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Efficient numerical algorithm for solving the Benjamin-Bona-Mahony partial differential equation using Fibonacci wavelets and advanced computational techniques

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Abstract. In this article, we have presented a novel and unified numerical strategy for addressing the Benjamin-Bona-Mahony (BBM) type partial differential equations with the use of the Fibonacci wavelets and collocation techniques. This technique is based on transforming the given PDEs into an equivalent integral equation via the wavelet basis approximation and collocation techniques to obtain the wavelet coefficients. Convergence analysis in the form of the theorems was also discussed to prove the demonstrated that the estimation of a function using Fibonacci wavelets converges uniformly to itself. It is anticipated that the proposed approach would be more efficient and suitable for solving a variety of nonlinear partial differential equations that occur in science and engineering. Examples and outcomes in tabulated form are given to show how the suggested wavelet method provides enhanced accuracy for a wide range of problems. MATLAB software is used to execute the computational operations.

1. Introduction

1.1. Emergence of the BBM equation

The BBM equation, proposed by Benjamin, Bona, and Mahony in 1972, was introduced as a modification of the KdV equation to address some of these issues. Specifically, the BBM equation aims to improve the description of wave propagation by incorporating a more accurate balance between dispersion and nonlinearity, making it a better model for certain physical phenomena. The BBM equation is particularly

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useful for modeling the propagation of long waves, where the effects of dispersion and dissipation play a significant role. It provides a more accurate description of how waves travel over long distances, integrating both nonlinear and dissipative phenomena more effectively. BBM equation has so many interesting and significant applications in wavelet analysis, such as studying acoustic waves with harmonic crystals, hydromagnetic waves with cold plasma, acoustic waves with compressible fluids, and surface waves in liquids with long wavelengths. The gesture of the PDEs of BBM type has not only drawn the attention of various academics from disciplines of applied sciences but several novel investigations have also been carried out. The dispersion relations for the different linearized equations may be used to examine the main mathematical differences between the KdV and BBM models. It is clear that these interactions only produce equivalent short-wave responses, and that their long-wave response are very different. Most scientific phenomena come up in practical or physical issues like those in many scientific and technical disciplines, including optical fibers, chemical kinematics, fluid mechanics, biology, solid-state, chemical, and plasma physics involve studying linear and nonlinear wave events [2]. Since many of these issues, with a few exceptions, cannot be resolved using conventional or analytical methods, it becomes essential to find solutions to these problems. To address these complexities, we turn to the field of numerical analysis. We can overcome these difficulties and achieve a solution with a certain degree of error by introducing novel techniques.

1.2. Wavelet methods in numerical computation

Wavelet analysis has significantly contributed to the numerical solutions of partial differential equations (PDEs) such as the Benjamin-Bona-Mahony (BBM) equation due to its ability to represent complex functions and solutions efficiently. Wavelet bases allow capturing both global and local properties of the solution and offer a natural framework for multiresolution analysis when solving the BBM equation at various levels of detail and its adaptivity. Wavelet bases refine the computational mesh in regions where the solution exhibits rapid changes or steep gradients. This adaptivity enhances the accuracy and efficiency of numerical solutions for the BBM equation. Wavelet-based numerical methods can enhance the numerical stability of solutions, particularly for nonlinear PDEs like the BBM equation. They also provide high accuracy due to their ability to localize both in time and frequency. Numerous mathematicians studied the Fibonacci wavelets to handle the differential [27] and integral equations [33] in order to gain advantages from the local property. Both non-linear Hunter-Saxton equations [9, 10] and time-fractional telegraph equations [4, 5] have seen extensive use of Fibonacci wavelets in numerical solutions. Vivek et al. [39] solved the Thomas-Fermi type differential equation under the use of Hermite Wavelets and the Green's functions.

Wavelet methods [35, 36] have been applied for solving partial differential equations (PDEs) since the 1990s. The exploration of approximate solutions for both linear and nonlinear partial differential equations is crucial in understanding the dynamics of linear and nonlinear physical phenomena. Linear and nonlinear wave phenomena manifest in diverse scientific and engineering domains, spanning plasma physics, fluid mechanics, biology, solid-state physics, optical fibers, chemical physics, and chemical kinetics. Furthermore, obtaining exact solutions for these problems remains largely unexplored.

After thoroughly studying the existing literature [29, 30, 37, 38], we concluded that wavelet analysis has significantly contributed to the numerical solution of nonlinear partial differential equations. However, there is a noticeable lack of contributions regarding the application of numerical methods using Fibonacci wavelets to the BBM nonlinear partial differential equation through direct approximation in the literature. This gap has continuously motivated us to contribute to wavelet analysis for the BBM equation using new methods. Some recent Literature is available dedicated to Fibonacci wavelets in [16, 17, 23].

In this research, our aim is to derive an approximate solution using the collocation method under the Fibonacci wavelet basis approximation for the most comprehensive and general form of the BBM linear and nonlinear partial differential equations represented by the following equation given in [26]:

$$\mathrm{A}U_{\beta}(\alpha,\beta)+\mathrm{B}U_{\alpha}(\alpha,\beta)+\mathrm{C}U(\alpha,\beta)U_{\alpha}(\alpha,\beta)-\mathrm{D}U_{\alpha\alpha\beta}(\alpha,\beta)=\zeta(\alpha,\beta), \tag{1}$$

where A, B, C and D are real constants and $\zeta(x, t)$ is a continuous function of real value on $[0, 1) \times [0, 1)$. Recently, considerable attention has been devoted to the literature on stable methods for the numerical solution of Benjamín–Bona–Mohany (BBM) equations. Moreover, many mathematicians have dedicated substantial effort to deriving both precise and approximate solutions for partial differential equations like the Benjamin–Bona–Mahony equations. Consequently, several effective, accurate, and powerful methods have been developed by these researchers. Some of these methods include: Ji-Huan He [19] introduced the Homotopy Perturbation Method, Shiralashetti and Hanaji [21] provided an approximate solution of the BBM equation using the Taylor Wavelet Method, Adomian G [40] gave an overview of the Decomposition Method for applied mathematics, Zaman Ziabakhsh-Ganji [3] presented an approximate solution of the BBM equation via the Exp-Function Method, Li and Liao [24] offered a solution for highly nonlinear problems using the Homotopy Analysis Method, Peter J. Olver [8] solved differential equations using the Lie Group Method, Shiralashetti et al. [1] provided a solution of the BBM equation using the Haar Wavelet Basis approximation method, Ryogo Hirota [18] solved the KdV equation for various collisions using the Bilinear Method, Rogers and Shadwich [12] discussed the Bäcklund Transformation and its applications, Abbasbandy and Shirzadi [22] solved a modified form of the BBM equation via an Integral Method, Shiralashetti and Kumbinarasaiah [25] introduced the Cardinal B-Spline Wavelets Method to solve the generalized Burgers–Huxley Equation, and Shiralashetti and Kumbinarasaiah [26] presented the solution of the BBM equation using the Laguerre Wavelet Basis in their study.

Alternative numerical algorithms based on wavelet bases have been extensively utilized for solving nonlinear partial differential equations, including the BBM equation. Examples include Chebyshev wavelets [13], Legendre wavelets [14], and Hermite wavelets. In this study, Benjamin-Bona-Mahony (BBM) type problems have been addressed using Fibonacci wavelets due to their simplicity, speed, and ability to achieve the desired accuracy for a reasonable number of grid points. The results are also compared with both the Taylor wavelet collocation method (TWCM) and the exact solution.

This article is organized into the following sections: Section 2 covers the fundamental characteristics of wavelets and Fibonacci wavelets, followed by the function approximation procedure. Section 3 is devoted to the main results in convergence analysis, including the uniform convergence of the Fibonacci wavelet basis approximation presented in the form of theorems. Section 4 explains the methodology for applying the Fibonacci wavelet approximation and the collocation technique to derive a set of algebraic equations. Section 5 presents the final approximate solution in a tabulated form for comparison purposes with the exact solution and existing methods. Finally, Section 6 provides a summary of the entire research study conducted.

