. Assume that a set $M \subset X$ is not $\mathring{B}$-connected, i.e., there exist $x \in X$ and $r > 0$ such that $\mathring{B} \cap M = A \sqcup C$, where $A.C$ are nonempty open-closed sets in $\mathring{B} := \mathring{B}(x, r)$. Then there exist a $\delta > 0$ and a ball $B_1 := B(z, r_1) \subset \mathring{B}$, $B_1 \cap A \neq \varnothing$, $B_1 \cap C \neq \varnothing$, such that*

$$\varrho(B_1 \cap A, B_1 \cap C) \geq \delta.$$

Proof. By the hypothesis, there exists a number $0 < r_1 < r$ such that the ball $\mathring{B}(x_1, r_1)$ intersects both A and C. If the ball $\mathring{B}(x_1, r_1)$ satisfies the hypotheses of the lemma, there is nothing to prove. Assuming the contrary, we may choose $0 < \delta < (r - r_1)/4$ such that

$$A_1 := A \cap \mathring{B}(x_1, r_1 - 2\delta) \neq \varnothing, \quad C_1 := C \cap \mathring{B}(x_1, r_1 - 2\delta) \neq \varnothing \ \text{ and } \ \varrho(A_1, C_1) < \delta.$$

Then there exist points $a_1 \in A_1$, $c_1 \in C_1$ such that $\varrho(a_1, c_1) < \delta$. For definiteness, we assume that $\varrho(a_1, c_1) = \|a_1 - c_1\,|$. We take c_1 for x_2 and define $r_2' := \delta$. Using the well-known equivalence

$$B(x, r) \subset \mathring{B}(x', r') \iff \|x - x'\,| < r' - r, \tag{2.1}$$

$$\mathring{B}(x, r) \subset \mathring{B}(x', r') \iff \|x - x'\,| \leq r' - r, \tag{2.2}$$

we have $B_2 := B(x_2, r_2') \subset \mathring{B}(x_1, r_1)$; in addition, it is clear that $\mathring{B}(x_1, r_2') \cap A \neq \varnothing$, $\mathring{B}(x_1, r_2') \cap C \neq \varnothing$. Arguing as above, we may find a number $0 < r_2 < r_2'$ such that the ball $\mathring{B}(x_2, r_2)$ intersects both A and C.

The induction step. There exists a number $0 < r_n < r_n'$ such that the ball $\mathring{B}(x_n, r_n)$ intersects both A and C. If the ball $B(x_n, r_n)$ satisfies the condition of the lemma, there is nothing to prove. Otherwise, we may choose $0 < \delta < (r_n' - r_n)/4$ so as to have

$$A_n := A \cap \mathring{B}(x_n, r_n' - 2\delta) \neq \varnothing, \quad C_n := C \cap \mathring{B}(x_n, r_n' - 2\delta) \neq \varnothing \ \text{ and } \ \varrho(A_n, C_n) < \delta.$$

[2)]This distance is sometimes referred to as the set-set distance.

Hence there exist points $a_n \in A_n$, $c_n \in C_n$ such that $\varrho(a_n, c_n) < \delta$. For definiteness, we assume that $\varrho(a_n, c_n) = \|a_n - c_n|$. We take c_n as x_{n+1}. Let $r'_{n+1} := \delta$. Now from (2.1) we have $B_{n+1} := B(x_{n+1}, r'_{n+1}) \subset \mathring{B}(x_n, r_n)$. Continuing the induction process, we obtain a sequence (x_n) such that

$$\|x_{n+1} - x_n| \le r_n - r_{n+1}, \tag{2.3}$$

and the ball $\mathring{B}(x_n, r_n)$ intersects both A and C. Inequality (2.3) means that (x_n) is a Cauchy sequence. By the assumption, the space is left-complete, and hence there exists a point $x \in X$ such that $\|x_n - x| \to 0$ as $n \to \infty$. Let $a_n \in \mathring{B}(x_n, r_n) \cap A$, $c_n \in \mathring{B}(x_n, r_n) \cap C$. We have

$$\|a_n - x| \le \|a_n - x_n| + \|x_n - x| \to 0, \qquad \|c_n - x| \le \|c_n - x_n| + \|x_n - x| \to 0.$$

The sets A and C are closed (with respect to the ball $\mathring{B}$), $x \in A \cap C$, but this contradicts the assumption $A \cap C = \varnothing$. Lemma 2.3 is proved. □

Definition 2.4. We say that the metric projection P_M is *ϱ-continuous* at a point x if the condition $\|x_n - x\|_{\mathrm{sym}} \to 0$ implies that

$$\varrho(P_M x, P_M x_n) \to 0.$$

In the case of (symmetric) normed linear spaces, ϱ-continuity was referred to as the continuity in the weak sense by Vlasov [39]; see also [12]. In the symmetric case, ϱ-continuity of the metric projection to convex closed sets was also studied by E. V. Oshchman and N. V. Nevesenko (see, for example, [12]). In particular, Nevesenko characterized the Banach spaces in which the metric projection onto any nonempty convex closed set is ϱ-continuous.

In the classical normed setting, there are several results that guarantee B- (or $\mathring{B}$-) connectedness for P- (or P_0-) connected sets (see, for example, [30], [6]). The first results in this direction are due to D. Wulbert and L. Vlasov (see, for example, [4]). For example, it is well known that in each Banach space any Chebyshev set with continuous metric projection is $\mathring{B}$-connected (i.e., its intersection with any open ball is connected). Among abundant extensions of the results of Wulbert, Vlasov, and many others in this direction, we mention the following one (see [4, § 5]):

> in any normed linear space, any P-connected set with upper semicontinuous metric projection is $\mathring{B}$-connected. (2.4)

A number of results in this direction are also known for symmetrizable asymmetric spaces (see [13], [8]).

In Theorems 2.5–2.17 that follow, we obtain results on $\mathring{B}$-connectedness of P-connected sets under various conditions. These results partially extend the corresponding results for normed spaces and symmetrizable asymmetric normed spaces to the case of arbitrary asymmetric spaces.

An analogue of the following result for Banach spaces was obtained by Vlasov [39].

Theorem 2.5. *Let X be a left-complete asymmetric normed space, $M \subset X$ be a P-connected set with ϱ-continuous metric projection. Then M is $\mathring{B}$-connected.*

Proof. Assume on the contrary that M is not $\mathring{B}$-connected. Then there exist $x \in X$ and $r > 0$ such that $\mathring{B} \cap M = A \sqcup C$, where $\mathring{B} := \mathring{B}(x, r)$, A, C are nonempty open-closed (in $\mathring{B}$) disjoint sets. By Lemma 2.3, there exist a $\delta > 0$ and a ball $B_1 := B(z, r_1) \subset \mathring{B}$ such that if

$$B_1 \cap A \ne \varnothing, \quad B_1 \cap C \ne \varnothing, \ \text{ then } \ \varrho(B_1 \cap A, B_1 \cap C) \ge \delta.$$

Let $a \in A$, $c \in C$. By the hypothesis, $P_M z$ is nonempty and connected, and hence $P_M z$ lies in one of the sets A or C. It can be assumed without loss of generality that $P_M z \subset C$. Let w be the farthest point from z on the interval $[z, a]$ such that $P_M v \subset C$ for each point $v \in [z, w)$ (such a farthest point w exists, because the restriction of the distance function $\rho(\,\cdot\,, M)$ to any interval is continuous). The set $P_M w$ is connected, and hence $P_M w$ lies either in C or in A. Assume that $P_M w \subset C$. There exists a sequence $w_n \subset (w, a]$ such that

$\|w_n - w| \to 0$ and $P_M w_n \subset A$. By ϱ-continuity of the metric projection, $\varrho(P_M w, P_M w_n) \to 0$. As a result, $\varrho(C, A) = 0$, a contradiction. Now we assume that $P_M w \subset A$. A similar analysis produces a sequence $w_n \subset (w, z]$ such that $\|w_n - w| \to 0$ and $P_M w_n \subset C$. As before, by ϱ-continuity of the metric projection we have $\varrho(P_M w, P_M w_n) \to 0$, which implies again that $\varrho(C, A) = 0$. This contradiction proves Theorem 2.5. □

Remark 2.6. In the actual fact, under the hypotheses of Theorem 2.5 it suffices to require that the restriction of the metric projection onto a P-connected set to any (finite) interval should be ϱ-continuous (or continuous).

