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A study on dual generalized Fibonacci matrices

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Abstract. In this paper, we introduce the dual generalized Fibonacci matrices. As special cases, we deal with dual Fibonacci and dual Lucas matrices. We present Binet's formulas, generating functions and the summation formulas for these matrices. Moreover, we give Catalan's, Cassini's, d'Ocagne's, Gelin-Cesàro's, Melham's identities and present matrices related with these sequences.

1. Introduction

In [17], it has been shown that there exist essentially three possible ways to generalize real numbers into real algebras of dimension 2. In fact, each possible system can be reduced to one of the following:

- numbers $x + yi$ with $i^2 = -1$ (complex numbers);
- numbers $x + yh$ with $h^2 = 1$, (hyperbolic numbers);
- numbers $x + y\varepsilon$ with $\varepsilon^2 = 0$, (dual numbers).

There are also other generalizations (extensions) of real numbers into real algebras of higher dimension. The hypercomplex numbers systems, [17], are extensions of real numbers. Some commutative examples of hypercomplex number systems are complex numbers, hyperbolic numbers, [21], and dual numbers, [12]. Some non-commutative examples of hypercomplex number systems are quaternions ([14],[15]), octonions [4] and sedenions [22]. The algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions) and $\mathbb{S}$ (sedenions) are real algebras obtained from the real numbers $\mathbb{R}$ by a doubling procedure called the Cayley-Dickson Process. This doubling process can be extended beyond the sedenions to form what are known as the 2^n-ions (see for example [5], [16], [18]).

Quaternions were invented by Irish mathematician W. R. Hamilton (1805-1865) ([14],[15]) as an extension to the complex numbers. Hyperbolic numbers with complex coefficients are introduced by J. Cockle in 1848, [9]. H. H. Cheng and S. Thompson [7] introduced dual numbers with complex coefficients and called complex dual numbers. Akar, Yüce and Şahin [1] introduced dual hyperbolic numbers.

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Here, we use the set of dual numbers. Dual numbers were first invented by W. K. Clifford [8] in 1873. The dual numbers which extend to the real numbers has the form

$$z = x + \varepsilon y$$

where ε is the dual unit and $\varepsilon^2 = 0$, $\varepsilon \neq 0$. The set

$$\mathbb{D} = \{z = x + \varepsilon y \mid \varepsilon \notin \mathbb{R},\ \varepsilon^2 = 0, x, y \in \mathbb{R}\}$$

is called dual number system and forms two dimensional commutative associative algebra over the real numbers. The algebra of dual numbers is a commutative ring (with identity) with the following addition and multiplication operations; for any two dual numbers z_1 and z_2,

$$\begin{aligned} z_1 + z_2 &= (x_1 + \varepsilon y_1) + (x_2 + \varepsilon y_2) = (x_1 + x_2) + \varepsilon (y_1 + y_2), \\ z_1 \times z_2 &= (x_1 + \varepsilon y_1) \times (x_2 + \varepsilon y_2) = x_1 x_2 + \varepsilon (x_1 y_2 + y_1 x_2). \end{aligned}$$

The equality of two dual numbers $z_1 = x_1 + \varepsilon y_1$ and $z_2 = x_2 + \varepsilon y_2$ is defined as;

$$z_1 = z_2 \text{ if and only if } x_1 = x_2 \text{ and } y_1 = y_2.$$

The division of two dual numbers provided $x_2 \neq 0$ is given by

$$\frac{z_1}{z_2} = \frac{x_1 + \varepsilon y_1}{x_2 + \varepsilon y_2} = \frac{(x_1 + \varepsilon y_1)(x_2 - \varepsilon y_2)}{(x_2 + \varepsilon y_2)(x_2 - \varepsilon y_2)} = \frac{x_1}{x_2} + \varepsilon \frac{-x_1 y_2 + y_1 x_2}{x_2^2}.$$

The conjugate of the dual number $z = x + \varepsilon y$ is defined by

$$\overline{z} = z^{\dagger} = x - \varepsilon y.$$

Note that for any dual numbers z_1, z_2, z we have

$$\begin{aligned} \overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2}, \\ \overline{z_1 \times z_2} &= \overline{z_1} \times \overline{z_2}, \\ \|z\|^2 &= z \times \overline{\overline{z}} = \sqrt{x^2} = |x|. \end{aligned}$$

More information on dual numbers may be found in [8] and ([14],[15]).

Now, we give background on dual matrices.