2. Fibonacci wavelets and function approximation

Wavelets are essentially a group of functions formed by dilating and translating a single function known as the mother wavelet. If we take translation and dilation as continuous parameters, x and y respectively, we get the following family of continuous wavelets [34]:

$$\varphi_{x,y}(\beta) = |x|^{-1/2}\varphi\left(\frac{\beta - y}{x}\right), \forall x, y \in \mathbb{R}, x \neq 0. \tag{2}$$

If we set $x = x_0^{-\theta}, y = \omega x_0^{-\theta} y_0$, with$x_0 > 1, y_0 > 1$, and where ω and θ are positive integers, the discrete family of wavelet bases in $L^2(\mathbb{R})$ is defined as:

$$\varphi_{\theta,\omega}(\beta) = |x_0|^{\frac{\theta}{2}}\, \varphi\left(x_0^{\theta}\beta - \omega y_0\right),$$

. This set of wavelets $\varphi_{\theta,\omega}(\beta)$ constitutes an orthonormal basis for the specific values $x_0 = 2$ and $y_0 = 1$.

2.1. Fibonacci wavelets

Fibonacci wavelets $\varphi_{\omega,r}(\beta) = \varphi(\theta, \widehat{\omega}, r, \beta)$ are defined by four variables; $\hat{\omega} = \omega - 1, \omega = 1, 2, 3, \ldots, 2^{\theta-1}$ for $\theta \in \mathbb{N}$. The variable r denotes the degree of the Fibonacci polynomials and β denotes the normalized time parameter. These wavelets are defined on $[0, 1]$ (see [5, 31, 32, 34]). Therefore

$$\varphi_{\omega,r}(\beta) = \begin{cases} \dfrac{2^{\frac{\theta-1}{2}}}{\sqrt{C_r}} F_r\left(2^{\theta-1}\beta - \hat{\omega}\right), & \frac{\hat{\omega}}{2^{\theta-1}} \leq \beta < \frac{\hat{\omega}+1}{2^{\theta-1}}, \\ 0, & \text{Otherwise} \end{cases} \tag{3}$$

with

$$C_r = \int_0^1 (F_r(\beta))^2 \, d\beta.$$

Here, $r = 0, 1, \ldots, \mu - 1$, represents the degree of the well-known Fibonacci polynomial $F_r(\beta)$ and the positive integer θ indicates the maximum resolution level, while $\omega = 1, 2, \ldots, 2^{\theta-1}$ denotes the translation parameter.

The solution of the following recurrence equation yield Fibonacci polynomials. Therefore, for every $\beta \in \mathbb{R}^+$:

$$F_{r+2}(\beta) = \beta F_{r+1}(\beta) + F_r(\beta), \quad \forall r \geq 0,$$

with

$$F_0(\beta) = 1, F_1(\beta) = \beta.$$

Moreover, the following closed-form formula can also define them:

$$F_{r-1}(\beta) = \frac{a^r - b^r}{a - b}, \quad \forall r \geq 1,$$

where a and b satisfy the equation $\left(\lambda^2 - \beta\lambda - 1\right) = 0$ when solved for λ. Furthermore, the polynomial expansion of the Fibonacci wavelets can also be expressed as [32]:

$$F_r(\beta) = \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r-i}{i} \beta^{r-2i}, \quad \forall r \geq 0$$

2.2. Function approximation

Let $y(\beta)$ be a function on the interval $[0, 1)$ then we can express this function using the Fibonacci wavelets basis as:

$$y(\beta) = \sum_{\omega=1}^{\infty} \sum_{r=0}^{\infty} d_{\omega,r} \varphi_{\omega,r}(\beta), \tag{4}$$

where $\varphi_{\omega,r}(\beta)$ are defined in (3) and $d_{\omega,r} = \langle y(\beta) \ \varphi_{\omega,r}(\beta) \rangle$ while $\langle , \rangle$ indicates the usual inner product. We truncate the series (4) so that,

$$y(\beta) \approx \sum_{\omega=1}^{2^{\theta-1}} \sum_{r=0}^{\mu-1} d_{\omega,r} \varphi_{\omega,r}(\beta) = E^T \varphi(\beta),$$

where E and $\varphi(\beta)$ are $2^{\theta-1}\mu \times 1$ a matrix,

$$E^T = \left[d_{1,0}, \ldots, d_{1,\mu-1}, d_{2,0}, \ldots, d_{2,\mu-1}, \ldots, d_{2^{\theta-1},0}, \ldots, d_{2^{\theta-1},\mu-1}\right],$$

$$\varphi(\beta) = \left[\varphi_{1,0}(\beta), \ldots, \varphi_{1,\mu-1}(\beta), \varphi_{2,0}(\beta), \ldots, \varphi_{2,\mu-1}(\beta), \ldots, \varphi_{2^{\theta-1},0}(\beta), \ldots, \varphi_{2^{\theta-1},\mu-1}(\beta)\right].$$

Similarly, we can expand any arbitrary space-time function having two variables $U(\alpha, \beta)$ on $[0, 1) \times [0, 1)$, using Fibonacci wavelets basis as:

$$U(\alpha,\beta) \approx \varphi^T(\beta)K\varphi(\alpha),$$

where

$$\varphi^\top(\beta) = \left[\varphi_{1,0}(\beta),\ldots,\varphi_{1,\mu-1}(\beta),\varphi_{2,0}(\beta),\ldots,\varphi_{2,\mu-1}(\beta),\ldots,\varphi_{2^{\theta-1},0}(\beta),\ldots,\varphi_{2^{\theta-1},\mu-1}(\beta)\right],$$

$$\varphi(\alpha) = \left[\varphi_{1,0}(\alpha),\ldots,\varphi_{1,\mu-1}(\alpha),\varphi_{2,0}(\alpha),\ldots,\varphi_{2,\mu-1}(\alpha),\ldots,\varphi_{2^{\theta-1},0}(\alpha),\ldots,\varphi_{2^{\theta-1},\mu-1}(\alpha)\right]^T,$$

and

$$K = \left[d_{i,j}\right]_{N\times N}, N \\ = 2^{\theta-1}\mu.$$

3. Fundamental results in convergence analysis

Functions Space $L^2(\mathbb{R})$: The collection of all functions f such that $|f(\beta)|^2$ integrable on $\mathbb{R}$.

Riesz Fischer theorem: If a sequence of functions $\{f_k\}_{k=1}^{\infty}$ in $L_2(\mathbb{R})$ converges itself in $L^2(\mathbb{R})$ then there exists a function $f \in L_2(\mathbb{R})$ such that $f_k - f \to 0$ as $k \to \infty$.

Theorem 3.1. *The Fibonacci wavelet expansion of a continuous, bounded function $U(\alpha,\beta) \in L^2(\mathbb{R}\times\mathbb{R})$, defined on $[0,1]\times[0,1]$, converges uniformly to itself.*

Proof. Let $U(\alpha,\beta) \in L^2(\mathbb{R}\times\mathbb{R})$ be a continuous function defined on $[0,1]\times[0,1]$ and bounded by a real number M. Consider the approximation of $U(\alpha,\beta)$ as follows:

$$U(\alpha,\beta) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} d_{i,j}\varphi_{i,j}(\alpha)\varphi_{i,j}(\beta)$$

where, $d_{i,j} = \langle U(\alpha,\beta), \varphi_{i,j}(\alpha)\varphi_{i,j}(\beta)\rangle$, and $\langle,\rangle$ denotes the inner product with $\varphi_{i,j}(\alpha)$ and $\varphi_{i,j}(\beta)$ being orthogonal functions on $[0,1]$. Then,

$$d_{i,j} = \int_0^1\int_0^1 U(\alpha,\beta)\varphi_{i,j}(\alpha)\varphi_{i,j}(\beta)d\alpha d\beta,$$

$$d_{i,j} = \int_0^1\int_I U(\alpha,\beta)\frac{2^{\frac{\theta-1}{2}}}{\sqrt{C_r}}F_r\left(2^{\theta-1}\alpha-\omega+1\right)\times\varphi_{i,j}(\beta)d\alpha d\beta,$$

where $I = \left[\frac{\omega-1}{2^{\theta-1}},\frac{\omega}{2^{\theta-1}}\right]$ and $C_r = \int_0^1 (F_r(\beta))^2\,d\beta$.
Put $2^{\theta-1}\alpha - \omega + 1 = \vartheta$. Then,

$$d_{i,j} = \frac{2^{\frac{\theta-1}{2}}}{\sqrt{C_r}}\int_0^1\int_0^1 U\left(\frac{\vartheta-1+\omega}{2^{\theta-1}},\beta\right)F_r(\vartheta)\frac{d\vartheta}{2^{\theta-1}}\times\varphi_{i,j}(\beta)d\beta,$$

$$d_{i,j} = \frac{2^{-\left(\frac{\theta-1}{2}\right)}}{\sqrt{C_r}}\int_0^1\left[\int_0^1 U\left(\frac{\vartheta-1+\omega}{2^{\theta-1}},\beta\right)F_r(\vartheta)d\vartheta\right]\times\varphi_{i,j}(\beta)d\beta.$$

