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# Harnack inequality for porous medium equations with a blow-up term along the Finsler-geometric flow

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**Abstract.** We prove global gradient estimates for positive bounded solutions of porous medium equations with a blow-up term

$$\partial_t u = \Delta_m u^p + cu^q$$

on a compact $n$-dimensional Finsler manifold $(M^n, F(t), m)$ evolving under the Finsler-geometric flow. Then, we obtain Harnack-type inequalities.

## 1. Introduction

Gradient estimates for solutions to the equations are very important tools in geometric analysis [27, 28]. Li and Yau [28] proved the classical parabolic type gradient estimate for the heat equation by use of the maximal principle, then they derived Harnack inequalities. Recently, Li-Yau type gradient estimates of nonlinear PDEs on manifolds have been extensively investigated. These methods are also used to geometric flows; for example, Hamilton [22] used them for the Riemannian-Ricci flow. Aronson and Bénilan [1] considered positive solutions to the porous medium equation (PME)

$$v_t = \Delta v^p$$

on $\mathbb{R}^n$ with $p > (1 - \frac{2}{n})_+$ and obtained some inequalities for them. These inequalities are called Aronson-Bénilan inequalities. Later, Vázquez [36] and Lu et al. [30] derived Aronson-Bénilan and Li-Yau type estimates for positive solutions to PME on complete manifolds with a fixed metric and some conditions on Ricci curvature. Huang et al. [23] generalized the results of [30] and obtained the Li-Xu type gradient and Li-Yau-Hamilton type estimates for positive solutions of PMEs on Riemannian manifolds.
In 2011, Mizoguchi et al. [32] studied the equation

$$u_t = \Delta u^p + u^q$$

with $p, q > 1$ on $\mathbb{R}^n \times (0, \infty)$ and in 2014, Duan et al. [19] investigated the solutions of

$$u_t = \Delta u^p + f(u)$$

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with $p > 1$ and smooth function $f$ on $\mathbb{R}^n \times (0, \infty)$. Also, Y. Z. Wang and X. Wang [37] obtained the gradient estimates for equations

$$v_t = \Delta_p v^a + bv^c$$

on compact Riemannian manifolds with nonnegative Ricci curvature where $a > 0$, $p > 1$, $b$ and $c$ are constants.

In recent years, many authors have used similar methods to find gradient estimates and Harnack inequalities for solutions of parabolic equations along the geometric flows, see for instance [2, 8, 13, 16, 20, 21, 24, 26, 29, 41] and the references therein. B. Ma and J. Li [31] proved Li-Yau type gradient estimates for PME along the Ricci flow. Cao and Zhu [17] and Wang et al. [38] derived Aronson-Bénilan estimates for solution to PME along the Ricci flow.

On the other hand on Finsler manifolds, Ohta and Sturm [34] proved a Li-Yau gradient estimate on compact Finsler manifolds. Then, Lakzian [25] obtained differential Harnack estimates for positive global solutions to $\partial_t v = \Delta_m v$ along Finsler-Ricci flow and later Zeng and He [40] generalized Lakzian's results. Zeng [39] extended their results with considered Finsler-geometric flow. Then, Cheng [18] proved gradient estimates for positive solutions of $\partial_t v = \Delta_m v$ along Finsler-Ricci flow regardless of the condition on $S$-curvature. Author [3] obtained gradient estimates for positive solutions of the parabolic equation

$$\partial_t u(x,t) = \Delta_m u(x,t) - A(x,t)u(x,t) - B(u(x,t))$$

under Finsler-geometric flow where $A$ is a function on $M \times [0, T]$ of $C^2$ in $x$-variables and $C^1$ in $t$-variable and $B(u)$ is a function of $C^2$ in $u$. Also, author in [4–7] studied gradient estimates for positive solutions to some equations under Finsler-geometric flow.

Let $(M^n, F(t), m)$, $t \in [0, T]$, be a compact Finsler manifold and its metric evolves by the Finsler-geometric flow

$$\frac{\partial}{\partial t} g(x,t) = 2h(x,t), \quad (x,t) \in M \times [0,T] \tag{1}$$

where $g(t)$ is the symmetric metric tensor associated with $F$, $h(t)$ is a general time-dependent symmetric $(0, 2)$-tensor, and $T$ is taken to be the maximum time of existence for the flow (1). When $h_{ij}(t) = -Ric_{ij}$, then the flow (1) is called the Finsler-Ricci flow [9, 11, 14]. Also, if $h_{ij}(t) = -\frac{1}{2}Hg_{ij}$ then flow (1) is called the Finsler-Yamabe flow [10, 15] where $H = g^{ij}Ric_{ij}$ is the scalar curvature. Also, let $\Delta_m$ be the Finslerian-Laplacian.

In the present paper, we prove various gradient estimates for positive solutions of PME with a blow-up term

$$\partial_t v = \Delta_m v^p + cv^q, \tag{2}$$

along the flow (1), where $p, q, c$ are constants such that $p > 1$, $q \geq 1$, and $c \geq 0$. A function $v$ on $M \times [0, T]$ is said to be a global solution to (2) if

$$v(x,t) \in L^2([0,T], H^1(M)) \bigcap H^1([0,T], H^{-1}(M))$$

and it satisfies

$$\int_M \phi \partial_t v dm = -p \int_M v^{p-1} D\phi(\nabla v) dm + c \int_M \phi v^q dm, \tag{3}$$

for any $\phi \in C^\infty(M)$ and for all $t \in [0, T]$. Let $v$ be a solution to (2) and $u = \frac{p}{p-1}v^{p-1}$. By direct computation, we have

$$\partial_t u = (p-1)u\Delta_m u + F^2(\nabla u) + au^{b+1}, \tag{4}$$

where $b = \frac{q-1}{p-1}$ and $a = c(p-1)^{\frac{p+q-2}{p-1}} p^{-\frac{q-1}{p-1}}$. Since we assume $v$ is positive on compact Finsler manifold $M$, we can set $D_1 \leq v(x,t) \leq D_2$ for some positive constants $D_1, D_2$. Then $\frac{p}{p-1}D_1^{p-1} \leq u(x,t) \leq \frac{p}{p-1}D_2^{p-1}$. For simplicity, let

$$\tilde{D}_i = \min\left\{\left(\frac{p}{p-1}D_1^{p-1}\right)^{b-i}, \left(\frac{p}{p-1}D_2^{p-1}\right)^{b-i}\right\}$$

and

$$\bar{D}_i = \max\left\{\left(\frac{p}{p-1}D_1^{p-1}\right)^{b-i}, \left(\frac{p}{p-1}D_2^{p-1}\right)^{b-i}\right\}$$

for $i = 0, 1, 2, 3$.

In this paper, we follow the works done on Finsler manifolds [7, 39] and obtain some gradients estimates for positive solutions of PME with a blow-up term under the Finsler-geometric flow. We now obtain the global space-time gradient estimates for PMEs with a blow-up term (4) along the Finsler-geometric flow (1).

