Filomat 39:1 (2025), 67–82
https://doi.org/10.2298/FIL2501067K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Study on HQBS-Submersions from almost quaternionic Hermitian manifolds

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Abstract. In this artical, h-quasi bi-slant submersions ($h - qbs$-submersions, in short) from almost quaternionic Hermitian manifolds onto Riemannian manifolds are defined and studied. In addition, the integrability of distributions, the geometry of foliations, the condition for such maps to be totally geodesic are investigated. Moreover, we have also worked out some non-trivial examples of the *hqbs*-submersion.

1. Introductions

The study of Riemannian submersions originated from the work of O'Neill [15] in 1966 and Gray [7] in 1967. Watson [27] studied almost Hermitian submersions. After that, this notion of almost Hermitian submersion has been extended to different kinds of sub-classes, according to the conditions on submersion by several geometers ([1], [10], [13], [16]-[19], [22], [25], [26]).

Quasi-bi-slant submersions are introduced by Prasad and others ([21], [23], [24]), and quasi-hemi-slant Riemannian submersions are studied by Longwap, Massamba and Homti [14].

Riemannian submersions have wide applications as: in Kaluza-Klein theory ([4], [11]), the Yang-Mills theory [5], Supergravity and superstring theories [12], robotic chains [2], etc. On the other hand, quaternionic manifolds have many applications in non-linear sigma-models with super symmetry [6], to obtain estimates for the Betti numbers of the manifold ([8],[9]) etc.

The this paper is organized as follows: In Section 2, basic definitions and properties of Riemannian submersions are mentioned. In Section 3, the definition of a h-qbs submersion is given, and its geometric properties are investigated. In addition, integrability of distributions and totally geodesic are also obtained. In the last section, proper examples for this notion are provided.

2020 *Mathematics Subject Classification*. Primary 53C15; Secondary 52C22, 53C26, 53C55, 55D17.
Keywords. Riemannian submersions, h-Quasi bi-slant submersions, integrable and totally geodesic.
Received: 09 January 2024; Accepted: 19 October 2024
Communicated by Ljubica Velimirović
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2. Preliminaries

Let $F : (N_1, g_{N_1}) \to (N_2, g_{N_2})$ be a Riemannian submersion [25]. Define O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ [15] by

$$\mathcal{A}_{Y_1}Y_2 = \mathcal{H}\nabla_{\mathcal{H}Y_1}\mathcal{V}Y_2 + \mathcal{V}\nabla_{\mathcal{H}Y_1}\mathcal{H}Y_2, \tag{1}$$

$$\mathcal{T}_{Y_1}Y_2 = \mathcal{H}\nabla_{\mathcal{V}Y_1}\mathcal{V}Y_2 + \mathcal{V}\nabla_{\mathcal{V}Y_1}\mathcal{H}Y_2 \tag{2}$$

for any vector fields Y_1, Y_2 on N_1, where ∇ is the Levi-Civita connection of g_{N_1}. It is easy to see that $\mathcal{T}_{Y_1}$ and $\mathcal{A}_{Y_1}$ are skew-symmetric operators on the tangent bundle of N_1 reversing the vertical and the horizontal distributions.

From (1) and (2), we get

$$\nabla_{Z_1}Z_2 = \mathcal{T}_{Z_1}Z_2 + \mathcal{V}\nabla_{Z_1}Z_2, \tag{3}$$

$$\nabla_{Z_1}U_1 = \mathcal{T}_{Z_1}U_1 + \mathcal{H}\nabla_{Z_1}U_1, \tag{4}$$

$$\nabla_{U_1}Z_1 = \mathcal{A}_{U_1}Z_1 + \mathcal{V}\nabla_{U_1}Z_1, \tag{5}$$

$$\nabla_{U_1}U_2 = \mathcal{H}\nabla_{U_1}U_2 + \mathcal{A}_{U_1}U_2 \tag{6}$$

for $Z_1, Z_2 \in \Gamma(\ker F_*)$ and $U_1, U_2 \in \Gamma(\ker F_*)^\perp$, where $\mathcal{H}\nabla_{Z_1}U_1 = \mathcal{A}_{U_1}Z_1$, if U_1 is basic [3]. Let (N_1, g_{N_1}) and (N_2, g_{N_2}) be Riemannian manifolds and $F : (N_1, g_{N_1}) \to (N_2, g_{N_2})$ be a C^∞ map then the second fundamental form of F is given by

$$(\nabla F_*)(Z_1, X_2) = \nabla^F_{Z_1}F_*(X_2) - F_*(\nabla^{N_1}_{Z_1}X_2) \tag{7}$$

for $Z_1, X_2 \in \Gamma(TN_1)$, where ∇^F is the pullback connection, and ∇ is the Riemannian connections of the metric g_{N_1}.

In addition, a differentiable map F between two Riemannian manifolds is totally geodesic if

$$(\nabla F_*)(Y_1, Z_2) = 0, \text{ for all } Y_1, Z_2 \in \Gamma(TN_1). \tag{8}$$

Lemma 2.1. *[3] Let (N_1, g_{N_1}) and (N_2, g_{N_2}) are Riemannian manifolds. If $F : N_1 \to N_2$ be a Riemannian submersion, then for any horizontal vector fields Z_1, Z_2 and vertical vector fields W_1, W_2 we have*

(*i*) $(\nabla F_*)(Z_1, Z_2) = 0$,
(*ii*) $(\nabla F_*)(W_1, W_2) = -F_*(\mathcal{T}_{W_1}W_2) = -F_*(\nabla_{W_1}W_2)$,
(*iii*) $(\nabla F_*)(Z_1, W_1) = -F_*(\nabla_{Z_1}W_1) = -F_*(\mathcal{A}_{Z_1}W_1)$.

Definition 2.2. *[20] Let (N_1, g_{N_1}, J) be an almost Hermitian manifold and (N_2, g_{N_2}) be a Riemannian manifold. A Riemannian submersion $F : N_1 \to N_2$ is called a quasi bi-slant Riemannian submersion (h-qbs submersions, in short) if there exist three mutually orthogonal distributions D, D_1 and D_2 such that*

(*i*) $\ker F_* = D \oplus D_1 \oplus D_2$,
(*ii*) $J(D) = D$ i.e., D is invariant,
(*iii*) $J(D_1) \perp D_2$ and $J(D_2) \perp D_1$,
(*iv*) for any non-zero vector field $Y_1 \in (D_1)_x$, $x \in N_1$, the angle θ_1 between JY_1 and $(D_1)_x$ is constant and independent of the choice of point x and Y_1 in $(D_1)_x$,
(*v*) for any non-zero vector field $Z_1 \in (D_2)_y$, $y \in N_1$, the angle θ_2 between JZ_1 and $(D_2)_y$ is constant and independent of the choice of point y and Z_1 in $(D_2)_y$,

These angles θ_1 and θ_2 are called slant angles of the submersion.