Using the generalized form of the mean value theorem for integrals,

$$d_{i,j} = \frac{2^{-\left(\frac{\theta-1}{2}\right)}}{\sqrt{C_r}}\int_0^1 U\left(\frac{\xi-1+\omega}{2^{\theta-1}},\beta\right)\varphi_{i,j}(\beta)d\beta\times\left[\int_0^1 F_r(\vartheta)d\vartheta\right],$$

where $\xi \in (0,1)$ and choose $\int_0^1 F_r(\vartheta)d\vartheta = Z$, then

$$d_{i,j} = \frac{Z2^{-\left(\frac{\theta-1}{2}\right)}}{\sqrt{C_r}} \int \frac{\omega^{\frac{\omega}{2^{\theta-1}}}}{2^{\theta-1}} U\left(\frac{\xi-1+\omega}{2^{\theta-1}}, \beta\right) \times \frac{2^{\frac{\theta-1}{2}}}{\sqrt{C_r}} F_r\left(2^{\theta-1}\beta - \omega + 1\right) d\beta.$$

Put $2^{\theta-1}\beta - \omega + 1 = \vartheta_1$. Then,

$$d_{i,j} = \frac{Z}{C_r} \int_0^1 U\left(\frac{\xi-1+\omega}{2^{\theta-1}}, \frac{\vartheta_1-1+\omega}{,} 2^{\theta-1}\right) F_r(\vartheta_1) \frac{d\vartheta_1}{2^{\theta-1}}$$

$$d_{i,j} = \frac{Z2^{-\theta+1}}{C_r} \int_0^1 U\left(\frac{\xi-1+\omega}{2^{\theta-1}}, \frac{\vartheta_1-1+\omega}{2^{\theta-1}}\right) F_r(\vartheta_1) d\vartheta_1.$$

Using the generalized form of the mean value theorem for integrals again,

$$d_{i,j} = \frac{Z2^{-\theta+1}}{C_r} U\left(\frac{\xi-1+\omega}{2^{\theta-1}}, \frac{\xi_1-1+\omega}{2^{\theta-1}}\right) \int_0^1 F_r(\vartheta_1) d\vartheta_1,$$

where, $\xi_1 \in (0,1)$ and $\int_0^1 F_r(\vartheta_1)d\vartheta_1 = X$. Thus

$$d_{i,j} = \frac{ZX2^{-\theta+1}}{C_r} U\left(\frac{\xi-1+\omega}{2^{\theta-1}}, \frac{\xi_1-1+\omega}{2^{\theta-1}}\right), \forall \xi, \xi_1 \in (0,1).$$

Therefore,

$$\left|d_{i,j}\right| = \left|\frac{ZX2^{-\theta+1}}{C_r}\right| \left|U\left(\frac{\xi-1+\omega}{2^{\theta-1}}, \frac{\xi_1-1+\omega}{2^{\theta-1}}\right)\right|,$$

Since U is bounded by M,

$$\left|d_{i,j}\right| = \frac{|Z||X|\left|2^{-\theta+1}\right| M}{|C_r|}.$$

Which implies that the series expansion of $U(\alpha,\beta)$ converges uniformly because the series $\sum_{i=1}^{\infty}\sum_{j=0}^{\infty} d_{i,j}$ is absolutely convergent. □

Theorem 3.2. *Suppose the Fibonacci wavelet sequence $\left\{\varphi^k_{\omega,r}(\alpha,\beta)\right\}_{k=1}^{\infty}$, consisting of continuous functions in $L^2(\mathbb{R})$ with respect to α on $[a,b]$, converges to the function $U(\alpha,\beta)$ uniformly with respect to α on $[a,b]$. Consequently, $U(\alpha,\beta)$ is continuous in $L^2(\mathbb{R})$ with respect to α on $[a,b]$.*

Proof. Since the Fibonacci wavelet sequence $\left\{\varphi^k_{\omega,r}(\alpha,\beta)\right\}_{k=1}^{\infty}$ converges uniformly to $U(\alpha,\beta)$ in $L^2(\mathbb{R})$, for every $\varepsilon > 0$, there exists an integer $k = k_\varepsilon$ such that

$$\varphi^k_{\omega,r}(\alpha,\beta) - U(\alpha,\beta) < \frac{\varepsilon}{3}, \quad \forall \beta \in [a,b]. \tag{5}$$

Moreover, since $\left\{\varphi^k_{\omega,r}(\alpha,\beta)\right\}$ is continuous in $L^2(\mathbb{R})$ with respect to $\beta \in [a,b]$, there exists a $\delta = \delta_\varepsilon$ such that

$$\varphi^k_{\omega,r}(\alpha,\beta') - \varphi^k_{\omega,r}(\alpha,\beta) < \frac{\varepsilon}{3}, \text{ whenever } \beta' - \beta < \delta \text{ for all } \beta', \beta \in [a,b].$$

Thus, we have

$$
\begin{aligned}
&U(\alpha,\beta') - U(\alpha,\beta)\\
&\quad = U(\alpha,\beta') - \varphi^{k}_{\omega,r}(\alpha,\beta') + \varphi^{k}_{\omega,r}(\alpha,\beta') - \varphi^{k}_{\omega,r}(\alpha,\beta) + \varphi^{k}_{\omega,r}(\alpha,\beta) - U(\alpha,\beta)\\
&\quad \le \left|U(\alpha,\beta') - \varphi^{k}_{\omega,r}(\alpha,\beta')\right| + \left|\varphi^{k}_{\omega,r}(\alpha,\beta') - \varphi^{k}_{\omega,r}(\alpha,\beta)\right| + \left|\varphi^{k}_{\omega,r}(\alpha,\beta) - U(\alpha,\beta)\right|\\
&\quad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}\\
&\quad = \varepsilon.
\end{aligned}
$$

Thus, $U(\alpha,\beta') - U(\alpha,\beta) < \varepsilon$ for all $\beta - \beta' < \delta$ with $\beta, \beta' \in [a,b]$. Hence, $U(\alpha,\beta)$ is continuous in $L^2(\mathbb{R})$ with respect to β on $[a,b]$. □

Theorem 3.3. *Suppose the Fibonacci wavelet sequence* $\left\{\varphi^{k}_{\omega,r}(\alpha,\beta)\right\}_{k=1}^{\infty}$ *converges uniformly in* $L_2(\mathbb{R})$ *with respect to* β *on* $[a,b]$*. Then there exists a function* $U(\alpha,\beta)$ *that is continuous in* $L_2(\mathbb{R})$ *with respect to* β *on* $[a,b]$*, such that* $\lim_{k\to\infty} \varphi^{k}_{\omega,r}(\alpha,\beta) = \varphi_{\omega,r}(\alpha,\beta)$ *for all* $\beta \in [a,b]$*.*

Proof. By applying Riesz-Fischer's theorem, we can find a function $U(\alpha,\beta)$ in $L^2(\mathbb{R})$ for each $\beta \in [a,b]$, such that

$$\lim_{k\to\infty} \varphi^{k}_{\omega,r}(\alpha,\beta) = U(\alpha,\beta). \tag{6}$$