**Theorem 1.1.** *Let $(M^n, F(t))$ be a complete solution to (1) in some time interval $[0, T]$. Assume that there exist some positive constants $k_1, k_2, k_3$ and $k_4$ such that $\Re \geq -k_1 g$, $-k_2 g \leq \mathfrak{h} \leq k_3 g$, $|\nabla \mathfrak{h}| \leq k_4$ for all $t \in [0, T]$, and S-curvature vanishes. Suppose that $v$ is any smooth positive global solution (2) such that $D_1 \leq v \leq D_2$ for some positive constants $D_1, D_2$. Assume that $p > 1$ and $u = \frac{p}{p-1}v^{p-1}$. For any $\alpha > 1$, $c \geq 0$, $q \geq 1$ on $M \times [0, T]$, and arbitrary constant $\epsilon \in (0, 1)$ the following estimate holds*

$$\frac{F^2(\nabla u)}{u} - \alpha\frac{\partial_t u}{u} + \alpha a u^b \leq \frac{n(p-1)\alpha^2}{n(p-1) + 2(1-\epsilon)}\left\{K_1 + K_2 + \frac{1}{t}\right\}, \tag{5}$$

*where*

$$K_1 = \frac{1}{2(\alpha-1)}\Big(k_3(\alpha-1) + \frac{p}{2}D_2^{p-1}\epsilon + pD_2^{p-1}k_1 + \frac{1}{2}ab(\alpha-1)\bar{D}_0 + ab\bar{D}_3\Big) + a\bar{D}_0 + ab\bar{D}_0,$$

*and*

$$K_2 = \left\{\frac{n(p-1) + 2(1-\epsilon)}{n\epsilon}\Big(\max\{k_2^2, k_3^2\} + 2k_4^2\Big)\right\}^{\frac{1}{2}}.$$

As an application of global gradient estimates obtained in Theorem 1.1, we have:

**Corollary 1.2.** *With the assumptions of Theorem 1.1, for $(x_1, t_1) \in M^n \times (0, T)$ and $(x_2, t_2) \in M^n \times (0, T)$ such that $t_1 < t_2$, the have*

$$\begin{aligned} u(x_1, t_1) \leq\ & u(x_2, t_2)\left(\frac{t_2}{t_1}\right)^{\frac{n(p-1)\alpha}{n(p-1)+2(1-\epsilon)}} \exp\left\{\int_0^1 \frac{\alpha p}{4(p-1)}D_2^{p-1}\frac{[F(\dot{\gamma}(s))]^2}{t_2 - t_1}ds \right.\\ & \left. + \frac{n(p-1)\alpha(t_2 - t_1)}{n(p-1) + 2(1-\epsilon)}\{K_1 + K_2\} + \alpha a\tilde{D}_0(t_2 - t_1)\right\}. \end{aligned}$$

*where $\gamma$ is a smooth curve connecting points $x_1$ and $x_2$ in $M$ with $\gamma(1) = x_1$ and $\gamma(0) = x_2$, and $F(\dot{\gamma}(s)))|_{\tau(s)}$ is the length of the vector $\dot{\gamma}(t)$ at time $\tau(s) = (1-s)t_2 + st_1$.*

**Remark 1.3.** *If in Theorem 1.1 and Corollary 1.2, we let the metric be Riemannian and the geometric flow be Ricci flow then the results of [31] are obtained for $c = 0$.*

**Remark 1.4.** *We use Zeng's method in [39] to prove our results. Zeng has used the auxiliary function* $\mathcal{G} = F^2(\nabla f) - \alpha\partial_t f$ *to obtain the gradient estimates of the positive global solution* $v = e^f$ *of the heat equation* $\partial_t v = \Delta_m v$. *In the sense of distribution, he computed* $\Delta_m^{\nabla f}\mathcal{G} - \partial_t\mathcal{G} + 2D\mathcal{G}(\nabla f)$ *and by used it obtained gradient estimates of the positive global solutions to the heat equation. In this paper, we use the auxiliary function* $\mathcal{P} = \frac{F^2(\nabla u)}{u} - \alpha\frac{\partial_t u}{u} + \alpha a u^b$ *to obtain the gradient estimates of the positive global solution of (2). In the sense of distribution, we calculate* $(p-1)u\Delta_m^{\nabla u}\mathcal{P} - \partial_t\mathcal{P} + 2pD\mathcal{P}(\nabla u)$ *and by use it we drive gradient estimates of the positive global solutions to (2). The calculations and results of this paper are new and completly different from [39].*

## 2. Preliminaries

We will briefly review the basic concepts of Finsler geometry that are required for the paper (see [12, 35]). Suppose that $M$ is an $n$-dimensional connected, smooth manifold and $\pi : TM \to M$ is the natural projection. Let $U \subset M$ be an open set, we denote the local coordinates on $U$ and the fiber-wise linear coordinates on $TU$ by $(x^i)_{i=1}^n$ and $(x^i, y^j)_{i,j=1}^n$, respectively, where $y = \sum_{j=1}^n y^j \frac{\partial}{\partial x^j}|_x \in T_xU,\ x \in U$.

A function $F : TM \to [0, \infty)$ is called a Finsler structure on $M$ whenever it satisfies the following properties:
(i) Regularity: $F \in C^\infty(TM \setminus \{0\})$;
(ii) Positive homogeneity: $F(x, \theta y) = \theta F(x, y)$ for any $(x, y) \in TM$ and $\theta > 0$;
(iii) Strong convexity: for any $(x, y) \in TM \setminus \{0\}$, $g_{ij}(y) := \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j}F^2(x, y)$ is positive definite.
The pair $(M, F)$ is said to be a Finsler manifold. A Finsler measure space is a triple $(M^n, F, m)$ such that $M$ is a smooth, connected manifold, $F$ is a Finsler structure on $M$, and $m$ is a measure on $M$. Let $V$ be a non-zero vector field, then a Riemannian structure $g_V$ defined by

$$g_V(X, Y) = \sum_{i,j=1}^n g_{ij}(V)X^iY^j, \quad X, Y \in T_xM$$

and a linear connection defined by

$$D^V_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \Gamma^k_{ij}\frac{\partial}{\partial x^k},$$

where

$$\Gamma^k_{ij} = \frac{1}{2}g^{il}\left(\frac{\delta g_{kl}}{\delta x^j} + \frac{\delta g_{jl}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^l}\right)$$

are the Chern connection coefficients. Here

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i\frac{\partial}{\partial y^j}, \ N^j_i = \frac{\partial G^j}{\partial y^i}, \ G^j = \frac{1}{4}g^{jl}\left\{[F^2]_{x^k y^l}y^k - [F^2]_{x^l}\right\},$$

and the pair $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\}$ is a frame for $TTM$. Let $\{dx^i, \delta y^i\}$ denotes the local frame dual to $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\}$, where $\delta y^i = dy^i + N^i_j dx^j$. The torsion freeness of Chern connection implies that

$$D^V_X Y - D^V_Y X = [X, Y],$$

and almost $g$-compatibility of Chern connection yields

$$Xg_V(Y, Z) = g_V(D^V_X Y, Z) + g_V(Y, D^V_X Z) + 2C_V(D^V_X V, Y, Z),$$

for $V \in T_xM \setminus \{0\}$, $X, Y, Z \in T_xM$ where $C_V$ denotes the Cartan tensor. The covariant derivative of any smooth local section $S = S^{ij}\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$ of $\pi^*TM \otimes \pi^*T^*M$ is given by

$$(\nabla S)^{ij} = S^{ij}_{|s}dx^s + S^{ij}_{;s}\frac{\delta y^s}{F},$$

where $S^{ij}_{|s}$ and $S^{ij}_{;s}$ are the horizontal and vertical covariant derivatives, respectively and are defined by

$$S^{ij}_{|s} = \frac{\delta S^{ij}}{\delta x^s} + S^{ki}\Gamma^j_{ks} + S^{kj}\Gamma^i_{ks}$$

and

$$S^{ij}_{;s} = F\frac{\partial S^{ij}}{\partial y^s},$$

respectively.

