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Fixed points of $(\epsilon - \delta)$ nonexpansive mappings

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Abstract. We obtain fixed point theorems for nonexpansive mappings by employing a new (ϵ, δ) condition. Our results contain the well-known fixed point theorems due to Meir and Keeler, and Banach as particular cases. The fixed-point sets and domains of the mappings satisfying our theorems have interesting algebraic, geometric and dynamical features. Various examples substantiate our results.

1. Introduction

Meir and Keeler [8] proved that a selfmapping f of a complete metric space (X, d) has a unique fixed point if it satisfies:

(a) given $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that

$$\epsilon \leq d(x, y) < \epsilon + \delta \Rightarrow d(fx, fy) < \epsilon.$$

In 1999 Pant [10] employed the condition:

(b) given $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that

$$\epsilon < \max\{d(x, fx), d(y, fy)\} < \epsilon + \delta \Rightarrow d(fx, fy) \leq \epsilon$$

to resolve the Rhoades' problem [15] on the existence of contractive mappings having discontinuity at the fixed point. Later, Pant and Pant [11] showed that condition (b) applies to nonexpansive mappings as well (see Theorem 2.9 [11]) and named such mappings as $(\epsilon - \delta)$ nonexpansive mappings. Condition (b) or its variants have been employed by researchers to find new solutions of Rhoades' problem, e. g., Bisht and Pant [2], Bisht and Rakocevic [3], Celik and Ozgur [4], Pant [12], Pant et al [13, 14], Tas and Ozgur [16], Zheng and Wang [18]. In the present paper, we replace condition (b) by a new $(\epsilon - \delta)$ condition that applies to contractive as well as nonexpansive mappings. Our result generalizes the fixed point results due to Meir and Keeler [8] and Banach [1].

Definition 1.1 ([5, 6]). *If f is a self-mapping of a set X then a point x in X is called an eventually fixed point of f if there exists a natural number N such that $f^{n+1}(x) = f^n(x)$ for $n \geq N$. If $fx = x$ then x is called a fixed point of f. A point x in X is called a periodic point of period n if $f^n x = x$. The least positive integer n for which $f^n x = x$ is called the prime period of x.*

Definition 1.2. *The set $\{x \in X : fx = x\}$ is called the fixed point set of the mapping $f : X \to X$.*

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2. Main Results

Theorem 2.1. *Let (X,d) be a complete metric space and $f : X \to X$ be such that for each x, y in X with $x \neq fx$ or $y \neq fy$ we have*

(i) Given $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that

$$\epsilon < d(x,y) < \epsilon + \delta \Rightarrow d(fx, fy) \leq \epsilon,$$

(ii) $d(fx, fy) < d(x,y)$.

Then f has a fixed point. Further, f has a unique fixed point if and only if condition (ii) is satisfied for each $x \neq y$ in X.

Proof. From (ii) it follows that $d(fx, fy) \leq d(x,y)$ for each x, y in X. Therefore, f is a nonexpansive mapping and, hence, continuous. Also, for any n points $x_1, x_2, \ldots, x_n$ we get

$$\begin{aligned} &d(fx_1, fx_2) + d(fx_2, fx_3) + \ldots + d(fx_{n-1}, fx_n) \\ &\leq d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{n-1}, x_n), \end{aligned} \tag{1}$$

and
$$\begin{aligned} &d(fx_1, fx_2) + \ldots + d(fx_{n-1}, fx_n) + d(fx_n, fx_1) \\ &\leq d(x_1, x_2) + \ldots + d(x_{n-1}, x_n) + d(x_n, x_1). \end{aligned} \tag{2}$$