Consider the subsequence $\left\{\varphi^{k_i}_{\omega,r}(\alpha,\beta)\right\}_{i=1}^{\infty}$ where

$$\varphi^{k_{i+1}}_{\omega,r}(\alpha,\beta) - \varphi^{k_i}_{\omega,r}(\alpha,\beta) < \frac{1}{2^i}, \quad \forall \beta \in [a,b]. \tag{7}$$

From (6), we have

$$
\begin{aligned}
U(\alpha,\beta) &= \lim_{p\to\infty} \varphi^{k_p}_{\omega,r}\\
&= \varphi^{k_i}_{\omega,r} + \left(\varphi^{k_{i+1}}_{\omega,r} - \varphi^{k_i}_{\omega,r}\right) + \left(\varphi^{k_{i+2}}_{\omega,r} - \varphi^{k_{i+1}}_{\omega,r}\right) + \cdots,
\end{aligned}
$$

and from (7)

$$
\begin{aligned}
U(\alpha,\beta) - \varphi^{k_i}_{\omega,r} &\le \varphi^{k_{i+1}}_{\omega,r} - \varphi^{k_i}_{\omega,r} + \varphi^{k_{i+2}}_{\omega,r} - \varphi^{k_{i+1}}_{\omega,r} + \cdots\\
&\le \frac{1}{2^i} + \frac{1}{2^{i+1}} + \cdots = \frac{1}{2^{i-1}}, \text{ for } i = 1,2,3,\ldots.
\end{aligned}
$$

This shows that the subsequence $\left\{\varphi^{k_i}_{\omega,r}(\alpha,\beta)\right\}$ uniformly converges to $U(\alpha,\beta)$ in $L^2(\mathbb{R})$ for $\beta \in [a,b]$. According to Theorem 3.3, the function $U(\alpha,\beta)$ is continuous in $L_2(\mathbb{R})$ for $\beta \in [a,b]$. □

4. Discussion on the proposed method

This section introduces the collocation method employing Fibonacci wavelets to address the nonlinear Benjamin-Bona-Mahony partial differential equation depicted below Examine the standard BBM equation, expressed as follows:

$$\mathrm{A}U_{\beta}(\alpha,\beta) + \mathrm{B}U_{\alpha}(\alpha,\beta) + \mathrm{C}U(\alpha,\beta)U_{\alpha}(\alpha,\beta) - \mathrm{D}U_{\alpha\alpha\beta}(\alpha,\beta) = \zeta(\alpha,\beta), \tag{8}$$

subject to the conditions given as,

$$U(\alpha,0) = \tilde{f}(\alpha), \alpha \in [0,1], \quad U(0,\beta) = \tilde{g}_0(\beta), \quad U(1,\beta) = \tilde{g}_1(\beta), \forall \beta \ge 0, \tag{9}$$

where A, B, C and D are real constants and $\tilde{f}(\alpha)$, $\tilde{g}_0(\beta)$, $\tilde{g}_1(\beta)$ and $\zeta(\alpha,\beta)$ are real-valued continuous functions. Now consider the following approximation,

$$U_{\alpha\alpha\beta}(\alpha,\beta) \approx \varphi^T(\beta)K\varphi(\alpha), \tag{10}$$

$$\varphi^T(\beta) = \left[\varphi_{1,0}(\beta),\ldots,\varphi_{1,\mu-1}(\beta),\varphi_{2,0}(\beta),,\ldots,\varphi_{2,\mu-1}(\beta),\ldots,\varphi_{2^{\theta-1},0}(\beta),\ldots,\varphi_{2^{\theta-1},\mu-1}(\beta)\right], \tag{11}$$

$$\varphi(\alpha) = \left[\varphi_{1,0}(\alpha),\ldots,\varphi_{1,\mu-1}(\alpha),\varphi_{2,0}(\alpha),\ldots,\varphi_{2,\mu-1}(\alpha),\ldots,\varphi_{2^{\theta-1},0}(\alpha),\ldots,\varphi_{2^{\theta-1},\mu-1}(\alpha)\right]^T, \tag{12}$$

$$K = \left[d_{j,j}\right]_{N\times N}, N = 2^{\theta-1}\mu. \tag{13}$$

Here K denotes $N\times N$ as a Fibonacci wavelets basis coefficient matrix that can be determined as follows. By integrating (13) first with respect to β and then with respect to α from 0 to β and α respectively, we obtain,

$$U_\alpha(\alpha,\beta) = U_\alpha(0,\beta) + U_\alpha(\alpha,0) - U_\alpha(0,0) + \int_0^\alpha\int_0^\beta \varphi^T(\beta)K\varphi(\alpha)d\beta d\alpha. \tag{14}$$

Again integrating (14) w.r.t α from 0 to α, we have

$$U(\alpha,\beta) = U(0,\beta) + \alpha\left(U_\alpha(0,\beta) - U_\alpha(0,0)\right) + U(\alpha,0) - U(0,0) + \int_0^\alpha\int_0^\alpha\int_0^\beta \varphi^T(\beta)K\varphi(\alpha)d\beta d\alpha d\alpha. \tag{15}$$

Put $\alpha = 1$ in (15) and apply the initial and boundary conditions given in (9) to get

$$U_\alpha(0,\beta) - U_\alpha(0,0) = \tilde{g}_1(\beta) - \tilde{g}_0(\beta) + \tilde{f}(0) - \tilde{f}(1) - \int_0^\alpha\int_0^\alpha\int_0^\beta \varphi^T(\beta)K\varphi(\alpha)d\beta d\alpha d\alpha\bigg|_{\alpha=1}. \tag{16}$$

Now substitute (16) in (15) and (14), so that

$$\begin{aligned} U_\alpha(\alpha,\beta) =& U_\alpha(\alpha,0) + \tilde{g}_1(\beta) - \tilde{g}_0(\beta) + \tilde{f}(0) - \tilde{f}(1) \\ &- \int_0^\alpha\int_0^\alpha\int_0^\beta \varphi^T(\beta)K\varphi(\alpha)d\beta d\alpha d\alpha\bigg|_{\alpha=1} \\ &+ \int_0^\alpha\int_0^\alpha\int_0^\beta \varphi^T(\beta)K\varphi(\alpha)d\beta d\alpha d\alpha, \end{aligned} \tag{17}$$

and

$$\begin{aligned} U(\alpha,\beta) =& U(0,\beta) + \alpha\left(\tilde{g}_1(\beta) - \tilde{g}_0(\beta) + \tilde{f}(0) - \tilde{f}(1)\right) \\ &- \int_0^\alpha\int_0^\alpha\int_0^\beta \varphi^T(\beta)K\varphi(\alpha)d\beta d\alpha d\alpha\bigg|_{\alpha=1} \\ &+ U(\alpha,0) - U(0,0) + \int_0^\alpha\int_0^\alpha\int_0^\beta \varphi^T(\beta)K\varphi(\alpha)d\beta d\alpha d\alpha. \end{aligned} \tag{18}$$

Now on differentiating (18) w.r.t β,

$$\begin{aligned} U_\beta(\alpha,\beta) =& U_\beta(0,\beta) + \alpha\left(\tilde{g}_1'(\beta) - \tilde{g}_0'(\beta)\right) - \int_0^\alpha\int_0^\alpha \varphi^T(\beta)K\varphi(\alpha)d\alpha d\alpha\bigg|_{\alpha=1} \\ &+ \int_0^\alpha\int_0^\alpha \varphi^T(\beta)K\varphi(\alpha)d\alpha d\alpha. \end{aligned} \tag{19}$$

Using the collocation points $\alpha_i, \beta_i = \frac{2i-1}{2\mu^2}, i = 1, 2, \ldots, \mu^2$, we can collocate the algebraic equation (8), obtained after substituting (19), (18), (17) and (10) in (8). Then, using either Newton's iterative approach or MATLAB's fsolve command, solve the resultant system of algebraic equations that was produced by the aforementioned procedure for the unknown wavelets coefficients $d[i, j]$, where $i, j = 1, 2, \ldots, 2^{\theta-1}\mu$. Now we are about to get a numerical solution of the concerned equation after putting these values of the Fibonacci wavelet coefficients in (18). At the end (18) will represent the numerical solution of the BBM problem (8) based on Fibonacci wavelets. If $U_e(\alpha, \beta)$ be the exact solution and $U_{app}(\alpha, \beta)$ be the approximate solution of a problem then $AE =| U_e(\alpha, \beta) - U_{app}(\alpha, \beta) |$ is called as absolute error.