The Legendre transform $\mathcal{L} : TM \to T^*M$ is given by

$$\mathcal{L}(V) = \begin{cases} g_V(V,.) & V \in T_xM \setminus \{0\}, \\ 0 & V = 0. \end{cases}$$

For any $V \in TM$, $F(V) = F^*(\mathcal{L}(V))$, where $F^*$ is the dual norm of $F$ on $T^*M$, and presented by

$$F^*(x,\xi) := \sup_{F(x,V)\leq 1} \xi(V)$$

for any $\xi \in T^*M$. Gradient vector of a smooth function $u : M \to \mathbb{R}$, is defined by

$$\nabla u(x) := \mathcal{L}^{-1}(du(x)) = \begin{cases} g^{ij}(x,\nabla u)\frac{\partial u}{\partial x^j}\frac{\partial}{\partial x^i} & du(x) \neq 0, \\ 0 & du(x) = 0. \end{cases}$$

Let $M_u := \{x \in M | du(x) \neq 0\}$. For $x \in M_u$, we define the Hessian of $u$ as follows

$$\nabla^2 u(X,Y) = g_{\nabla u}(D^{\nabla u}_X \nabla u, X).$$

The divergence of $V = V^i\frac{\partial}{\partial x^i} \in TM$ with respect to measure $dm = e^{\Phi}dx^1dx^2...dx^n$ is given by

$$\mathrm{div}_m V = \sum_{i=1}^{n}\left(\frac{\partial V^i}{\partial x^i} + V^i\frac{\partial \Phi}{\partial x^i}\right).$$

Then, given a function $u \in W^{1,2}(M)$ the Finsler-Laplacian of $u$ is given by $\Delta_m u := \mathrm{div}_m(\nabla u)$, where $W^{1,2}(M)$ denotes the completion of $C^\infty(M)$. For any vector field $V$, the weighted Laplacian on the weighted Riemannian manifold $(M, g_V, m)$ is defined by $\Delta^V_m u := \mathrm{div}_m(\nabla^V u)$ where

$$\nabla^V u(x) := \begin{cases} g^{ij}(x,V)\frac{\partial u}{\partial x^j}\frac{\partial}{\partial x^i} & du(x) \neq 0, \\ 0 & du(x) = 0. \end{cases}$$

We note that $\Delta^{\nabla u}_m u = \Delta_m u$.
For any two linearly independent vectors $X, Y \in T_xM \setminus \{0\}$, the flag curvature is given by

$$\mathcal{K}^V(X,Y) = \frac{g_V(R^V(X,Y)Y,X)}{g_V(X,X)g_V(Y,Y) - g_V(X,Y)^2},$$

where

$$R^V(X,Y)Z := D^V_X D^V_Y Z - D^V_Y D^V_X Z - D^V_{[X,Y]}Z.$$

Let $e_1, ..., e_{n-1}, \frac{V}{F(V)}$ be an orthonormal basis of $T_xM$ with respect to $g_V$, the Ricci curvature is determined by

$$Ric(V) := \sum_{i=1}^{n-1}\mathcal{K}^V(V, e_i).$$

For a vector field $V \in T_xM \setminus \{0\}$, $\tau(V) := \ln \frac{\sqrt{\det(g_{ij}(V))}}{\sigma_F(x)}$ is called the distortion of $(M, F, m)$, where $\sigma_F(x)$ is the volume ratio

$$\sigma_F(x) = \frac{\mathrm{Vol}(B_{\mathbb{R}^n}(1))}{\mathrm{Vol}\left((a_i) \in \mathbb{R}^n | F(\sum a_i \frac{\partial}{\partial x^i}) < 1\right)},$$

and

$$S(y) := \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i}(\ln \sigma_F(x))$$

is said to be a $S$-curvature. From [33] we have:

**Definition 2.1.** *Suppose that $(M, F, m)$ is a Finsler measure space, for any $V \in T_xM \setminus \{0\}$, let $\gamma : (-\epsilon, \epsilon) \to M$ be the geodesic such that $\gamma(0) = x$, $\dot{\gamma}(0) = V$. Define*

$$\dot{S}(V) := F^{-2}(V) \frac{d}{dt}[S(\gamma(t), \dot{\gamma}(t)]_{t=0}$$

*where $S(V)$ is the $S$-curvature. Then weighted Ricci curvature of $(M, F, m)$ is defined by*

$$\begin{cases} Ric_n(V) := \begin{cases} Ric(V) + \dot{S}(V) & S(V) = 0, \\ -\infty & S(V) \neq 0, \end{cases} \\ Ric_N(V) := Ric(V) + \dot{S}(V) + \frac{S(V)^2}{(N-n)F(V)^2} \qquad \forall N \in (n, \infty), \\ Ric_\infty(V) := Ric(V) + \dot{S}(V). \end{cases}$$

Also Akbarzadeh's Ricci tensor $Ric_{ij}$ is given by

$$Ric_{ij} := \frac{\partial^2}{\partial y^i \partial y^j}\left(\frac{F^2 Ric}{2}\right).$$

The second order contravariant tensor of Akbarzadeh's Ricci tensor is determined by $\mathfrak{R} := R^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$ where $Ric^{ij} = g^{ik} g^{jl} Ric_{kl}$. Similarly we denote the second order contravariant tensor of $h$ by $\mathfrak{h} := h^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$ where $h^{ij} = g^{ik} g^{jl} h_{kl}$.

## 3. Proof of our results

In this section, we prove our main results. We use the Bochner-Weitzenböck formula, proved by Ohta-Sturm [34] to prove Theorem 1.1 and it is as follows.

**Theorem 3.1.** *[34][Bochner-Weitzenböck formula] Suppose that $(M^n, F, m)$ is a Finsler manifold with volume form $dm$. Then for any $v \in C^\infty(M)$, we get*

$$\frac{1}{2}\Delta_m^{\nabla v}(F^2(\nabla v)) = D(\Delta_m v)(\nabla v) + Ric_\infty(\nabla v) + |\nabla^2 v|^2_{HS(\nabla v)}, \tag{6}$$

*point-wise on $M_u$. Also, for a given $v \in W^{2,2}_{loc}(M) \bigcap C^1(M)$ with $\Delta_m v \in W^{1,2}_{loc}(M)$, we have*

$$-\frac{1}{2}\int_M D\psi\left(\nabla^{\nabla v}(F^2(\nabla v))\right) dm = \int_M \psi\Big(D(\Delta_m v)(\nabla v) + Ric_\infty(\nabla v) + |\nabla^2 v|^2_{HS(\nabla v)}\Big) dm, \tag{7}$$

*for all nonnegative functions $\psi \in W^{1,2}_c(M) \bigcap L^\infty(M)$, where $|\nabla^2 v|^2_{HS(\nabla u)}$ is the Hilbert-Schmidt norm with respect to $g_{\nabla v}$.*

We also require the following lemma from [39] for establish our results.

**Lemma 3.2.** *Suppose that $h(t)$ is a symmetric $(0,2)$-tensor field on $(M^n, F(t), m)$. Thus, we have*

$$(h^{ij}(\nabla v))_i v_j = h^{ij}_{|i} v_j + h^{ij}_{;k} \frac{v_j}{F} (\nabla^2 v)^k_i, \tag{8}$$

*where $h^{ij}_{|i}$ and $h^{ij}_{;k}$ are the horizontal and vertical covariant derivatives of $h^{ij}$, respectively.*

From [39], we have:

**Lemma 3.3.** *Suppose that $(M^n, F(t))$ is a complete solution to (1). Then, we have*

$$\partial_t (F^2(\nabla v)) = -2h^{ij}(\nabla v) v_i v_j + 2D(\partial_t v)(\nabla v). \tag{9}$$

To prove Theorem 1.1, we state and prove the following lemma.

**Lemma 3.4.** *Suppose that the function $u$ satisfies in (4), $\alpha > 1$ is a constant, and*

$$\mathcal{P}(x,t) = \frac{F^2(\nabla u)}{u}(x,t) - \alpha \frac{\partial_t u}{u}(x,t) + \alpha a u^b.$$

*In the sense of distribution and under flow (1), we have*

$$\begin{aligned} \Box \mathcal{P} = & -\frac{2(\alpha-1)}{u} h^{ij}(\nabla u) u_i u_j - 2(p-1)\alpha h^{ij}(\nabla u) u_{ij} - 2(p-1)\alpha h^{ij}_{|i} u_j \\ & -2(p-1)\alpha h^{ij}_{;k} \frac{u_j}{F} (\nabla^2 u)^k_i + 2(p-1) Ric(\nabla u) + 2(p-1)|\nabla^2 u|^2_{HS(\nabla u f)} \\ & +((p-1)\Delta_m u)^2 + (\alpha-1)(\frac{\partial_t u}{u} - a u^b)^2 + \alpha(p-1) a b u^b \Delta_m u \\ & +\alpha(p-1) a b^2 u^{b-1} F^2(\nabla u) + \alpha(p+1) a b u^{b-1} F^2(\nabla v) + 2 a u^{b-1} F^2(\nabla u) \\ & -2 a b u^{b-2} F^2(\nabla u) - a u^b \mathcal{P}, \end{aligned} \tag{10}$$