Let $\mathbb{R}_{m\times n}$ be the set of $m \times n$ real matrices. Let $\mathbb{D}_{m\times n}$ be the set of $m \times n$ matrices with dual number entries. A dual matrix $Z \in \mathbb{D}_{m\times n}$ can be written in the following form:

$$\begin{aligned} Z &= (z_{ij})_{m\times n} = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{m1} & z_{m2} & \cdots & z_{mn} \end{pmatrix}_{m\times n} = \begin{pmatrix} x_{11} + \varepsilon y_{11} & x_{12} + \varepsilon y_{12} & \cdots & x_{1n} + \varepsilon y_{1n} \\ x_{21} + \varepsilon y_{21} & x_{22} + \varepsilon y_{22} & \cdots & x_{2n} + \varepsilon y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} + \varepsilon y_{m1} & x_{m2} + \varepsilon y_{m2} & \cdots & x_{mn} + \varepsilon y_{mn} \end{pmatrix}_{m\times n} \\ &= \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}_{m\times n} + \varepsilon \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{pmatrix}_{m\times n} \\ &= X + \varepsilon Y \end{aligned}$$

where

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}_{m\times n},$$

$$Y = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{pmatrix}_{m\times n}.$$

Therefore the set of dual matrices can be given as

$$\mathbb{D}_{m\times n} = \{Z = A + \varepsilon B \mid \varepsilon \notin \mathbb{R},\ \varepsilon^2 = 0,\ A, B \in \mathbb{R}_{m\times n}\}.$$

For more information on dual matrices, see for example [10]. For $m = n = 2$, a dual matrix Z can be written in the following form:

$$\begin{aligned} Z &= (z_{ij})_{2x2} = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \\ &= \begin{pmatrix} x_{11} + \varepsilon y_{11} & x_{12} + \varepsilon y_{12} \\ x_{21} + \varepsilon y_{21} & x_{22} + \varepsilon y_{22} \end{pmatrix} \\ &= \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} + \varepsilon \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \\ &= X + \varepsilon Y \end{aligned}$$

where

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix},$$

$$Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}.$$

Now let us recall the definition of generalized Fibonacci numbers.

A generalized Fibonacci sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relations

$$W_n = W_{n-1} + W_{n-2};\ \ W_0 = a,\ W_1 = b,\ \ (n \geq 2) \tag{1}$$

with the initial values W_0, W_1 not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-1)} + W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1) holds for all integer n.

The first few generalized Fibonacci numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Fibonacci numbers

n	W_n	W_{-n}
0	W_0	...
1	W_1	$-W_0 + W_1$
2	$W_0 + W_1$	$2W_0 - W_1$
3	$W_0 + 2W_1$	$-3W_0 + 2W_1$
4	$2W_0 + 3W_1$	$5W_0 - 3W_1$
5	$3W_0 + 5W_1$	$-8W_0 + 5W_1$
6	$5W_0 + 8W_1$	$13W_0 - 8W_1$

If we set $W_0 = 0, W_1 = 1$ then $\{W_n\}$ is the well-known Fibonacci sequence and if we set $W_0 = 2, W_1 = 1$ then $\{W_n\}$ is the well-known Lucas sequence. In other words, Fibonacci sequence $\{F_n\}_{n\geq 0}$ (OEIS: A000045, [20]) and Lucas sequence $\{L_n\}_{n\geq 0}$ (OEIS: A000032, [20]) are defined by the second-order recurrence relations

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, F_1 = 1 \tag{2}$$

and

$$L_n = L_{n-1} + L_{n-2}, \quad L_0 = 2, L_1 = 1. \tag{3}$$

The sequences $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$F_{-n} = -F_{-(n-1)} + F_{-(n-2)}$$

and

$$L_{-n} = -L_{-(n-1)} + L_{-(n-2)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (2) and (3) hold for all integer n.

We can list some important properties of generalized Fibonacci numbers that are needed.

- Binet formula of generalized Fibonacci sequence can be calculated using its characteristic equation which is given as

  $$z^2 - z - 1 = 0.$$

  The roots of characteristic equation are

  $$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}.$$

  Using these roots and the recurrence relation, Binet formula can be given as

  $$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta} \tag{4}$$

  where $A = W_1 - W_0\beta$ and $B = W_1 - W_0\alpha$.

- Binet formula of Fibonacci and Lucas sequences are

  $$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

  and

  $$L_n = \alpha^n + \beta^n$$

  respectively.

- The generating function for generalized Fibonacci numbers is

  $$g(z) = \frac{W_0 + (W_1 - W_0)z}{1 - z - z^2}. \tag{5}$$

- The Cassini identity for generalized Fibonacci numbers is

  $$W_{n+1}W_{n-1} - W_n^2 = (W_0W_1 - W_1^2 - W_0^2). \tag{6}$$

- $$\begin{aligned} A\alpha^n &= \alpha W_n + W_{n-1}, & (7)\\ B\beta^n &= \beta W_n + W_{n-1}. & (8) \end{aligned}$$

In this paper, we define the dual generalized Fibonacci matrices in the next section and give some properties of them.

2. Dual Generalized Fibonacci Matrices and their Generating Functions and Binet's Formulas

In this section, we define dual generalized Fibonacci matrices and present generating functions and Binet formulas for them. In [19] and [13], the authors defined and investigated dual Fibonacci and Lucas numbers. In [2], the authors defined and investigated dual generalized Pell numbers. In [23], the authors defined and studied dual generalized Fibonacci numbers. Aydın [3] studied dual generalized Jacobsthal sequence and Cerda-Morales [6] studied dual third-order Jacobsthal numbers.