Let x_0 be any point in X and $\{x_n\}$ be the sequence defined by $x_n = fx_{n-1}$, that is, $x_n = f^n x_0$. If $x_n = x_{n+1}$ for some n, then x_n is a fixed point of f and the theorem holds. Therefore, assume that $x_n \neq x_{n+1}$ for each $n \geq 0$. Then using (ii), for each $n \geq 1$ and $p \geq 1$ we have

$$d(x_n, x_{n+p}) = d(fx_{n-1}, fx_{n+p-1}) < d(x_{n-1}, x_{n+p-1}).$$

This implies that $\{d(x_n, x_{n+p})\}$ is a strictly decreasing sequence and, hence, tends to a limit $r \geq 0$. If $r > 0$, then there exists a natural number N such that

$$n \geq N \Rightarrow r < d(x_n, x_{n+p}) < r + \delta(r). \tag{3}$$

By virtue of (i) this implies that $d(fx_n, fx_{n+p}) \leq r$, that is, $d(x_{n+1}, x_{n+p+1}) \leq r$, which contradicts (3). Hence, $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ and $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point z in X such that $\lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} f^k x_n = z$ for each integer $k \geq 1$. Continuity of f yields $\lim_{n\to\infty} fx_n = fz$. This implies $z = fz$ and z is a fixed point.

Now let y be any point in X. Then, since $f^n x_0 = x_n$ is not a fixed point, using (ii) we get

$$d(f^n y, f^n x_0) < d(f^{n-1} y, f^{n-1} x_0).$$

This shows that $\{d(f^n y, f^n x_0)\}$ is a strictly decreasing sequence that will tend to a limit $t \geq 0$. If $t > 0$, then there exists a natural number N such that

$$n \geq N \Rightarrow t < d(f^n y, f^n x_0) < t + \delta(t). \tag{4}$$

Using (i), we get $d(ff^n y, ff^n x_0) = d(f^{n+1} y, f^{n+1} x_0) \leq t$. This contradicts (4). Hence $\lim_{n\to\infty} d(f^n y, f^n x_0) = 0$, that is, $\lim_{n\to\infty} f^n y = z$. Thus, if there exists a point x_0 such that $f^{n+1} x_0 \neq f^n x_0$ for each n, then for each y in X the sequence of iterates $\{f^n y\}$ converges to z and z will be the unique fixed point. Thus $f^{n+1} x_0 \neq f^n x_0, n \geq 0$, for some x_0 implies uniqueness of the fixed point. Now, if condition (ii) is satisfied for all x, y in X then f can have only one fixed point. Conversely, suppose that f has a unique fixed point. Then for distinct x, y we have $x \neq fx$ or $y \neq fy$ which implies that condition (ii) holds. This proves the theorem. □

Example 2.2. *Let $X = [1, \infty)$ and d be the Euclidean metric. Let $f : X \to X$ be the signum function $fx = sgn\, x$ defined as*

$$fx = -1 \text{ if } x < 0,\ f0 = 0,\ fx = 1 \text{ if } x > 0.$$

Then $fx = 1$ for each x and f is a contraction mapping. f satisfies condition (ii) for all x, y in X, satisfies (i) with $\delta(\epsilon) = \epsilon$ and has a unique fixed point $x = 1$. If $x \neq 1$ then $fx = f^2x$ and x is an eventually fixed point.

Example 2.3. *Let $X = (-\infty, -1] \cup [1, \infty)$ and d be the Euclidean metric on X. Let $f : X \to X$ be the signum function $fx = sgn\, x$ defined as in Example 2.2.*

Then f satisfies the conditions of Theorem 2.1 and has two fixed points -1 and 1. The mapping f satisfies condition (i) with $\delta(\epsilon) = 2 - \epsilon$ if $\epsilon < 2$ and $\delta(\epsilon) = \epsilon$ if $\epsilon \geq 2$.

Example 2.4. *Consider the region of the complex plane defined by $z = re^{i\theta} = |z|e^{i\theta}, r \geq 1$, where r, θ and $|z|$ have their usual meaning. Let X be the set of points of intersection of this region with the three rays beginning at the origin and respectively making angles $0, \frac{2\pi}{3}, \frac{4\pi}{3}$ measured counter clockwise with the positive real axis. Let d be usual metric on X. Define $f : X \to X$ by*

$$fz = \frac{z}{|z|}.$$

Then f satisfies condition (i) with $\delta(\epsilon) = \sqrt{3} - \epsilon$ if $\epsilon < \sqrt{3}$ and $\delta(\epsilon) = \epsilon$ if $\epsilon \geq \sqrt{3}$, and f satisfies $d(fz_1, fz_2) < d(z_1, z_2)$ if $z_1 \neq fz_1$ or $z_2 \neq fz_2$. Hence f satisfies the conditions of Theorem 2.1 and has three fixed points $e^{i0}, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}$.