The following definition is new.

Definition 2.7. A set M will be said to be *externally strongly complete* if, for each Cauchy sequence $(z_n) \subset X$ approaching the set M (i.e., $\rho(z_n, M) \to 0$), there exists a point $\widehat{z} \in M$ such that $\|z_n - \widehat{z}\|_{\mathrm{sym}} \to 0$.

The following result can be useful.

Proposition 2.8. *Let one of the following two conditions be satisfied*: (1) *X is left-complete with closed unit ball and let $M \subset X$ be closed*; (2) *M is externally strongly complete. Assume that M is not $\mathring{B}$-connected, i.e., there exists a ball $\mathring{B}(x, r)$ such that $\mathring{B}(x, r) \cap M = A \sqcup C$, where A, C are nonempty open-closed subsets of $\mathring{B}(x, r)$. Then there exists an $\varepsilon_0 > 0$ such that no ball $\mathring{B}(u, \varepsilon_0)$, $u \in X$, can intersect both A and C.*

Proof. Assume the contrary. For any point x_1 such that $\|x_1 - x| < r/2$ by (2.1) we have $B(x_1, r/2) \subset \mathring{B}(x, r)$. Let $r_1 := r/2$. By the assumption, the ball $\mathring{B}(x_1, r_1)$ intersects both A and C. Proceeding by induction, we will construct a sequence (x_n) such that $\|x_{n+1} - x_n| < r_1 := r/2^n$ and that the ball $\mathring{B}(x_n, r_n)$ intersects both A and C. In case (1), since $\mathring{B}(x_n, r_n) \cap (A \sqcup C) \neq \varnothing$, $A, C \subset M$ and $r_n \to 0$, since X is left-complete, and since the ball $B(0, 1)$ is closed, there exists an $x \in B(x_1, r/2) \subset \mathring{B}(x, r)$ such that $\|x_n - x| \to 0$. Let $a_n \in (\mathring{B}(x_n, r_n) \cap A)$, $c_n \in (\mathring{B}(x_n, r_n) \cap C)$. Then

$$\|a_n - x| \leq \|a_n - x_n| + \|x_n - x| \to 0 \tag{2.5}$$

and, similarly, $\|c_n - x| \to 0$. By the hypothesis, A is closed, and so from (2.5) we have $x \in A$. Similarly $x \in C$. However, this contradicts the fact that $A \cap C = \varnothing$. In case (2), there exists an $x \in M$ such that $\|x_n - x\|_{\mathrm{sym}} \to 0$. Now a similar argument shows that $x \in A \cap C$. Proposition 2.8 is proved. □

Definition 2.9. A set M is called *right- (left-) approximatively compact* if the conditions $(y_n) \subset M$, $\|y_n - x| \to \rho(x, M)$ (respectively, $\|x - y_n| \to \rho^-(x, M)$) imply that there exists a subsequence (y_{n_k}), (left-) converging to a point $\widehat{y} \in M$, i.e., $\|y_{n_k} - \widehat{y}| \to 0$. The corresponding point x is a point of *right* (*left*) *approximative compactness* for M.

Below $\mathrm{AC}(M)$ is the set of points of (right) approximative compactness for M.

Remark 2.10. If x is a point of right approximative compactness for M, then the point $\widehat{y}$ from Definition 2.9 may fail to be a nearest point from M for x (the case $\widehat{y} \notin P_M x$, of course, is impossible in the symmetrizable case). See also Remark 2.12 below.

Correspondingly, to exclude the "improper" case $\widehat{y} \notin P_M x$, where $x \in \mathrm{AC}(M)$, we introduce the following definition.

Definition 2.11. A set M is said to be *regularly* (*right-*) *approximatively compact* if the conditions $(y_n) \subset M$, $\|y_n - x| \to \rho(x, M)$, imply that there exist a point $\widehat{y} \in P_M x$ and a subsequence (y_{n_k}) converging to the point $\widehat{y}$, i.e., $\|y_{n_k} - \widehat{y}| \to 0$. The corresponding point x is called a point of regular (right-) approximative compactness.

A set $M \subset X$ is *regularly left-approximatively compact* if the conditions $(y_n) \subset M$, $\|x - y_n| \to \rho^-(x, M)$, imply that there exist a point $\widehat{y} \in P^-_M x$ and a subsequence (y_{n_k}) (left-) converging to the point $\widehat{y}$, i.e., $\|y_{n_k} - \widehat{y}| \to 0$.

Remark 2.12. If x is a point of left approximative compactness for M, then the inclusion $\widehat{y} \in P^-_M x$ always holds. As a corollary, a left-approximatively compact set is necessarily regularly left-approximatively compact.

Remark 2.13. It is easily checked that if the ball $B(0,1)$ of a space X is closed[3)], then in X

right-approximative compactness of a set is equivalent to its regular right-approximative compactness.

In [8, Theorem 2 and Remark 5], we showed, in particular, that if X is an asymmetric normed space in which the ball $B(0,1)$ is closed, and $M \subset X$ is (right-) approximatively compact and $\mathring{B}$-complete, then P-connectedness of M implies its B-connectedness. The following result partially strengthens this fact.

Corollary 2.14. *Let X be a left-complete asymmetric normed space, $M \subset X$ be a P-connected regularly right-approximatively compact set. Then M is $\mathring{B}$-connected.*

Proof. We choose an interval $[a,z]$ as in the proof of Theorem 2.5. If the restriction of the metric projection to any interval is ϱ-continuous, then the required result follows from Theorem 2.5 and Remark 2.6. Assume now that the restriction of the metric projection P_M to the interval $[a,z]$ is not ϱ-continuous. Then there exist a point $x \in [a,z]$ and a sequence $(x_n) \subset [z,a]$, $\|x_n - x\|_{\mathrm{sym}} \to 0$ such that

$$\varrho(P_M x, P_M x_n) \geq \sigma \tag{2.6}$$

for some $\sigma > 0$ for all sufficiently large n. Let $y_n \in P_M x_n$ be arbitrary. We have

$$\rho(x,M) \leq \|y_n - x| \leq \|y_n - x_n| + \|x_n - x| = \rho(x_n,M) + \|x_n - x| \to \rho(x,M).$$

This shows that (y_n) is a minimizing sequence for x. By the assumption, M is regularly right-approximatively compact, and hence there exists a subsequence (n_k) such that $\|y_{n_k} - y^*| \to 0$ for some point $y^* \in P_M x$. We have

$$0 < \sigma \overset{(2.6)}{\leq} \lim_{k\to\infty} \varrho(P_M x, P_M x_{n_k}) = 0,$$

which is absurd. As a result, the metric projection is ϱ-continuous on $[a,z]$. Corollary 2.14 is proved. □

The following result partially extends Vlasov's result (2.4), and also Theorem 2.5 and Corollary 2.14.

Theorem 2.15. *Let X be an asymmetric normed space in which the ball $B(0,1)$ is closed and let $M \subset X$ be right-approximatively compact. Then P-connectedness of M implies its $\mathring{B}$-connectedness.*

Proof. Assume that M is not $\mathring{B}$-connected. Then there exists a ball $\mathring{B}(x,r)$ such that $\mathring{B}(x,r) \cap M = A \sqcup C$, where A, C are nonempty open-closed (in the intersection $\mathring{B}(x,r) \cap M$) disjoint sets. Without loss of generality we assume that $\rho(x,A) \geq \rho(x,C)$. Let $a \in A$. The restrictions of the distance functions $\rho(\,\cdot\,,A)$ and $\rho(\,\cdot\,,C)$ to the interval $[a,x]$ are continuous, and hence there exists a point $x_0 \in [a,x]$ such that $\rho(x_0,A) = \rho(x_0,C) =: r_0 > 0$ and $B(x_0,r_0) \subset \mathring{B}(x,r)$. It follows that

$$\text{the set } P_M x_0 \text{ is disconnected.} \tag{2.7}$$

Indeed, let (a_k) and (c_k) be minimizing sequences, respectively, from the sets A and C for the point x_0. By the assumption, M is approximatively compact, and hence some subsequences $(a_{k_m}) \subset (a_k)$ and $(c_{k_m}) \subset (c_k)$ converge, respectively, to points a' and c' from M. By the hypothesis, the ball B is closed, and hence $a' \in P_M x_0$, $c' \in P_M x_0$ (see Remark 2.13). We have $B(x_0,r_0) \subset \mathring{B}(x,r)$, and so $B(x_0,\rho(x_0,M)+\delta) \subset \mathring{B}(x,r)$ for sufficiently small $\delta > 0$. Further, the sets A and C are closed relative to $M \cap \mathring{B}(x,r)$. Hence $a' \in A$ and $c' \in C$, which proves (2.7). However (2.7) contradicts P-connectedness of the set M. Theorem 2.15 is proved. □

3)The condition that the ball $B(0,1)$ is closed in X is equivalent to the continuity of the norm $\|\cdot| : (X,\|\cdot|) \to (\mathbb{R},\|\cdot|)$ and implies regularity of the space $X = (X,\|\cdot|)$ (see [18]). We also note that there is an example of an asymmetric Hausdorff space X in which the ball $B(0,1)$ is not closed [18].