The nth dual generalized Fibonacci matrix is

$$DW_n = \begin{pmatrix} W_{n+1}+\varepsilon W_{n+2} & W_n+\varepsilon W_{n+1} \\ W_n+\varepsilon W_{n+1} & W_{n-1}+\varepsilon W_n \end{pmatrix} = \begin{pmatrix} W_{n+1}+\varepsilon (W_{n+1}+W_n) & W_n+\varepsilon W_{n+1} \\ W_n+\varepsilon W_{n+1} & W_{n+1}-W_n+\varepsilon W_n \end{pmatrix} \tag{9}$$

with initial conditions

$$\begin{aligned} DW_0 &= \begin{pmatrix} W_1+\varepsilon (W_0+W_1) & W_0+\varepsilon W_1 \\ W_0+\varepsilon W_1 & W_1-W_0+\varepsilon W_0 \end{pmatrix}, \\ DW_1 &= \begin{pmatrix} W_0+W_1+\varepsilon (W_0+2W_1) & W_1+\varepsilon (W_0+W_1) \\ W_1+\varepsilon (W_0+W_1) & W_0+\varepsilon W_1 \end{pmatrix}, \end{aligned}$$

where $\varepsilon^2 = 0$. As special cases, the nth dual Fibonacci matrix and the nth dual Lucas matrix are given as

$$DF_n = \begin{pmatrix} F_{n+1}+\varepsilon F_{n+2} & F_n+\varepsilon F_{n+1} \\ F_n+\varepsilon F_{n+1} & F_{n-1}+\varepsilon F_n \end{pmatrix}$$

and

$$DL_n = \begin{pmatrix} L_{n+1}+\varepsilon L_{n+2} & L_n+\varepsilon L_{n+1} \\ L_n+\varepsilon L_{n+1} & L_{n-1}+\varepsilon L_n \end{pmatrix}$$

respectively. It can be easily shown that

$$DW_n = DW_{n-1} + DW_{n-2}. \tag{10}$$

The sequence $\{DW_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$DW_{-n} = -DW_{-(n-1)} + DW_{-(n-2)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (10) holds for all integer n. Note that

$$DW_{-n} = \begin{pmatrix} W_{-n+1}+\varepsilon W_{-n+2} & W_{-n}+\varepsilon W_{-n+1} \\ W_{-n}+\varepsilon W_{-n+1} & W_{-n-1}+\varepsilon W_{-n} \end{pmatrix}.$$

Note also that

$$DW_n\varepsilon = \begin{pmatrix} W_{n+1}\varepsilon & W_n\varepsilon \\ W_n\varepsilon & W_{n-1}\varepsilon \end{pmatrix} = \varepsilon \begin{pmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{pmatrix}.$$

The first few dual generalized Fibonacci matrices with positive subscript and negative subscript are given in the following Table 2 and Table 3.

Table 2. A few dual generalized Fibonacci matrices with positive subscript

n	DW_n
0	$\begin{pmatrix} W_1+\varepsilon(W_0+W_1) & W_0+\varepsilon W_1 \\ W_0+\varepsilon W_1 & W_1-W_0+\varepsilon W_0 \end{pmatrix}$
1	$\begin{pmatrix} W_0+W_1+\varepsilon(W_0+2W_1) & W_1+\varepsilon(W_0+W_1) \\ W_1+\varepsilon(W_0+W_1) & W_0+\varepsilon W_1 \end{pmatrix}$
2	$\begin{pmatrix} W_0+2W_1+\varepsilon(2W_0+3W_1) & W_0+W_1+\varepsilon(W_0+2W_1) \\ W_0+W_1+\varepsilon(W_0+2W_1) & W_1+\varepsilon(W_0+W_1) \end{pmatrix}$
3	$\begin{pmatrix} 2W_0+3W_1+\varepsilon(3W_0+5W_1) & W_0+2W_1+\varepsilon(2W_0+3W_1) \\ W_0+2W_1+\varepsilon(2W_0+3W_1) & W_0+W_1+\varepsilon(W_0+2W_1) \end{pmatrix}$
4	$\begin{pmatrix} 3W_0+5W_1+\varepsilon(5W_0+8W_1) & 2W_0+3W_1+\varepsilon(3W_0+5W_1) \\ 2W_0+3W_1+\varepsilon(3W_0+5W_1) & W_0+2W_1+\varepsilon(2W_0+3W_1) \end{pmatrix}$