In this example if we take any four points z_1, z_2, z_3, z_4 then, in addition to condition (ii), we get inequality (2) for $n = 4$ and

$$\begin{aligned} &\epsilon < d(z_1, z_2) + d(z_2, z_3) + d(z_3, z_4) + d(z_4, z_1) < \epsilon + \delta(\epsilon) \\ \Rightarrow\ &d(fz_1, fz_2) + d(fz_2, fz_3) + d(fz_3, fz_4) + d(fz_4, fz_1) \leq \epsilon, \end{aligned}$$

with $\delta(\epsilon) = 3 + \sqrt{3} - \epsilon$ if $\epsilon < 3 + \sqrt{3}$ and $\delta(\epsilon) = \epsilon$ if $\epsilon \geq 3 + \sqrt{3}$.

Example 2.5. *In analogy with Example 2.4, if we consider the set of points of intersection of the region $z = re^{i\theta}, r \geq 1$, with four rays beginning at the origin and respectively making angles $0, \frac{\pi}{2}, \pi, 3\frac{\pi}{2}$ measured counter clockwise with the positive real axis then f satisfies conditions (i) and (ii) and we get four fixed points $e^{i0}, e^{i\frac{\pi}{2}}, e^{i\pi}, e^{i(3\frac{\pi}{2})}$. If we take any five points z_1, z_2, z_3, z_4, z_5 then, analogous to (2), we get the inequalities*

$$\begin{aligned} &d(fz_1, fz_2) + d(fz_2, fz_3) + d(fz_3, fz_4) + d(fz_4, fz_5) + d(fz_5, fz_1) \\ &< d(z_1, z_2) + d(z_2, z_3) + d(z_3, z_4) + d(z_4, z_5) + d(z_5, z_1), \end{aligned} \tag{5}$$

$$\begin{aligned} \text{and} \qquad &\epsilon < d(z_1, z_2) + d(z_2, z_3) + d(z_3, z_4) + d(z_4, z_5) + d(z_5, z_1) < \epsilon + \delta(\epsilon) \\ &\Longrightarrow\ d(fz_1, fz_2) + d(fz_2, fz_3) + d(fz_3, fz_4) + d(fz_4, fz_5) + d(fz_5, fz_1) \leq \epsilon. \end{aligned}$$

In a similar manner, if we take the intersection of the region $z = re^{i\theta}, r \geq 1$, with two rays beginning at the origin and making angles $0, \pi$ respectively with the positive real axis then we get Example 2.3 given above. Likewise, if we take intersection of the region $z = re^{i\theta}, r \geq 1$, with the positive real axis then we get Example 2.2 above.

Example 2.6. *If we consider the set of points of intersection of the region $z = re^{i\theta}, r \geq 1$, with N rays beginning at the origin and respectively making angles $0, \frac{2\pi}{N}, 2(\frac{2\pi}{N}), 3(\frac{2\pi}{N}), \ldots, (N-1)(\frac{2\pi}{N})$ measured counter clockwise with the positive real axis, then for the function $fz = \frac{z}{|z|}$ we will get N fixed points $e^{i0}, e^{i\frac{2\pi}{N}}, e^{i2(\frac{2\pi}{N})}, e^{i3(\frac{2\pi}{N})}, \ldots, e^{i(N-1)(\frac{2\pi}{N})}$. Also, for any $(N+1)$ points we will get inequalities analogous to (2).*