Remark 2.16. If, under the hypotheses of Theorem 2.15, the set M is regularly right approximatively compact, then the condition in Theorem 2.15 that the ball $B = B(0,1)$ of the space X is closed becomes superfluous (see also Remark 2.13). This result strengthens Corollary 2.14, because now it is not assumed that the space is left-complete.

We require the following auxiliary result (see [8, Corollary 1]).

Lemma 2.A. *Let M be a $\mathring{B}$-complete subset of an asymmetric space. Then*

$$M \in (\mathring{B}) \cap (P_0) \iff M \in (B). \tag{2.8}$$

In [8, Theorem 2] it was shown that if X is a symmetrizable asymmetric normed space and $M \subset X$ is approximatively compact and $\mathring{B}$-complete, then P-connectedness of M implies its B-connectedness and $\mathring{B}$-path-connectedness. The following result partially extends this result to the case of arbitrary (not necessarily symmetrizable) asymmetric T_1-spaces.

Theorem 2.17. *Assume that at least one of the following conditions is satisfied*:
(1) *X is an asymmetric normed space and $M \subset X$ regularly right-approximatively compact;*
(2) *X is an asymmetric normed space with closed ball B and $M \subset X$ is right-approximatively compact.*
Let M be $\mathring{B}$-complete. Then P-connectedness of M implies its B-connectedness.

Proof. In case (1), the required result follows from Remark 2.16 and Lemma 2.A, and in case (2), from Lemma 2.A and Theorem 2.15. □

3. Density of points of approximative uniqueness in uniformly convex asymmetric spaces

Sets (and points) of approximative uniqueness (see, for example, [34], [33], [6]) have been studied rather intensively over the recent years. We recall the corresponding definitions (see [31] and [34]).

Definition 3.1. Let $M \subset X$ and x be a point of (right-) approximative compactness for M. If, for x, there exists a unique point $y \in M$ such that $\|y - y_n| \to 0$ as $n \to \infty$ for each sequence $(y_n) \subset M$ such that $\|y_n - x| \to \rho(x, M)$, then $x \in X$ is called a *point of (right-) approximative uniqueness* for M (written, $x \in \mathrm{AU}(M)$).

Definition 3.2. A subset M of an asymmetric normed space $X = (X, \|\cdot|)$ is called *left-strongly closed* if, for each sequence $(x_n) \subset M$, the condition $\|x_n - x| \to 0$ $(n \to \infty)$ implies that $x \in M$ and $\|x_n - x\|_{\mathrm{sym}} \to 0$ $(n \to \infty)$.

Following [32], [34], we recall the definition of a uniformly convex asymmetric space.
Given $x, y \in X$, we set

$$\Delta(a) := \|x - ay| + a\|y| - \|x|, \quad a \in [0,1].$$

Definition 3.3. An asymmetric space $X = (X, \|\cdot|)$ is called *uniformly convex* if, for all $\varepsilon > 0$ and $a \in (0,1]$, there exists a $\delta > 0$ such that, for all $x, y \in X$, $\|x| = \|y| = 1$, the condition $\Delta(a) < \delta$ implies that $x \in B(\mu y, \varepsilon)$ for some $\mu \in [1-\varepsilon, 1]$.

From this definition it follows that, in any uniformly convex X, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that, for all $x, y \in X$, $\|x| = \|y| = 1$, the condition $\|(x+y)/2| \geqslant 1 - \delta$ implies that $\|x - \mu y| \leqslant \varepsilon$) for some $\mu \in [1-\varepsilon, 1]$.

Remark 3.4. In relation to Definition 3.3 it is worth pointing out that, in the asymmetric case, definitions of uniformly convex bodies (or, what is the same, of uniformly convex functions — their level surfaces are, of course, uniformly convex bodies) were given by many authors (E. S. Levitin, C. Zanco, A. Zucchi, B. T. Polyak, M. V. Balashov, P. A. Borodin, etc; see, for example, [16] and [13]), but *only* in the symmetrizable case. Definition 3.3 is undoubtedly superior to the previous ones because the former applies to arbitrary (not necessarily symmetrizable) spaces with asymmetric norm and is capable of delivering direct asymmetric analogues of classical results for symmetric uniformly convex spaces.

Definition 3.5. An asymmetric space $X = (X, \|\cdot|)$ is called *locally uniformly convex* (see [34]) if, for each $y \in S$ (S is the unit sphere), $\varepsilon > 0$, and $a \in (0,1]$, there exists a $\delta > 0$ such that, for each $x \in S$, the condition $\|x - ay| + a\|y| - \|x| < \delta$ implies that $\|x - y| < \varepsilon$.

The following auxiliary result was proved in [34, Lemma 1].

Lemma 3.B. *Let X be a uniformly convex asymmetric seminormed space, $\Delta \in (0,1)$, $M \subset X$, $g_0 \in X \setminus M$, $r = \rho(g_0, M) > 0$. Then, for each $\varepsilon \in (0,1)$, there exists a $\delta_0 \in (0, \varepsilon/8)$ such that*

$$u \in B(\mu(\widetilde{u}_0 - g_0) + g_0, \varepsilon) \ \textit{for some} \ \ \mu \in [1-\varepsilon, 1]$$

for an arbitrary $\widetilde{u}_0 \in M$ such that $\|\widetilde{u}_0 - g_0| < r + \delta_0$, and any $u \in M$, $\|u - g_1| < \rho(g_1, M) + \delta_0$, where $g_1 := g_0 + \Delta(\widetilde{u}_0 - g_0)/\|\widetilde{u}_0 - g_0|$.

We require the following auxiliary result of independent interest.

Lemma 3.6. *Let X be a uniformly convex asymmetric normed space complete with respect to the symmetrization norm, and let $\varnothing \neq M \subset X$ be an externally strongly complete set. Then any $\|\cdot\|_{\mathrm{sym}}$-neighborhood of an arbitrary point $x \in X \setminus M$ contains a point of existence for M.*

Proof. We argue as in the proof of Lemma 3.B. It can be assumed without loss of generality that $\rho(x, M) = 1$. We claim that any arbitrary neighborhood $\mathscr{O}_\sigma^{\mathrm{sym}}(x) := \{u \mid \|x - u\|_{\mathrm{sym}} \leq \sigma\}$ contains a required point of existence $v_0 \in X \setminus M$ for M (i.e., $v_0 \in \mathrm{E}(M)$).

Let $\Delta \in (0, \min\{1/3, \sigma/3\})$. We proceed by induction to construct $\|\cdot\|_{\mathrm{sym}}$-Cauchy sequences $(y_n) \subset M$ and $(x_n) \subset X$. For any sufficiently small $\varepsilon \in (0,1)$, we define $\varepsilon_n = \frac{\varepsilon}{2^n}$, $\delta_0 = \varepsilon$ and $\sigma_0 = \Delta/2$. We also set $x_0 = x$.