Table 3. A few dual generalized Fibonacci matrices with negative subscript

n	DW_{-n}
0	$\begin{pmatrix} W_1+\varepsilon(W_0+W_1) & W_0+\varepsilon W_1 \\ W_0+\varepsilon W_1 & W_1-W_0+\varepsilon W_0 \end{pmatrix}$
1	$\begin{pmatrix} W_0+\varepsilon W_1 & W_1-W_0+\varepsilon W_0 \\ W_1-W_0+\varepsilon W_0 & 2W_0-W_1-\varepsilon(W_0-W_1) \end{pmatrix}$
2	$\begin{pmatrix} W_1-W_0+\varepsilon W_0 & 2W_0-W_1-\varepsilon(W_0-W_1) \\ 2W_0-W_1-\varepsilon(W_0-W_1) & 2W_1-3W_0-\varepsilon(W_1-2W_0) \end{pmatrix}$
3	$\begin{pmatrix} 2W_0-W_1-\varepsilon(W_0-W_1) & 2W_1-3W_0-\varepsilon(W_1-2W_0) \\ 2W_1-3W_0-\varepsilon(W_1-2W_0) & 5W_0-3W_1-\varepsilon(3W_0-2W_1) \end{pmatrix}$
4	$\begin{pmatrix} 2W_1-3W_0-\varepsilon(W_1-2W_0) & 5W_0-3W_1-\varepsilon(3W_0-2W_1) \\ 5W_0-3W_1-\varepsilon(3W_0-2W_1) & 5W_1-8W_0-\varepsilon(3W_1-5W_0) \end{pmatrix}$

For dual Fibonacci numbers (taking $W_n = F_n, F_0 = 0, F_1 = 1$) we get

$$\begin{aligned} DF_0 &= \begin{pmatrix} 1+\varepsilon & \varepsilon \\ \varepsilon & 1 \end{pmatrix}, \\ DF_1 &= \begin{pmatrix} 1+2\varepsilon & 1+\varepsilon \\ 1+\varepsilon & \varepsilon \end{pmatrix}, \end{aligned}$$

and for dual Lucas numbers (taking $W_n = L_n, L_0 = 2, L_1 = 1$) we get

$$\begin{aligned} DL_0 &= \begin{pmatrix} 1+3\varepsilon & 2+\varepsilon \\ 2+\varepsilon & -1+2\varepsilon \end{pmatrix}, \\ DL_1 &= \begin{pmatrix} 3+4\varepsilon & 1+3\varepsilon \\ 1+3\varepsilon & 2+\varepsilon \end{pmatrix}. \end{aligned}$$

A few dual Fibonacci matrices and dual Lucas matrices with positive subscript and negative subscript are given in the following Table 4 and Table 5.

Table 4. Dual Fibonacci matrices

n	DF_n	DF_{-n}
0	$\begin{pmatrix} 1+\varepsilon & \varepsilon \\ \varepsilon & 1 \end{pmatrix}$	$\begin{pmatrix} 1+\varepsilon & \varepsilon \\ \varepsilon & 1 \end{pmatrix}$
1	$\begin{pmatrix} 1+2\varepsilon & 1+\varepsilon \\ 1+\varepsilon & \varepsilon \end{pmatrix}$	$\begin{pmatrix} \varepsilon & 1 \\ 1 & -1+\varepsilon \end{pmatrix}$
2	$\begin{pmatrix} 2+3\varepsilon & 1+2\varepsilon \\ 1+2\varepsilon & 1+\varepsilon \end{pmatrix}$	$\begin{pmatrix} 1 & -1+\varepsilon \\ -1+\varepsilon & -\varepsilon+2 \end{pmatrix}$
3	$\begin{pmatrix} 3+5\varepsilon & 2+3\varepsilon \\ 2+3\varepsilon & 1+2\varepsilon \end{pmatrix}$	$\begin{pmatrix} -1+\varepsilon & -\varepsilon+2 \\ -\varepsilon+2 & -3+2\varepsilon \end{pmatrix}$
4	$\begin{pmatrix} 5+8\varepsilon & 3+5\varepsilon \\ 3+5\varepsilon & 2+3\varepsilon \end{pmatrix}$	$\begin{pmatrix} 2-\varepsilon & -3+2\varepsilon \\ -3+2\varepsilon & 5-3\varepsilon \end{pmatrix}$
5	$\begin{pmatrix} 8+13\varepsilon & 5+8\varepsilon \\ 5+8\varepsilon & 3+5\varepsilon \end{pmatrix}$	$\begin{pmatrix} -3+2\varepsilon & 5-3\varepsilon \\ 5-3\varepsilon & -8+5\varepsilon \end{pmatrix}$

