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On $\mathfrak{q}$-statistical approximation of wavelets aided Kantorovich $\mathfrak{q}$-Baskakov operators

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Abstract. The aim of this research is to examine various statistical approximation properties of Kantorovich $\mathfrak{q}$-Baskakov operators using wavelets. We discuss and investigate the weighted statistical approximation employing a Bohman-Korovkin type theorem as well as a statistical rate of convergence applying a weighted modulus of smoothness ω_{ρ_α} correlated with the space $B_{\rho\alpha}(\mathbb{R}_+)$ and Lipschitz type maximal functions.

1. Preliminaries and introduction

In 1995, Agratini [1] introduced a class of Szász-type operators by means of compactly supported wavelets of Daubechies. Later on in 1997, Gonska and Zhou [20] used the Daubechies' compactly-supported wavelets to establish a new class of Baskakov-type operators. This technique of employing wavelets in modifying the classical operators is very useful which provides a tool to achieve the local information of approximation by such operators. In [28], Nasiruzzaman *et al.* further modified the operators of Gonska and Zhou [20] by defining their $\mathfrak{q}$-analog to get a better rate of convergence. In this article, our focus is to study various approximation properties exhibited by the operators described in [28]. Our proposed study aims to further enhance our understanding of these operators and their potential applications.

Note that that the Bernstein polynomials [14] converge uniformly to the value $\mathrm{g}(x)$ for every continuous function g, where x is any real value between 0 and 1. The following defines the Bernstein polynomials:

$$\left(\mathcal{B}^*_{\mathrm{r}}\mathrm{g}\right)(x) = \sum_{s=0}^{\mathrm{r}} \binom{\mathrm{r}}{s} x^s (1-x)^{\mathrm{r}-s} \mathrm{g}\left(\frac{s}{\mathrm{r}}\right), \tag{1}$$

where $\binom{r}{i}$ refers to the binomial coefficients.

The Szász [34] as well as Baskakov [13] operators were formed in approximating the continuous functions which were defined for the unbounded interval $[0,\infty)$. Here, the Baskakov operators are written as

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$$\left(\mathcal{B}_r\, g\right)(x)=\sum_{s=0}^{\infty}\binom{r+s-1}{s}\frac{x^s}{(1+x)^{r+s}}\,g\left(\frac{s}{r}\right).$$

Bernstein operators were modified by Kantorovich [23] and were called Bernstein-Kantorovich operators. These operators are utilized in approximating the functions of broader classes as opposed to continuous functions. Moreover, the following are the operators that define Bernstein-Kantorovich operators:

$$\left(\mathcal{K}_r\, g\right)(x)=(r+1)\sum_{s=0}^{r}\binom{r}{s}x^s(1-x)^{r-s}\int_{\frac{s}{r+1}}^{\frac{s+1}{r+1}} g(t)dt, \tag{2}$$

for functions $g\in L_p[0,1]$ $(1\le p<\infty)$.

To determine the L_p-approximation, Ditzian and Totik [17] provided the Kantorovich modification of Baskakov operators, which is called the Baskakov-Kantorovich operators written as

$$\left(\mathcal{K}_m\, g\right)(x)=m\sum_{l=0}^{\infty}\binom{m+l-1}{l}\frac{x^l}{(1+x)^{m+l}}\int_{\frac{l}{m}}^{\frac{l+1}{m}} g(t)\,dt. \tag{3}$$

There are various modifications and generalizations of these operators which have been studied by several authors to get better and better approximation, e.g. [4, 6, 7, 9–12, 29, 31, 33]. The q-calculus application appeared as a relatively new research field in the approximation theory. Here, the first q-analogue of the famous Bernstein polynomials was established by Lupaş [24] by employing the concept of q-integers. On the other hand, in 1997, Phillips [30] took into consideration a different q-analogue of the classical Bernstein polynomials. Subsequently, numerous researchers investigated the q-generalizations with regard to a variety of operators by examining their approximation properties, e.g. [8, 12, 26, 27]. For instance, the q-variant of Baskakov operators [5] is defined as

$$\left(\mathcal{V}_{m,q}\, g\right)(x)=\sum_{l=0}^{\infty}B_{m,l,q}(x)\,g\left(\frac{[l]_q}{q^{l-1}[m]_q}\right), \tag{4}$$

where

$$B_{m,q}(x)=\begin{bmatrix} m+l-1 \\ l \end{bmatrix}_q\frac{x^l}{(1+x)_q^{m+l}}q^{\frac{l(l-1)}{2}},$$

while the q-Baskakov-Kantorovich operators [21] are defined by

$$\left(\mathcal{T}_{m,q}\, g\right)(x)=[m]_q\sum_{l=0}^{\infty}q^{l-1}B_{m,l,q}(x)\int_{\frac{q[l]_q}{[m]_q}}^{\frac{[l+1]_q}{[m]_q}} g\left(q^{1-l}\,t\right)d_qt. \tag{5}$$

Lemma 1.1. *With respect to the test functions given by $e_j=t^j$, $j=0,1,2$, it follows that*

(1) $\left(\mathcal{V}_{m,q}\,e_0\right)(x) = 1,$

(2) $\left(\mathcal{V}_{m,q}\,e_1\right)(x) = x,$

(3) $\left(\mathcal{V}_{m,q}\,e_2\right)(x) = x^2+\dfrac{x}{[m]_q}\left(1+\dfrac{x}{q}\right).$

1.1. *Basics of* $\mathfrak{q}$*-Calculus*

The $\mathfrak{q}$-integer $[m]_{\mathfrak{q}}$, the $\mathfrak{q}$-factorial $[m]_{\mathfrak{q}}!$ as well as the $\mathfrak{q}$-binomial coefficient are given as below (see [22]) :

$$[m]_{\mathfrak{q}} := \begin{cases} \frac{1-\mathfrak{q}^m}{1-\mathfrak{q}}, & \text{if } \mathfrak{q} \in \mathbb{R}^+ \setminus \{1\} \\ m, & \text{if } \mathfrak{q} = 1, \end{cases} \quad \text{for } m \in \mathbb{N} \text{ and } [0]_{\mathfrak{q}} = 0,$$

$$[m]_{\mathfrak{q}}! := \begin{cases} [m]_{\mathfrak{q}}[m-1]_{\mathfrak{q}} \cdots [1]_{\mathfrak{q}}, & m \geq 1, \\ 1, & m = 0, \end{cases}$$

$$\begin{bmatrix} m \\ \mathfrak{l} \end{bmatrix}_{\mathfrak{q}} := \frac{[m]_{\mathfrak{q}}!}{[\mathfrak{l}]_{\mathfrak{q}}![m-\mathfrak{l}]_{\mathfrak{q}}!},$$

accordingly. Here, the $\mathfrak{q}$-analogue of $(1+x)^m$ is given by the polynomial

$$(1+x)^m_{\mathfrak{q}} := \begin{cases} (1+x)(1+\mathfrak{q}x)\cdots(1+\mathfrak{q}^{m-1}x) & m = 1, 2, 3, \cdots \\ 1 & n = 0. \end{cases}$$