*where $\Box \mathcal{P} = (p-1) u \Delta_m^{\nabla u} \mathcal{P} - \partial_t \mathcal{P} + 2pD\mathcal{P}(\nabla u)$.*

*Proof.* Firstly, we assume that $\mathcal{I}(x,t) = \frac{\partial_t u}{u}(x,t) - a u^b(x,t)$ and we compute

$$(p-1) u \Delta_m^{\nabla u} \mathcal{I} - \partial_t \mathcal{I} + 2pD\mathcal{I}(\nabla u).$$

The function $\mathcal{I}(x,t) \in H^1(M)$ is Hölder continuous in both space and time. For every non-negative test function $\phi \in H^1_0(M \times [0,T])$ such that its support is included in the domain of the local coordinate, we get

$$\partial_t \Big( D(\phi)(\nabla u) \Big) = \partial_t (g^{ij})(\nabla u) \phi_i u_j + \frac{\partial g^{ij}}{\partial y^k} (\partial_t u^k) \phi_i u_j + g^{ij}(\nabla u)(\partial_t \phi))_i u_j + g^{ij}(\nabla u) \phi_i (\partial_t u)_j.$$

Using equation $\partial_t g^{ij} = -2h^{ij}$ and $C_V(V,X,Y) = 0$, we deduce

$$\partial_t \Big( D(\phi)(\nabla u) \Big) = -2h^{ij}(\nabla u)(\phi)_i v_j + D(\partial_t(\phi))(\nabla u) + D(\phi) \Big( \nabla^{\nabla u}(\partial_t u) \Big). \tag{11}$$

Multiplying $(p-1) u \Delta_m^{\nabla f} \mathcal{I} - \partial_t \mathcal{I} + 2pD\mathcal{I}(\nabla u)$ by $\phi$ and integrating, we infer

$$\mathcal{A} := \int_0^T \int_M \phi \Big( (p-1) u \Delta_m^{\nabla u} \mathcal{I} - \partial_t \mathcal{I} + 2pD\mathcal{I}(\nabla u) \Big) dm dt.$$

Integration by part to part yields

$$\mathcal{A} = \int_0^T \int_M \Big( -(p-1) u D\phi(\nabla^{\nabla u} \mathcal{I}) - (p-1)\phi Du(\nabla^{\nabla u} \mathcal{I}) + \mathcal{I} \partial_t \phi \Big) dm dt + \int_0^T \int_M \Big( 2\phi p D\mathcal{I}(\nabla u) \Big) dm dt.$$

Inserting $\mathcal{I}(x,t) = \frac{\partial_t u}{u}(x,t) - au^b$ in the last equation, we obtain

$$\begin{aligned}\mathcal{A} &= \int_0^T \int_M \Big( -(p-1)uD(\phi)(\nabla^{\nabla u}(\frac{\partial_t u}{u})) - (p-1)\phi D(u)(\nabla^{\nabla u}(\frac{\partial_t u}{u})) \Big) dmdt \\ &+ \int_0^T \int_M \Big( \partial_t(\phi)(\frac{\partial_t u}{u}) + 2p\phi D(\frac{\partial_t u}{u})(\nabla u) \Big) dmdt \\ &+ \int_0^T \int_M \Big( (p-1)auD(\phi)\nabla u^b - au^b\partial_t\phi - (p+1)a\phi bu^{b-1}F^2(\nabla u) \Big) dmdt.\end{aligned}$$

By direct computation, we conclude that

$$\begin{aligned}\mathcal{A} &= \int_0^T \int_M \Big( -(p-1)D(\phi)(\nabla^{\nabla u}(\partial_t u)) + (p-1)\frac{\partial_t u}{u} D(\phi)(\nabla^{\nabla u}(u)) \Big) dmdt \\ &+ \int_0^T \int_M \Big( -(p-1)\frac{\phi}{u} Du)(\nabla^{\nabla u}(\partial_t u)) + (p-1)\frac{\phi \partial_t u}{u^2} F^2(\nabla u) \Big) dmdt \\ &+ \int_0^T \int_M \Big( \partial_t(\phi)(\frac{\partial_t u}{u}) + 2p\phi D(\frac{\partial_t u}{u})(\nabla u) \Big) dmdt \\ &+ \int_0^T \int_M \Big( (p-1)auD(\phi)\nabla u^b - au^b\partial_t\phi - (p+1)a\phi bu^{b-1}F^2(\nabla u) \Big) dmdt.\end{aligned}$$

Applying (11) and

$$\int_0^T \int_M \partial_t \Big( D(\phi)(\nabla u) \Big) dmdt = 0,$$

we infer

$$\begin{aligned}\mathcal{A} &= \int_0^T \int_M \Big( -2(p-1)h^{ij}(\nabla u)(\phi)_i u_j + (p-1)D(\partial_t\phi)(\nabla u) \Big) dmdt \\ &+ \int_0^T \int_M \Big( (p-1)\frac{\partial_t u}{u} D(\phi)(\nabla^{\nabla u}(u)) \Big) dmdt \\ &+ \int_0^T \int_M \Big( -(p-1)\frac{\phi}{u} D(u)(\nabla^{\nabla v}(\partial_t u)) + (p-1)\frac{\phi \partial_t u}{u^2} F^2(\nabla u) \Big) dmdt \\ &+ \int_0^T \int_M \Big( (p-1)auD(\phi)\nabla u^b - (p+1)a\phi bu^{b-1}F^2(\nabla u) \Big) dmdt \\ &+ \int_0^T \int_M \Big( \partial_t(\phi)(\frac{\partial_t u}{u} - au^b) + 2p\phi D(\frac{\partial_t u}{u})(\nabla u) \Big) dmdt.\end{aligned}$$

Substituting $\frac{\partial_t u}{u} = (p-1)\Delta_m u + \frac{F^2(\nabla u)}{u} + au^b$ in the above equation, we arrive at

$$\begin{aligned}\mathcal{A} &= \int_0^T \int_M \Big( -2(p-1)h^{ij}(\nabla u)(\phi)_i u_j + (p-1)\frac{\partial_t u}{u} D(\phi)(\nabla^{\nabla u}(u)) \Big) dmdt \\ &+ \int_0^T \int_M \Big( (p-1)auD(\phi)\nabla u^b - (p+1)a\phi bu^{b-1}F^2(\nabla u) \Big) dmdt \\ &+ \int_0^T \int_M \Big( \frac{F^2(\nabla u)}{u}\partial_t(\phi) + (p+1)\phi D(\frac{\partial_t u}{u})(\nabla u) \Big) dmdt.\end{aligned}$$

Formula (8) of Lemma 3.2 leads to

$$\mathcal{A} = \int_0^T \int_M \phi \Big( 2(p-1)h^{ij}_{|i} u_j + 2(p-1)h^{ij}_{;k} \frac{u_j}{F}(\nabla^2 u)^k_i + 2(p-1)h^{ij}(\nabla u)u_{ij} \Big) dmdt$$

$$+\int_0^T\int_M \phi\Big((p-1)\frac{\partial_t u}{u}D(\phi)(\nabla^{\nabla u}(u))\Big)dmdt$$
$$+\int_0^T\int_M \Big((p-1)auD(\phi)\nabla u^b-(p+1)a\phi bu^{b-1}F^2(\nabla u)\Big)dmdt$$
$$+\int_0^T\int_M \Big(\frac{F^2(\nabla u)}{u}\partial_t(\phi)+(p+1)\phi D(\frac{\partial_t u}{u})(\nabla u)\Big)dmdt.$$