Table 5. Dual Lucas matrices

n	$\widetilde{L_n}$	$\widetilde{L_{-n}}$
0	$\begin{pmatrix} 1+3\varepsilon & 2+\varepsilon \\ 2+\varepsilon & -1+2\varepsilon \end{pmatrix}$	$\begin{pmatrix} 1+3\varepsilon & 2+\varepsilon \\ 2+\varepsilon & -1+2\varepsilon \end{pmatrix}$
1	$\begin{pmatrix} 3+4\varepsilon & 1+3\varepsilon \\ 1+3\varepsilon & 2+\varepsilon \end{pmatrix}$	$\begin{pmatrix} +2+\varepsilon & -1+2\varepsilon \\ -1+2\varepsilon & 3-\varepsilon \end{pmatrix}$
2	$\begin{pmatrix} 7\varepsilon+4 & 4\varepsilon+3 \\ 4\varepsilon+3 & 3\varepsilon+1 \end{pmatrix}$	$\begin{pmatrix} -1+2\varepsilon & 3-\varepsilon \\ 3-\varepsilon & -4+3\varepsilon \end{pmatrix}$
3	$\begin{pmatrix} 7+11\varepsilon & 4+7\varepsilon \\ 4+7\varepsilon & 3+4\varepsilon \end{pmatrix}$	$\begin{pmatrix} 3-\varepsilon & -4+3\varepsilon \\ -4+3\varepsilon & 7-4\varepsilon \end{pmatrix}$
4	$\begin{pmatrix} 11+18\varepsilon & 7+11\varepsilon \\ 7+11\varepsilon & 4+7\varepsilon \end{pmatrix}$	$\begin{pmatrix} -4+3\varepsilon & 7-4\varepsilon \\ 7-4\varepsilon & -11+7\varepsilon \end{pmatrix}$
5	$\begin{pmatrix} 18+29\varepsilon & 11+18\varepsilon \\ 11+18\varepsilon & 7+11\varepsilon \end{pmatrix}$	$\begin{pmatrix} 7-4\varepsilon & -11+7\varepsilon \\ -11+7\varepsilon & 18-11\varepsilon \end{pmatrix}$

Now, we will state Binet's formula for the dual generalized Fibonacci matrices and in the rest of the paper, we fix the following notations:

$$\begin{aligned} \widetilde{\alpha} &= (1+\varepsilon\alpha)\begin{pmatrix} \alpha^1 & 1 \\ 1 & \alpha^{-1} \end{pmatrix}, \\ \widetilde{\beta} &= (1+\varepsilon\beta)\begin{pmatrix} \beta^1 & 1 \\ 1 & \beta^{-1} \end{pmatrix}. \end{aligned}$$

Note that we have the following identities:

$$\begin{aligned} \widetilde{\alpha}\widetilde{\beta} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ \widetilde{\alpha}^m\widetilde{\beta}^n &= (\widetilde{\alpha}\widetilde{\beta})\widetilde{\alpha}^{m-1}\widetilde{\beta}^{n-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ for all integers } m \neq 0, n \neq 0, \\ \widetilde{\alpha}^2 &= \frac{\alpha+2+2\varepsilon(3\alpha+1)}{\alpha+1}\begin{pmatrix} \alpha+1 & \alpha \\ \alpha & 1 \end{pmatrix}, \\ \widetilde{\beta}^2 &= \frac{\beta+2+2\varepsilon(3\beta+1)}{\beta+1}\begin{pmatrix} \beta+1 & \beta \\ \beta & 1 \end{pmatrix}, \\ \widetilde{\alpha}^3 &= \frac{\alpha+2+3\varepsilon(3\alpha+1)}{2\alpha+1}\begin{pmatrix} 4\alpha+3 & 3\alpha+1 \\ 3\alpha+1 & \alpha+2 \end{pmatrix}, \\ \widetilde{\beta}^3 &= \frac{\beta+2+3\varepsilon(3\beta+1)}{2\beta+1}\begin{pmatrix} 4\beta+3 & 3\beta+1 \\ 3\beta+1 & \beta+2 \end{pmatrix}. \end{aligned}$$

Theorem 2.1. *(Binet's Formula) For any integer n, the nth dual generalized Fibonacci matrix is*

$$DW_n = \frac{A\widetilde{\alpha}\alpha^n - B\widetilde{\beta}\beta^n}{\alpha - \beta}. \tag{11}$$

Proof. Using Binet's formula

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}$$

of the generalized Fibonacci numbers, we obtain

$$\begin{aligned} DW_n &= \begin{pmatrix} W_{n+1} + \varepsilon W_{n+2} & W_n + \varepsilon W_{n+1} \\ W_n + \varepsilon W_{n+1} & W_{n-1} + \varepsilon W_n \end{pmatrix} \\ &= \frac{1}{\alpha - \beta}\left(A\left(1 + \varepsilon\alpha\right)\begin{pmatrix} \alpha^1 & 1 \\ 1 & \alpha^{-1} \end{pmatrix}\alpha^n - B\left(1 + \varepsilon\beta\right)\begin{pmatrix} \beta^1 & 1 \\ 1 & \beta^{-1} \end{pmatrix}\beta^n\right) \\ &= \frac{1}{\alpha - \beta}(A\widetilde{\alpha}\alpha^n - B\widetilde{\beta}\beta^n) \end{aligned}$$

This proves (11). □

As special cases, for any integer n, the Binet's Formula of nth dual Fibonacci number is

$$DF_n = \frac{\widetilde{\alpha}\alpha^n - \widetilde{\beta}\beta^n}{\alpha - \beta}$$

and the Binet's Formula of nth dual Lucas number is

$$DL_n = \widetilde{\alpha}\alpha^n + \widetilde{\beta}\beta^n.$$

Next, we present generating function.