The Gauss binomial formula is written as

$$(x+a)^m_{\mathfrak{q}} = \sum_{\mathfrak{l}=0}^{m} \begin{bmatrix} m \\ \mathfrak{l} \end{bmatrix}_{\mathfrak{q}} \mathfrak{q}^{\mathfrak{l}(\mathfrak{l}-1)/2} a^{\mathfrak{l}} x^{m-\mathfrak{l}}.$$

On the other hand, the $\mathfrak{q}$-derivative $D_{\mathfrak{q}}\, \mathrm{g}$ of a function g is as follows

$$(D_{\mathfrak{q}}\mathrm{g})(x) = \frac{\mathrm{g}(x) - \mathrm{g}(\mathfrak{q}x)}{(1-\mathfrak{q})x},\ x \neq 0,$$

as well as $(D_{\mathfrak{q}}\mathrm{g})(0) = \mathrm{g}'(0)$, provided that $\mathrm{g}'(0)$ exists. If g is differentiable, then

$$\lim_{\mathfrak{q}\to 1} D_{\mathfrak{q}}\mathrm{g}(x) = \lim_{\mathfrak{q}\to 1} \frac{\mathrm{g}(x) - \mathrm{g}(\mathfrak{q}x)}{(1-\mathfrak{q})x} = \frac{d\mathrm{g}(x)}{dx}.$$

For $m \geq 1$,

$$D_{\mathfrak{q}}(1+x)^m_{\mathfrak{q}} = [m]_{\mathfrak{q}}(1+\mathfrak{q}x)^{m-1}_{\mathfrak{q}},\ D_{\mathfrak{q}}\left(\frac{1}{(1+x)^m_{\mathfrak{q}}}\right) = -\frac{[m]_{\mathfrak{q}}}{(1+x)^{m+1}_{\mathfrak{q}}},$$

$$D_{\mathfrak{q}}\left(\frac{u(x)}{v(x)}\right) = \frac{v(\mathfrak{q}x)D_{\mathfrak{q}}u(x) - u(\mathfrak{q}x)D_{\mathfrak{q}}v(x)}{v(x)v(\mathfrak{q}x)}.$$

The $\mathfrak{q}$-Jackson definite integral is given by

$$\int_0^{\infty/A} f(x)d_{\mathfrak{q}}x = (1-\mathfrak{q}) \sum_{n=-\infty}^{\infty} f\left(\frac{\mathfrak{q}^n}{A}\right)\frac{\mathfrak{q}^n}{A} \qquad (A \in \mathbb{R} - \{0\}).$$

1.2. $\mathfrak{q}$*-Statistical convergence*

The definition of $\mathfrak{q}$-analog of Cesàro matrix C_1 is not unique (see [2], [3]). Here, we may take into consideration the $\mathfrak{q}$-Cesàro matrix, $C_1(\mathfrak{q}) = (c^1_{nk}(\mathfrak{q}^k))^{\infty}_{n,k=0}$ expressed by

$$c^1_{nk}(\mathfrak{q}^k) = \begin{cases} \frac{\mathfrak{q}^k}{[n+1]_{\mathfrak{q}}} & \text{if } k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

which is regular for $\mathfrak{q} \geq 1$.

Suppose $\mathcal{K} \subseteq \mathbb{N}$ (the set of natural numbers). Then, $\delta(\mathcal{K}) = \lim_{\mathfrak{r}} \frac{1}{\mathfrak{r}} \#\{k \leq \mathfrak{r} \ : \ k \in \mathcal{K}\}$ is known as the asymptotic density of $\mathcal{K}$, in which # denotes the cardinality of the enclosed set. Moreover, a sequence $\eta = (\eta_k)$ is known as statistically convergent to the number $\mathfrak{s}$ provided that $\delta(\mathcal{K}_\varepsilon) = 0$ for every $\varepsilon > 0$, in which $\mathcal{K}_\varepsilon = \{k \leq \mathfrak{r} : |\eta_k - \mathfrak{s}| > \varepsilon\}$ (refer to [19]).

In the recent past, Aktuğlu and Bekar [3] defined $\mathfrak{q}$-density as well as $\mathfrak{q}$-statistical convergence. The $\mathfrak{q}$-density is defined as

$$\delta_{\mathfrak{q}}(\mathcal{K}) = \delta_{C_1^{\mathfrak{q}}}(\mathcal{K}) = \lim \inf_{n\to\infty} (C_1^{\mathfrak{q}} \chi_{\mathcal{K}})_n = \lim \inf_{n\to\infty} \sum_{k\in K} \frac{\mathfrak{q}^{k-1}}{[n]}, \ \mathfrak{q} \geq 1.$$

A sequence $\eta = (\eta_k)$ is known to be $\mathfrak{q}$-statistically convergent to $\mathcal{L}$ provided that $\delta_{\mathfrak{q}}(\mathcal{K}_\varepsilon) = 0$, in which $\mathcal{K}_\varepsilon = \{k \leq n : |\eta_k - \mathcal{L}| \geq \varepsilon\}$ for every $\varepsilon > 0$. In other words, for each $\varepsilon > 0$,.

$$\lim_n \frac{1}{[n]} \#\{k \leq n : \mathfrak{q}^{k-1} |\eta_k - \mathcal{L}| \geq \varepsilon\} = 0.$$

In this case we write $St_{\mathfrak{q}} - \lim \eta_k = \mathcal{L}$.

Note that if $\delta(\mathcal{K}) = 0 \implies \delta_{\mathfrak{q}}(\mathcal{K}) = 0$. Therefore, statistical convergence [19, Example 15] implies $\mathfrak{q}$-statistical convergence but not conversely (refer to [Example 15][3]).