Using (9) in last equation, we obtain

$$\begin{aligned}\mathcal{A} &= \int_0^T\int_M \phi\Big(2(p-1)h^{ij}_{|i}u_j+2(p-1)h^{ij}_{;k}\frac{u_j}{F}(\nabla^2 u)^k_i+2(p-1)h^{ij}(\nabla u)u_{ij}\Big)dmdt\\ &\quad+\int_0^T\int_M \phi\Big(\frac{2}{u}h^{ij}(\nabla u)u_iu_j-\frac{\partial_t u}{u^2}F^2(\nabla u)-(p-1)\frac{\partial_t u}{v}\Delta_m u\Big)dmdt\\ &\quad-\int_0^T\int_M \phi\Big(((p-1)b+(p+1))abu^{b-1}F^2(\nabla u)+(p-1)abu^b\Delta_m u\Big)dmdt. \end{aligned} \tag{12}$$

Now for compute $(p-1)u\Delta_m^{\nabla u}\mathcal{P}-\partial_t\mathcal{P}+2pD\mathcal{P}(\nabla u)$, multiplying it by $\phi$ and integrating, we have

$$\mathcal{B} := \int_0^T\int_M \phi\Big((p-1)u\Delta_m^{\nabla f}\mathcal{P}-\partial_t\mathcal{P}+2pD\mathcal{P}(\nabla u)\Big)dmdt.$$

Integration by parts to parts implies that

$$\mathcal{B} = \int_0^T\int_M \Big(-(p-1)uD\phi(\nabla^{\nabla v}\mathcal{P})-(p-1)\phi Du(\nabla^{\nabla v}\mathcal{P})-\phi\partial_t\mathcal{P}+2p\phi D\mathcal{P}(\nabla u)\Big)dmdt.$$

Using definition of $\mathcal{A}$, we get

$$\begin{aligned}\mathcal{B} &= -\alpha\mathcal{A}+\int_0^T\int_M \Big(-(p-1)uD\phi(\nabla^{\nabla u}\frac{F^2(\nabla u)}{u})-(p-1)\phi Du(\nabla^{\nabla u}\frac{F^2(\nabla u)}{u})\Big)dmdt\\ &\quad+\int_0^T\int_M \Big(-\phi\partial_t(\frac{F^2(\nabla u)}{u})+2p\phi D(\frac{F^2(\nabla u)}{u})(\nabla u)\Big)dmdt.\end{aligned}$$

From Lemma 3.3, we conclude

$$\begin{aligned}\mathcal{B} &= -\alpha\mathcal{A}+\int_0^T\int_M \Big(-(p-1)uD\phi(\nabla^{\nabla u}\frac{F^2(\nabla u)}{u})\Big)dmdt\\ &\quad+\int_0^T\int_M \Big((p+1)\phi Du(\nabla^{\nabla u}\frac{F^2(\nabla u)}{u})\Big)dmdt\\ &\quad+\int_0^T\int_M \Big(2\frac{\phi}{u}h^{ij}(\nabla u)u_iu_j-2\frac{\phi}{u}D(\partial_t u)(\nabla u)+\phi\frac{\partial_t u}{u^2}F^2(\nabla u)\Big)dmdt.\end{aligned}$$

Substituting $\frac{\partial_t u}{u}=(p-1)\Delta_m u+\frac{F^2(\nabla u)}{u}+au^b$ in last equation, we derive

$$\begin{aligned}\mathcal{B} &= -\alpha\mathcal{A}+\int_0^T\int_M \Big(-(p-1)D\phi(\nabla^{\nabla u}F^2(\nabla u))\Big)dmdt\\ &\quad+\int_0^T\int_M \Big(-2(p-1)\frac{\phi}{u}D(u\Delta_m u)(\nabla u)\Big)dmdt\\ &\quad+\int_0^T\int_M \Big(-2\frac{\phi}{u}D(F^2(\nabla u))(\nabla u)+(p-1)\frac{F^2(\nabla u)}{u}D\phi(\nabla u)\Big)dmdt\end{aligned}$$

$$+\int_0^T\int_M\left((p+1)\phi Du(\nabla^{\nabla u}\frac{F^2(\nabla u)}{u})\right)dmdt$$
$$+\int_0^T\int_M\left(\phi\left[(p-1)\Delta_m u+\frac{F^2(\nabla u)}{u}+au^b\right]\frac{F^2(\nabla u)}{u}\right)dmdt$$
$$+\int_0^T\int_M\left(2\frac{\phi}{u}h^{ij}(\nabla u)u_iu_j-\frac{2\phi}{u}D(au^b)\nabla u\right)dmdt.$$

Using th formula (7) and $S=0$, we have $Ric_\infty(V)=Ric(V)$ and

$$\begin{aligned}\mathcal{B} &= -\alpha\mathcal{A}+\int_0^T\int_M\phi\left(2(p-1)Ric(\nabla u)+2(p-1)|\nabla^2u|^2_{HS(\nabla u)}\right)dmdt\\
&+\int_0^T\int_M\phi\left(-2(p-1)\frac{1}{u}\Delta_m uF^2(\nabla u)\right)dmdt\\
&+\int_0^T\int_M\left(-2\frac{\phi}{u}D(F^2(\nabla u))(\nabla u)+(p-1)\frac{F^2(\nabla u)}{u}D\phi(\nabla u)\right)dmdt\\
&+\int_0^T\int_M\left((p+1)\phi Du(\nabla^{\nabla u}\frac{F^2(\nabla u)}{u})\right)dmdt\\
&+\int_0^T\int_M\left(\phi\left[(p-1)\Delta_m u+\frac{F^2(\nabla u)}{u}+au^b\right]\frac{F^2(\nabla u)}{u}\right)dmdt\\
&+\int_0^T\int_M\left(2\frac{\phi}{u}h^{ij}(\nabla u)u_iu_j-\frac{2\phi}{u}D(au^b)\nabla u\right)dmdt.\end{aligned}$$

Integration by part to part leads to

$$\begin{aligned}\mathcal{B} &= -\alpha\mathcal{A}+\int_0^T\int_M\phi\left(2(p-1)Ric(\nabla u)+2(p-1)|\nabla^2u|^2_{HS(\nabla u)}\right)dmdt\\
&+\int_0^T\int_M\phi\left(-2(p-1)\frac{1}{u}\Delta_m uF^2(\nabla u)+2\frac{1}{v}h^{ij}(\nabla u)u_iu_j\right)dmdt\\
&-\int_0^T\int_M\left(\phi\frac{1}{u^2}F^4(\nabla u)-a\phi u^{b-1}F^2(\nabla u)+2ab\phi u^{b-2}F^2(\nabla u)\right)dmdt.\end{aligned}\tag{13}$$

Replacing $\mathcal{A}$ from (12) into (13), we deduce

$$\begin{aligned}&(p-1)u\Delta_m^{\nabla u}\mathcal{P}-\partial_t\mathcal{P}+2pD\mathcal{P}(\nabla u)\\
=\ &-2(p-1)\alpha h^{ij}_{|i}u_j-2(p-1)\alpha h^{ij}_{;k}\frac{u_j}{F}(\nabla^2u)^k_i\\
&-2(p-1)\alpha h^{ij}(\nabla u)u_{ij}-\frac{2(\alpha-1)}{u}h^{ij}(\nabla u)u_iu_j\\
&+\alpha\frac{\partial_t u}{u^2}F^2(\nabla u)+(p-1)\alpha\frac{\partial_t u}{u}\Delta_m u\\
&+2(p-1)Ric(\nabla u)+2(p-1)|\nabla^2u|^2_{HS(\nabla u)}\\
&-2(p-1)\frac{1}{u}\Delta_m uF^2(\nabla u)-\frac{1}{u^2}F^4(\nabla u)\\
&+\alpha((p-1)b+(p+1))abu^{b-1}F^2(\nabla u)+\alpha(p-1)abu^b\Delta_m u\\
&+au^{b-1}F^2(\nabla u)-2abu^{b-2}F^2(\nabla u).\end{aligned}\tag{14}$$

Noticing

$$\begin{aligned}
&\alpha\frac{\partial_t u}{u^2}F^2(\nabla u)+(p-1)\alpha\frac{\partial_t u}{u}\Delta_m u-2(p-1)\frac{1}{u}\Delta_m uF^2(\nabla u)-\frac{1}{u^2}F^4(\nabla u)\\
&=\alpha\frac{\partial_t u}{u}\left(\frac{F^2(\nabla u)}{u}+(p-1)\Delta_m u\right)-\left(\frac{F^2(\nabla u)}{u}+(p-1)\Delta_m u\right)^2+((p-1)\Delta_m u)^2\\
&=(\alpha-1)(\frac{\partial_t u}{u}-au^b)^2+((p-1)\Delta_m u)^2+\alpha au^b(\frac{\partial_t u}{u}-au^b). \qquad (15)
\end{aligned}$$