Let (N_1, E, g_{N_1}) is an almost quaternionic Hermitian manifold [10] with local basis $\{J_1, J_2, J_3\}$ of sections of E and 1-forms $\omega_1, \omega_2, \omega_3$ such that for $\alpha \in \{1, 2, 3\}$ on U

$$J_\alpha^2 = -id, \; J_\alpha J_{\alpha+1} = -J_{\alpha+1}J_\alpha = J_{\alpha+2}, \tag{9}$$

$$g_{N_1}(J_\alpha Z_1, J_\alpha Z_2) = g_{N_1}(Z_1, Z_2), \tag{10}$$

$$\nabla_{Z_1}J_\alpha = \omega_{\alpha+2}(Z_1)J_{\alpha+1} - \omega_{\alpha+1}(Z_1)J_{\alpha+2} \tag{11}$$

$\forall\, Z_1, Z_2 \in \Gamma(TN_1)$.

3. H-Quasi bi-slant submersions

In this section, h-qbs submersions F from an almost quaternionic Hermitian manifold (N_1, I, J, K, g_{N_1}) onto a Riemannian manifold (N_2, g_{N_2}) is defined and studied.

Definition 3.1. *$F : (N_1, E, g_{N_1}) \to (N_2, g_{N_2})$ is said to be an h-qbs submersion if $q \in N_1$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U so that for any $R \in \{I, J, K\}$, there is a distribution $D \subset (\ker F_*)$ on U that satisfies the condition*

(a) $\ker F_* = D \oplus D_1 \oplus D_2$,

(b) $R(D) = D$ i.e., D is invariant,

(c) $R(D_1) \perp D_2$ and $R(D_2) \perp D_1$

(d) for any non-zero vector field $Y_1 \in (D_1)_x$, $x \in N_1$, the angle θ_R^1 between RY_1 and $(D_1)_x$ is constant and independent of the choice of point x and Y_1 in $(D_1)_x$.

(e) for any non-zero vector field $Y_2 \in (D_2)_y$, $y \in N_1$, the angle θ_R^2 between RY_2 and $(D_2)_y$ is constant and independent of the choice of point y and Y_2 in $(D_2)_y$,

Where $(\ker F_*)$ admits three orthogonal complementary distributions D, D_1 and D_2 such that D is invariant, D_1 is slant with angle θ_R^1 and D_2 is slant with angle θ_R^2.

We call basis $\{I, J, K\}$ is said to be an h-qbs basis and the angles $\{\theta_I^1, \theta_J^1, \theta_K^1\}$ and $\{\theta_I^2, \theta_J^2, \theta_K^2\}$ are said to be h-qbs angles.

Moreover, if

$$\theta_I^1 = \theta_J^1 = \theta_K^1 = \theta^1, \theta_I^2 = \theta_J^2 = \theta_K^2 = \theta^2,$$

then we call $F : (N_1, E, g_{N_1}) \to (N_2, g_{N_2})$ a strictly h-qbs submersion, $\{I, J, K\}$ strictly h-qbs basis and θ^2 strictly h-qbs angle.

Definition 3.2. *$F : (N_1, E, g_{N_1}) \to (N_2, g_{N_2})$ is said to be an almost h-qbs submersion if $p \in N_1$ with a neighborhood U, there exists a quaternionic Hermitian $\{I, J, K\}$ of sections of E on U such that for any $R \in \{I, J, K\}$, there is a distribution $D^R \subset (\ker F_*)$ on U such that*

(a) $\ker F_* = D^R \oplus D_1^R \oplus D_2^R$,

(b) $R(D^R) = D^R$ i.e., D^R is invariant,

(c) $R(D_1^R) \perp D_2^R$ and $R(D_2^R) \perp D_1^R$,

(d) for any non-zero vector field $V_1 \in (D_1^R)_x$, $x \in N_1$, the angle θ_R^1 between RV_1 and $(D_1^R)_x$ is constant and independent of the choice of point x and V_1 in $(D_1)_x$,

(e) for any non-zero vector field $V_2 \in (D_2^R)_y$, $y \in N_1$, the angle θ_R^2 between RV_2 and $(D_2^R)_y$ is constant and independent of the choice of point y and V_2 in $(D_2^R)_y$,

Where the vertical distribution $(\ker F_*)$ admits three orthogonal complementary distributions D^R, D_1^R and D_2^R such that D^R is invariant, D_1^R is slant with angle θ_R^1 and D_2^R is slant with angle θ_R^2.

We call such basis $\{I, J, K\}$ an almost h-qbs basis and the angles $\{\theta_I^1, \theta_J^1, \theta_K^1\}$ and $\{\theta_I^2, \theta_J^2, \theta_K^2\}$ almost h-qbs angles.

Let F be h-qbs submersion from an almost quaternionic Hermitian manifold (N_1, I, J, K, g_{N_1}) onto a Riemannian manifold (N_2, g_{N_2}). Then, we have

$$TN_1 = \ker F_* \oplus (\ker F_*)^{\perp}. \tag{12}$$

Now, for $Z_1 \in \Gamma(\ker F_*)$, we put

$$Z_1 = P_R Z_1 + Q_R Z_1 + S_R Z_1, \tag{13}$$

where P, Q and S are projection morphisms of $\ker F_*$ onto D^R, D_1^R and D_2^R, respectively.

For $U_1 \in (\Gamma \ker F_*)$, we set

$$RU_1 = \phi_R U_1 + \omega_R U_1, \tag{14}$$

where $\phi_R U_1 \in (\Gamma \ker F_*)$ and $\omega_R U_1 \in (\Gamma \ker F_*)^\perp$.

From (13) and (14), we get

$$\begin{aligned} RZ_1 &= R(PZ_1) + R(QZ_1) + R(SZ_1), \\ &= \phi_R(PZ_1) + \omega_R(PZ_1) + \phi_R(QZ_1) + \omega_R(QZ_1) + \phi_R(SZ_1) + \omega_R(SZ_1). \end{aligned}$$

Since $RD^R = D^R$, we get $\omega_R PZ_1 = 0$.

Hence above equation reduces to

$$RZ_1 = \phi_R(PZ_1) + \phi_R QZ_1 + \omega_R QZ_1 + \phi_R SZ_1 + \omega_R SZ_1. \tag{15}$$

Thus we get

$$R(\ker F_*) = D^R \oplus (\phi D_1^R \oplus \phi D_2^R) \oplus (\omega D_1^R \oplus \omega D_2^R), \tag{16}$$

where $\oplus$ denotes orthogonal direct sum.

Further, let $W_1 \in \Gamma(D_1^R)$ and $W_2 \in \Gamma(D_2^R)$. Then $g_{N_1}(W_1, W_2) = 0$.

From definition 3.2(c), we have

$$\begin{aligned} g_{N_1}(RW_1, W_2) &= g_{N_1}(W_1, RW_2) = 0, \\ g_{N_1}(\phi_R W_1, W_2) &= g_{N_1}(W_1, \phi_R W_2) = 0. \end{aligned}$$

Let $V_1 \in \Gamma(D^R)$ and $U_1 \in \Gamma(D_1^R)$. Then, we have

$$g_{N_1}(\phi_R U_1, V_1) = 0,$$

as D^R is invariant i.e., $RV_1 \in \Gamma(D^R)$.