Example 2.7. *Let us consider a family of concentric circles $z = re^{i\theta} = |z|e^{i\theta}, r = 4^n, n = 0, 1, 2, \ldots$, in the complex plane, where r, θ and $|z|$ have their usual meaning. Let X be the set of points of intersection of these circles with the three rays beginning at the origin and respectively making angles $0, \frac{2\pi}{3}, \frac{4\pi}{3}$ measured counter clockwise with the positive real axis. Let d be usual metric on X. Define $f : X \to X$ by $fz = \frac{z}{|z|}$.*

Then f has three fixed points $e^{i0}, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}$ and every other point in X is an eventually fixed point since $f^2z = fz$ for such points. f satisfies condition (i) with $\delta(\epsilon) = \sqrt{3} - \epsilon$ if $\epsilon < \sqrt{3}$ and $\delta(\epsilon) = \epsilon$ if $\epsilon \geq \sqrt{3}$. If $z_1 \neq fz_1$ or $z_2 \neq fz_2$ then we have $d(fz_1, fz_2) \leq (\frac{2}{3})d(z_1, z_2)$ and condition (ii) is satisfied. For any four points z_1, z_2, z_3, z_4 we shall get $d(fz_1, fz_2) + d(fz_2, fz_3) + d(fz_3, fz_4) + d(fz_4, fz_1) \leq \frac{2}{3}[d(z_1, z_2) + d(z_2, z_3) + d(z_3, z_4) + d(z_4, z_1)]$.

Example 2.8. *Let $X = \{4^n e^{i\theta} : 0 \leq \theta \leq 2\pi, n = 0, 1, 2, \ldots\}$ be the self-similar family of concentric circles, each lying within larger circles having radii in a geometric progression, in the XY-plane and let d be the usual metric on X. Define $f : X \to X$ by $fz = \frac{z}{|z|}$.*

Then each point on the unit circle $z = e^{i\theta}$ is a fixed point while every other point is an eventually fixed point. In this example, the unit circle is a fixed circle. Fixed circles are presently an active area of study (see [7, 9, 17]). If $x \neq fx$ or $y \neq fy$ then $d(fx, fy) \leq \frac{2}{3}d(x, y)$ and, therefore, conditions (i) and (ii) hold.

Example 2.9. *Let (X, d) be a metric space and f be the identity mapping on X, that is, $fx = x$ for each x in X. Then f satisfies conditions (i) and (ii) of Theorem 2.1 and each point is a fixed point.*

Remark 2.10. *If a selfmapping f of a complete metric space (X, d) satisfies the condition (a) of the Meir-Keeler theorem then f has a unique fixed point and consequently satisfies the conditions of Theorem 2.1 also. Hence Theorem 2.1 contains the Meir-Keeler theorem as a particular case. This further implies that Theorem 2.1 contains the Banach contraction theorem since the Meir-Keeler theorem contains the Banach contraction theorem.*

Remark 2.11. *In Example 2.6 the fixed point set consists of N fixed points $e^{i0}, e^{i\frac{2\pi}{N}}, e^{i2(\frac{2\pi}{N})}, e^{i3(\frac{2\pi}{N})}, \ldots, e^{i(N-1)(\frac{2\pi}{N})}$. Some interesting features of this set are:*

- *A. These fixed points are the N^{th} roots of unity, lie on the unit circle, form a cyclic group under multiplication,*
- *B. These points are the vertices of a regular polygon of N sides.*
- *C. If $N = 2^n - 1$ then the fixed point set is identical with the periodic points of period n for the doubling map which is important in dynamics of complex functions (see [5], [6]).*

Similarly, the fixed points in Examples 2.4 and 2.5 respectively represent the cube roots and 4^{th} roots of unity and the set of fixed points in Example 2.4 is identical with the set of periodic points of period 2 for the doubling map.

Remark 2.12. *The domain of a mapping satisfying Theorem 2.1 may possess interesting geometric features. For example, the domain of the mapping in Example 2.8 is a self-similar family of circles.*

3. Applications

We now give an application of condition (ii) in determining the cardinality of the fixed point set of a mapping for which Theorem 2.1 holds.