5. Numerical experiments

Test question 1. Consider the following non-homogeneous partial differential equation [27] of the form (1) with A = 1, B = 1, C = 0, and D = 2:

$$U_\beta(\alpha, \beta) - 2U_{\alpha\alpha\beta}(\alpha, \beta) + U_\alpha(\alpha, \beta) = \zeta(\alpha, \beta), \tag{20}$$

under the given conditions as,

$$\zeta(\alpha, \beta) = 0, \quad U(\alpha, 0) = e^{-\alpha} \quad 0 \le \alpha \le 1,$$

$$U(0, \beta) = e^{-\beta}, \quad U(1, \beta) = e^{-1-\beta}, \quad \forall \beta \ge 0.$$

which has the exact solution $U(\alpha, \beta) = e^{-\alpha-\beta}$. Now using the proposed Fibonacci wavelet collocation method (FWCM) for $\theta = 1$ and $\mu = 9$, we find an approximate solution to this problem. Numerical values of $U(\alpha, \beta)$ obtained by FWCM are compared with the TWCM [21] and the exact values in Table 1, while Table 2 shows a comparison of the absolute error of the current method at different values of β. Figure (1) shows the obtained Fibonacci wavelet approximate solution. By increasing the number of collocation points, we can get more accuracy. In Figure (2), error variation for different values of α and β for the current method can be observed. Comparison of the estimated solution for different values to β and α are shown in Figure (3) and (4) respectively. Figure (5) represents the absolute error (AE) that occurred for the current method in the 2-D plane for different values of β.

Table 1: Comparison of the FWCM with TWCM and the exact solution for the question 1 at $N = 9$.

α	$\beta = 0.1$			$\beta = 0.3$		
	Exact Solution	TWCM[21]	FWCM	Exact Solution	TWCM[21]	FWCM
0	9.048374180e-1	9.04837418e-1	9.048374180e-1	7.40818220e-1	7.4081822e-1	7.40818220e-1
0.1	8.187307530e-1	8.15262011e-1	8.207284279e-1	6.70320046e-1	6.60796272e-1	6.75942531e-1
0.2	7.408182206e-1	7.34120745e-1	7.448266262e-1	6.06530659e-1	5.88016343e-1	6.17850814e-1
0.3	6.703200460e-1	6.60881116e-1	6.763292929e-1	5.48811636e-1	5.22565888e-1	5.65892412e-1
0.4	6.065306597e-1	5.95064024e-1	6.144499998e-1	4.96585303e-1	4.64536601e-1	5.19292034e-1
0.5	5.488116360e-1	5.36233872e-1	5.583950319e-1	4.49328964e-1	4.14014567e-1	4.77088594e-1
0.6	4.965853037e-1	4.83985287e-1	5.073344151e-1	4.06569659e-1	3.71050411e-1	4.38053325e-1
0.7	4.493289641e-1	4.37927638e-1	4.603658104e-1	3.67879441e-1	3.35624548e-1	4.00581027e-1
0.8	4.493289641e-1	3.97668539e-1	4.603658104e-1	3.67879441e-1	3.07608600e-1	4.00581027e-1
0.9	3.678794411e-1	3.62797522e-1	3.744507010e-1	3.01194211e-1	2.86728231e-1	3.21121949e-1
1.0	3.328710836e-1	3.32871083e-1	3.328710836e-1	2.72531793e-1	2.72531793e-1	2.72531793e-1

Table 2: Comparision of the FWCM errors for the question 1.

α	$\beta = 0.1$		$\beta = 0.3$	
	AE	AE	AE	AE
	$N = 9$	$N = 8$	$N = 9$	$N = 8$
	FWCM	TWCM[21]	FWCM	TWCM[21]
0.0	0	0	0	0
0.1	1.99767491e-03	346874144e-03	5.62248533e-03	952377308e-03
0.2	4.00840557e-03	669747519e-03	1.13201551e-02	185138181e-03
0.3	6.00924694e-03	943892945e-03	1.70807767e-02	262457492e-04
0.4	7.91934013e-03	114666352e-02	2.27067309e-02	320487027e-04
0.5	9.58339589e-03	125777638e-02	2.77596302e-02	353143961e-03
0.6	1.07491113e-02	126000168e-02	3.14836654e-02	355192482e-02
0.7	1.10368463e-02	114013256e-02	327015861e-02	322548994e-03
0.8	1.10368463e-02	890112025e-03	3.27015861e-02	875988598e-04
0.9	6.57125987e-03	508191883e-03	1.99277371e-02	852624831e-03
1.0	0	555111511e-17	0	555115123e-17

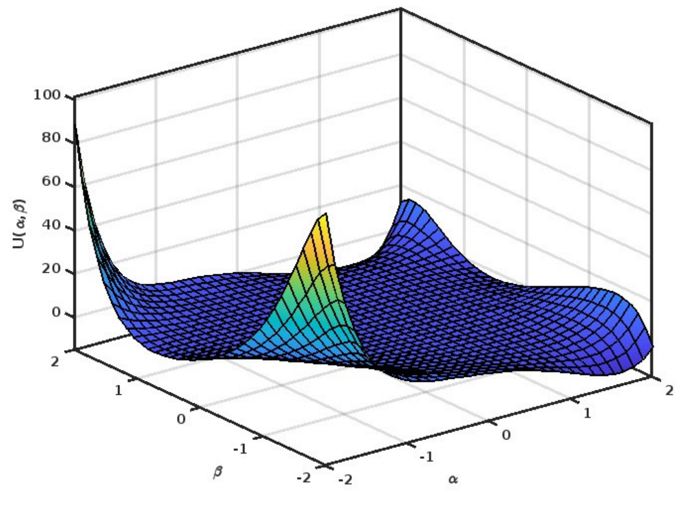

Figure 1: Approximate solution using Fibonacci wavelets for the question 1.

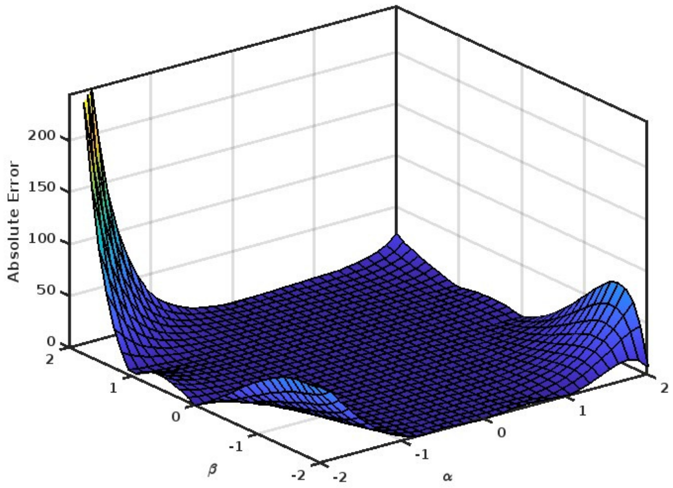

Figure 2: Error comparison for the question 1.