Similarly, for $V_1 \in \Gamma(D^R)$ and $V_2 \in \Gamma(D_2^R)$, we obtain

$$g_{N_1}(\phi_R V_2, V_1) = 0,$$

From above equations, we have

$$g_{N_1}(\phi_R Z_1, \phi_R Z_2) = 0, g_{N_1}(\omega_R Z_1, \omega_R Z_2) = 0,$$

for all $Z_1 \in \Gamma(D_1^R)$ and $Z_2 \in \Gamma(D_2^R)$.

So, we can write

$$\phi D_1^R \cap \phi D_2^R = \{0\}, \omega D_1^R \cap \omega D_2^R = \{0\}.$$

If $\theta_2 = \frac{\pi}{2}$, then $\phi_R S = 0$ and D_2^R is anti-invariant, i.e., $R(D_2^R) \subseteq (\ker F_*)^\perp$. In this instance, D_2^R is denoted by $(D^R)^\perp$. In addition, we get

$$R(\ker F_*) = D^R \oplus \phi_R D_1^R \oplus \omega_R D_1^R \oplus R(D^R)^\perp. \tag{17}$$

Since $\omega D_1^R \subseteq (\ker F_*)^\perp$, $\omega D_2^R \subseteq (\ker F_*)^\perp$. So we can write

$$(\ker F_*)^\perp = \omega_R D_1^R \oplus \omega_R D_2^R \oplus \mu,$$

where μ is orthogonal complement of $(\omega D_1^R \oplus \omega D_2^R)$ in $(\ker F_*)^\perp$. Also for $X_1 \in \Gamma(\ker F_*)^\perp$, we get

$$RX_1 = B_R X_1 + C_R X_1, \tag{18}$$

where $B_R X_1 \in \Gamma(\ker F_*)$ and $C_R X_1 \in \Gamma(\ker F_*)^\perp$.

We will denote a submersion from an almost quaternionic Hermitian manifold (N_1, I, J, K, g_{N_1}) onto a Riemannian manifold (N_2, g_{N_2}) such that (I, J, K) is an almost h-qbs basis by F.

Lemma 3.3. *If F be a h-qbs submersion then we have*

$$\phi_R^2 V_1 + B_R\omega_R V_1 = -V_1, \omega_R\phi_R V_1 + C_R\omega_R V_1 = 0,$$

$$\omega_R B_R V_2 + C_R^2 V_2 = -V_2, \phi_R B_R V_2 + B_R C_R V_2 = 0,$$

for all $V_1 \in \Gamma(\ker F_*)$ and $V_2 \in \Gamma(\ker F_*)^\perp$ and $R \in \{I, J, K\}$.

Proof. Using (9), (14) and (18) , we have Lemma 3.3. □

The proof of the following Lemma is the same as Lemma in [20] so, we skip the proof.

Lemma 3.4. *If F be an almost h-qbs submersion then we have*

(*i*) $\phi_R^2 V_i = -(\cos^2\theta_R^i)V_i$,
(*ii*) $g_{N_1}(\phi_R V_i, \phi_R Z_i) = \cos^2\theta_R^i g_{N_1}(V_i, Z_i)$,
(*iii*) $g_{N_1}(\omega_R V_i, \omega_R Z_i) = \sin^2\theta_R^i g_{N_1}(V_i, Z_i)$,
for all $V_i, Z_i \in \Gamma(D_i^R)$, where $i = 1, 2$ and $R \in \{I, , J, K\}$.

Lemma 3.5. *If F be a h-qbs submersion then we get*

$$\mathcal{V}\nabla_{V_1}\phi_R V_2 + \mathcal{T}_{V_1}\omega_R V_2 = \phi_R\mathcal{V}\nabla_{V_1}V_2 + B_R\mathcal{T}_{V_1}V_2, \tag{19}$$

$$\mathcal{T}_{V_1}\phi_R V_2 + \mathcal{H}\nabla_{V_1}\omega_R V_2 = \omega_R\mathcal{V}\nabla_{V_1}V_2 + C_R\mathcal{T}_{V_1}V_2, \tag{20}$$

$$\mathcal{V}\nabla_{Z_1}B_R Z_2 + \mathcal{A}_{Z_1}C_R Z_2 = \phi_R\mathcal{A}_{Z_1}Z_2 + B_R\mathcal{H}\nabla_{Z_1}Z_2, \tag{21}$$

$$\mathcal{A}_{Z_1}B_R Z_2 + \mathcal{H}\nabla_{Z_1}C_R Z_2 = \omega_R\mathcal{A}_{Z_1}Z_2 + C_R\mathcal{H}\nabla_{Z_1}Z_2, \tag{22}$$

$$\mathcal{V}\nabla_{V_1}B_R Z_1 + \mathcal{T}_{V_1}C_R Z_1 = \phi_R\mathcal{T}_{V_1}Z_1 + B_R\mathcal{H}\nabla_{V_1}Z_1, \tag{23}$$

$$\mathcal{T}_{V_1}B_R Z_1 + \mathcal{H}\nabla_{V_1}C_R Z_1 = \omega_R\mathcal{T}_{V_1}Z_1 + C_R\mathcal{H}\nabla_{V_1}Z_1 \tag{24}$$

$$\mathcal{V}\nabla_{Z_1}\phi_R V_1 + \mathcal{A}_{Z_1}\omega_R V_1 = B_R\mathcal{A}_{Z_1}V_1 + \phi_R\mathcal{V}\nabla_{Z_1}V_1, \tag{25}$$

$$\mathcal{A}_{Z_1}\phi_R V_1 + \mathcal{H}\nabla_{Z_1}\omega_R V_1 = C_R\mathcal{A}_{Z_1}V_1 + \omega_R\mathcal{V}\nabla_{Z_1}V_1 \tag{26}$$

for any $V_1, V_2 \in \Gamma(\ker F_*)$ and $Z_1, Z_2 \in \Gamma(\ker F_*)^\perp$.

Proof. Using (3)-(6), (14) and (18), we get (19)-(26). □

Now, we define

$$(\nabla_{Z_1}\phi_R)Z_2 = \mathcal{V}\nabla_{Z_1}\phi_R Z_2 - \phi_R\mathcal{V}\nabla_{Z_1}Z_2, \tag{27}$$

$$(\nabla_{Z_1}\omega_R)Z_2 = \mathcal{H}\nabla_{Z_1}\omega_R Z_2 - \omega_R\mathcal{V}\nabla_{Z_1}Z_2, \tag{28}$$

$$(\nabla_{W_1}C_R)W_2 = \mathcal{H}\nabla_{W_1}C_R W_2 - C_R\mathcal{H}\nabla_{W_1}W_2, \tag{29}$$

$$(\nabla_{W_1}B_R)W_2 = \mathcal{V}\nabla_{W_1}B_R W_2 - B_R\mathcal{H}\nabla_{W_1}W_2, \tag{30}$$

for any $Z_1, Z_2 \in \Gamma(\ker F_*)$ and $W_1, W_2 \in \Gamma(\ker F_*)^\perp$.