1^0. By Lemma 3.B, we can find a number $\delta_1 \in (0, \varepsilon_1/8)$, a point $y_1 \in M$, $\|y_1 - x| < \rho(x, M) + \delta_1$, and a point $x_1 \in [x, y_1]$, $\|x_1 - x\|_{\mathrm{sym}} = \Delta_1 := \frac{1}{2}\Delta$ (in this case, $\|y_1 - x_1| < \rho(x_1, M) + \delta_1$) such that, for an arbitrary $u \in M$, $\|u - x_1| < \rho(x_1, M) + 3\delta_1$, we have $u \in B(\mu_1(y_1 - x) + x, \varepsilon_1)$ for some $\mu_1 \in [1 - \varepsilon_1, 1]$. We choose $\sigma_1 \in (0, \sigma)$ so that $\sigma_1 < \delta_0/2$ and that $\rho(x_1, M) \leqslant \rho(z, M) + \delta_0/2$ for each $z \in B(x_1, \sigma_1)$.

2^0. Suppose that points $(y_k)_{k=1}^n \subset M$, $(x_k)_{k=1}^n \subset X$ and numbers $(\delta_k)_{k=1}^n$, $(\mu_k)_{k=1}^n$, $(\sigma_k)_{k=1}^n$ $(n \geqslant 2)$ are constructed so that, for all k and $n \geq 2$,

$$\begin{aligned} &y_k \in M \quad \text{and} \quad \|y_k - x_{k-1}| < \rho(x_{k-1}, M) + \delta_k, \\ &x_k \in [x_{k-1}, y_k] \quad \text{and} \quad \|x_k - x_{k-1}\|_{\mathrm{sym}} = \Delta_k := \frac{1}{2}\min\{\Delta_{k-1}, \sigma_{k-1}, \delta_{k-1}\} \end{aligned}$$

(in this case $\|y_k - x_k| < \rho(x_k, M) + \delta_k$), and in addition, for each $u \in M$ such that $\|u - x_k| < \rho(x_k, M) + 3\delta_k$, we have $u \in B(\mu_k(y_k - x_{k-1}) + x_{k-1}, \varepsilon_k)$ for some $\mu_k \in [1 - \varepsilon_k, 1]$. Hence $y_{k+1} \in B(\mu_k(y_k - x_{k-1}) + x_{k-1}, \varepsilon_k)$ $(k < n)$. Moreover, $\sigma_k < \delta_{k-1}/2$ and, for each $z \in B(x_k, \sigma_k)$, we have $\rho(x_k, M) \leqslant \rho(z, M) + \delta_{k-1}/2$.

By Lemma 3.B, there exist a $\delta_{n+1} \in (0, \min_{k=1,n} \delta_k)$ and a point $y_{n+1} \in M$, $x_{n+1} \in [x_n, y_{n+1}]$, such that

$$\|y_{n+1} - x_n| < \rho(x_n, M) + \delta_{n+1}, \quad \|x_{n+1} - x_n\|_{\mathrm{sym}} = \Delta_{n+1} := \frac{1}{2}\min\{\Delta_n, \sigma_n, \delta_n\} \tag{3.1}$$

and

$$u \in B(\mu_{n+1}(y_{n+1} - x_n) + x_n, \varepsilon_{n+1}) \ \text{ for some } \ \mu_{n+1} \in [1 - \varepsilon_{n+1}, 1].$$

for each $u \in M$, $\|u - x_{n+1}| < \rho(x_{n+1}, M) + 3\delta_{n+1}$. It is clear that

$$\|y_{n+1} - x_{n+1}| < \rho(x_{n+1}, M) + \delta_{n+1}. \tag{3.2}$$

In addition, $y_{n+1} \in B(\mu_n(y_n - x_{n-1}) + x_{n-1}, \varepsilon_n)$. We choose $\sigma_{n+1} \in (0, \min_{k=1,n} \sigma_k)$ so as to have $\sigma_{n+1} < \delta_n/2$ and that $\rho(x_{n+1}, M) \leqslant \rho(z, M) + \delta_n/2$ for each $z \in B(x_{n+1}, \sigma_{n+1})$.

3^0. We have

$$\Big\|\Big(\prod_{k\geqslant n+1}\mu_k\Big)y_{n+1}-\Big(\prod_{k\geqslant n}\mu_k\Big)y_n\Big| = \Big(\prod_{k\geqslant n+1}\mu_k\Big)\Big\|y_{n+1}-\mu_n y_n\Big|$$
$$\leqslant \|y_{n+1}-(\mu_n(y_n-x_{n-1})+x_{n-1})| + \|\mu_n(y_n-x_{n-1})+x_{n-1}-\mu_n y_n|$$
$$\leqslant \varepsilon_n + \|(1-\mu_n)x_{n-1}| \leqslant 2\varepsilon_n.$$

So, $\widehat{y_n} := (\prod_{k\geqslant n}\mu_k)y_n$ is a Cauchy sequence in X. Further, by (3.1) the series $\sum_n \|x_{n+1}-x_n\|_{\mathrm{sym}}$ converges, and hence (x_n) is a Cauchy sequence with respect to the symmetrization norm $\|\cdot\|_{\mathrm{sym}}$, Therefore, since the space is complete, there exists a point $\bar{x}\in X$ such that $\|x_n-\bar{x}\|_{\mathrm{sym}}\to 0$ $(n\to\infty)$. It is well known (see, for example, [17, Proposition 1.1.12] and [10]) that if $\varnothing\neq N\subset X$ and $u,w\in X$, $N\subset X$, then

$$\rho(u,N)\leqslant \|w-u\,| + \rho(w,N), \tag{3.3}$$

Now using (3.3), we have

$$\|y_n\,| \leq \|y_n - x_n\,| + \|x_n\,| \overset{(3.2)}{\leq} \rho(x_n,M)+\delta_n+\|x_n\,| \overset{(3.3)}{\leq} \rho(\bar{x},M)+\|\bar{x}-x_n\,|+\delta_n+\|x_n\,|,$$

and hence since $\|x_n-\bar{x}\|_{\mathrm{sym}}\to 0$ and since by construction the sequence (δ_n) is bounded, it follows that the sequence $(\|y_n|)$ is also bounded. Further,

$$\rho(\widehat{y_n},M)\leq \|y_n-\widehat{y_n}\,| = \Big(1-\prod_{k\geqslant n}\mu_k\Big)\|y_n| \to 0 \tag{3.4}$$

(here, we used the fact that $\varepsilon_n = \frac{\varepsilon}{2^n}$, $\mu_n\in[1-\varepsilon_n,1]$ and that the sequence $(\|y_n\,|)$ is bounded). By the assumption, M is externally strongly complete, and now, in view of (3.4) there exists a $y\in M$ such that

$$\|\widehat{y_n}-y\|_{\mathrm{sym}}\to 0 \qquad (n\to\infty). \tag{3.5}$$

Next, $\widehat{y_n} := (\prod_{k\geqslant n}\mu_k)y_n$ and $\prod_{k\geqslant n}\mu_k\to 1$, and now from (3.5) we have

$$\|y_n-y\|_{\mathrm{sym}}\to 0 \qquad (n\to\infty). \tag{3.6}$$

Hence (y_n) is a Cauchy sequence with respect to $\|\cdot\|_{\mathrm{sym}}$. Consequently,

$$\|y_n-y| \leqslant \|y_n-\widehat{y_n}| + \|\widehat{y_n}-y| \to 0 \quad (n\to\infty).$$

and

$$\rho(\bar{x},M)\leqslant \|y_n-\bar{x}| \leqslant \|y_n-x_n| + \|x_n-\bar{x}| = \rho(x_n,M)+\delta_n+\|x_n-\bar{x}|$$

for all $n\in\mathbb{N}$. As a corollary, $\bar{x}\in\mathscr{O}_\sigma^{\mathrm{sym}}(x)$ and $\rho(\bar{x},M)+\frac{\delta_n}{2}\geqslant\rho(x_n,M)$. Further,

$$\|y_n-\bar{x}\,|\to\rho(\bar{x},M) \quad\text{and}\quad \rho(x_n,M)\to\rho(\bar{x},M)\ \ (n\to\infty). \tag{3.7}$$

So,

$$\rho(\bar{x},M)\leqslant \|y-\bar{x}| \leqslant \|y-y_n| + \|y_n-\bar{x}| \overset{(3.6),(3.7)}{\to} \rho(\bar{x},M) \quad (n\to\infty), \tag{3.8}$$

i.e., $y\in M$ is a nearest point from M for $\bar{x}$. Lemma 3.6 is proved. □

Remark 3.7. The proof of Lemma 3.6 shows that in the actual fact the condition that M is externally strongly complete in Lemma 3.6 can be replaced by the condition that there exists a point x_0 such that the set

$$\mathrm{con}(x_0,M) := \{w\in X \mid w\in[x_0,z]\ \text{ for some } z\in M\}$$

is externally strongly complete.