Using (14) and (15), we get (10). □

*Proof.* [Proof of Theorem 1.1] We consider $\mathcal{W}=t\mathcal{P}$, $\mu=\frac{F^2(\nabla u)}{u\mathcal{P}}$, and

$$\square\mathcal{W}=(p-1)u\Delta_m^{\nabla u}\mathcal{W}-\partial_t\mathcal{W}+2pD\mathcal{W}(\nabla u).$$

Applying (10) and $\alpha(p-1)\Delta_m u=-\mathcal{P}-(\alpha-1)\frac{F^2(\nabla u)}{u}$, we have

$$\begin{aligned}
\square\mathcal{W} &= t\left(\Delta_m^{\nabla u}\mathcal{P}-\partial_t\mathcal{P}+2D\mathcal{P}(\nabla u)\right)-\frac{\mathcal{W}}{t}\\
&= -\frac{2t(\alpha-1)}{u}h^{ij}(\nabla u)u_iu_j-2t(p-1)\alpha h^{ij}(\nabla u)u_{ij}-2t(p-1)\alpha h^{ij}_{|i}u_j\\
&\quad -2t(p-1)\alpha h^{ij}_{;k}\frac{u_j}{F}(\nabla^2u)^k_i+2t(p-1)Ric(\nabla u)+2t(p-1)|\nabla^2u|^2_{HS(\nabla u)}\\
&\quad +t((p-1)\Delta_m u)^2+t(\alpha-1)(\frac{\partial_t u}{u}-au^b)^2-abu^b\mathcal{W}-tabu^b(\alpha-1)\frac{F^2(\nabla u)}{u}\\
&\quad +t\alpha(p-1)ab^2u^{b-1}F^2(\nabla u)+t\alpha(p+1)abu^{b-1}F^2(\nabla u)+2tau^{b-1}F^2(\nabla u)\\
&\quad -2tabu^{b-2}F^2(\nabla u)-(au^b+\frac{1}{t})\mathcal{W}.
\end{aligned}$$

For any $\epsilon\in(0,1)$ Young's inequality implies that

$$2\alpha h^{ij}(\nabla u)u_{ij}\le\epsilon u_{ij}^2+\frac{\alpha^2}{\epsilon}h_{ij}^2\le\epsilon u_{ij}^2+\frac{\alpha^2}{\epsilon}\max\{k_2^2,k_3^2\}. \qquad (16)$$

Notice also that

$$2\alpha h^{ij}_{|i}u_j\le\epsilon F^2(\nabla u)+\frac{\alpha^2}{\epsilon}(h^{ij}_{|i})^2\le\epsilon F^2(\nabla u)+\frac{\alpha^2}{\epsilon}k_4^2, \qquad (17)$$

$$2\alpha h^{ij}_{;k}\frac{u_j}{F}(\nabla^2u)^k_i\le\epsilon u_{ik}^2+\frac{\alpha^2}{\epsilon}(h^{ij}_{;k}\frac{u_j}{F})^2\le\epsilon u_{ik}^2+\frac{\alpha^2}{\epsilon}(h^{ij}_{;k})^2\le\epsilon u_{ik}^2+\frac{\alpha^2}{\epsilon}k_4^2, \qquad (18)$$

and

$$2(\alpha-1)h^{ij}(\nabla u)u_iu_j\le2(\alpha-1)k_3F^2(\nabla u). \qquad (19)$$

Also, we have

$$\begin{aligned}
&\left[-ab(\alpha-1)u^b+\alpha(p-1)ab^2u^{b-2}+\alpha(p+1)abu^{b-2}+2au^{b-2}-2abu^{b-3}\right]\frac{F^2(\nabla u)}{u}\\
&\ge-\left[ab(\alpha-1)\bar{D}_0-\alpha(p-1)ab^2\tilde{D}_2-\alpha(p+1)ab\tilde{D}_2-2a\tilde{D}_2+2ab\bar{D}_3\right]\frac{F^2(\nabla u)}{u}\\
&\ge-\left[(\alpha-1)ab\bar{D}_0+2ab\bar{D}_3\right]\frac{F^2(\nabla u)}{u}. \qquad (20)
\end{aligned}$$

Inserting (16)-(20) into (16) gives

$$\begin{aligned}\Box\mathcal{W} \geq & -\frac{2t(\alpha-1)}{u}k_3F^2(\nabla u)-t(p-1)\epsilon u_{ij}^2-t(p-1)\frac{\alpha^2}{\epsilon}\max\{k_2^2,k_3^2\}\\ & -t(p-1)\epsilon F^2(\nabla u)-t(p-1)\frac{\alpha^2}{\epsilon}k_4^2-t(p-1)\epsilon u_{ik}^2\\ & -t(p-1)\frac{\alpha^2}{\epsilon}k_4^2-2t(p-1)k_1F^2(\nabla u)+2t(p-1)u_{ij}^2\\ & -t\left[ab(\alpha-1)\bar{D}_0+2ab\bar{D}_3\right]\frac{F^2(\nabla u)}{u}\\ & +t((p-1)\Delta_m u)^2-\frac{\mathcal{W}}{t}-(1+b)a\bar{D}_0\mathcal{W}.\end{aligned}$$

Since $u \leq \frac{p}{p-1}D_2^{p-1}$, we derive

$$\begin{aligned}\Box\mathcal{W} \geq & -tC_1\frac{F^2(\nabla u)}{u}-t(p-1)\alpha^2C_2-2t(p-1)(\epsilon-1)u_{ij}^2\\ & +t((p-1)\Delta_m u)^2-\frac{\mathcal{W}}{t}-(1+b)a\bar{D}_0\mathcal{W},\end{aligned}$$

where

$$C_1=2\Big(k_3(\alpha-1)+\frac{p}{2}D_2^{p-1}\epsilon+pk_1D_2^{p-1}+\frac{1}{2}ab(\alpha-1)\bar{D}_0+ab\bar{D}_3\Big)$$

and

$$C_2=\frac{1}{\epsilon}\max\{k_2^2,k_3^2\}+\frac{2}{\epsilon}k_4^2.$$

Applying the Cauchy inequality

$$u_{ij}^2 \geq \frac{1}{n}(\Delta_m u)^2$$

into last inequality, we conclude

$$\begin{aligned}\Box\mathcal{W} \geq & -tC_1\frac{F^2(\nabla u)}{u}-t(p-1)\alpha^2C_2\\ & +t\frac{n(p-1)+2(1-\epsilon)}{n(p-1)}((p-1)\Delta_m u)^2-\frac{\mathcal{W}}{t}-(1+b)a\bar{D}_0\mathcal{W}.\end{aligned} \tag{21}$$

Thus, since

$$(p-1)\Delta_m u=-\frac{1}{\alpha}\left(\frac{\mathcal{W}}{t}+(\alpha-1)\frac{F^2(\nabla u)}{u}\right)$$

the equation (21) becomes

$$\begin{aligned}\Box\mathcal{W} \geq & \ t\frac{n(p-1)+2(1-\epsilon)}{n(p-1)\alpha^2}\left(\frac{\mathcal{W}}{t}+(\alpha-1)\frac{F^2(\nabla u)}{u}\right)^2-\frac{\mathcal{W}}{t}-(1+b)a\bar{D}_0\mathcal{W}\\ & -tC_1\frac{F^2(\nabla u)}{u}-t(p-1)\alpha^2C_2.\end{aligned} \tag{22}$$