Lemma 3.6. *If F be an almost h-qbs submersion then we get*

$$(\nabla_{Z_1}\phi_R)Z_2 = B_R\mathcal{T}_{Z_1}Z_2 - \mathcal{T}_{Z_1}\omega_R Z_2,$$

$$(\nabla_{Z_1}\omega_R)Z_2 = C_R\mathcal{T}_{Z_1}Z_2 - \mathcal{T}_{Z_1}\phi_R Z_2,$$

$$(\nabla_{W_1}C_R)W_2 = \omega_R\mathcal{A}_{W_1}W_2 - \mathcal{A}_{W_1}B_R W_2,$$

$$(\nabla_{W_1}B_R)W_2 = \phi_R\mathcal{A}_{W_1}W_2 - \mathcal{A}_{W_1}C_R W_2$$

for any vectors $Z_1, Z_2 \in \Gamma(\ker F_*)$ and $W_1, W_2 \in \Gamma(\ker F_*)^\perp$.

Proof. Using (19)−(22) and (27)−(30) we get Lemma 3.6. □

If the tensors ϕ_R and ω_R are parallel with respect to the linear connection ∇ on N_1, respectively, then

$$B_R\mathcal{T}_{Z_1}Z_2 = \mathcal{T}_{Z_1}\omega_R Z_2, C_R\mathcal{T}_{Z_1}Z_2 = \mathcal{T}_{Z_1}\phi_R Z_2,$$

for any $Z_1, Z_2 \in \Gamma(TN_1)$.

We will denote a h-qbs submersion from a hyperkähler manifold (N_1, I, J, K, g_{N_1}) onto a Riemannian manifold (N_2, g_{N_2}) such that (I, J, K) is a h-qbs basis by F.

Theorem 3.7. *For F, the following conditions are equivalent:*

(a) invariant distribution D^R is integrable.

(b)

$$\begin{aligned} & g_{N_1}(\mathcal{T}_{X_2}\phi_I X_1 - \mathcal{T}_{X_1}\phi_I X_2, \omega_I QV_1 + \omega_I SV_1) \\ = \; & g_{N_1}(\mathcal{V}\nabla_{X_1}\phi_I X_2 - \mathcal{V}\nabla_{X_2}\phi_I X_1, \phi_I QV_1 + \phi_I SV_1) \end{aligned}$$

for $X_1, X_2 \in \Gamma(D^I)$ and $V_1 \in \Gamma(D_1^I \oplus D_2^I)$.

(c)

$$\begin{aligned} & g_{N_1}(\mathcal{T}_{X_2}\phi_J X_1 - \mathcal{T}_{X_1}\phi_J X_2, \omega_J QV_1 + \omega_J SV_1) \\ = \; & g_{N_1}(\mathcal{V}\nabla_{X_1}\phi_J X_2 - \mathcal{V}\nabla_{X_2}\phi_J X_1, \phi_J QV_1 + \phi_J SV_1) \end{aligned}$$

for $X_1, X_2 \in \Gamma(D^J)$ and $V_1 \in \Gamma(D_1^J \oplus D_2^J)$.

(d)

$$\begin{aligned} & g_{N_1}(\mathcal{T}_{X_2}\phi_K X_1 - \mathcal{T}_{X_1}\phi_K X_2, \omega_K QV_1 + \omega_K SV_1) \\ = \; & g_{N_1}(\mathcal{V}\nabla_{X_1}\phi_K X_2 - \mathcal{V}\nabla_{X_2}\phi_K X_1, \phi_K QV_1 + \phi_K SV_1) \end{aligned}$$

for $X_1, X_2 \in \Gamma(D^K)$ and $V_1 \in \Gamma(D_1^K \oplus D_2^K)$.

Proof. For $X_1, X_2 \in \Gamma(D^R)$, $V_1 \in \Gamma(D_1^R \oplus D_2^R)$ and $R \in \{I, J, K\}$. Using equations (3), (10), (13) and (14), we have

$$\begin{aligned} & g_{N_1}([X_1, X_2], V_1) \\ = \; & g_{N_1}(\nabla_{X_1}RX_2, RV_1) - g_{N_1}(\nabla_{X_2}RX_1, RV_1), \\ = \; & g_{N_1}(\nabla_{X_1}\phi_R X_2, RV_1) - g_{N_1}(\nabla_{X_2}\phi_R X_1, RV_1), \\ = \; & g_{N_1}(\mathcal{T}_{X_1}\phi_R X_2 - \mathcal{T}_{X_2}\phi_R X_1, \omega_R QV_1 + \omega_R SV_1) - \\ & g_{N_1}(\mathcal{V}\nabla_{X_1}\phi_R X_2 - \mathcal{V}\nabla_{X_2}\phi_R X_1, \phi_R QV_1 + \phi_R SV_1), \end{aligned}$$

which completes the proof. □

Theorem 3.8. *For F, the following conditions are equivalent:*

(a) invariant distribution D_1^R is integrable.
(b)

$$\begin{aligned} & g_{N_1}(\mathcal{T}_{X_1}\omega_I\phi_I Y_2 - \mathcal{T}_{Y_2}\omega_I\phi_I X_1, W_2) \\ = \; & g_{N_1}(\mathcal{T}_{X_1}\omega_I Y_2 - \mathcal{T}_{Y_2}\omega_I X_1, IPW_2 + \phi_I SW_2) + \\ & g_{N_1}(\mathcal{H}\nabla_{X_1}\omega_I Y_2 - \mathcal{H}\nabla_{Y_2}\omega_I X_1, \omega_I SW_2) \end{aligned}$$

for all $X_1, Y_2 \in \Gamma(D_1^I)$ and $W_2 \in \Gamma(D^I \oplus D_2^I)$.
(c)

$$\begin{aligned} & g_{N_1}(\mathcal{T}_{X_1}\omega_J\phi_J Y_2 - \mathcal{T}_{Y_2}\omega_J\phi_J X_1, W_2) \\ = \; & g_{N_1}(\mathcal{T}_{X_1}\omega_J Y_2 - \mathcal{T}_{Y_2}\omega_J X_1, JPW_2 + \phi_J SW_2) + \\ & g_{N_1}(\mathcal{H}\nabla_{X_1}\omega_J Y_2 - \mathcal{H}\nabla_{Y_2}\omega_J X_1, \omega_J SW_2) \end{aligned}$$

for all $X_1, Y_2 \in \Gamma(D_1^J)$ and $W_2 \in \Gamma(D^J \oplus D_2^J)$.
(d)

$$\begin{aligned} & g_{N_1}(\mathcal{T}_{X_1}\omega_K\phi_R Y_2 - \mathcal{T}_{Y_2}\omega_K\phi_K X_1, W_2) \\ = \; & g_{N_1}(\mathcal{T}_{X_1}\omega_K Y_2 - \mathcal{T}_{Y_2}\omega_K X_1, RPW_2 + \phi_K SW_2) + \\ & g_{N_1}(\mathcal{H}\nabla_{X_1}\omega_K Y_2 - \mathcal{H}\nabla_{Y_2}\omega_K X_1, \omega_K SW_2) \end{aligned}$$

for all $X_1, Y_2 \in \Gamma(D_1^K)$ and $W_2 \in \Gamma(D^K \oplus D_2^K)$.