2. Wavelets aided $\mathfrak{q}$-Baskakov-Kantorovich operators

We now recall some basic properties of wavelets [15, 25]. Here, the wavelets denotes the set of functions of the form

$$\Psi_{\mu,\nu}(x) = \mu^{-\frac{1}{2}} \Psi\left(\frac{x-\nu}{\mu}\right) \ \mu > 0, \ \nu \in \mathbb{R},$$

which are formed by translations and dilations of a single function Ψ, which is called the mother wavelet or basic wavelet. Moreover, following the Franklin-Stro̎mberg theory, the constant μ may be substituted by 2^i while ν may be substituted by $2^i \mathfrak{l}$ having i and $\mathfrak{l}$ to be the integers. For an arbitrary function $\mathrm{g} \in L_2(\mathbb{R})$, the wavelets have a crucial part in the orthonormal basis, in which the g function is given as:

$$\mathrm{g}(x) = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \gamma(i, \mathfrak{l}) \Psi_{i,\mathfrak{l}}(x),$$

in which

$$\gamma(i, \mathfrak{l}) = 2^{\frac{i}{2}} \Psi_{i,\mathfrak{l}}(x) \int_{\mathbb{R}} f(x) \Psi(2^i x - \mathfrak{l}) \mathrm{d}x.$$

Daubechies [16] constructed an orthonormal basis for $L_2(\mathbb{R})$ of the form

$$2^{\frac{i}{2}} \Psi_s(x)(2^i x - \mathfrak{l}),$$

where s refers to the non-negative integer, $i, \mathfrak{l}$ denote the integers as well as the support of Ψ_s is $[0, 2s+1]$. For a positive constant ξ, if Ψ_s has ξ_s order of continuous derivatives, then for any $0 \leq \mathfrak{l} \leq s$, $s \in \mathbb{N}$, we have

$$\int_{\mathbb{R}} x^{\mathfrak{l}} \Psi_s(x) \mathrm{d}x = 0. \tag{6}$$

Evidently, when $s = 0$, the system is reduced to the Haar system. Here, with regard to any $\Psi \in L_\infty(\mathbb{R})$, we now have the conditions given by: (i) a finite positive ξ having the property $\sup \Psi \subset [0, \xi]$, while (ii) its first s moment vanishes. Furthermore, for $1 \leq l \leq s,\ s \in \mathbb{N}$, we have $\int_{\mathbb{R}} t^{l}\Psi(t)dt = 0$ and $\int_{\mathbb{R}} \Psi(t)dt = 1$. Therefore, by employing the Haar basis, the Baskakov type operators are expressed as [1]:

$$(\mathcal{L}_m\, g)(x) = m \sum_{l=0}^{\infty} \binom{m+l-1}{l} \frac{x^{l}}{(1+x)^{m+l}} \int_{\mathbb{R}} g(t)\, \Psi(mt - l)\, dt, \tag{7}$$

in which the operators $\mathcal{L}_m$ refer to the extensions of Baskakov-Kantorovich operators. By considering the $\sup \Psi \subset [0, \xi]$, the operators $\mathcal{L}_m$ are given as [1]:

$$(\mathcal{L}_m\, g)(x) = \sum_{l=0}^{\infty} \binom{m+l-1}{l} \frac{x^{l}}{(1+x)^{m+l}} \int_{0}^{\xi} g\left(\frac{t+l}{m}\right) \Psi(t) dt. \tag{8}$$

Now, we recall the q-Baskakov type operators by employing compactly-supported wavelets of Daubechies constructed in [28].
Let $\int_{\mathbb{R}} x^{s}\Psi_k(x) d_q x = 0$ when $0 \leq s \leq k$ for $k \in \mathbb{N}$ as well as $q > 0$.

With regard to $\Psi \in L_\infty(\mathbb{R})$, we assume the conditions given below in terms of wavelets: (i) a finite positive ξ having the property $\sup \Psi \subset [0, \xi]$; and (ii) its first k moment vanishes. For $1 \leq s \leq k$ and $k \in \mathbb{N}$, we now obtain $\int_{\mathbb{R}} t^{s}\Psi(t)d_q t = 0$ as well as $\int_{\mathbb{R}} \Psi(t)d_q t = 1$. Therefore, for all $1 \leq s \leq k,\ k \in \mathbb{N}$ as well as $0 < q < 1$, Nasiruzzaman *et al.* [28] constructed the q-analogue of Baskakov-Kantorovich type wavelets operators given by:

$$\left(\mathcal{S}_{r,q}\, g\right)(x) = [r]_q \sum_{s=0}^{\infty} q^{s-1} B_{r,s,q}(x) \int_{\mathbb{R}} g(t)\, \Psi\left(q^{s-1}[r]_q t - [s]_q\right) d_q t. \tag{9}$$

Thus, these operators $\mathcal{S}_{r,q}(g;x)$ extend the q-Baskakov-Kantorovich operators given by (5). For the choices of $k = 0$ as well as Ψ Haar basis, we obtain the q-Baskakov-Kantorovich operators $\mathcal{T}_{r,q}(g;x)$ by (5). Additionally, for the choices $k = 0,\ q = 1$ as well as Ψ Haar basis, we get the Baskakov-Kantorovich operators $\mathcal{K}_{r,q}(g;x)$ by (3). Considering the $\sup \Psi \subset [0, \xi]$, the operators $\mathcal{S}_{r,q}(g;x)$ we get the following operators:

$$\left(\mathcal{S}_{r,q}\, g\right)(x) = \sum_{s=0}^{\infty} B_{r,s,q}(x) \int_{0}^{\xi} g\left(\frac{t + [s]_q}{q^{s-1}[r]_q}\right) \Psi(t)\, d_q t. \tag{10}$$

It is evident that by choosing $q = 1$, we obtain classical Baskakov-Kantorovich wavelets operators $\mathcal{L}_{r,s}$ by (7) as well as (8).

We need the following result of [28]:

Theorem 2.1. *Suppose $e_j = t^j$ when $0 \leq j \leq k$ and $k \in \mathbb{N}$. Then, we obtain*

$$\left(\mathcal{S}_{r,q}\, e_j\right)(x) = \left(\mathcal{V}_{r,q}\, e_j\right)(x),$$

in which $x \in [0, \infty)$ as well as the operators $\left(\mathcal{V}_{r,q}\, g\right)(x)$ are defined as above.

3. Weighted q-Statistical approximation

This section presents the statistical approximation of wavelets Kantorovich q-Baskakov operators $\mathcal{S}_{r,q}$ defined by (9) employing a Bohman-Korovkin type theorem [18].

Suppose N_g is the constant depending on the function g and $B_\rho(\mathbb{R})$ represents the weighted space of a real valued function g with the property that $|g(x)| \leq N_g \rho(x)$ for all $x \in \mathbb{R}$. Now, we take into consideration the weighted subspace $C_\rho(\mathbb{R})$ of $B_\rho(\mathbb{R})$ which is defined as

$$C_\rho(\mathbb{R}) = \{g \in B_\rho(\mathbb{R}) : g \text{ continuous in } \mathbb{R}\}.$$

with the norm $\| g \|_\rho = \sup_{x\in\mathbb{R}} \frac{|g(x)|}{\rho(x)}$ and both $C_\rho(\mathbb{R})$ and $B_\rho(\mathbb{R})$ are Banach spaces. By the use of A-statistical convergence, Duman and Orhan [18] proved the theorem given below, which is useful in proving our main result.