Using $\mu=\frac{F^2(\nabla u)}{u\mathcal{G}}$ in (23), we have

$$\begin{aligned}\Box\mathcal{W} \geq & \ \frac{n(p-1)+2(1-\epsilon)}{tn(p-1)\alpha^2}\left(1+(\alpha-1)\mu\right)^2\mathcal{W}^2-\frac{\mathcal{W}}{t}-(1+b)a\bar{D}_0\mathcal{W}\\ & -C_1\mu\mathcal{W}-t(p-1)\alpha^2C_2.\end{aligned} \tag{23}$$

If $\mu < 0$ then the theorem holds, then we have $\mu \geq 0$ and

$$\square\mathcal{W} \geq \frac{n(p-1)+2(1-\epsilon)}{tn(p-1)\alpha^2}\left(1+(\alpha-1)\mu\right)^2\mathcal{W}^2 - \left[C_1\mu + \frac{1}{t} + (1+b)a\bar{D}_0\right]\mathcal{W} - t(p-1)\alpha^2 C_2. \tag{24}$$

For a fixed $t \in (0, T]$, we assume $\mathcal{W}$ achieves its maximum at the point $(x_0, t_0) \in M \times (0, T]$. For the case $\mathcal{W}(x_0, t_0) \leq 0$, the proof is trivial. Thus, we can consider $\mathcal{W}(x_0, t_0) > 0$. We prove that $\square\mathcal{W}(x_0, t_0) \leq 0$. Assume that $\square\mathcal{W}(x_0, t_0) > 0$. Hence, on a neighborhood of $(x_0, t_0)$ we have $\square\mathcal{W} > 0$ and the function $\mathcal{W}$ would be a strict sub-solution. Therefore, on the boundary of any small parabolic cylinder $[t_0 - \delta, t_0] \times \{z \in M | d(x, z) < \delta\}$ the value of $\mathcal{W}(x_0, t_0)$ is strictly less than the supremum of $\mathcal{W}$. In particular, $\mathcal{W}$ could not be maximum at $(x_0, t_0)$, which is a contradiction. Thus, $\square\mathcal{W}(x_0, t_0) \leq 0$, that is at $(x_0, t_0)$ we have

$$0 \geq \frac{n(p-1)+2(1-\epsilon)}{t_0 n(p-1)\alpha^2}\left(1+(\alpha-1)\mu\right)^2\mathcal{W}^2 - \left[C_1\mu + \frac{1}{t_0} + (1+b)a\bar{D}_0\right]\mathcal{W} - t_0(p-1)\alpha^2 C_2. \tag{25}$$

Solving the last inequality of $\mathcal{W}$ in (25), we derive

$$\begin{aligned}\mathcal{W} \leq\ & \frac{t_0 n(p-1)\alpha^2}{2(n(p-1)+2(1-\epsilon))\left(1+(\alpha-1)\mu\right)^2}\Bigg\{C_1\mu + \frac{1}{t_0} + (1+b)a\bar{D}_0 \\ & + \left\{\left[C_1\mu + \frac{1}{t_0} + (1+b)a\bar{D}_0\right]^2 + 4\frac{n(p-1)+2(1-\epsilon)}{n}\left(1+(\alpha-1)\mu\right)^2 C_2\right\}^{\frac{1}{2}}\Bigg\}.\end{aligned}$$

Using inequality $\sqrt{A+B} \leq \sqrt{A} + \sqrt{B}$ for $A, B \geq 0$, we have

$$\begin{aligned}\mathcal{W} \leq\ & \frac{t_0 n(p-1)\alpha^2}{(n(p-1)+2(1-\epsilon))\left(1+(\alpha-1)\mu\right)^2}\Bigg\{C_1\mu + \frac{1}{t_0} + (1+b)a\bar{D}_0 \\ & + \left(1+(\alpha-1)\mu\right)\left\{\frac{n(p-1)+2(1-\epsilon)}{n}C_2\right\}^{\frac{1}{2}}\Bigg\}.\end{aligned} \tag{26}$$

At the point $(x_0, t_0)$, we have $\frac{\mu}{\left(1+(\alpha-1)\mu\right)^2} \leq \frac{1}{4(\alpha-1)}$ and $\frac{1}{\left(1+(\alpha-1)\mu\right)^2} \leq 1$. Therefore (26) yields

$$\mathcal{W} \leq \frac{t_0 n(p-1)\alpha^2}{n(p-1)+2(1-\epsilon)}\left\{\frac{1}{4(\alpha-1)}C_1 + \frac{1}{t_0} + (1+b)a\bar{D}_0 + \left\{\frac{n(p-1)+2(1-\epsilon)}{n}C_2\right\}^{\frac{1}{2}}\right\}.$$

Since $t \geq t_0$, we conclude

$$\begin{aligned}\mathcal{W}(x,t) &\leq \mathcal{W}(x_0, t_0) \\ &\leq \frac{tn(p-1)\alpha^2}{n(p-1)+2(1-\epsilon)}\left\{\frac{1}{4(\alpha-1)}C_1 + \frac{1}{t} + (1+b)a\bar{D}_0 + \left\{\frac{n(p-1)+2(1-\epsilon)}{n}C_2\right\}^{\frac{1}{2}}\right\}.\end{aligned}$$

Therefore,

$$\mathcal{P}(x,t) \leq \frac{n(p-1)\alpha^2}{n(p-1)+2(1-\epsilon)}\left\{\frac{1}{4(\alpha-1)}C_1 + \frac{1}{t} + (1+b)a\bar{D}_0 + \left\{\frac{n(p-1)+2(1-\epsilon)}{n}C_2\right\}^{\frac{1}{2}}\right\}.$$

Since $t \in [0, T]$ is arbitrary, we get (5). This completes the proof of theorem. □

*Proof.* [Proof of Corollary 1.2] Let $f = \ln u$. Inequality (5) yields

$$\partial_t f \geq \frac{p-1}{\alpha p}D_2^{1-p}F^2(\nabla f) + \alpha a\tilde{D}_0 - \frac{n(p-1)\alpha}{n(p-1)+2(1-\epsilon)}\left\{K_1 + K_2 + \frac{1}{t}\right\}. \tag{27}$$

For a curve $\eta$ in $M^n \times (0,\infty)$, $\eta : [t_1, t_2] \to M^n \times (0,\infty)$ by $\eta(t) = (\gamma(t), t)$, we get

$$\begin{aligned} f(x_1,t_1) - f(x_2,t_2) &= \int_0^1 \frac{d}{ds}\Big(f(\gamma(s),\tau(s))\Big)ds \\ &= \int_0^1 (t_2-t_1)\Big(\frac{Df(\dot\gamma(s))}{t_2-t_1} - \partial_t v\Big)ds \\ &\leq \int_0^1 (t_2-t_1)\Big(\frac{F(\dot\gamma(s))F(\nabla f)}{t_2-t_1} - \partial_t f\Big)ds. \end{aligned}$$

Using (27), we have

$$\begin{aligned} f(x_1,t_1) - f(x_2,t_2) &\leq \int_0^1 (t_2-t_1)\Big(\frac{F(\dot\gamma(s))F(\nabla f)}{t_2-t_1} - \frac{p-1}{\alpha p}D_2^{1-p}F^2(\nabla f) + \alpha a\tilde{D}_0\Big)ds \\ &\quad + \int_0^1 (t_2-t_1)\Big(\frac{n(p-1)\alpha}{n(p-1)+2(1-\epsilon)}\Big\{K_1+K_2+\frac{1}{t}\Big\}\Big)ds. \end{aligned}$$

Applying inequality $-Ax^2 + Bx \leq \frac{B^2}{4A}$, we derive

$$\begin{aligned} f(x_1,t_1) - f(x_2,t_2) &\leq \int_0^1 \frac{\alpha p}{4(p-1)}D_2^{p-1}\frac{[F(\dot\gamma(s))]^2}{t_2-t_1}ds \\ &\quad + \frac{n(p-1)\alpha(t_2-t_1)}{n(p-1)+2(1-\epsilon)}\{K_1+K_2\} + \alpha a\tilde{D}_0(t_2-t_1) \\ &\quad + \frac{n(p-1)\alpha}{n(p-1)+2(1-\epsilon)}\ln(\frac{t_2}{t_1}). \end{aligned}$$

By exponentiation, we conclude

$$\begin{aligned} v(x_1,t_1) &\leq v(x_2,t_2)(\frac{t_2}{t_1})^{\frac{n(p-1)\alpha}{n(p-1)+2(1-\epsilon)}} \exp\Bigg\{\int_0^1 \frac{\alpha p}{4(p-1)}D_2^{p-1}\frac{[F(\dot\gamma(s))]^2}{t_2-t_1}ds \\ &\quad + \frac{n(p-1)\alpha(t_2-t_1)}{n(p-1)+2(1-\epsilon)}\{K_1+K_2\} + \alpha a\tilde{D}_0(t_2-t_1)\Bigg\}. \end{aligned}$$

This completes the proof of corollary. □

## 4. Applications to Finsler-Ricci flow and Finsler-Yamabe flow

In this section, we use our results to the Finsler-Ricci flow and the Finsler-Yamabe flow.