Theorem 2.2. *The generating function for the dual generalized Fibonacci matrices is*

$$\sum_{n=0}^{\infty} DW_n x^n = \frac{DW_0 + (DW_1 - DW_0)x}{1 - x - x^2}.$$

Proof. Let

$$g(x) = \sum_{n=0}^{\infty} DW_n x^n$$

be generating function of the dual generalized Fibonacci matrices. Then, using the definition of the dual generalized Fibonacci matrices, and substracting $xg(x)$ and $x^2g(x)$ from $g(x)$, we obtain (note the shift in the index n in the third line)

$$\begin{aligned} (1 - x - x^2)g(x) &= \sum_{n=0}^{\infty} DW_n x^n - x\sum_{n=0}^{\infty} DW_n x^n - x^2\sum_{n=0}^{\infty} DW_n x^n \\ &= \sum_{n=0}^{\infty} DW_n x^n - \sum_{n=0}^{\infty} DW_n x^{n+1} - \sum_{n=0}^{\infty} DW_n x^{n+2} \\ &= \sum_{n=0}^{\infty} DW_n x^n - \sum_{n=1}^{\infty} DW_{n-1} x^n - \sum_{n=2}^{\infty} DW_{n-2} x^n \\ &= (DW_0 + DW_1 x) - DW_0 x + \sum_{n=2}^{\infty}(DW_n - DW_{n-1} - DW_{n-2})x^n \\ &= (DW_0 + DW_1 x) - DW_0 x = DW_0 + (DW_1 - DW_0)x. \end{aligned}$$

Note that we used the recurrence relation $DW_n = DW_{n-1} + DW_{n-2}$. Rearranging above equation, we get

$$g(x) = \frac{DW_0 + (DW_1 - DW_0)x}{1 - x - x^2}.$$

□

As special cases, the generating functions for the dual Fibonacci and dual Lucas matrices are

$$\sum_{n=0}^{\infty} DF_n x^n = \frac{\begin{pmatrix} 1+\varepsilon & \varepsilon \\ \varepsilon & 1 \end{pmatrix} + \begin{pmatrix} \varepsilon & 1 \\ 1 & -1+\varepsilon \end{pmatrix} x}{1 - x - x^2}$$

and

$$\sum_{n=0}^{\infty} DL_n x^n = \frac{\begin{pmatrix} 1+3\varepsilon & 2+\varepsilon \\ 2+\varepsilon & -1+2\varepsilon \end{pmatrix} + \begin{pmatrix} 2+\varepsilon & -1+2\varepsilon \\ -1+2\varepsilon & 3-\varepsilon \end{pmatrix} x}{1 - x - x^2}$$

respectively.

3. Some Identities

We now present a few special identities for the dual generalized Fibonacci sequence $\{DW_n\}$. The following theorem presents the Catalan's identity for the dual generalized Fibonacci numbers.

Theorem 3.1. *(Catalan's identity) For all integers n and m, the following identity holds*

$$DW_{n+m} DW_{n-m} - DW_n^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. Using the Binet Formula

$$DW_n = \frac{A\widetilde{\alpha}\alpha^n - B\widetilde{\beta}\beta^n}{\alpha - \beta},$$

we get

$$\begin{aligned} DW_{n+m} DW_{n-m} - DW_n^2 &= -\beta^n \alpha^n AB \frac{(\alpha^m - \beta^m)^2}{\beta^m \alpha^m (\alpha - \beta)^2} \widetilde{\alpha\beta} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

since $\widetilde{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. □

As special cases of the above theorem, we give Catalan's identity of dual Fibonacci and dual Lucas matrices.

Corollary 3.2. *(Catalan's identity for the dual Fibonacci and Lucas matrices) For all integers n and m, the following identities hold*

$$\begin{aligned} DF_{n+m} DF_{n-m} - DF_n^2 &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ DL_{n+m} DL_{n-m} - DL_n^2 &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Note that for $m = 1$ in Catalan's identity, we get the Cassini's identity for the dual generalized Fibonacci matrices.

Corollary 3.3. *(Cassini's identity) For all integers n, the following identity holds*

$$DW_{n+1}DW_{n-1} - DW_n^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

As special cases of Cassini's identity, we give Cassini's identity of dual Fibonacci and dual Lucas matrices.

Corollary 3.4. *(Cassini's identity of dual Fibonacci and Lucas matrices) For all integers n, the following identities hold:*

$$\begin{aligned} DF_{n+1}DF_{n-1} - DF_n^2 &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ DL_{n+1}DL_{n-1} - DL_n^2 &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

The d'Ocagne's, Gelin-Cesàro's and Melham' identities can also be obtained by using the Binet's formula of the dual generalized Fibonacci matrices:

$$DW_n = \frac{A\widetilde{\alpha}\alpha^n - B\widetilde{\beta}\beta^n}{\alpha - \beta}.$$

The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of the dual generalized Fibonacci matrix sequence $\{DW_n\}$.

Theorem 3.5. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$DW_{m+1}DW_n - DW_mDW_{n+1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(b) *(Gelin-Cesàro's identity)*

$$DW_{n+2}DW_{n+1}DW_{n-1}DW_{n-2} - DW_n^4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(c) *(Melham's identity)*

$$DW_{n+1}DW_{n+2}DW_{n+6} - DW_{n+3}^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof.