Theorem 3.1. *(Duman and Orhan [18]). If $A = (a_{jɼ})_{j,ɼ}$ is a positive regular summability matrix, and let $(L_ɼ)_ɼ$ denote a sequence of positive linear operators from $C_{\rho_1}(\mathbb{R})$ to $B_{\rho_2}(\mathbb{R})$, in which ρ_1 as well as ρ_2 satisfies $\lim_{|x|\to\infty} \frac{\rho_1}{\rho_2} = 0$. Then*

$$st_A - \lim_{ɼ} \| L_ɼ ɋ - ɋ \|_{\rho_2} = 0, \ \forall ɋ \in C_{\rho_1}(\mathbb{R})$$

if and only if

$$st_A - \lim_{ɼ} \| L_ɼ H_v - H_v \|_{\rho_1} = 0 \ for \ v = 0, 1, 2,$$

in which $H_v = \frac{x^v \rho_1(x)}{1 + x^2}$.

By examining this result, it is clear that if $\mathbb{R}$ is substituted by $\mathbb{R}_+$, then the theorem holds true. Also, by analyzing Lemma 1.1, we see that the sequence of operators $(\mathcal{S}_{ɼ,ɋ})_ɼ$ fails to satisfy the properties of Bohman-Korovkin theorem. Now, let us take into consideration the weight functions $\rho_0(x) = 1 + x^2$ and $\rho_\alpha(x) = 1 + x^{2+\alpha}$ for $x \in \mathbb{R}_+$ and $\alpha > 0$ together with the remark below.

Remark 3.2. *It is true that for $ɋ \in (0, 1)$, $\lim_{ɼ\to\infty} [ɼ]_ɋ = 0$ or $\frac{1}{1 - ɋ}$. Now, we consider the sequence $(ɋ_ɼ)_ɼ$ for $ɋ_ɼ \in (0, 1)$ with the property that $st - \lim_{ɼ\to\infty} ɋ_ɼ = 1$ and $st - \lim_{ɼ\to\infty} ɋ_ɼ^ɼ = 1$. Based on these facts, we have $\lim_{ɼ\to\infty} [ɼ]_ɋ = \infty$. This will lead to check the convergence of the operators (9). Thus, we now obtain the theorem stated as:*

Theorem 3.3. *Suppose that the sequence $(ɋ_ɼ)_ɼ$ satisfies Remark 3.2 above and $\mathcal{S}_{ɼ,ɋ}$ be a positive linear operator. Then, we have:*

$$St_ɋ - \lim_{ɼ} \| (\mathcal{S}_{ɼ,ɋ}(g) - g \|_{\rho_\alpha} = 0, \ \forall g \in C_{\rho_0}(\mathbb{R}_+).$$

Proof. Based on Lemma 1.1(i) and Theorem 2.1, we have:

$$\begin{aligned} \| (\mathcal{S}_{ɼ,ɋ}(g) - g \|_{\rho_0} &= \sup_{x\in\mathbb{R}} \frac{|(\mathcal{S}_{ɼ,ɋ_ɼ} e_0)(x) - e_0(x)|}{1 + x^2}, \\ &= \sup_{x\in\mathbb{R}} \frac{|1 - 1|}{1 + x^2}, \\ &= 0. \end{aligned}$$

In other words,

$$St_ɋ - \lim_{ɼ} \| (\mathcal{S}_{ɼ,ɋ}(g) - g \|_{\rho_0} = 0.$$

Again, based on Lemma 1.1 (ii) and Theorem 2.1, we have:

$$\begin{aligned}\| (\mathcal{S}_{\mathrm{r},\mathrm{q}}(\mathrm{g}) - \mathrm{g} \|_{\rho_0} &= \sup_{x\in\mathbb{R}} \frac{|(\mathcal{S}_{\mathrm{r},\mathrm{q}_\mathrm{r}} e_1)(x) - e_1(x)|}{1+x^2}, \\ &= \sup_{x\in\mathbb{R}} \frac{|x-x|}{1+x^2}, \\ &= 0.\end{aligned}$$

Using Lemma 1.1 and Theorem 2.1, we have:

$$\begin{aligned}\| (\mathcal{S}_{\mathrm{r},\mathrm{q}}(\mathrm{g}) - \mathrm{g} \|_{\rho_0} &= \sup_{x\in\mathbb{R}} \frac{\left|(\mathcal{S}_{\mathrm{r},\mathrm{q}_\mathrm{r}} e_2)(x) - e_2(x)\right|}{1+x^2}, \\ &= \sup_{x\in\mathbb{R}} \frac{\left|\left(x^2 + x\frac{1}{[\mathrm{r}]_{\mathrm{q}_\mathrm{r}}}\left(1+\frac{1}{\mathrm{q}_\mathrm{r}}x\right)\right) - x^2\right|}{1+x^2}, \\ &= \sup_{x\in\mathbb{R}} \frac{\left|\left(1+\frac{1}{\mathrm{q}_\mathrm{r}\,[\mathrm{r}]_{\mathrm{q}_n}} - 1\right)x^2 + x\frac{1}{[\mathrm{r}]_{\mathrm{q}_\mathrm{r}}}\right|}{1+x^2}, \\ &\le \sup_{x\in\mathbb{R}} \left|\frac{1}{\mathrm{q}_\mathrm{r}\,[\mathrm{r}]_{\mathrm{q}_\mathrm{r}}}x^2 + x\frac{1}{[\mathrm{r}]_{\mathrm{q}_\mathrm{r}}}\right|, \\ &\le \sup_{x\in\mathbb{R}} \left(\left|x^2\right| \frac{1}{\mathrm{q}_\mathrm{r}\,[\mathrm{r}]_{\mathrm{q}_\mathrm{r}}} + |x|\,\frac{1}{[\mathrm{r}]_{\mathrm{q}_\mathrm{r}}}\right), \\ &= \left(\| e_2 \|_{\rho_0} \frac{1}{\mathrm{q}_\mathrm{r}\,[\mathrm{r}]_{\mathrm{q}_\mathrm{r}}} + \| e_1 \|_{\rho_0} \frac{1}{[\mathrm{r}]_{\mathrm{q}_\mathrm{r}}}\right), \\ &\le \left(\frac{1}{\mathrm{q}_\mathrm{r}\,[\mathrm{r}]_{\mathrm{q}_\mathrm{r}}} + \frac{1}{[\mathrm{r}]_{\mathrm{q}_\mathrm{r}}}\right).\end{aligned}$$

From Remark 3.2, we have $st - \lim_{\mathrm{r}\to\infty} \mathrm{q}_\mathrm{r} = 1$. Furthermore, we also obtain $\lim_{\mathrm{r}\to\infty}[\mathrm{r}]_{\mathrm{q}} = \infty$. Consequently

$$St_{\mathrm{q}} - \lim_{\mathrm{r}} \| (\mathcal{S}_{\mathrm{r},\mathrm{q}}(\mathrm{g}) - \mathrm{g} \|_{\rho_0} = 0.$$