Recall that a set of *first category* (or a *meager set*) is a union of countably many nowhere dense sets. The class of subsets of X of first category will be denoted by $\mathrm{I}(X)$, and the class of sets of *second category* (i.e., complements of sets of first category) in X will be denoted by $\mathrm{II}(X)$. The class of sets of second category with respect to the symmetrization norm is denoted by $\mathrm{II}_{\mathrm{sym}}(X)$.

In the normed space setting it is well known (see, for example, [4, § 4]) that

in Efimov–Stechkin spaces and, in particular, in complete uniformly convex spaces, the set of points of approximative compactness of a closed set is a set of second category. (3.9)

The following two theorems partially extend (3.9) to the asymmetric case (for the first result, see [34, § 3]).

Theorem 3.A. *Let X be a complete (or complete with respect to the symmetrization norm) locally uniformly convex asymmetric space, $M \subset X$ be a closed set for which the set of points of existence for M in X is dense in X with respect to the asymmetric norm of X (or the symmetrization norm). Then the set of points of approximative uniqueness for M is of second category with respect to the asymmetric norm of X (of second category with respect to the symmetrization norm), i.e.,*

$$\mathrm{AU}(M) \in \mathrm{II}(X) \quad \textit{for } M \subset X : \overline{\mathrm{E}(M)} = X,$$
$$\mathrm{AU}(M) \in \mathrm{II}_{\mathrm{sym}}(X) \quad \textit{for } M \subset X : \overline{\mathrm{E}_{\mathrm{sym}}(M)} = X.$$

From Lemma 3.6 and Theorem 3.A we have the following result.

Theorem 3.8. *Let X be a uniformly convex asymmetric space complete with respect to the symmetrization norm and let $\varnothing \neq M \subset X$ is an externally strongly complete set. Then the set of point of approximative uniqueness for M is of second category with respect to the symmetrization norm i.e.,*

$$\mathrm{AU}(M) \in \mathrm{II}_{\mathrm{sym}}(X), \quad \textit{where } \varnothing \neq M \subset X \textit{ is externally strongly complete.}$$

4. Stability of best and near-best approximants at points of approximative compactness

In this section, we study stability of best approximants in asymmetric spaces. Several known results for normed spaces are carried over to the case of arbitrary asymmetric spaces.

Theorem 4.1. *Let X be an asymmetric space, M be a nonempty set, $x \notin M$, $x \in \mathrm{AC}(M)$, $y \in P_M x$. Then*

$$[x, y] \subset \mathrm{AC}(M),$$

i.e., any point from the interval $[x, y]$ is a point of approximative compactness for M.

Proof. As in the normed case, it is easily checked that if

$$N \subset X, \quad u \in X, \quad v \in P_M x \text{ and } w \in [u, v], \text{ then } v \in P_M w. \tag{4.1}$$

Indeed, using (3.3), we have

$$\rho(w, N) \leq \|w - v\| = \|u - v\| - \|u - w\| = \rho(u, N) - \|u - w\| \leq \rho(w, N),$$

i.e., $\|w - v\| = \rho(w, N)$ and $v \in P_N w$.

Now let x, y satisfy the hypotheses of Theorem 4.1, $z \in (x, y)$. By (4.1), we have $y \in P_M z$. Let $(v_n) \subset M$ be a minimizing sequence from M for z, i.e., by definition $\|v_n - z| \to \rho(z, M) + 0$. It is clear that

$$\|y - x| \leq \|v_n - x| \leq \|v_n - z| + \|z - x| \to \|y - z| + \|z - x| = \|y - x|.$$

This shows that $\|v_n - x| \to \|y - x| = \rho(x, M)$, i.e., (v_n) is a minimizing sequence from M for z. Since $x \in \mathrm{AC}(M)$, by definition, (v_n) contains a converging subsequence to some point from M. $\square$

The following definition has no analogue in the symmetrizable spaces (and, of course, in the case of normed linear spaces), because in this case the distance function to a set (the metric function) is always 1-Lipschitz, and hence, continuous.

Below, $\mathrm{C}(\rho(M))$ is the set of points of continuity of the distance function $\rho(\,\cdot\,,M)$ for a given nonempty set $M\subset X$ (i.e., $x\in \mathrm{C}(\rho(M))$ if and only if $\rho(x_n,M)\to\rho(x,M)$ for each sequence (x_n) such that $\|x_n-x|\to 0$).

Remark 4.2. It is worth pointing out that in asymmetric spaces (unlike the classical normed spaces case) the continuity of the metric projection onto M does not generally imply continuity of the distance function $\rho(\,\cdot\,,M)$ (for more details, see [36]).

Theorem 4.3. *Let X be an asymmetric space in the which the ball $B(0,1)$ is closed, M be a nonempty left-strongly closed set, $x\notin M$, $P_Mx=\{y\}$. Assume that x is a point of continuity of the distance function and is a point of approximative compactness for M (i.e., $x\in \mathrm{C}(\rho(M))\cap \mathrm{AC}(M)$). Then any point $z\in[x,y]$ is a point of continuity of the distance function and point of approximative compactness, i.e.,*

$$[x,y]\subset \mathrm{C}(\rho(M))\cap \mathrm{AC}(M).$$

Remark 4.4. In Theorem 4.3 we can omit the requirement that the ball $B(0,1)$ is closed if we assume instead that each point x is a point of regular approximative compactness for M (in the sense of Definition 2.11).

Proof. Let a sequence (z_n) left-converge to point $z\in[x,y]$, i.e., $\|z_n-z|\to 0$. We define the point x_n from the condition that the vectors $\overrightarrow{xx_n}$ and $\overrightarrow{zz_n}$ have the same direction and the triangles $\triangle zyz_n$ and $\triangle xyx_n$ are similar. By similarity,

$$\rho(z_n,M)\le\|y-z_n|=\frac{\|y-z|\cdot\|y-x_n|}{\|y-x|}=\alpha\|y-x_n|,\ \text{ where }\ \alpha:=\frac{\|y-z|}{\|y-x|}. \tag{4.2}$$

Let $y_n'\in M$ be a "almost nearest" point from M for the point x_n, i.e.,

$$\|y_n'-x_n|\le\rho(x_n,M)+1/n. \tag{4.3}$$

By the assumption x, is a point of continuity of the distance function, i.e., $\rho(x_n,M)\to\rho(x,M)$ as $\|x_n-x|\to 0$ (note that $\|x_n-x|\to 0$, since by the assumption $\|z_n-z|\to 0$ and the vectors $\overrightarrow{xx_n}$ and $\overrightarrow{zz_n}$ have the same direction). Hence from (4.3) and the clear inequality $\rho(x_n,M)\le\|y_n'-x_n|$ we have

$$\|y_n'-x_n|\to\rho(x,M). \tag{4.4}$$

By the assumption, x is a point of right approximative compactness for M, and so from (4.4) and Definition 2.9 it follows that the sequence (y_n') contains a subsequence (y_{n_k}') (which we identify without loss of generality with (y_n')) left-converging to some point $\widehat{y}\in M$, i.e., $\|y_n'-\widehat{y}|\to 0$. By Remark 2.13, $\widehat{y}\in P_Mx$ (here, we used the fact that the ball $B(0,1)$ is closed), and since $P_Mx=\{y\}$ by the assumption, we have $\widehat{y}=y$. Further, by the assumption M is left-strongly closed, and hence the convergence $\|y_n'-y|\to 0$ implies that

$$\|y_n'-y\|_{\mathrm{sym}}\to 0. \tag{4.5}$$

Applying (4.4) and (4.5), we obtain

$$\|y-x_n|\le\|y_n'-x_n|+\|y-y_n'|\to\rho(x,M) \tag{4.6}$$

Now from (4.6) and (4.2) and since $\rho(x,M)=\|y-x|$, we have

$$\varlimsup_{n\to\infty}\rho(z_n,M)\le\alpha\rho(x,M)=\|y-z|\le\rho(z,M),$$
$$\varliminf_{n\to\infty}\rho(z_n,M)\ge\rho(z,M)$$