Suppose (X, d) is a complete metric space and Theorem 2.1 holds for $f : X \to X$. Then f has one or more fixed points. We have seen in Theorem 2.1 that if condition (ii) is satisfied for all $x, y, x \neq y$, in X then f has a unique fixed point. If $u \neq v$ are fixed points of f then we obviously get $d(fu, fv) = d(u, v)$.

Suppose each set of $n + 1$ points $y_1, y_2, \ldots, y_{n+1}$ in X satisfies

$$d(fy_1, fy_2) + d(fy_2, fy_3) + \ldots + d(fy_n, fy_{n+1}) + d(fy_{n+1}, fy_1)$$
$$< d(y_1, y_2) + d(y_2, y_3) + \ldots + d(y_n, y_{n+1}) + d(y_{n+1}, y_1).$$

Then, the number of fixed points of f cannot exceed n. For, if f has $n+1$ fixed points, say $z_1, z_2, \ldots, z_{n+1}$, then we get

$$d(fz_1, fz_2) + d(fz_2, fz_3) + \ldots + d(fz_n, fz_{n+1}) + d(fz_{n+1}, fz_1)$$
$$= d(z_1, z_2) + d(z_2, z_3) + \ldots + d(z_n, z_{n+1}) + d(z_{n+1}, z_1),$$

which contradicts our assumption.

Now, suppose there exists a set of n points $x_1, x_2, \ldots, x_n$ in X such that f does not satisfy

$$d(fx_1, fx_2) + d(fx_2, fx_3) + \ldots + d(fx_{n-1}, fx_n) + d(fx_n, fx_1)$$
$$< d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{n-1}, x_n) + d(x_n, x_1).$$

This condition implies that each of $x_1, x_2, \ldots, x_n$ is a fixed point of f. To see this, suppose $x_1, x_2, \ldots, x_{n-1}$ are fixed points of but not x_n. Then

$$d(fx_1, fx_2) + d(fx_2, fx_3) + \ldots + d(fx_{n-1}, fx_n) + d(fx_n, fx_1)$$
$$= d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{n-2}, x_{n-1}) + d(fx_{n-1}, fx_n) + d(fx_n, fx_1)$$

Using (ii) we get $d(fx_{n-1}, fx_n) + d(fx_n, fx_1) < d(x_{n-1}, x_n) + d(x_n, x_1)$ which implies

$$d(fx_1, fx_2) + d(fx_2, fx_3) + \ldots + d(fx_{n-1}, fx_n) + d(fx_n, fx_1)$$
$$< d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{n-1}, x_n) + d(x_n, x_1).$$

This contradicts our assumption. Hence each of $x_1, x_2, \ldots, x_n$ is a fixed point of f. This can be summarised as:

Theorem 3.1. *The cardinality of the set of fixed point of a selfmapping f satisfying the conditions of Theorem 2.1 equals n if and only if for each set of $n+1$ points $y_1, y_2, \ldots, y_{n+1}$ we have*

$$d(fy_1, fy_2) + d(fy_2, fy_3) + \ldots + d(fy_n, fy_{n+1}) + d(fy_{n+1}, fy_1)$$
$$< d(y_1, y_2) + d(y_2, y_3) + \ldots + d(y_n, y_{n+1}) + d(y_{n+1}, y_1), \quad (6)$$

while there exists a set of n points $x_1, x_2, \ldots, x_n$ in X that does not satisfy

$$d(fx_1, fx_2) + d(fx_2, fx_3) + \ldots + d(fx_{n-1}, fx_n) + d(fx_n, fx_1)$$
$$< d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{n-1}, x_n) + d(x_n, x_1). \quad (7)$$

Remark 3.2. *The proof of Theorem 2.1 shows that if for some x in X we have $f^n x \neq f^{n+1} x$ for each $n \geq 0$ then f has a unique fixed point. This implies that if f has more than one fixed point then the orbit $\{f^n x : n = 0, 1, \ldots\}$ of each x in X is a finite set, that is, starting the iteration with any initial point we reach the fixed point in a finite number of steps. This simplifies the search for fixed points. If f has a finite number of fixed points then inequalities (6) and (7) will help in finding cardinality of the fixed point set.*

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