By employing Lemma 1.1 and also selecting $A = C_1$, known as the Cesáro matrix of order one, $\rho_0(x) = 1 + x^2$, $\rho_\alpha(x) = 1 + x^{2+\alpha}$ for $x \in \mathbb{R}_+$ and $\alpha > 0$, the proof is immediate from Theorem 3.1. □

4. The Rate of Convergence

In this section, we present the rate of statistical convergence of the operators $\mathcal{S}_{\mathrm{r},\mathrm{q}}$ (9) by means of weighted modulus of smoothness and Lipschitz type maximal functions. The weighted modulus of smoothness ω_{ρ_α} correlated to the space $B_{\rho\alpha}(\mathbb{R}_+)$ of a function g is defined as:

$$\omega_{\rho_\alpha}(\mathrm{g};\delta) = \sup_{x\ge 0,\ 0<i<\delta} \frac{|\mathrm{g}(x+i) - \mathrm{g}(x)|}{1+(x+i)^{2+\alpha}},\ \delta > 0, \alpha \ge 0. \tag{11}$$

It satisfies the following three axioms.

(a) $\omega_{\rho_\alpha}(g;\beta\delta) \leq (\beta+1)\omega_{\rho_\alpha}(g;\delta)$ for $\delta > 0$ and $\beta > 0$.

(b) $\omega_{\rho_\alpha}(g;\mathfrak{r}\delta) \leq \mathfrak{r}\omega_{\rho_\alpha}(g;\delta)$ for $\delta > 0$ and $\mathfrak{r} \in \mathbb{N}$.

(c) $\lim\limits_{\delta\to\infty} \omega_{\rho_\alpha}(g;\delta) = 0$.

The following theorem gives an error estimate of an operator $\mathcal{S}_{\mathfrak{r},\mathfrak{q}}$ for the unbounded function h by means of weighted modulus of smoothness correlated to the space $B_{\rho\alpha}(\mathbb{R}_+)$.

Theorem 4.1. *Suppose that* $\mathfrak{q} \in (0,1)$ *and* $\alpha \geq 0$. *Then, for any* $g \in B_{\rho\alpha}(\mathbb{R}_+)$, *we have*

$$\left|(\mathcal{S}_{\mathfrak{r},\mathfrak{q}}\, g)(x) - g(x)\right| \leq \sqrt{\mathcal{S}_{\mathfrak{r},\mathfrak{q}}(\mu_{x,\alpha}^2;x)}\left(1 + \frac{1}{\delta}\sqrt{\mathcal{S}_{\mathfrak{r},\mathfrak{q}}(\phi_x^2;x)}\right)\omega_{\rho_\alpha}(g;\delta),$$

where $\mu_{x,\alpha}(y) = 1 + \left(x + \left|y-x\right|\right)^{2+\alpha}$ *as well as* $\phi_x(y) = \left|y-x\right|$ *for* $y \geq 0$.

Proof. Suppose that $\mathfrak{r} \in \mathbb{N}$ and $g \in B_{\rho\alpha}(\mathbb{R}_+)$. Using equality (11) and axiom (a) above, we can write that

$$\begin{aligned}\left|g(y) - g(x)\right| &\leq \left(1 + (x + \left|y-x\right|)^{2+\alpha}\right)\left(1 + \frac{1}{\delta}\left|y-x\right|\right)\omega_{\rho_\alpha}(g;\delta),\\ &= \mu_{x,\alpha}(y)\left(1 + \frac{1}{\delta}\phi_x(y)\right)\omega_{\rho_\alpha}(g;\delta).\end{aligned}$$

Next, using the Cauchy inequality of the positive linear operators yields

$$\begin{aligned}\left|(\mathcal{S}_{\mathfrak{r},\mathfrak{q}}\, g)(x) - g(x)\right| &\leq [\mathfrak{r}]_{\mathfrak{q}}\sum_{s=0}^{\infty}\mathfrak{q}^{s-1}v_{s,\mathfrak{r}}^{\mathfrak{q}}(x)\int_{\mathbb{R}}\left|g(y)-g(x)\right|\Psi\left([\mathfrak{r}]_{\mathfrak{q}}\frac{\mathfrak{q}^{s-1}}{1}y - [s]_{\mathfrak{q}}\right)d_{\mathfrak{q}}y,\\ &\leq \left(\mathcal{S}_{\mathfrak{r},s,\mathfrak{q}}(\mu_{x,\alpha};x) + \frac{1}{\delta}\mathcal{S}_{\mathfrak{r},s,\mathfrak{q}}(\mu_{x,\alpha}\phi_x;x)\right)\omega_{\rho_\alpha}(g;\delta),\\ &\leq \sqrt{\mathcal{S}_{\mathfrak{r},s,\mathfrak{q}}(\mu_{x,\alpha}^2;x)}\left(1 + \frac{1}{\delta}\sqrt{\mathcal{S}_{\mathfrak{r},s,\mathfrak{q}}(\phi_x^2;x)}\right)\omega_{\rho_\alpha}(g;\delta).\end{aligned}$$

□

Now, we introduce the lemma given below, which may facilitate in proving the primary findings for this research, since it is one of the facts which ensure that $(\mathcal{S}_{\mathfrak{r},\mathfrak{q}}g)(x) \in B_{\rho\alpha}(\mathbb{R}_+)$.

Lemma 4.2. *Suppose that* $0 < \mathfrak{q} \leq 1$, *then for* $i, \mathfrak{r} \in \mathbb{N}$ *and* $x \in \mathbb{R}_+$, *we obtain*

$$(\mathcal{V}_{\mathfrak{r},\mathfrak{q}}\, e_i)(x) \leq \frac{1}{[\mathfrak{r}]_{\mathfrak{q}}^{i-1}(1+x)_{\mathfrak{q}}^{\mathfrak{r}}}x + \frac{2^{i-1}}{\mathfrak{q}^{i-1}}x(\mathcal{V}_{\mathfrak{r}+1,\mathfrak{q}}\, e_{i-1})(x). \tag{12}$$