Proof. For $X_1, Y_2 \in \Gamma(D_1^R)$ and $W_2 \in \Gamma(D^R \oplus D_2^R)$, we have

$$g_{N_1}([X_1, Y_2], W_2) = g_{N_1}(\nabla_{X_1} Y_2, W_2) - g_{N_1}(\nabla_{Y_2} X_1, W_2).$$

Using equations (3), (4), (10), (13), (14) and Lemma 3.4, we get

$$\begin{aligned} & g_{N_1}([X_1, Y_2], W_2) \\ = \; & g_{N_1}(\nabla_{X_1} RY_2, RW_2) - g_{N_1}(\nabla_{Y_2} RX_1, RW_2), \\ = \; & g_{N_1}(\nabla_{X_1}\phi_R Y_2, RW_2) + g_{N_1}(\nabla_{X_1}\omega_R Y_2, RW_2) - g_{N_1}(\nabla_{Y_2}\phi_R X_1, RW_2) - \\ & g_{N_1}(\nabla_{Y_2}\omega_R X_1, RW_2), \\ = \; & \cos^2\theta_R^1 g_{N_1}(\nabla_{X_1} Y_2, W_2) - \cos^2\theta_R^1 g_{N_1}(\nabla_{Y_2} X_1, W_2) - g_{N_1}(\mathcal{T}_{X_1}\omega_R\phi_R Y_2 - \\ & \mathcal{T}_{Y_2}\omega_R\phi_R X_1, W_2) + g_{N_1}(\mathcal{H}\nabla_{X_1}\omega_R Y_2 + \mathcal{T}_{X_1}\omega_R Y_2, RPW_2 + \phi_R SW_2 + \omega_R SW_2) \\ & -g_{N_1}(\mathcal{H}\nabla_{Y_2}\omega_R X_1 + \mathcal{T}_{Y_2}\omega_R X_1, RPW_2 + \phi_R SW_2 + \omega_R SW_2). \end{aligned}$$

Now, we have

$$\begin{aligned} & \sin^2\theta_R^1 g_{N_1}([X_1, Y_2], W_2) \\ = \; & g_{N_1}(\mathcal{T}_{X_1}\omega_R Y_2 - \mathcal{T}_{Y_2}\omega_R X_1, RPW_2 + \phi_R SW_2) + \\ & g_{N_1}(\mathcal{H}\nabla_{X_1}\omega_R Y_2 - \mathcal{H}\nabla_{Y_2}\omega_R X_1, \omega_R SW_2) - \\ & g_{N_1}(\mathcal{T}_{X_1}\omega_R\phi_R Y_2 - \mathcal{T}_{Y_2}\omega_R\phi_R X_1, W_2), \end{aligned}$$

which completes the proof. □

As above theorem one can easily obtain the following theorem:

Theorem 3.9. *For F, the following conditions are equivalent:*

(a) the slant distribution D_2^R is integrable.
(b)

$$\begin{aligned} & g_{N_1}(\mathcal{T}_{V_1}\omega_I\phi_I V_2 - \mathcal{T}_{V_2}\omega_I\phi_I V_1, Z_1) \\ = \;& g_{N_1}(\mathcal{H}\nabla_{V_1}\omega_I V_2 - \mathcal{H}\nabla_{V_2}\omega_I V_1, \omega_I S Z_1) + \\ & g_{N_1}(\mathcal{T}_{V_1}\omega_I V_2 - \mathcal{T}_{V_2}\omega_I V_1, IPZ_1 + \phi_I S Z_1) \end{aligned}$$

for $V_1, V_2 \in \Gamma(D_2^I)$ and $Z_1 \in \Gamma(D^I \oplus D_1^I)$.
(c)

$$\begin{aligned} & g_{N_1}(\mathcal{T}_{V_1}\omega_J\phi_J V_2 - \mathcal{T}_{V_2}\omega_J\phi_J V_1, Z_1) \\ = \;& g_{N_1}(\mathcal{H}\nabla_{V_1}\omega_J V_2 - \mathcal{H}\nabla_{V_2}\omega_J V_1, \omega_J S Z_1) + \\ & g_{N_1}(\mathcal{T}_{V_1}\omega_J V_2 - \mathcal{T}_{V_2}\omega_J V_1, JPZ_1 + \phi_J S Z_1) \end{aligned}$$

for $V_1, V_2 \in \Gamma(D_2^J)$ and $Z_1 \in \Gamma(D^J \oplus D_1^J)$.
(d)

$$\begin{aligned} & g_{N_1}(\mathcal{T}_{V_1}\omega_K\phi_K V_2 - \mathcal{T}_{V_2}\omega_K\phi_K V_1, Z_1) \\ = \;& g_{N_1}(\mathcal{H}\nabla_{V_1}\omega_K V_2 - \mathcal{H}\nabla_{V_2}\omega_K V_1, \omega_K S Z_1) + \\ & g_{N_1}(\mathcal{T}_{V_1}\omega_K V_2 - \mathcal{T}_{V_2}\omega_K V_1, KPZ_1 + \phi_K S Z_1) \end{aligned}$$

for $V_1, V_2 \in \Gamma(D_2^K)$ and $Z_1 \in \Gamma(D^K \oplus D_1^K)$.

Theorem 3.10. *For F the following conditions are equivalent:*

(a) the horizontal distribution $(\ker F_*)^\perp$ defines a totally geodesic.
(b)

$$\begin{aligned} & g_{N_1}(\mathcal{A}_{V_1}V_2, PX_1 + \cos^2\theta_I^1 QX_1 + \cos^2\theta_I^2 SX_1) \\ = \;& g_{N_1}(\mathcal{H}\nabla_{V_1}V_2, \omega_I\phi_I PX_1 + \omega_I\phi_I QX_1 + \omega_I\phi_I SX_1) + \\ & g_{N_1}(\mathcal{A}_{V_1}B_I V_2 + \mathcal{H}\nabla_{V_1}C_I V_2, \omega_I X_1) \end{aligned}$$

for $V_1, V_2 \in \Gamma(\ker F_*)^\perp$ and $X_1 \in \Gamma(\ker F_*)$.
(c)

$$\begin{aligned} & g_{N_1}(\mathcal{A}_{V_1}V_2, PX_1 + \cos^2\theta_J^1 QX_1 + \cos^2\theta_J^2 SX_1) \\ = \;& g_{N_1}(\mathcal{H}\nabla_{V_1}V_2, \omega_J\phi_J PX_1 + \omega_J\phi_J QX_1 + \omega_J\phi_J SX_1) + \\ & g_{N_1}(\mathcal{A}_{V_1}B_J V_2 + \mathcal{H}\nabla_{V_1}C_J V_2, \omega_J X_1) \end{aligned}$$

for $V_1, V_2 \in \Gamma(\ker F_*)^\perp$ and $X_1 \in \Gamma(\ker F_*)$.
(d)

$$\begin{aligned} & g_{N_1}(\mathcal{A}_{V_1}V_2, PX_1 + \cos^2\theta_K^1 QX_1 + \cos^2\theta_K^2 SX_1) \\ = \;& g_{N_1}(\mathcal{H}\nabla_{V_1}V_2, \omega_K\phi_K PX_1 + \omega_K\phi_K QX_1 + \omega_K\phi_K SX_1) + \\ & g_{N_1}(\mathcal{A}_{V_1}B_K V_2 + \mathcal{H}\nabla_{V_1}C_K V_2, \omega_K X_1) \end{aligned}$$

for $V_1, V_2 \in \Gamma(\ker F_*)^\perp$ and $X_1 \in \Gamma(\ker F_*)$.