(in the second inequality, we used the lower semicontinuity of the distance function $\rho(\,\cdot\,,M)$; see [32, p. 146]). Now it follows that $\rho(z_n,M)\to\rho(z,M)$. The inclusion $[x,y]\subset \mathrm{AC}(M)$ is secured by Theorem 4.1. Theorem 4.3 is proved. □

Recall that the *one-sided Hausdorff distance* (or the *deviation*) between sets M and N is defined as follows:

$$d(M,N) := \sup\{\rho(x,N) \mid x \in M\}.$$

Theorem 4.5. *Let X be an asymmetric space in which the unit ball $B(0,1)$ is closed, M be a nonempty left-strongly closed set, $x \notin M$, $P_M x = \{y\}$. Assume that x is a point of continuity of the distance function and a point of right approximative compactness (approximative uniqueness) for M. Then each point $z \in [x,y]$ is a point of continuity of the metric projection in the following sense:*

$$d(P_M^{\delta_n} z_n, \{y\}) \to 0 \quad as \ \ \|z_n - z| \to 0, \ \ \delta_n \to 0 \ \ in \ X \times \mathbb{R}, \ \delta_n \geq 0. \tag{4.7}$$

Proof. By Theorem 4.3, $z \in \mathrm{C}(\rho(M))$. It is clear that $P_M z = \{y\}$. Let $(z_n, \delta_n) \to (z, 0)$ in the sense of (4.7) and let $v_n \in P_M^{\delta_n} z_n$, i.e., $\|v_n - z_n| \leq \rho(z_n, M) + \delta_n$. Then

$$\|v_n - z| \leq \|v_n - z_n| + \|z_n - z| \leq \rho(z_n, M) + \delta_n + \|z_n - z| \to \rho(z, M),$$

i.e., (v_n) is a minimizing sequence from M for z. By the assumption, $x \in \mathrm{AC}(M)$, and hence $z \in \mathrm{AC}(M)$ by Theorem 4.3. Hence there exists a subsequence (v_{n_k}) (left-) converging to some point $\widehat{v} \in M$, i.e., $\|v_{n_k} - \widehat{v}| \to 0$. By the assumption, the unit ball $B(0,1)$ is closed, and hence by Remark 2.13, $\widehat{v} \in P_M z = \{y\}$. Theorem 4.5 is proved. □

Theorem 4.6. *Let X be an asymmetric space, M be a nonempty left-strongly closed set, $x \notin M$, $P_M x \neq \varnothing$. Assume that x is a point of continuity of the distance function and a point of right regular approximative compactness for M. Then the metric projection is continuous at x in the following sense:*

$$d(P_M^{\delta_n} x_n, P_M x) \to 0 \quad as \ \ (x_n, \delta_n) \to (x, 0) \ in \ X \times \mathbb{R}, \ \delta_n \geq 0.$$

Corollary 4.7. *Under the hypotheses of Theorem* 4.6, *any point $z \in [x,y]$ is a point of continuity of the restriction of the metric projection to the set of existence points* $\mathrm{E}(M)$.

5. Connectedness of intersections of sets with balls in uniformly convex asymmetric spaces

The following Vlasov's result is well known in the symmetrical setting (see [39]): *in a complete uniformly convex Banach space each P-connected set is B-connected*. Tsar'kov [30, Theorem 3] extended this result as follows: in any reflexive (CLUR)-space[4)] (and, in particular, in any complete uniformly convex space) each closed P_0-connected set is B-connected. On the other hand, in any nonreflexive Banach space, as an example of a disconnected P_0-connected one may take the union of two distinct parallel hyperplanes generated by a non-norm-attaining functional.) To the case of symmetrizable uniformly convex spaces, Vlasov's result was extended by Borodin [13]. In [8], the authors of the present paper showed that, in any symmetrizable asymmetric Efimov–Stechkin space, each closed P_0-connected set is B-path connected (see also [37]). In the following result, we partially extend the above results to the case of general (not necessarily symmetrizable) asymmetric normed spaces.

Definition 5.1. A set X is a *semilinear space* (or *cone*) over $\mathbb{R}$ if X is equipped with operations of addition of elements and multiplication by nonnegative numbers, and if, for arbitrary $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}_+$,

1) $x + y = y + x$;
2) $x + (y + z) = (x + y) + z$;
3) there exists a unique $\theta \in X$ such that $x + \theta = x$;
4) $\alpha(x + y) = \alpha x + \alpha y$;
5) $(\alpha + \beta)x = \alpha x + \beta y$;
6) $\alpha(\beta x) = (\alpha\beta)x$;
7) $0 \cdot x = \theta$; $1 \cdot x = x$.

[4)](CLUR) (compactly locally uniformly rotund) is the class of spaces such that $x \in S$, $y_n \in S$, $\|x + y_n\|/2 \to 1$ imply that (y_n) has a convergent subsequence. Any reflexive (CLUR)-space is an Efimov–Stechkin space. For more on (CLUR)-spaces, see [30].

Theorem 5.2. *Let K be a right-complete semilinear space, $X \subset K$ be a uniformly convex asymmetric linear space complete with respect to the symmetrization norm, whose right completion is K. Let $M \subset X$ be an externally strongly complete set. Assume that M is P_0-connected in K. Then M is $\mathring{B}$-connected in X.*

In addition, if M is $\mathring{B}$-complete, then M is B-connected.

Remark 5.3. In [37], some conditions are obtained under which P-connectedness of a set implies that it has connected (or path-connected) intersections with balls.

Proof. We need some auxiliary results. Let us first verify that if $x \in X$, $r > 0$ and $M \cap B(x,r) = A_1 \sqcup C_1$, where $A_1, C_1 \in \mathscr{F}(B(x,r))$, $\max\{\rho(x,A_1), \rho(x,C_1)\} < r$, then there exist $x_1 \in X$ and $r_1 > 0$ such that

$$B(x_1,r_1) \subset B(x,r) \quad \text{and} \quad \rho(x_1,M) = \rho(x_1,A_1) = \rho(x_1,C_1) < r_1 \tag{5.1}$$

(analogues of (5.1) in the normed space setting can be found in the papers by L. P. Vlasov, V. A. Koshcheev, I. G. Tsar'kov [30], etc.). Here, as above, $\mathscr{F}(N)$ is the class of closed subsets of a set N.

Let us prove (5.1). For definiteness, we assume that $\rho(x,A_1) < \rho(x,C_1)$ (the required result in the case of the equality is clear). Consider a point $y \in C_1$ such that $\rho(x,A_1) < \|y-x| < r$. Then the function $\varphi(\cdot) := \rho(\cdot,A_1) - \rho(\cdot,C_1)$ at the points x and y has values of different signs. The restriction of the function φ to the interval $[x,y]$ is continuous, and, therefore, there exists a point $x_1 \in [x,y]$ such that $\rho(x_1,A_1) = \rho(x_1,C_1) =: r_0$. Since $\|x_1-x| = \|y-x| - \|y-x_1| < r-r_0$, we have $B(x_1,r_1) \subset B(x, r_1+\|x-x_1|) = B(x,r)$ for $r_1 := r - \|x_1-x| > r_0$, which completes the proof of (5.1).