Proof. For $s \in \mathbb{N}$ as well as $0 < \mathfrak{q} \leq 1$, we have the inequality given below:

$$1 \leq [s+1]_{\mathfrak{q}} \leq 2[s]_{\mathfrak{q}}. \tag{13}$$

Now, let $i \in \mathbb{N}$. Using Equation (4), we have:

$$
\begin{aligned}
(\mathcal{V}_{\mathrm{r},\mathrm{q}}e_i)(x) &= \sum_{s=0}^{\infty} \upsilon_{\mathrm{r},s}^{\mathrm{q}}(x) e_i\left(\frac{[s]_{\mathrm{q}}}{\mathrm{q}^{s-1}[\mathrm{r}]_{\mathrm{q}}}\right), \\
&= \sum_{s=0}^{\infty} \upsilon_{\mathrm{r},s}^{\mathrm{q}}(x)\left(\frac{[s]_{\mathrm{q}}}{\mathrm{q}^{s-1}[\mathrm{r}]_{\mathrm{q}}}\right)^i, \\
&= \sum_{s=0}^{\infty} \upsilon_{\mathrm{r},s}^{\mathrm{q}}(x)\frac{[s]_{\mathrm{q}}^i}{\mathrm{q}^{(s-1)i}[\mathrm{r}]_{\mathrm{q}}^i}, \\
&= \sum_{s=1}^{\infty} x\upsilon_{\mathrm{r}+1,s-1}^{\mathrm{q}}(x)\frac{[s]_{\mathrm{q}}^{i-1}}{\mathrm{q}^{(s-1)(i-1)}[\mathrm{r}]_{\mathrm{q}}^{i-1}}, \\
&= \sum_{s=0}^{\infty} x\upsilon_{\mathrm{r}+1,s}^{\mathrm{q}}(x)\frac{[s+1]_{\mathrm{q}}^{i-1}}{\mathrm{q}^{s(i-1)}[\mathrm{r}]_{\mathrm{q}}^{i-1}}, \\
&= \frac{1}{[\mathrm{r}]_{\mathrm{q}}^{i-1}(1+x)_{\mathrm{q}}^{\mathrm{r}}}x + x\sum_{s=1}^{\infty} \upsilon_{\mathrm{r}+1,s}^{\mathrm{q}}(x)\frac{[s+1]_{\mathrm{q}}^{i-1}}{\mathrm{q}^{s(i-1)}[\mathrm{r}]_{\mathrm{q}}^{i-1}}.
\end{aligned}
$$

Using Inequality (13), we have,

$$
\begin{aligned}
(\mathcal{V}_{\mathrm{r},\mathrm{q}}\,e_i)(x) &\leq \frac{x}{[\mathrm{r}]_{\mathrm{q}}^{i-1}(1+x)_{\mathrm{q}}^{\mathrm{r}}} + x\sum_{s=1}^{\infty} \upsilon_{\mathrm{r}+1,s}^{\mathrm{q}}(x)\frac{(2\,[s]_{\mathrm{q}})^{i-1}}{\mathrm{q}^{s(i-1)}[\mathrm{r}]_{\mathrm{q}}^{i-1}}, \\
&= \frac{x}{[\mathrm{r}]_{\mathrm{q}}^{i-1}(1+x)_{\mathrm{q}}^{\mathrm{r}}} + \frac{2^{i-1}}{\mathrm{q}^{i-1}}x\sum_{s=1}^{\infty} \upsilon_{\mathrm{r}+1,s}^{\mathrm{q}}(x)\frac{[s]_{\mathrm{q}}^{i-1}}{\mathrm{q}^{(s-1)(i-1)}[\mathrm{r}]_{\mathrm{q}}^{i-1}}.
\end{aligned}
$$

Based on Equation (4), we have that:

$$
(\mathcal{V}_{\mathrm{r}+1,\mathrm{q}}\,e_{i-1})(x) = \sum_{s=1}^{\infty} \upsilon_{\mathrm{r}+1,s}^{\mathrm{q}}(x)\frac{[s]_{\mathrm{q}}^{i-1}}{\mathrm{q}^{(s-1)(i-1)}[\mathrm{r}]_{\mathrm{q}}^{i-1}}.
$$

Consequently,

$$
(\mathcal{V}_{\mathrm{r},\mathrm{q}}\,e_i)(x) \leq \frac{1}{[\mathrm{r}]_{\mathrm{q}}^{i-1}(1+x)_{\mathrm{q}}^{\mathrm{r}}}x + \frac{2^{i-1}}{\mathrm{q}^{i-1}}x(\mathcal{V}_{\mathrm{r}+1,\mathrm{q}}e_{i-1})(x).
$$

□

Remark 4.3. *Any positive and linear operator is monotone. Theorem 2.1 and Lemma 12 ensure that* $(\mathcal{S}_{\mathrm{r},\mathrm{q}}\mathrm{g})(x) \in B_{\rho\alpha}(\mathbb{R}_+)$ *for any* $\mathrm{g} \in B_{\rho\alpha}(\mathbb{R}_+)$ *and* $\alpha \in \mathbb{N}_0$*, where* $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

We may state the major outcome of this section as follows:

Theorem 4.4. *Let* $(\mathrm{q}_{\mathrm{r}})_{\mathrm{r}}$ *be the sequence satisfying Remark 3.2 above and* $\alpha \in \mathbb{N}_0$*. Then, for every* $\mathrm{g} \in B_{\rho\alpha}(\mathbb{R}_+)$*, we have*

$$
\lim_{\mathrm{r}} \| (\mathcal{S}_{\mathrm{r},\mathrm{q}_{\mathrm{r}}}\mathrm{g})(x) - \mathrm{g}(x) \|_{\rho_\alpha} \leq 3C_\alpha \omega_{\rho_\alpha}(\mathrm{g};\delta_{\mathrm{r}}),
$$

where $C_\alpha > 0$ *is a constant and* $\delta_{\mathrm{r}} = \sqrt{\dfrac{1}{\mathrm{q}_{\mathrm{r}}[\mathrm{r}]_{\mathrm{q}_{\mathrm{r}}}}}$.

Proof. From Lemma 1.1, we have the following:

$$\mathcal{S}_{\Gamma,\mathfrak{q}_\Gamma}(\phi_x^2;x) = \left(x^2 + x\frac{1}{[\Gamma]_{\mathfrak{q}_\Gamma}}\left(1+\frac{1}{\mathfrak{q}_\Gamma}x\right)\right) - x^2,$$
$$= \left(1+\frac{1}{\mathfrak{q}_\Gamma[\Gamma]_{\mathfrak{q}_\Gamma}} - 1\right)x^2 + x\frac{1}{[\Gamma]_{\mathfrak{q}_\Gamma}},$$
$$= \frac{1}{\mathfrak{q}_\Gamma[\Gamma]_{\mathfrak{q}_\Gamma}}x^2 + \frac{1}{[\Gamma]_{\mathfrak{q}_\Gamma}}x.$$

Consequently, we have the inequality:

$$\mathcal{S}_{\Gamma,\mathfrak{q}_\Gamma}(\phi_x^2;x) \leq \frac{1}{\mathfrak{q}_\Gamma[\Gamma]_{\mathfrak{q}_\Gamma}}x^2 + \frac{3}{[\Gamma]_{\mathfrak{q}_\Gamma}}x. \tag{14}$$