Proof. For $V_1, V_2 \in \Gamma(\ker F_*)^\perp$ and $X_1 \in \Gamma(\ker F_*)$, we have

$$g_{N_1}(\nabla_{V_1}V_2, X_1) = g_{N_1}(\nabla_{V_1}V_2, PX_1 + QX_1 + SX_1).$$

Using equations (5), (6), (10), (13), (14), (18) and Lemma 3.4, we get

$$\begin{aligned} & g_{N_1}(\nabla_{V_1}V_2, X_1) \\ = \; & g_{N_1}(\nabla_{V_1}RV_2, RPX_1) + g_{N_1}(\nabla_{V_1}RV_2, RQX_1) + g_{N_1}(\nabla_{V_1}RV_2, RSX_1), \\ = \; & g_{N_1}(\mathcal{A}_{V_1}V_2, PX_1 + \cos^2\theta_R^1 QX_1 + \cos^2\theta_R^2 SX_1) \\ & -g_{N_1}(\mathcal{H}\nabla_{V_1}V_2, \omega_R\phi_R PX_1 + \omega_R\phi_R QX_1 + \omega_R\phi_R SX_1) \\ & +g_{N_1}(\mathcal{A}_{V_1}B_R V_2 + \mathcal{H}\nabla_{V_1}C_R V_2, \omega_R PX_1 + \omega_R QX_1 + \omega_R SX_1). \end{aligned}$$

Now, since $\omega_R PX_1 + \omega_R QX_1 + \omega_R RX_1 = \omega_R X_1$ and $\omega_R PX_1 = 0$, one obtains

$$\begin{aligned} & g_{N_1}(\nabla_{V_1}V_2, X_1) \\ = \; & g_{N_1}(\mathcal{A}_{V_1}V_2, PX_1 + \cos^2\theta_R^1 QX_1 + \cos^2\theta_R^2 RX_1) \\ & -g_{N_1}(\mathcal{H}\nabla_{V_1}V_2, \omega_R\phi_R PX_1 + \omega_R\phi_R QX_1 + \omega_R\phi_R RX_1) \\ & +g_{N_1}(\mathcal{A}_{V_1}B_R V_2 + \mathcal{H}\nabla_{V_1}C_R V_2, \omega_R X_1). \end{aligned}$$

□

Theorem 3.11. *For F the following conditions are equivalent:*

(a) the vertical distribution $(\ker F_*)$ defines a totally geodesic.

(b)

$$\begin{aligned} & g_{N_1}(\mathcal{T}_{Z_1}Z_2, Y_1) + \cos^2\theta_I^1 g_{N_1}(\mathcal{T}_{Z_1}QZ_2, Y_1) + \cos^2\theta_I^2 g_{N_1}(\mathcal{T}_{Z_1}SZ_2, Y_1) \\ = \; & g_{N_1}(\mathcal{H}\nabla_{Z_1}\omega_I\phi_I PZ_2 + \mathcal{H}\nabla_{Z_1}\omega_I\phi_I QZ_2 + \mathcal{H}\nabla_{Z_1}\omega_I\phi_I SZ_2, Y_1) + \\ & g_{N_1}(\mathcal{T}_{Z_1}\omega_I Z_2, B_I Y_1) + g_{N_1}(\mathcal{H}\nabla_{Z_1}\omega_I Z_2, C_I Y_1) \end{aligned}$$

for $Z_2, Z_2 \in \Gamma(\ker F_*)$ and $Y_1 \in \Gamma(\ker F_*)^\perp$.

(c)

$$\begin{aligned} & g_{N_1}(\mathcal{T}_{Z_1}Z_2, Y_1) + \cos^2\theta_J^1 g_{N_1}(\mathcal{T}_{Z_1}QZ_2, Y_1) + \cos^2\theta_J^2 g_{N_1}(\mathcal{T}_{Z_1}SZ_2, Y_1) \\ = \; & g_{N_1}(\mathcal{H}\nabla_{Z_1}\omega_J\phi_J PZ_2 + \mathcal{H}\nabla_{Z_1}\omega_J\phi_J QZ_2 + \mathcal{H}\nabla_{Z_1}\omega_J\phi S_J Z_2, Y_1) + \\ & g_{N_1}(\mathcal{T}_{Z_1}\omega_J Z_2, B_J Y_1) + g_{N_1}(\mathcal{H}\nabla_{Z_1}\omega_J Z_2, C_J Y_1) \end{aligned}$$

for $Z_2, Z_2 \in \Gamma(\ker F_*)$ and $Y_1 \in \Gamma(\ker F_*)^\perp$

(d)

$$\begin{aligned} & g_{N_1}(\mathcal{T}_{Z_1}Z_2, Y_1) + \cos^2\theta_K^1 g_{N_1}(\mathcal{T}_{Z_1}QZ_2, Y_1) + \cos^2\theta_K^2 g_{N_1}(\mathcal{T}_{Z_1}SZ_2, Y_1) \\ = \; & g_{N_1}(\mathcal{H}\nabla_{Z_1}\omega_K\phi_K PZ_2 + \mathcal{H}\nabla_{Z_1}\omega_K\phi_K QZ_2 + \mathcal{H}\nabla_{Z_1}\omega_K\phi_K SZ_2, Y_1) + \\ & g_{N_1}(\mathcal{T}_{Z_1}\omega_K Z_2, B_K Y_1) + g_{N_1}(\mathcal{H}\nabla_{Z_1}\omega_K Z_2, C_K Y_1) \end{aligned}$$

for $Z_1, Z_2 \in \Gamma(\ker F_*)$ and $Y_1 \in \Gamma(\ker F_*)^\perp$.