(a) We obtain (a) since

$$DW_{m+1}DW_n - DW_mDW_{n+1} = \frac{AB\,(\alpha^n\beta^m - \alpha^m\beta^n)}{(\alpha - \beta)}\widetilde{\alpha\beta}$$

and since $\widetilde{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(b) We get (b) since

$$DW_{n+2}DW_{n+1}DW_{n-1}DW_{n-2} - DW_n^4 = -\frac{AB\alpha^n\beta^n}{(\alpha-\beta)^2}\frac{(\alpha^2+\beta^2+3\alpha\beta)(A^2\widetilde{\alpha}^2\alpha^{2n}+B^2\widetilde{\beta}^2\beta^{2n})}{\alpha^2\beta^2}\widetilde{\alpha\beta}$$

and $\widetilde{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(c) We obtain (c) since

$$DW_{n+1}DW_{n+2}DW_{n+6} - DW_{n+3}^3 = -AB\frac{(\alpha\beta)^{n+1}}{\alpha-\beta}\left(A\widetilde{\alpha}(1+\beta)\alpha^{n+2} - B\widetilde{\beta}(1+\alpha)\beta^{n+2}\right)\widetilde{\alpha\beta}$$

and since $\widetilde{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. □

As special cases of the above theorem, we give the d'Ocagne's, Gelin-Cesàro's and Melham' identities of dual Fibonacci and dual Lucas matrices. Firstly, we present the d'Ocagne's, Gelin-Cesàro's and Melham' identities of dual Fibonacci matrices.

Corollary 3.6. *Let n and m be any integers. Then, for the dual Fibonacci matrices, the following identities are true:*

(a) *(d'Ocagne's identity)*

$$DF_{m+1}DF_n - DF_mDF_{n+1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(b) *(Gelin-Cesàro's identity)*

$$DF_{n+2}DF_{n+1}DF_{n-1}DF_{n-2} - DF_n^4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(c) *(Melham's identity)*

$$DF_{n+1}DF_{n+2}DF_{n+6} - DF_{n+3}^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Secondly, we present the d'Ocagne's, Gelin-Cesàro's and Melham' identities of dual Lucas matrices.

Corollary 3.7. *Let n and m be any integers. Then, for the dual Lucas matrices, the following identities are true:*

(a) *(d'Ocagne's identity)*

$$DL_{m+1}DL_n - DL_mDL_{n+1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(b) *(Gelin-Cesàro's identity)*

$$DL_{n+2}DL_{n+1}DL_{n-1}DL_{n-2} - DL_n^4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(c) *(Melham's identity)*

$$DL_{n+1}DL_{n+2}DL_{n+6} - DL_{n+3}^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

4. Linear Sums

In this section, we give the summation formulas of the dual generalized Fibonacci matrices with positive and negative subscripts. Next, we present the formulas which give the summation of the dual generalized Fibonacci matrices.

Theorem 4.1. *For $n \geq 0$, dual generalized Fibonacci matrices have the following formulas:*

(a) $\sum_{k=0}^{n} DW_k = DW_{n+2} - DW_1$.

(b) $\sum_{k=0}^{n} DW_{2k} = DW_{2n+1} - DW_1 + DW_0$.

(c) $\sum_{k=0}^{n} DW_{2k+1} = DW_{2n+2} - DW_0$.

Proof. Use mathematical induction on n. □

As a first special case of the above theorem, we have the following summation formulas for dual Fibonacci matrices:

Corollary 4.2. *For $n \geq 0$, dual Fibonacci matrices have the following properties:*

(a) $\sum_{k=0}^{n} DF_k = DF_{n+2} - DF_1 = \begin{pmatrix} F_{n+3} - 1 + \varepsilon(F_{n+4} - 2) & F_{n+2} - 1 + \varepsilon(F_{n+3} - 1) \\ F_{n+2} - 1 + \varepsilon(F_{n+3} - 1) & F_{n+1} + \varepsilon(F_{n+2} - 1) \end{pmatrix}$.

(b) $\sum_{k=0}^{n} DF_{2k} = DF_{2n+1} - DF_1 + DF_0 = \begin{pmatrix} F_{2n+2} + \varepsilon(F_{2n+3} - 1) & F_{2n+1} - 1 + \varepsilon F_{2n+2} \\ F_{2n+1} - 1 + \varepsilon F_{2n+2} & F_{2n} + 1 + \varepsilon(F_{2n+1} - 1) \end{pmatrix}$.

(c) $\sum_{k=0}^{n} DF_{2k+1} = DF_{2n+2} - DF_0 = \begin{pmatrix} F_{2n+3} - 1 + \varepsilon(F_{2n+4} - 1) & F_{2n+2} + \varepsilon(F_{2n+3} - 1) \\ F_{2n+2} + \varepsilon(F_{2n+3} - 1) & F_{2n+1} - 1 + \varepsilon F_{2n+2} \end{pmatrix}$.