Let $\alpha \geq 0$ be a constant and $g \in B_{\rho\alpha}(\mathbb{R}_+)$. Using Theorem 4.1 as well as the inequality in (14) above, we get the following:

$$\begin{aligned}\lim_{\Gamma} \| (\mathcal{S}_{\Gamma,\mathfrak{q}}g)(x) - g(x) \|_{\rho_\alpha} &= \frac{\left|(\mathcal{S}_{\Gamma,\mathfrak{q}}g)(x) - g(x)\right|}{1+x^{2+\alpha}},\\
&\leq \sqrt{\frac{\mathcal{S}_{\Gamma,\mathfrak{q}}(\mu_{x,\alpha}^2;x)}{1+x^{2+\alpha}}}\left(1+\frac{1}{\delta}\sqrt{\frac{\mathcal{S}_{\Gamma,\mathfrak{q}}(\phi_x^2;x)}{1+x^{1+\alpha}}}\right)\omega_{\rho_\alpha}(g;\delta),\\
&\leq \sqrt{\frac{\mathcal{S}_{\Gamma,\mathfrak{q}}(\mu_{x,\alpha}^2;x)}{1+x^{2+\alpha}}}\left(1+\frac{1}{\delta}\sqrt{\left|\frac{1}{\mathfrak{q}_\Gamma[\Gamma]_{\mathfrak{q}_\Gamma}}x^2 + \frac{3}{[\Gamma]_{\mathfrak{q}_\Gamma}}x\right|}\right),\\
&\times \omega_{\rho_\alpha}(g;\delta),\\
&\leq \sqrt{\frac{\mathcal{S}_{\Gamma,\mathfrak{q}}(\mu_{x,\alpha}^2;x)}{1+x^{2+\alpha}}}\left(1+\frac{1}{\delta}\sqrt{\frac{1}{\mathfrak{q}_\Gamma[\Gamma]_{\mathfrak{q}_\Gamma}}\| e_2 \|_{\rho_\alpha} + \frac{3}{[\Gamma]_{\mathfrak{q}_\Gamma}}\| e_2 \|_{\rho_\alpha}}\right)\\
&\times \omega_{\rho_\alpha}(g;\delta).\end{aligned}$$

Furthermore,

$$\begin{aligned}\lim_{\Gamma} \| (\mathcal{S}_{\Gamma,\mathfrak{q}_\Gamma}g)(x) - g(x) \|_{\rho_\alpha} &\leq \sqrt{\frac{\mathcal{S}_{\Gamma,\mathfrak{q}}(\mu_{x,\alpha}^2;x)}{1+x^{2+\alpha}}}\left(1+\frac{2}{\delta}\sqrt{\frac{1}{\mathfrak{q}_\Gamma[\Gamma]_{\mathfrak{q}_\Gamma}}}\right)\omega_{\rho_\alpha}(g;\delta),\\
&\leq \| \mathcal{S}_{\Gamma,\mathfrak{q}}(\mu_{x,\alpha}^2;x) \|_{\delta\alpha}\left(1+\frac{2}{\delta}\sqrt{\frac{1}{\mathfrak{q}_\Gamma[\Gamma]_{\mathfrak{q}_\Gamma}}}\right)\omega_{\rho_\alpha}(g;\delta).\end{aligned}$$

Let $C_\alpha = \| \mathcal{S}_{\Gamma,\mathfrak{q}}(\mu_{x,\alpha}^2;x) \|_{\delta\alpha}$ and choose $\delta = \sqrt{\frac{1}{\mathfrak{q}_\Gamma[\Gamma]_{\mathfrak{q}_\Gamma}}}$, we have:

$$\lim_{\Gamma} \| (\mathcal{S}_{\Gamma,\mathfrak{q}_\Gamma}g)(x) - g(x) \|_{\rho_\alpha} \leq 3C_\alpha\omega_{\rho_\alpha}(g;\delta_\Gamma).$$

□

Remark 4.5. *Since $(\mathfrak{q}_\Gamma)_\Gamma$ satisfies Remark 3.2, the sequence $(\delta_\Gamma)_\Gamma$ is statistically null, that is $st - \lim_{\Gamma}\omega_{\rho_\alpha}(g;\delta_\Gamma) = 0$. Therefore, Theorem 4.4 above gives the statistical rate of convergence of $\mathcal{S}_{\Gamma,\mathfrak{q}_\Gamma}(x)$ to g .*

5. Graphical analysis

Using computer software, we will demonstrate some numerical examples with illustrative graphics.

Example 5.1. *Let* $g(x) = (x - \frac{1}{5})(x - \frac{4}{9})$, $\mathfrak{q} = 0.95$ *and* $n \in \{10, 30, 80\}$. *The convergence of the operator towards the function* $g(x)$ *is shown in Figure 1.*

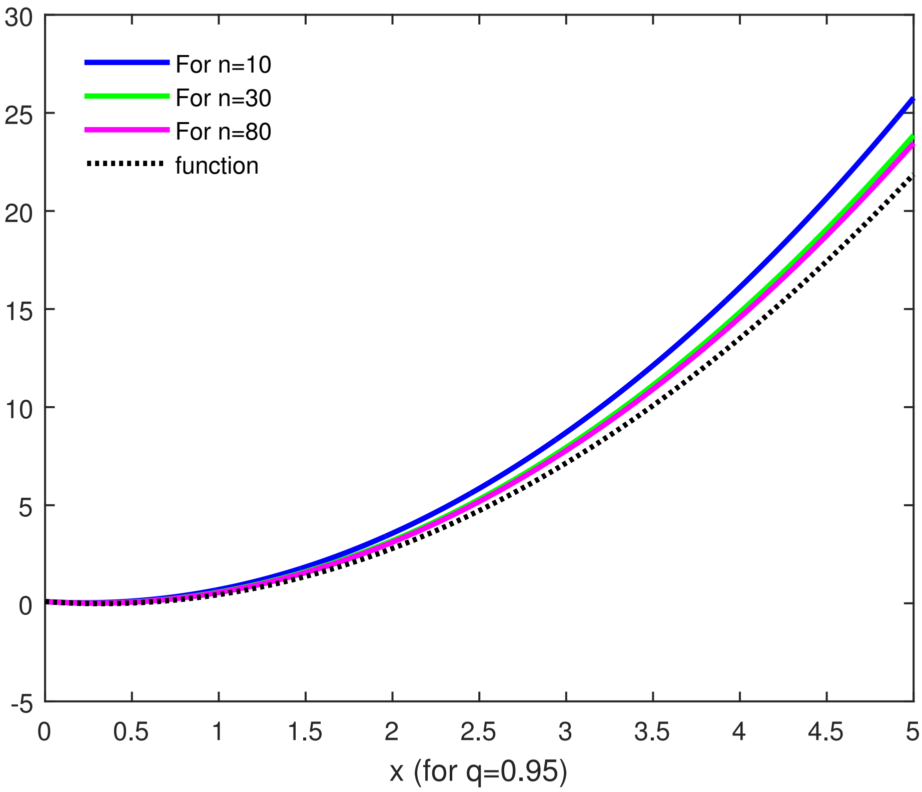

Figure 1: convergence of the operator towards the function $g(x) = (x - \frac{1}{5})(x - \frac{4}{9})$