Proof. For $Z_1, Z_2 \in \Gamma(\ker F_*)$ and $Y_1 \in \Gamma(\ker F_*)^\perp$, from equations (10) and (13), we get

$$\begin{aligned} & g_{N_1}(\nabla_{Z_1}Z_2, Y_1) \\ = \; & g_{N_1}(\nabla_{Z_1}RPZ_2, RY_1) + g_{N_1}(\nabla_{Z_1}RQZ_2, RY_1) + g_{N_1}(\nabla_{Z_1}RSZ_2, RY_1). \end{aligned}$$

Now, using equations (3), (4), (14) and Lemma and 3.4, we obtain

$$\begin{aligned} & g_{N_1}(\nabla_{Z_1}Z_2, Y_1) \\ = \; & g_{N_1}(\mathcal{T}_{Z_1}Z_2, Y_1) + \cos^2\theta_R^1 g_{N_1}(\mathcal{T}_{Z_1}QZ_2, Y_1) + \cos^2\theta_R^2 g_{N_1}(\mathcal{T}_{Z_1}SZ_2, Y_1) \\ & -g_{N_1}(\mathcal{H}\nabla_{Z_1}\omega_R\phi_R PZ_2 + \mathcal{H}\nabla_{Z_1}\omega_R\phi_R QZ_2 + \mathcal{H}\nabla_{Z_1}\omega_R\phi_R SZ_2, Y_1) \\ & +g_{N_1}(\nabla_{Z_1}\omega_R PZ_2 + \nabla_{Z_1}\omega_R QZ_2 + \nabla_{Z_1}\omega_R SZ_2, JY_1). \end{aligned}$$

Now, since $\omega_R PZ_2 + \omega_R QZ_2 + \omega_R SZ_2 = \omega_R Z_2$ and $\omega_R PZ_2 = 0$, we get

$$
\begin{aligned}
& g_{N_1}(\nabla_{Z_1} Z_2, Y_1) \\
= \; & g_{N_1}(\mathcal{T}_{Z_1} Z_2, Y_1) + \cos^2\theta_R^1 g_{N_1}(\mathcal{T}_{Z_1} QZ_2, Y_1) + \cos^2\theta_R^2 g_{N_1}(\mathcal{T}_{Z_1} SZ_2, Y_1) \\
& - g_{N_1}(\mathcal{H}\nabla_{Z_1}\omega_R\phi_R PZ_2 + \mathcal{H}\nabla_{Z_1}\omega_R\phi_R QZ_2 + \mathcal{H}\nabla_{Z_1}\omega_R\phi_R SZ_2, Y_1) \\
& + g_{N_1}(\mathcal{T}_{Z_1}\omega_R Z_2, B_R Y_1) + g_{N_1}(\mathcal{H}\nabla_{Z_1}\omega_R Z_2, C_R Y_1).
\end{aligned}
$$

□

Theorem 3.12. *For F the following conditions are equivalent:*

(a) the invariant distribution D^R defines a totally geodesic.

(b)

$$
\begin{aligned}
g_{N_1}(\mathcal{T}_{V_1}\phi_I PZ_2, \omega_I QU_1 + \omega_I SU_1) &= -g_{N_1}(\mathcal{V}\nabla_{V_1}\phi_I PZ_2, \phi_I QU_1 + \phi_I SU_1), \\
g_{N_1}(\mathcal{T}_{V_1}\phi_I PZ_2, C_I U_2) &= -g_{N_1}(\mathcal{V}\nabla_{V_1}\phi_I PZ_2, B_I U_2)
\end{aligned}
$$

for $V_1, Z_2 \in \Gamma(D^I)$, $U_1 \in \Gamma(D_1^I \oplus D_2^I)$ and $U_2 \in \Gamma(\ker F_*)^\perp$.

(c)

$$
\begin{aligned}
g_{N_1}(\mathcal{T}_{V_1}\phi_J PZ_2, \omega_J QU_1 + \omega_J SU_1) &= -g_{N_1}(\mathcal{V}\nabla_{V_1}\phi_J PZ_2, \phi_J QU_1 + \phi_J SU_1), \\
g_{N_1}(\mathcal{T}_{V_1}\phi_J PZ_2, C_J U_2) &= -g_{N_1}(\mathcal{V}\nabla_{V_1}\phi_J PZ_2, B_J U_2)
\end{aligned}
$$

for $V_1, Z_2 \in \Gamma(D^J)$, $U_1 \in \Gamma(D_1^J \oplus D_2^J)$ and $U_2 \in \Gamma(\ker F_*)^\perp$.

(d)

$$
\begin{aligned}
g_{N_1}(\mathcal{T}_{V_1}\phi_K PZ_2, \omega_K QU_1 + \omega_K SU_1) &= -g_{N_1}(\mathcal{V}\nabla_{V_1}\phi_K PZ_2, \phi_K QU_1 + \phi_K SU_1), \\
g_{N_1}(\mathcal{T}_{V_1}\phi_K PZ_2, C_K U_2) &= -g_{N_1}(\mathcal{V}\nabla_{V_1}\phi_K PZ_2, B_K U_2)
\end{aligned}
$$

for $V_1, Z_2 \in \Gamma(D^K)$, $U_1 \in \Gamma(D_1^K \oplus D_2^K)$ and $U_2 \in \Gamma(\ker F_*)^\perp$.

Proof. For $V_1, Z_2 \in \Gamma(D^R)$, $U_1 \in \Gamma(D_1^R \oplus D_2^R)$ and $U_2 \in \Gamma(\ker F_*)^\perp$. Using equations (3), (10), (13) and (14), we have

$$
\begin{aligned}
& g_{N_1}(\nabla_{V_1} Z_2, U_1) \\
= \; & g_{N_1}(\nabla_{V_1} RZ_2, RU_1), \\
= \; & g_{N_1}(\nabla_{V_1} RPZ_2, RQU_1 + RSU_1), \\
= \; & g_{N_1}(\mathcal{T}_{V_1}\phi_R PZ_2, \omega_R QU_1 + \omega_R SU_1) + g_{N_1}(\mathcal{V}\nabla_{V_1}\phi_R PZ_2, \phi_R QU_1 + \phi_R SU_1).
\end{aligned}
$$

Now, again using equations (3), (10), (13), (14) and (18), we obtain

$$
\begin{aligned}
& g_{N_1}(\nabla_{V_1} Z_2, U_2) \\
= \; & g_{N_1}(\nabla_{V_1} RZ_2, RU_2), \\
= \; & g_{N_1}(\nabla_{V_1}\phi_R PZ_2, B_R U_2 + C_R U_2), \\
= \; & g_{N_1}(\mathcal{V}\nabla_{V_1}\phi_R PZ_2, B_R U_2) + g_{N_1}(\mathcal{T}_{V_1}\phi_R PZ_2, C_R U_2),
\end{aligned}
$$

which completes the proof. □

Theorem 3.13. *For F the following conditions are equivalent:*

(a) the slant distribution D_1^R defines a totally geodesic.