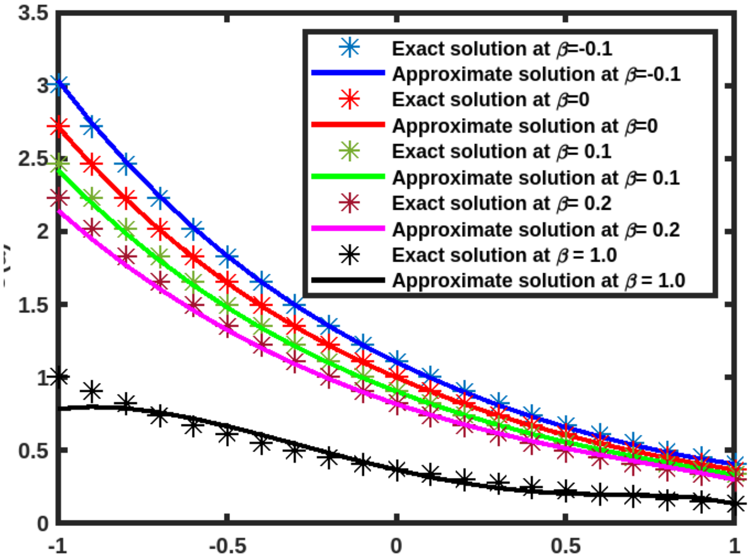

Figure 3: Fibonacci wavelet solution for several values of β of question 1.

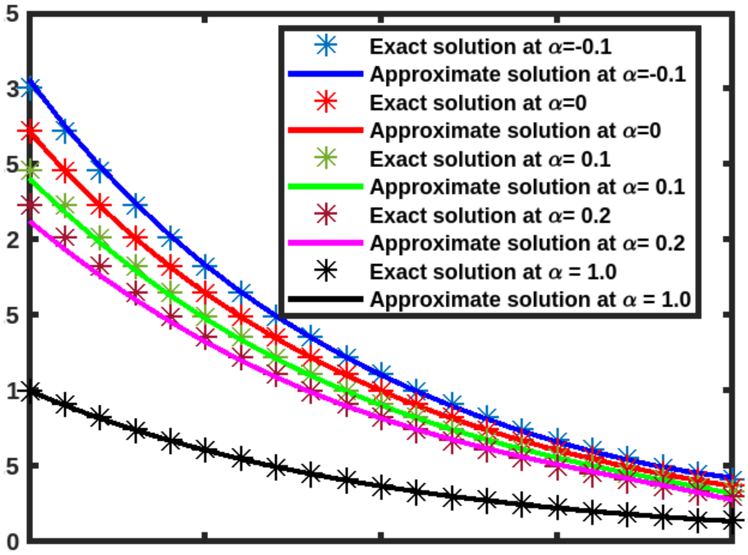

Figure 4: Fibonacci wavelet solution for several values of α of question 1.

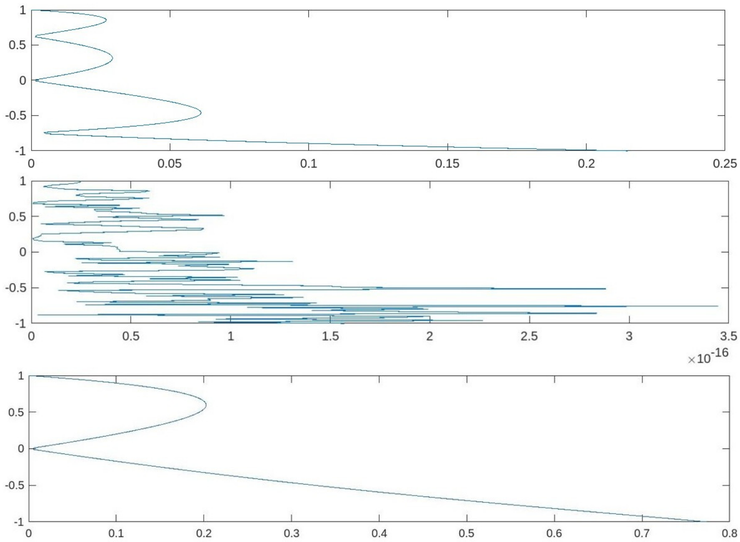

Figure 5: Error at different values of β for the question 1.

Test question 2. Consider the following linear non-homogeneous BBM partial differential equation of the form [27],

$$U_\beta(\alpha,\beta) - 2U_{\alpha\alpha\beta}(\alpha,\beta) + e^{\alpha+\beta} = 0 \tag{21}$$

under the given conditions,

$$U(\alpha,0) = e^{\alpha} \quad 0 \le \alpha \le 1$$

$$U(0,\beta) = e^{\beta}, \quad U(1,\beta) = e^{1+\beta}, \quad \forall\beta \ge 0.$$

which has the exact solution $e^{-\alpha-\beta}$. Now using the proposed Fibonacci wavelet collocation method (FWCM), we find an approximate solution to this problem. Numerical values obtained by this method and the exact values are compared in Table 3. In contrast, Table 4 shows the absolute error comparison of the current method for several values of β. Figure (6) presents an approximate solution for problem 2. Increase the number of collocation points to get more accuracy. In Figure (7), error variation for different values of α and β using the current method can be observed. A comparison of the approximate solution of problem 2 for different values to β and α is shown in Figure (8) and (9) respectively. Figure (10) represents the absolute error representation in the 2-D plane for different values of β.

Table 3: Comparison of FWCM and the exact solution for the question 2.

α	$\beta = 0.1$			$\beta = 0.3$		
	Exact solution	TWCM[21]	FWCM	Exact solution	TWCM[21]	FWCM
0.1	1.22140275816	1.10510127712	1.22158955635	1.49182469764	1.10524058029	1.49252798901
0.2	1.34985880757	1.22127267535	1.35020103811	1.64872127070	1.22153292411	1.65039560014
0.3	1.49182469764	1.49157245087	1.49222548543	1.82211880039	1.35001164625	1.82470380647
0.4	1.64872127070	1.34970612331	1.64903315076	2.01375270747	1.59207720107	2.01690290325
0.5	1.82211880039	1.64809190303	1.82217131857	2.22554092849	1.64935127363	2.22870248969
0.6	2.01375270747	1.82060736665	2.01339697410	2.45960311115	1.82363222043	2.46213548640
0.7	2.22554092849	2.01090827650	2.22471683251	2.71828182845	2.01660259161	2.71963873703
0.8	2.22554092849	2.2195158646	2.22471683251	2.71828182845	2.22914179706	2.71963873703
0.9	2.71828182845	2.45779565416	2.71722055307	3.32011692273	2.46142660049	3.31925420884
1.0	3.00416602394	2.71825464577	3.00416602394	3.66929666761	2.71830901141	3.66929666761

Table 4: Error comparison of FWCM and several other methods for the question 2.

α	$\beta = 0.1$		$\beta = 0.3$	
	AE $N = 9$ FWCM	AE $N = 8$ TWCM[21]	AE $N = 9$ FWCM	AE $N = 8$ TWCM[21]
0.1	1.8679819e-04	5.85892922e-05	7.03291377e-04	5.86104522e-05
0.2	3.4223053e-04	1.17868835e-04	1.67432944e-03	1.17951864e-04
0.3	4.0078779e-04	1.39185735e-04	2.58500608e-03	1.39340023e-04
0.4	3.1188006e-04	2.37328598e-04	3.15019578 e-03	2.37585115e-04
0.5	5.2518188e-05	6.12880530e-04	3.16156120e-03	6.13515637e-04
0.6	3.5573336e-04	1.49321263e-03	2.53237524e-03	1.49519876e-03
0.7	8.24095979e-04	2.82429353e-03	1.35690857e-03	2.82974651e-03
0.8	8.24095979e-04	3.56708672e-03	1.35690857e-03	3.57861305e-03
0.9	1.06127537e-03	1.78286108e-03	8.62713894e-04	1.79889318e-03
1.0	0	4.44089209e-16	0	8.88178419e-16

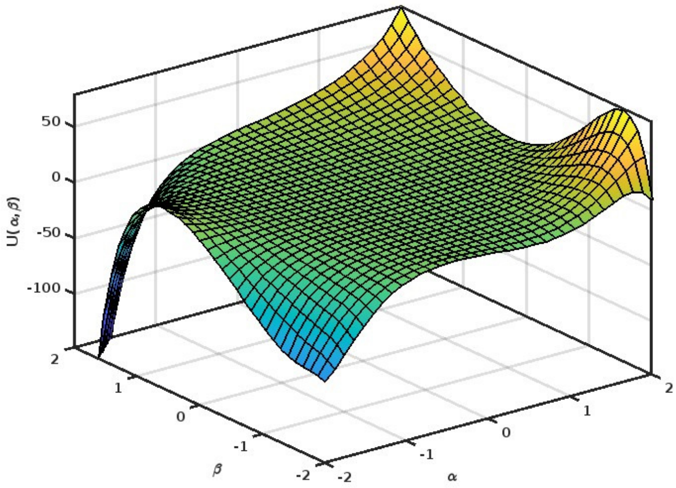

Figure 6: Approximate solution using Fibonacci wavelets for the question 2.