Example 5.2. *Let* $g(x) = x^2 - 1$, $\mathfrak{q} = 1$ *and* $n \in \{10, 30, 60\}$. *The convergence of the operator towards the function* $g(x)$ *is shown in Figure 2.*

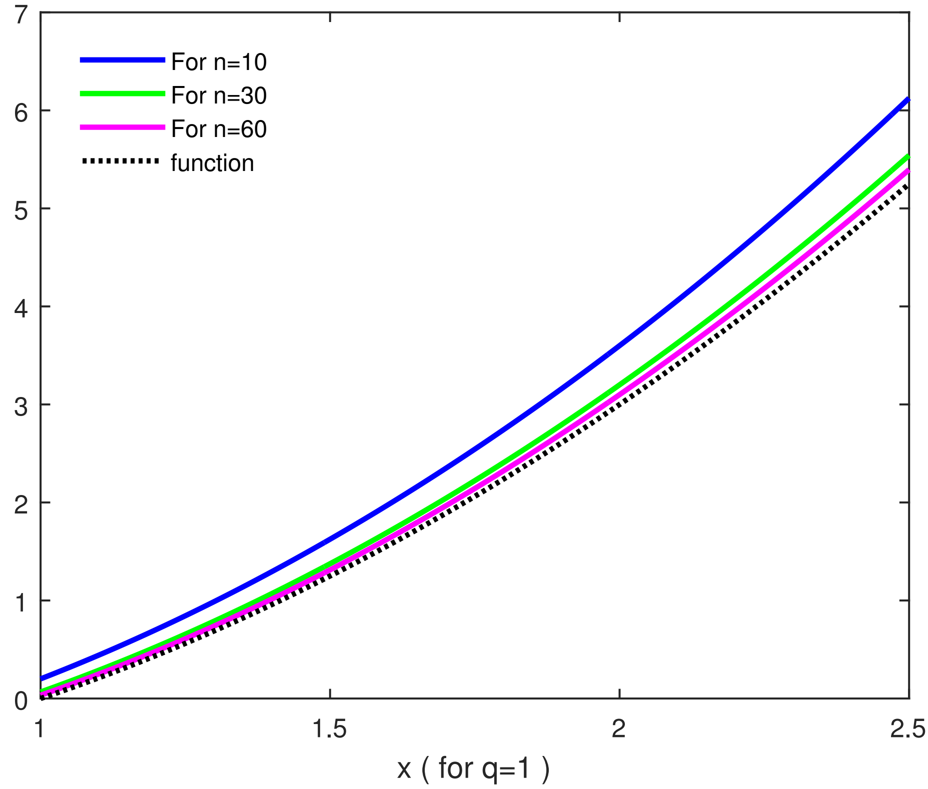

Figure 2: convergence of the operator towards the function $g(x) = x^2 - 1$

Example 5.3. *Let* $f(x) = x^2 - 4x + 3$*. For* $n = 50$ *and different values of* $\mathfrak{q}$*, the convergence of the operator towards the function* $f(x)$ *is shown in Figure 3.*

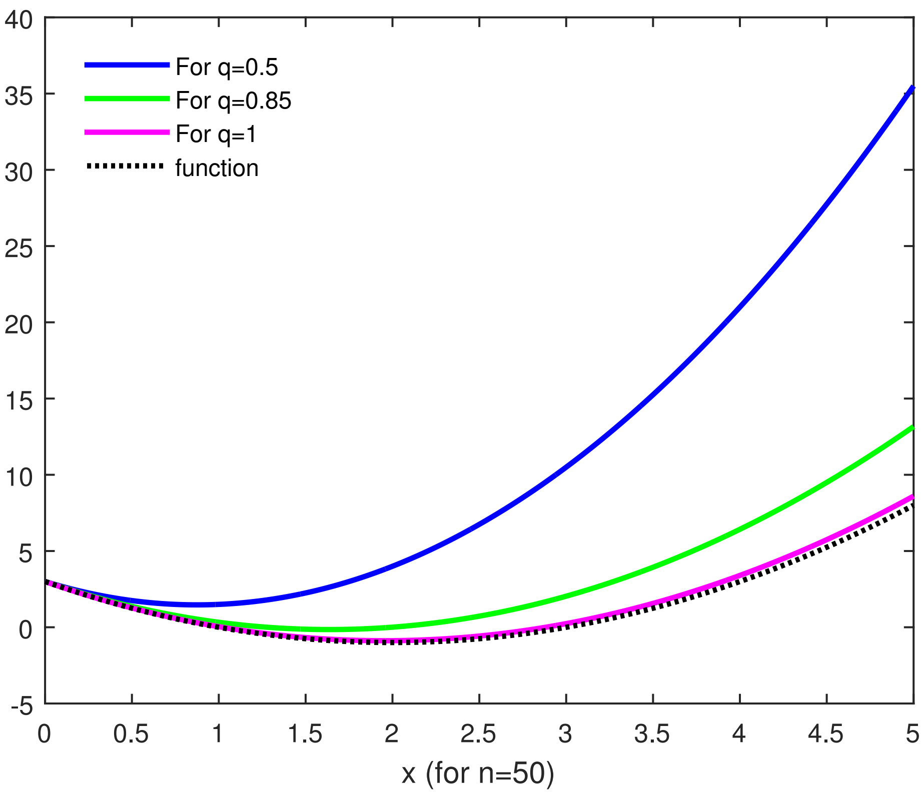

Figure 3: Convergence of the operator for different values of $\mathfrak{q}$

6. Conclusion

With the facilitation of Bohman Korovkin-type theorem, the investigation on weighted statistical approximation behavior of wavelets Kantorovich $\mathfrak{q}$-Baskakov operators $\mathcal{S}_{\mathfrak{r},\mathfrak{q}}$ is discussed under this study. Moreover, the statistical rate of the operators $\mathcal{S}_{\mathfrak{r},\mathfrak{q}}$ is provided in this research with regard to the weighted modulus of smoothness correlated to the space $B_{\rho\alpha}(\mathbb{R}_+)$. The statistical approximation properties discussed in this study are the same as those of classical $\mathfrak{q}$-Baskakov operators defined by (4) since they share the same moments.

Declarations

Ethical Approval

Not Applicable

Availability of supporting data

Not Applicable

Competing interests

Not Applicable

Funding

Not Applicable

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