Let us first prove the following result. Let $M \subset X$ satisfy the hypotheses of the theorem, $x \in X$, $r > 0$ and let $M \cap B(x,r) = A_1 \sqcup C_1$, where $A_1, C_1 \in \mathscr{F}(B(x,r))$ and

$$d := \rho(x,A_1) = \rho(x,C_1) < r. \tag{5.2}$$

Then, for each $\varepsilon > 0$, there exist $x_0 \in X$ and $r_0 > 0$ such that

$$\begin{aligned} &B(x_0,r_0) \subset \mathring{B}(x,r), \quad \rho(x_0,A_1) = \rho(x_0,C_1) < r_0 \ \text{ and} \\ &A_1 \cap B(x_0,r_0) \subset \mathring{B}(y_0,\varepsilon) \ \text{ for some point } \ y_0 \in A_1. \end{aligned} \tag{5.3}$$

Let us verify (5.3). Let a point $z \in C_1$ be such that $\|z-x| < d + (r-d)/10$. For $\delta := \min\{(r-d)/10, \frac{1}{3}\rho(x,A_1)\}$, since the restriction of the function $\rho(\cdot,A_1) - \rho(\cdot,C_1)$ to the interval $[x,z]$ is continuous, there exists a point $z_1 \in (x,z)$ such that

$$\rho(z_1,C_1) = \rho(z_1,A_1) - 2\delta. \tag{5.4}$$

Let us show that A_1 is externally strongly complete. Let $(z_n) \subset X$ be a Cauchy sequence approaching the set A_1, i.e., $\rho(z_n,A_1) \to 0$. By the assumption M is externally strongly complete, and hence, there exists a point $z \in M$ such that $\|z_n - z\|_{\mathrm{sym}} \to 0$. Let $y_n \in A_1$: $\|y_n - z_n| \le \rho(z_n,A_1) + 1/n$. We have $\|y_n - z| \le \|y_n - z_n| + \|z_n - z| \to 0$. Since A_1 is closed, $z \in A_1$, which proves that A_1 is externally strongly complete. A similar analysis shows that C_1 is externally strongly complete. By Theorem 3.8, there exists a point $z_2 \in \mathrm{AU}(A_1)$ such that $\|z_2 - z_1\|_{\mathrm{sym}} \le \delta$. Let us show that

$$B(z_2, \rho(z_2,A_1) + \delta) \subset \mathring{B}(x,r). \tag{5.5}$$

We have $\rho(u,N) \leqslant \|v-u| + \rho(v,N)$ for arbitrary $u,v \in X$ and $N \subset X$ (see, for example, [10]), and hence

$$\begin{aligned} \rho(z_2,A_1) + \delta &\le \rho(z_1,A_1) + 2\delta = \rho(z_1,C_1) + 4\delta \\ &\le \|z-z_1| + 4\delta = \|z-x| - \|z_1-x| + 4\delta \\ &\le \|z-x| - (\|z_2-x| - \|z_2-z_1|) + 4\delta \\ &\le \|z-x| - \|z_2-x| + 5\delta \\ &\le d + \frac{r-d}{10} - \|z_2-x| + 5\cdot\frac{r-d}{10} \\ &= \frac{1}{10}(6r+4d) - \|z_2-x| < r - \|z_2-x|, \end{aligned}$$

which in view of (2.1) implies (5.5).

For $\delta > 0$ we set

$$P^{\delta}_{M}x := \Big\{y \in M \mid \|y - x| \le \rho(x, M) + \delta\Big\}.$$

By the assumption, X is uniformly convex, and hence from the definition of a point of approximative uniqueness we see that, for arbitrary $x \in \mathrm{AU}(M)$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$P^{\delta}_{M}x \subset \mathring{B}(y, \varepsilon), \qquad \text{where } \ P_M x = \{y\}. \tag{5.6}$$

Correspondingly, we have $z_2 \in \mathrm{AU}(A_1)$, and hence from (5.6) for some $\delta = \delta(\varepsilon) > 0$ we have

$$P^{\delta}_{A_1}z_2 \subset \mathring{B}(y_1, \varepsilon), \qquad \text{where } \ P_{A_1} z_2 = \{y_1\}. \tag{5.7}$$

Let $\alpha_1 = \alpha_1(\varepsilon)$ be such that $\rho(z_2, A_1) < \alpha_1 < \rho(z_2, A_1) + \delta$.

Setting $z_3 \in \mathring{B}(z_2, \alpha_1) \cap A_1$, we have

$$\rho(z_2, C_1) - \rho(z_2, A_1) < (\rho(z_1, C_1) + \delta) - (\rho(z_1, A_1) - \delta) \overset{(5.4)}{=} 0,$$
$$\rho(z_3, C_1) - \rho(z_3, A_1) = \rho(z_3, C_1) > 0,$$

and hence, since the restriction of the function $\rho(\,\cdot\,, C_1) - \rho(\,\cdot\,, A_1)$ to the interval $[z_2, z_3]$ is continuous, there exists a point $x_0 \in [z_2, z_3]$ such that $\rho(x_0, A_1) = \rho(x_0, C_1)$. Setting $r_0 := \alpha_1 - \|x_0 - z_2|$ (this quantity is positive, because $\rho(z_2, A_1) < \alpha_1$ and $x_0 \in [z_2, z_3]$), we have

$$B(x_0, r_0) \subset B(z_2, \alpha_1) \subset B(z_2, \rho(z_2, A_1) + \delta) \subset \mathring{B}(x, r). \tag{5.8}$$

Consequently, $(B(x_0, r_0) \cap A_1) \overset{(5.8)}{\subset} (B(z_2, \alpha_1) \cap A_1) \overset{(5.7)}{\subset} \mathring{B}(y_1, \varepsilon)$ and

$$\rho(x_0, A_1) = \rho(x_0, C_1) \le \|z_3 - x_0| = \|z_3 - z_2| - \|x_0 - z_2| < \alpha_1 - \|x_0 - z_2| = r_0,$$

which proves (5.3) with $y_0 = y_1$.

Let us proceed with the proof of Theorem 5.2. By (5.1), there exist $x_1 \in X$, $r_1 > 0$ such that

$$B(x_1, r_1) \subset B(x, r) \quad \text{and} \quad \rho(x_1, M) = \rho(x_1, A_0) = \rho(x_1, C_0).$$

We set

$$A_1 = B(x_1, r_1) \cap A_0, \quad C_1 = B(x_1, r_1) \cap C_0.$$

We argue by induction on i. Assume that there exist $x_i \in X$ and $r_i > 0$ such that $B(x_i, r_i) \cap M = A_i \sqcup C_i$, where $A_i, C_i \in \mathscr{F}(X)$ and $\rho(x_i, A_i) = \rho(x_i, C_i) < r_i$.

By (5.3), there exist $x_{i+1} \in X$ and $r_{i+1} > 0$ such that

$$\begin{aligned} &B(x_{i+1}, r_{i+1}) \subset \mathring{B}(x_i, r_i), \qquad \rho(x_{i+1}, A_i) = \rho(x_{i+1}, C_i) < r_{i+1},\\ &C_{i+1} := (B(x_{i+1}, r_{i+1}) \cap A_i) \subset \mathring{B}(y_i, r/2^{(i+1)}), \ \text{ where } \ y_i \in A_i. \end{aligned} \tag{5.9}$$

We set $A_{i+1} = B(x_{i+1}, r_{i+1}) \cap C_i$. For $m > n$,

$$\|x_m - x_n| \le \sum_{k=n}^{m-1} \|x_{k+1} - x_k| \overset{(2.1),(5.9)}{\le} \sum_{k=n}^{m-1} (r_{k+1} - r_k) = r_m - r_n,$$

and hence, (x_i) is a Cauchy sequence. Let x_0 be the right limit of this sequence is the right completion K of the space X, i.e., $\|x_0 - x_i| \to 0$. The sequence (r_i) $(r_i > r_{i+1} \ge 0)$ converges to some number $r_0 > 0$ $(r_0 \ne 0$, because $A_0 \cap C_0 = \varnothing)$. It is clear that

$$B(x_0, r_0) = \bigcap_{i=1}^{\infty} \mathring{B}(x_i, r_i). \tag{5.10}$$

Each of the sequences $(A_{2k})_{k=1}^{\infty}$, $(A_{2k-1})_{k=1}^{\infty}$ consists of nested closed sets, and in view of (5.9), for each $k \in \mathbb{N}$ we have $y_{2k-1} \in A_{2k-1} \subset \mathring{B}(y_{2k-1}, r/2^{2k})$. Hence because (y_{2k-1}) is a Cauchy sequence and M is externally strongly complete, the sequence (y_{2k-1}) $\|\cdot\|_{\mathrm{sym}}$-converges to some point $u' \in M$. Similarly, for each $k \in \mathbb{N}$, we have $y_{2k} \in A_{2k} \subset \mathring{B}(y_{2k}, r/2^{2k-1})$. Hence, since the space is complete, the sequence (y_{2k}) converges to some point $u'' \in M$. It is clear that $\widetilde{A} := \bigcap_k A_{2k-1} = \{u'\}$, $\widetilde{C} := \bigcap_k A_{2k} = \{u''\}$. By (5.10), we have $\widetilde{A} = B(x_0, r_0) \cap A_0$, $\widetilde{C} = B(x_0, r_0) \cap C_0$. Now, since the ball $B(0, 1)$ is closed, we have

$$\rho(x_0, \widetilde{A}) = \rho(x_0, A_0) = \rho(x_0, \widetilde{C}) = \rho(x_0, C_0) = \rho(x_0, M).$$