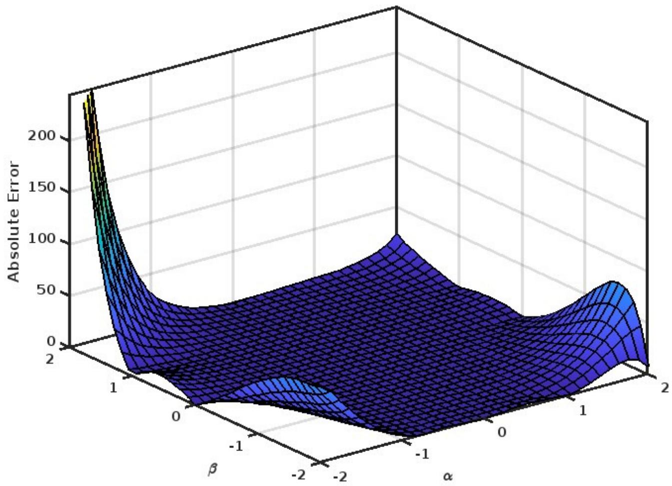

Figure 7: Error representation for the question 2.

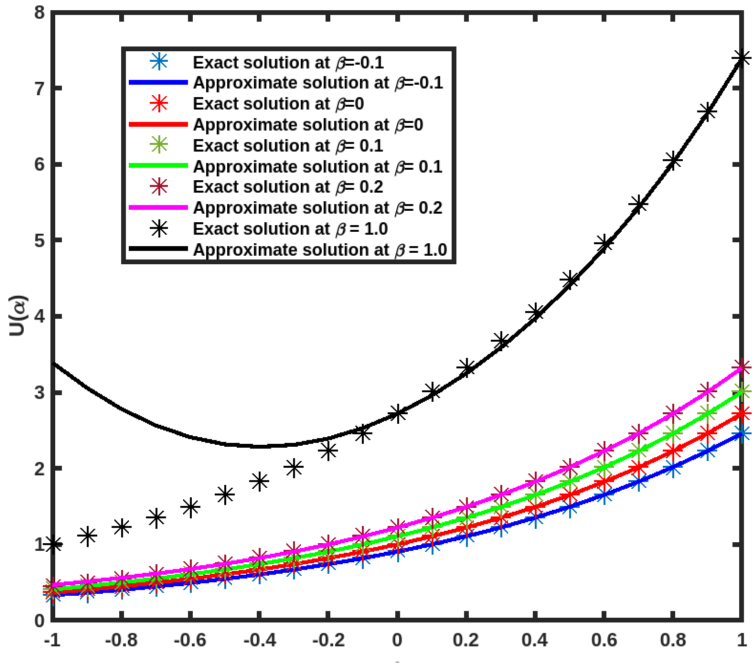

Figure 8: Approximate solution using Fibonacci wavelets for different values of β for the question 2.

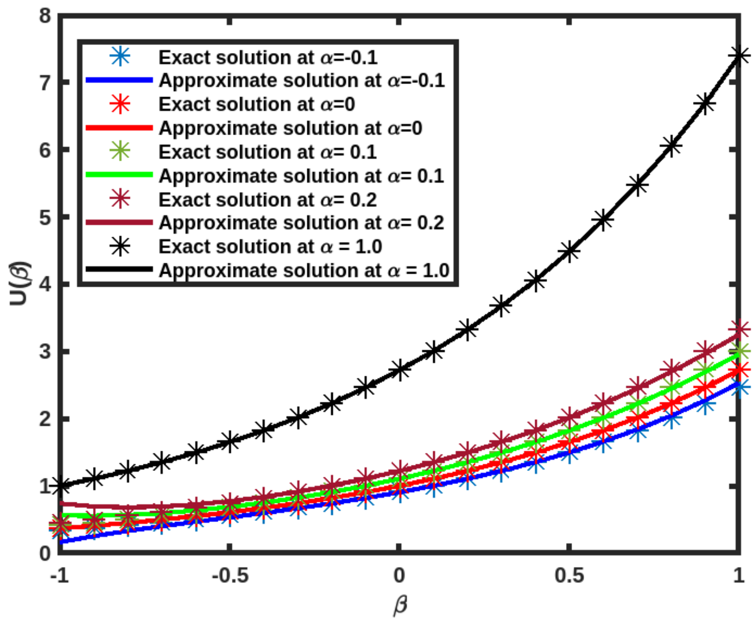

Figure 9: Approximate solution using Fibonacci wavelets for different values of α for the question 2.

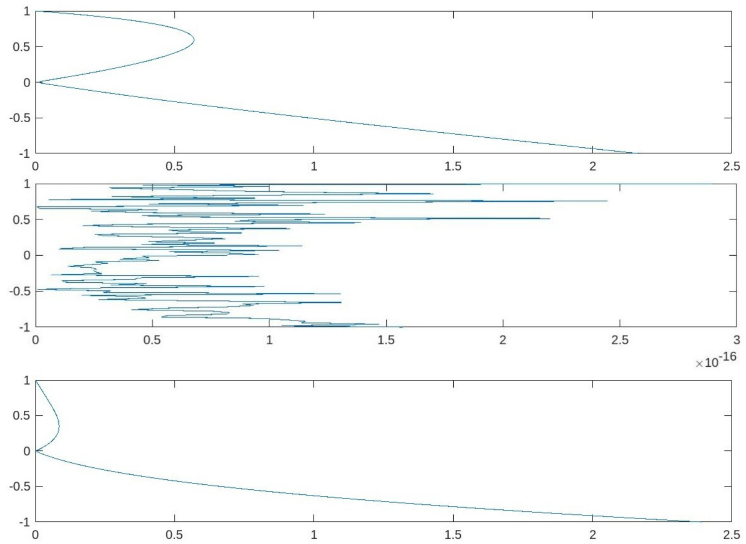

Figure 10: Absolute error for several values of β for the question 2.

Test Question 3. Consider the following nonlinear homogeneous BBM partial differential equation of the form [27],

$$U_\beta(\alpha,\beta) - U_{\alpha\alpha\beta}(\alpha,\beta) + U(\alpha,\beta)U_\alpha(\alpha,\beta) = 0, \tag{22}$$

under the conditions given,

$$U(\alpha,0) = 0 \quad 0 \le \alpha \le 1$$

and

$$U(0,\beta) = 0, \quad U(1,\beta) = \frac{1}{1+\beta}, \quad \forall\beta \ge 0$$

Which has the exact solution $\frac{\alpha}{1+\beta}$. The Fibonacci wavelet solution of this problem is the same as the exact solution for a small value of N, which demonstrates the efficiency, applicability, and many more than those existing methods. Figure (11) describes the surface plot of the Fibonacci wavelet approximate solution for $\theta = 1, \mu = 6$. Error comparison and comparison of solution with several other existing solutions are shown in Tables 5,6 and 7.

Table 5: Comparison of FWCM and many other existing solutions for the question 3.