As a second special case of the above theorem, we have the following summation formulas for dual Lucas matrices:

Corollary 4.3. *For $n \geq 0$, dual Lucas matrices have the following properties.*

(a) $\sum_{k=0}^{n} DL_k = DL_{n+2} - DL_1 = \begin{pmatrix} L_{n+3} - 3 + \varepsilon(L_{n+4} - 4) & L_{n+2} - 1 + \varepsilon(L_{n+3} - 3) \\ L_{n+2} - 1 + \varepsilon(L_{n+3} - 3) & L_{n+1} - 2 + \varepsilon(L_{n+2} - 1) \end{pmatrix}$.

(b) $\sum_{k=0}^{n} DL_{2k} = DL_{2n+1} - DL_1 + DL_0 = \begin{pmatrix} L_{2n+2} - 2 + \varepsilon(L_{2n+3} - 1) & L_{2n+1} + 1 + \varepsilon(L_{2n+2} - 2) \\ L_{2n+1} + 1 + \varepsilon(L_{2n+2} - 2) & L_{2n} - 3 + \varepsilon(L_{2n+1} + 1) \end{pmatrix}$.

(c) $\sum_{k=0}^{n} DL_{2k+1} = DL_{2n+2} - DL_0 = \begin{pmatrix} L_{2n+3} - 1 + \varepsilon(L_{2n+4} - 3) & L_{2n+2} - 2 + \varepsilon(L_{2n+3} - 1) \\ L_{2n+2} - 2 + \varepsilon(L_{2n+3} - 1) & L_{2n+1} + 1 + \varepsilon(L_{2n+2} - 2) \end{pmatrix}$.

5. Determinant, Inverse, Eigenvalues and Eigenvectors of Dual Generalized Fibonacci Matrices

Note that we get

$$W_{n+1}^2 - W_n^2 - W_{n+1}W_n = -W_n^2 + W_{n-1}W_{n+1} = W_{n-1}W_{n+2} - W_nW_{n+1}$$

by using the identities

$$\begin{aligned} W_{n+2} &= W_{n+1} + W_n, \\ W_{n-1} &= W_{n+1} - W_n. \end{aligned}$$

Next, we present determinant, inverse, eigenvalues and eigenvectors of DW_n.

Theorem 5.1. *For all integers n the following properties hold:*

(a) *Determinant of DW_n is*

$$\begin{aligned}\det(DW_n) &= (\varepsilon+1)(-W_n^2+W_{n-1}W_{n+1})\\ &= (\varepsilon+1)(W_{n+1}^2-W_n^2-W_{n+1}W_n).\end{aligned}$$

(b) *Inverse of DW_n is*

$$\begin{aligned}(DW_n)^{-1} &= \frac{1}{(\varepsilon+1)(-W_n^2+W_{n-1}W_{n+1})}\begin{pmatrix} W_{n-1}+\varepsilon W_n & -W_n-\varepsilon W_{n+1}\\ -W_n-\varepsilon W_{n+1} & W_{n+1}+\varepsilon W_{n+2}\end{pmatrix}\\ &= \frac{1}{(\varepsilon+1)(W_{n+1}^2-W_n^2-W_{n+1}W_n)}\begin{pmatrix} W_{n+1}-W_n+\varepsilon W_n & -W_n-\varepsilon W_{n+1}\\ -W_n-\varepsilon W_{n+1} & W_{n+1}+\varepsilon(W_{n+1}+W_n)\end{pmatrix}\end{aligned}$$

provided that $(\varepsilon+1)(W_{n+1}^2-W_n^2-W_{n+1}W_n)=(\varepsilon+1)(-W_n^2+W_{n-1}W_{n+1})\neq 0.$

(c) *Eigenvalues of DW_n are given as*

$$\begin{aligned}\lambda_1 &= \frac{(2W_{n+1}+(\sqrt{5}-1)W_n)+\varepsilon((\sqrt{5}+1)W_{n+1}+2W_n)}{2}\\ &= \left(\frac{\sqrt{5}+1}{4}\right)\left(\varepsilon+\frac{-1+\sqrt{5}}{2}\right)\left(2W_{n+1}+(\sqrt{5}-1)W_n\right)\end{aligned}$$

and

$$\begin{aligned}\lambda_2 &= \frac{(2W_{n+1}+(-\sqrt{5}-1)W_n)+\varepsilon((-\sqrt{5}+1)W_{n+1}+2W_n)}{2}\\ &= \left(\frac{\sqrt{5}-1}{4}\right)\left(\varepsilon+\frac{-1-\sqrt{5}}{2}\right)\left(-2W_{n+1}+(\sqrt{5}+1)W_n\right).\end{aligned}$$

(d) *Eigenvalues of DW_n associated to eigenvalues λ_1 and λ_2, respectively, are*

$$\mu_1=\left\{\begin{array}{c}\frac{1+\sqrt{5}}{2}\\ 1\end{array}\right\}$$

and

$$\mu_2=\left\{\begin{array}{c}\frac{1-\sqrt{5}}{2}\\ 1\end{array}\right\}.$$

6. Declaration of competing interest

There is no competing interest.

7. Data availability

No data was used for the research described in the article.

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