### 4.1. *The Finsler-Ricci flow*

When $h = -Ric$, the flow (1) is the Finsler-Ricci flow. In this situation we have

$$(Ric^{ij}(\nabla u))_i u_j = Ric^{ij}_{|i}u_j + Ric^{ij}_{;k}\frac{v_j}{F}(\nabla^2 u)^k_i$$

and

$$\partial_t(F^2(\nabla u)) = 2Ric^{ij}(\nabla u)u_iu_j + 2D(\partial_t u)(\nabla u).$$

Hence, similar to Lemma 3.4 along the Finsler-Ricci flow we have

$$\Box\mathcal{P} = \frac{2(\alpha-1)}{u}Ric^{ij}(\nabla u)u_iu_j + 2(p-1)\alpha Ric^{ij}(\nabla u)u_{ij} + 2(p-1)\alpha Ric^{ij}_{|i}u_j$$

$$+2(p-1)\alpha Ric^{ij}_{;k}\frac{u_j}{F}(\nabla^2 u)^k_i + 2(p-1)Ric(\nabla u) + 2(p-1)|\nabla^2 u|^2_{HS(\nabla u)}$$
$$+((p-1)\Delta_m u)^2 + (\alpha-1)(\frac{\partial_t u}{u} - au^b)^2 + \alpha(p-1)abu^b\Delta_m u$$
$$+\alpha(p-1)ab^2u^{b-1}F^2(\nabla u) + \alpha(p+1)abu^{b-1}F^2(\nabla u) + 2au^{b-1}F^2(\nabla u)$$
$$-2abu^{b-2}F^2(\nabla u) - au^b\mathcal{P}.$$

Therefore, similar to Theorem 1.1, we can state the following theorem, whose proof is similar to the proof of Theorem 1.1.

**Theorem 4.1.** *Suppose that $(M^n, F(t))$ is a complete solution to the Finsler-Ricci flow in some time interval $[0,T]$. Suppose that there exist some positive constants $k_5, k_6$ and $k_7$ such that $-k_5 g \leq \Re \leq k_6 g$, $|\nabla\Re| \leq k_7$ for all $t \in [0,T]$, and S-curvature vanishes. Let $v$ be any smooth positive global solution (2) such that $D_1 \leq v \leq D_2$ for some positive constants $D_1, D_2$. Assume that $p > 1$ and $u = \frac{p}{p-1}u^{p-1}$. Then for any $\alpha > 1$, $c \geq 0$ and $q \geq 1$ on $M \times [0,T]$, the following estimate holds*

$$\frac{F^2(\nabla u)}{u} - \alpha\frac{\partial_t u}{u} + \alpha au^b \leq \frac{n(p-1)\alpha^2}{n(p-1)+2(1-\epsilon)}\left\{K_3 + K_4 + \frac{1}{t}\right\}, \tag{28}$$

*where $\epsilon \in (0,1)$ is an arbitrary constant,*

$$K_3 = \frac{1}{2(\alpha-1)}\Big(k_5(\alpha-1) + \frac{p}{2}D_2^{p-1}\epsilon + pD_2^{p-1}k_5 + \frac{1}{2}ab(\alpha-1)\bar{D}_0 + ab\bar{D}_3\Big) + a\bar{D}_0 + ab\bar{D}_0,$$

*and*

$$K_4 = \left\{\frac{n(p-1)+2(1-\epsilon)}{n\epsilon}\Big(\max\{k_5^2, k_6^2\} + 2k_7^2\Big)\right\}^{\frac{1}{2}}.$$

*4.2. The Finsler-Yamabe flow*

When $h = -\frac{1}{2}Hg$, the flow (1) is the Finsler-Yamabe flow where $H = g^{ij}Ric_{ij}$ is scalar curvature. In this case we get

$$(H(\nabla v))_i v_j = H_{|j}u_j + H_{;k}\frac{u_j}{F}(\nabla^2 u)^k_j, \qquad (g^{ij}(\nabla u))_i u_j = 0,$$

and

$$\partial_t(F^2(\nabla u)) = HF^2(\nabla u) + 2D(\partial_t u)(\nabla u).$$

Hence, similar to Lemma 3.4, in the sense of distributions, $\mathcal{P}(x,t)$ satisfies the differential equality

$$\Box\mathcal{P} = \frac{(\alpha-1)}{u}HF^2(\nabla u) + (p-1)\alpha Hg^{ij}(\nabla u)u_{ij} + (p-1)\alpha H_{|j}u_j$$
$$+(p-1)\alpha H_{;k}\frac{u_j}{F}(\nabla^2 u)^k_j + 2(p-1)Ric(\nabla u) + 2(p-1)|\nabla^2 u|^2_{HS(\nabla f)}$$
$$+((p-1)\Delta_m u)^2 + (\alpha-1)(\frac{\partial_t u}{u} - au^b)^2 + \alpha(p-1)abu^b\Delta_m u$$
$$+\alpha(p-1)ab^2u^{b-1}F^2(\nabla u) + \alpha(p+1)abu^{b-1}F^2(\nabla v) + 2au^{b-1}F^2(\nabla u)$$
$$-2abu^{b-2}F^2(\nabla u) - au^b\mathcal{P}.$$

Therefore we can rewrite Theorem 1.1 as follows:

**Theorem 4.2.** *Suppose that $(M^n, F(t))$ is a complete solution to the Finsler-Yamabe flow (1) in some time interval $[0, T]$. Suppose that there exist some positive constants $k_8, k_9, k_{10}$ and $k_{11}$ such that $\Re \geq -k_8 g$, $-k_9 g \leq H \leq k_{10} g$, $|\nabla H| \leq k_{11}$ for all $t \in [0, T]$, and S-curvature vanishes. Let $v$ be any smooth positive global solution (2) such that $D_1 \leq v \leq D_2$ for some positive constants $D_1, D_2$. Assume that $p > 1$ and $u = \frac{p}{p-1} v^{p-1}$. Then for any $\alpha > 1$, $c \geq 0$ and $q \geq 1$ on $M \times [0, T]$, the following estimate holds*

$$\frac{F^2(\nabla u)}{u} - \alpha \frac{\partial_t u}{u} + \alpha a u^b \quad \leq \quad \frac{n(p-1)\alpha^2}{n(p-1) + 2(1-\epsilon)} \left\{ K_5 + K_6 + \frac{1}{t} \right\}, \tag{29}$$

*where $\epsilon \in (0, 1)$ is an arbitrary constant,*

$$K_5 \quad = \quad \frac{1}{2(\alpha - 1)} \Big( \frac{1}{2} k_9 (\alpha - 1) + \frac{p}{2} D_2^{p-1} \epsilon + p D_2^{p-1} k_8 + \frac{1}{2} ab(\alpha - 1) \bar{D}_0 + ab \bar{D}_3 \Big) + a \bar{D}_0 + ab \bar{D}_0,$$

*and*

$$K_6 = \left\{ \frac{n(p-1) + 2(1-\epsilon)}{4n\epsilon} \left( \max\{k_9^2, k_{10}^2\} + \frac{1}{2} k_{11}^2 \right) \right\}^{\frac{1}{2}}.$$

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