α	$\beta = 0.01$			$\beta = -0.1$		
	Exact Solution	TWCM[21]	FWCM	Exact Solution	TWCM[21]	FWCM
0.1	0.099009900	0.090818202	0.090909090	0.099900099	0.099081820	0.099900099
0.2	0.181818181	0.199069715	0.181818181	0.179978098	0.199069714	0.179978098
0.3	0.272727272	0.300152337	0.272727272	0.279898389	0.300152337	0.279898389
0.4	0.363636363	0.401789158	0.363636363	0.384740915	0.401789152	0.384740915
0.5	0.454545454	0.503226870	0.454545454	0.489142274	0.503226870	0.489142274
0.6	0.545454545	0.603819955	0.545454545	0.588780594	0.603819955	0.588780594
0.7	0.636363636	0.703155236	0.636363636	0.681695951	0.703155236	0.681695951
0.8	0.727272727	0.800980588	0.727272727	0.767661780	0.800980588	0.767661780
0.9	0.727272727	0.896937715	0.727272727	0.845607295	0.896937715	0.845607295
1.0	0.909090909	0.990099009	0.909090909	0.909090909	0.990099009	0.909090909

Table 6: Comparison of FWCM and several other existing methods for the question 3.

α	Analytical Solution	FWCM	TWCM	CWCS	Haar [27]	B-Spline [15]
0.1	0.0833333333	0.0833333333	0.0833333569	0.0833359813	0.0833917702	0.831005374
0.2	0.1666666666	0.1666666666	0.1666666103	0.1666690130	0.1663423613	0.1669571332
0.3	0.2500000000	0.2500000000	0.2500000981	0.2500076193	0.2502058351	0.2506686123
0.4	0.3333333333	0.3333333333	0.3333333568	0.3333392661	0.3337555328	0.3331309564
0.5	0.4166666666	0.4166666666	0.4166666916	0.4166600928	0.4164872697	0.4167964138
0.6	0.5000000000	0.5000000000	0.5000000763	0.4999950089	0.5001894034	0.5001078627
0.7	0.5833333333	0.5833333333	0.5833335611	0.5833399843	0.5837781324	0.5831276593
0.8	0.6666666666	0.6666666666	0.6666666801	0.6666662985	0.6609932537	0.6609999361
0.9	0.7500000000	0.7500000000	0.7500000906	0.7500000679	0.7508893129	0.7508928911
1.0	0.8333333333	0.8333333333	0.8333333867	0.8333334582	0.8333333333	0.8333333333

Table 7: Error comparison of FWCM with several other existing methods for the question 3.

α	TWCM $N = 16$ AE	Haar [27] AE	Chebyshev wavelet [7] $N = 16$ AE	B-Spline [15] AE	FWCM
0.1	7.30706490e − 08	1.89002025e − 04	7.49731169e − 05	8.38002352e − 04	0
0.2	1.01801692e − 08	1.91201587e − 03	1.50234900e − 06	9.44141500e − 05	0
0.3	4.70463471e − 08	3.94005912e − 03	2.36513300e − 06	5.26852330e − 04	0
0.4	1.33609860e − 09	2.37417460e − 04	4.12525088e − 06	3.59602891e − 05	0
0.5	7.21280037e − 09	7.89492800e − 03	1.25842681e − 06	5.14403077e − 04	0
0.6	2.13306307e − 07	8.90258870e − 03	6.86674486e − 06	6.84603292e − 04	0
0.7	5.85607735e − 08	1.02631992e − 04	1.49035965e − 06	8.77403481e − 04	0
0.8	3.42861970e − 08	2.25690270e − 03	4.02494006e − 06	1.09433653e − 04	0
1.9	2.90601174e − 08	4.25716120e − 04	6.91048528e − 07	8.34723711e − 04	0
1.0	4.22097516e − 17	8.53000895e − 09	7.07601845e − 16	1.38976145e − 14	0

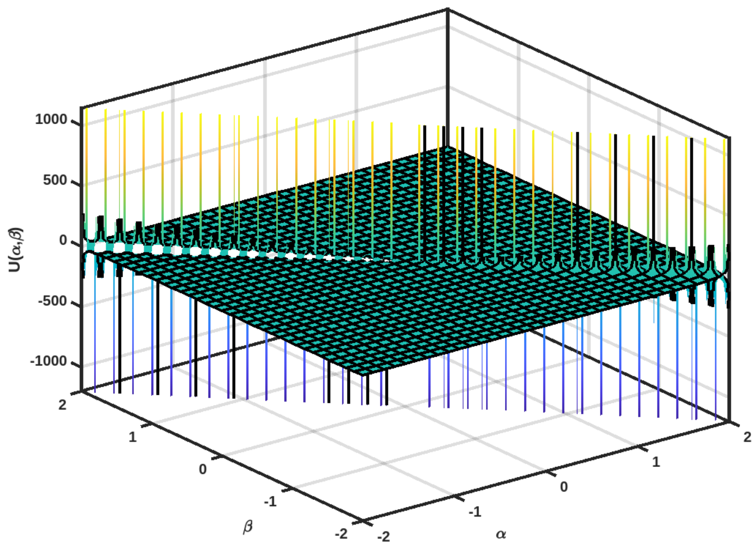

Figure 11: Approximate solution using Fibonacci wavelets for the question 3.

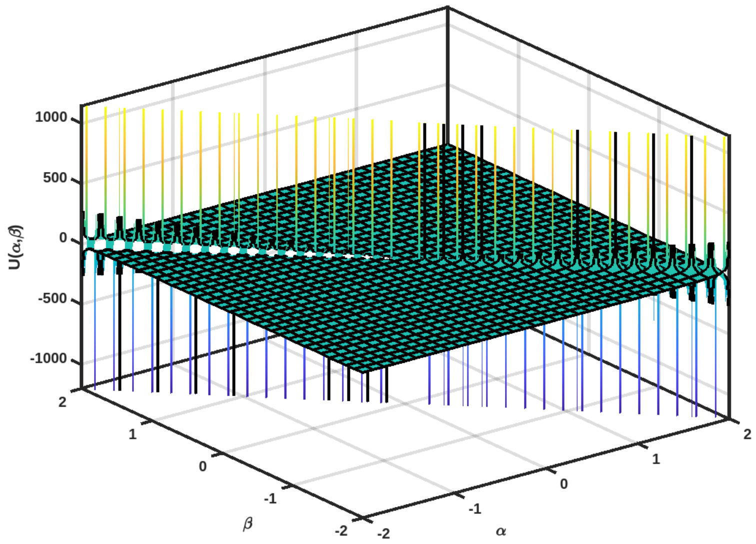

Figure 12: Exact solution for the question 3.

6. Conclusion

In this paper, we have presented the general form and the practical application domain of the Benjamin Bona Mahony (BBM) partial differential equation, along with a new and more comprehensive numerical method to determine its approximate solution. The application of this method to the problem under various conditions demonstrates that the proposed methodology is not only innovative but also significantly contributory and computationally efficient on a large scale in the field of numerical analysis, leveraging the remarkable features of wavelet analysis. The outcomes of the method are displayed in the form of tables and figures for different values of α, β, highlighting the rapid convergence and improved error accuracy compared to other methods. Extensive computer implementation of the methodology confirms that increasing the level of resolution and the number of collocation points can be effective for obtaining more accurate solutions to problems encountered in science and engineering. Finally, we provide the following conclusions from our investigation:

1. The introduced numerical approach with the Fibonacci wavelet provides a more accurate solution than the existing method.
2. This method is easy to implement in software applications, and we can extend it to higher orders with minor modifications to the existing approach.
3. The proposed method is also fairly straightforward to implement, and the computational results achieved show that it is highly efficient for addressing the previously mentioned issues numerically, as well as for solving other partial differential equations.
4. Abstract discussions are utilized to elucidate the properties of Fibonacci wavelets and their convergence analysis. Augmenting the number of truncated terms might encounter challenges in computer programming, which could be considered a limitation of the method.

7. Declaration

7.1. *Availability of data and material*

NA

7.2. Competing Interests

All authors assert that they do not possess any conflicts of interest.

7.3. Authors' contributions

Vivek, one of the authors, executed the computational tasks using MATLAB software and carried out programming-related operations, implementing the code design within MATLAB. Prof. H. M. Srivastava inspired, advised on the formation of the article, corrected grammatical errors, and provided guidance at every stage of the manuscript preparation. Manoj Kumar, one of the authors, carried out the convergence analysis of the proposed method. Also, all the authors made significant contributions to the design and preparation of the manuscript.

7.4. Acknowledgements

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