Finally, for the point x_0 there are precisely two different nearest points from M, where $u' \in A_0$, $u'' \in C_0$, which contradicts the P_0-connectedness of the set M in the cone space K. Theorem 5.2 is proved. □

Corollary 5.4. *Let K be a right-complete semilinear space, $X \subset K$ is a uniformly convex asymmetric linear space complete with respect to the symmetrization norm, whose right completion is K. Then each externally strongly complete set $M \subset X$ which is a uniqueness set (a Chebyshev set) in K is $\mathring{B}$-connected in X.*

Acknowledgment

The authors are grateful to the referees for advice and helpful criticism

References

[1] A. R. Alimov, *Convexity of Chebyshev sets contained in a subspace*, Math. Notes **78** (2005), no. 1, 3–13.
[2] A. R. Alimov, *Universality theorems for asymmetric spaces*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **26** (2023), no, 2, 2250017.
[3] A. R. Alimov, *Strict protosuns in asymmetric spaces of continuous functions*, Results in Mathematics, **78** (2023) Art. no. 95
[4] A. R. Alimov, I. G. Tsar'kov, *Connectedness and solarity in problems of best and near-best approximation*, Russian Math. Surveys **71** (2016), no. 1, 1–77.
[5] A.R. Alimov, I. G. Tsar'kov, *Smoothness of subspace sections of the unit balls of $C(Q)$ and L^1*, J. Approx. Theory **265** (2021), 105552–8.
[6] A.R. Alimov, I. G. Tsar'kov, *Geometric Approximation Theory*, Springer, Cham, 2021.
[7] A. R. Alimov, I. G. Tsar'kov, *Suns, moons, and $\mathring{B}$-complete sets in asymmetric spaces*, Set-Valued Variat. Anal. (2022).
[8] A. R. Alimov, I. G. Tsarkov, *$\mathring{B}$-complete sets: approximative and structural properties*, Siberian Math. J. **63** (2022), 412–420.
[9] A. R. Alimov, I. G. Tsar'kov, *Ball-complete sets and solar properties of sets in asymmetric spaces*, Results Math. **77** (2022) Art. no. 86.
[10] A. R. Alimov, I. G. Tsar'kov, *Some classical problems of geometric approximation theory in asymmetric spaces*, Math. Notes **112** (2022), no. 1, 3–16.
[11] A. R. Alimov, I. G. Tsar'kov, *Solarity and proximinality in generalized rational approximation in spaces $C(Q)$ and L^p*, Russian J. Math. Physics **29** (2022), no. 3 291–305.
[12] V. S. Balaganskii, L. P. Vlasov, *The problem of convexity of Chebyshev sets*, Russian Math. Surveys **51** (1996), no. 6, 1127–1190.
[13] P. A. Borodin, *On the convexity of N-Chebyshev sets*, Izv. Math. **75** (2011), no. 5, 889–914.
[14] S. Cobzaş, *Compact bilinear operators on asymmetric normed spaces*, Topol. Appl. **306** (2022), 107922.
[15] S. Cobzaş, *Ekeland, Takahashi and Caristi principles in preordered quasi-metric spaces*, Quaest. Math. (2022), 1–22.
[16] S. Cobzaş, *Geometric properties of Banach spaces and the existence of nearest and farthest points*, Abstr. Appl. Anal. (2005), no. 3, 259–285.
[17] Ş. Cobzaş, Functional analysis in asymmetric normed spaces, Front. Math. Birkhäuser/Springer Basel AG, Basel (2013).
[18] V. Donjuán, N. Jonard-Pérez, *Separation axioms and covering dimension of asymmetric normed spaces*, Quaest. Math. **43** 2020, no. 4, 467–491.
[19] T. Drezner, Z. Drezner, *Asymmetric distance location model*, INFOR: Information Systems and Operational Research **59** (2021), no. 1, 102–110.
[20] S. I. Dudov, E. S. Polovinkin, V. V. Abramova, *Properties of the distance function to strongly and weakly convex sets in a nonsymmetrical space*, Russian Math. (Iz. VUZ) **64** (2020), no. 5, 17–30.
[21] L.M. García Raffi, S. Romaguera, E. A. Sánchez Pérez, *On Hausdorff asymmetric normed linear spaces*, Houston. J. Math. **29** (2003), 717–728.
[22] Th. Jahn, Ch. Richter, *Coproximinality of linear subspaces in generalized Minkowski spaces*, J. Math. Anal. Appl. **504** (2021), no. 1, 12535.
[23] M.G. Krein, *The L-problem in an abstract normed linear space* in: *Some Problems of the Theory of Moments* eds N. I. Akhiezer, M. G. Krein (Gos. Nauch.Tekh. Izd. Ukrainy, Kharkov (1938), 171–199; AMS, Providence, Rhode Island (1962), 175–204).
[24] B.G. Merino, T. Jahn, C. Richter, *Uniqueness of circumcenters in generalized Minkowski spaces*, J. Approx. Theory **237** (2019), 153–159.
[25] O. O. Otafudua, M. S. Zwenia, *W-convexity on the Isbell-convex hull of an asymmetrically normed real vector space*, Filomat **37**(6) (2023), 1975–1988.

[26] F. Plastria, *Using the power of ideal solutions: simple proofs of some old and new results in location theory*, 4OR-Q J Oper Res **19** (2021), 449–467.

[27] P.L. Papini, J. Puerto, *Location problems with different norms for different points*, J. Optimiz. Theory Appl. **125** (2005), no. 3, 673–695.

[28] S. Romaguera, P. Tirado, *A characterization of Smyth complete quasi-metric spaces via Caristi's fixed point theorem*, Fixed Point Theory Appl. (2015), 1–13.

[29] N.M. Tran, P. Burdejová, M. Osipenko, W.K. Härdle, *Principal component analysis in an asymmetric norm*, J. Multivariate Analysis **171**, (2019), 1–21.

[30] I. G. Tsar'kov, *Relations between certain classes of sets in Banach spaces*, Math. Notes **40** (1986), no. 2, 597–610.

[31] I. G. Tsar'kov, *Uniform convexity in nonsymmetric spaces*, Math. Notes **110** (2021), no. 5, 773–783.

[32] I. G. Tsarkov, *Uniformly and locally convex asymmetric spaces*, Russ. J. Math. Phys. **29** (2022), no. 1, 141–148.

[33] I. G. Tsar'kov, *Continuity of a metric function and projection in asymmetric spaces*, Math. Notes **111** (2022), no. 4, 616–623.

[34] I. G. Tsar'kov, *Uniformly and locally rotund asymmetric spaces,* Sb. Math. **213:10** (2022), 1444–1469.

[35] I. G. Tsar'kov, *Geometry of the singular set of hypersurfaces and the eikonal equation*, Russ. J. Math. Physics **29** (2022), no. 2, 240–248.

[36] I. G. Tsar'kov, *Density of the points of continuity of the metric function and projection in asymmetric spaces* Math.Notes, **112** (2022), no. 2, 1017–1024.

[37] I. G. Tsar'kov, *Connectedness in asymmetric spaces*, J. Math. Anal. Appl. **527** (2023), 127381.

[38] I. G. Tsar'kov, *Smoothness of solutions of the eikonal equation and regular points of their level surfaces*, Russ. J. Math. Phys. **30** (2023), no. 2, 259–269.

[39] L. P. Vlasov, *Chebyshev sets and some generalizations of them*, Math. Notes **3** (1968), no. 1, 36–41.