(b)

$$\begin{aligned} g_{N_1}(\mathcal{T}_{Y_1}\omega_I\phi_I Y_2, Z_1) &= g_{N_1}(\mathcal{T}_{Y_1}\omega_I Q Y_2, RPZ_1 + \phi_I S Z_1) + g_{N_1}(\mathcal{H}\nabla_{Y_1}\omega_I Q Y_2, \omega_I S Z_1), \\ g_{N_1}(\mathcal{H}\nabla_{Y_1}\omega_I\phi_I Y_2, Z_2) &= g_{N_1}(\mathcal{H}\nabla_{Y_1}\omega_I Y_2, C_I Z_2) + g_{N_1}(\mathcal{T}_{Y_1}\omega_I Y_2, B_I Z_2) \end{aligned}$$

for $Y_1, Y_2 \in \Gamma(D_1^I), Z_1 \in \Gamma(D^I \oplus D_2^I)$ and $Z_2 \in \Gamma(\ker F_*)^\perp$

(c)

$$\begin{aligned} g_{N_1}(\mathcal{T}_{Y_1}\omega_J\phi_J Y_2, Z_1) &= g_{N_1}(\mathcal{T}_{Y_1}\omega_J Q Y_2, RPZ_1 + \phi_J S Z_1) + g_{N_1}(\mathcal{H}\nabla_{Y_1}\omega_J Q Y_2, \omega_J S Z_1), \\ g_{N_1}(\mathcal{H}\nabla_{Y_1}\omega_J\phi_J Y_2, Z_2) &= g_{N_1}(\mathcal{H}\nabla_{Y_1}\omega_J Y_2, C_J Z_2) + g_{N_1}(\mathcal{T}_{Y_1}\omega_J Y_2, B_J Z_2) \end{aligned}$$

for $Y_1, Y_2 \in \Gamma(D_1^J), Z_1 \in \Gamma(D^J \oplus D_2^J)$ and $Z_2 \in \Gamma(\ker F_*)^\perp$

(d)

$$\begin{aligned} g_{N_1}(\mathcal{T}_{Y_1}\omega_K\phi_K Y_2, Z_1) &= g_{N_1}(\mathcal{T}_{Y_1}\omega_K Q Y_2, RPZ_1 + \phi_K S Z_1) + g_{N_1}(\mathcal{H}\nabla_{Y_1}\omega_K Q Y_2, \omega_K S Z_1), \\ g_{N_1}(\mathcal{H}\nabla_{Y_1}\omega_K\phi_K Y_2, Z_2) &= g_{N_1}(\mathcal{H}\nabla_{Y_1}\omega_K Y_2, C_K Z_2) + g_{N_1}(\mathcal{T}_{Y_1}\omega_K Y_2, B_K Z_2) \end{aligned}$$

for $Y_1, Y_2 \in \Gamma(D_1^K), Z_1 \in \Gamma(D^K \oplus D_2^K)$ and $Z_2 \in \Gamma(\ker F_*)^\perp$.

Proof. For $Y_1, Y_2 \in \Gamma(D_1^R), Z_1 \in \Gamma(D^R \oplus D_2^R)$ and $Z_2 \in \Gamma(\ker F_*)^\perp$. Using equations (3), (4), (10), (13), (14) and Lemma 3.4, we have

$$\begin{aligned} & g_{N_1}(\nabla_{Y_1} Y_2, Z_1) \\ = \ & g_{N_1}(\nabla_{Y_1}\phi_R Y_2, RZ_1) + g_{N_1}(\nabla_{Y_1}\omega_R Y_2, RZ_1), \\ = \ & \cos^2\theta_R^1 g_{N_1}(\nabla_{Y_1} Y_2, Z_1) - g_{N_1}(\mathcal{T}_{Y_1}\omega_R\phi_R Y_2, Z_1) \\ & + g_{N_1}(\mathcal{T}_{Y_1}\omega_R Q Y_2, RPZ_1 + \phi_R S Z_1) + g_{N_1}(\mathcal{H}\nabla_{Y_1}\omega_R Q Y_2, \omega_R S Z_1). \end{aligned}$$

Now, we have

$$\begin{aligned} & \sin^2\theta_R^1 g_{N_1}(\nabla_{Y_1} Y_2, Z_1) \\ = \ & -g_{N_1}(\mathcal{T}_{Y_1}\omega_R\phi_R Y_2, Z_1) + g_{N_1}(\mathcal{T}_{Y_1}\omega_R Q Y_2, RPZ_1 + \phi_R S Z_1) \\ & + g_{N_1}(\mathcal{H}\nabla_{Y_1}\omega_R Q Y_2, \omega_R S Z_1). \end{aligned}$$

Next, from equations (3), (4), (10), (13), (14), (18) and Lemma 3.4, we obtain

$$\begin{aligned} & g_{N_1}(\nabla_{Y_1} Y_2, Z_2) \\ = \ & g_{N_1}(\nabla_{Y_1}\phi_R Y_2, RZ_2) + g_{N_1}(\nabla_{Y_1}\omega_R Y_2, RZ_2), \\ = \ & \cos^2\theta_R^1 g_{N_1}(\nabla_{Y_1} Y_2, Z_2) - g_{N_1}(\mathcal{H}\nabla_{Y_1}\omega_R\phi_R Y_2, Z_2) \\ & + g_{N_1}(\mathcal{H}\nabla_{Y_1}\omega_R Y_2, C_R Z_2) + g_{N_1}(\mathcal{T}_{Y_1}\omega_R Y_2, B_R Z_2). \end{aligned}$$

Now, we have

$$\begin{aligned} & \sin^2\theta_R^1 g_{N_1}(\nabla_{Y_1} Y_2, Z_2) \\ = \ & -g_{N_1}(\mathcal{H}\nabla_{Y_1}\omega_R\phi_R Y_2, Z_2) + g_{N_1}(\mathcal{H}\nabla_{Y_1}\omega_R Y_2, C_R Z_2) + g_{N_1}(\mathcal{T}_{Y_1}\omega_R Y_2, B_R Z_2). \end{aligned}$$

□

As above theorem one can easily obtain the following theorem:

Theorem 3.14. *For F the following conditions are equivalent:*

(a) the slant distribution D_2^R defines a totally geodesic.

(b)

$$\begin{aligned} g_{N_1}(\mathcal{T}_{Z_1}\omega_I\phi_I Z_2, X_1) &= g_{N_1}(\mathcal{T}_{Z_1}\omega_I QZ_2, RPX_1 + \phi_I SX_1) + g_{N_1}(\mathcal{H}\nabla_{Z_1}\omega_I QZ_2, \omega_I SX_1), \\ g_{N_1}(\mathcal{H}\nabla_{Z_1}\omega_I\phi_I Z_2, X_2) &= g_{N_1}(\mathcal{H}\nabla_{Z_1}\omega_I Z_2, C_I X_2) + g_{N_1}(\mathcal{T}_{Z_1}\omega_I Z_2, B_I X_2) \end{aligned}$$

for $Z_1, Z_2 \in \Gamma(D_2^I)$, $X_1 \in \Gamma(D^I \oplus D_1^I)$ and $X_2 \in \Gamma(\ker F_*)^\perp$.

(c)

$$\begin{aligned} g_{N_1}(\mathcal{T}_{Z_1}\omega_J\phi_J Z_2, X_1) &= g_{N_1}(\mathcal{T}_{Z_1}\omega_J QZ_2, RPX_1 + \phi_J SX_1) + g_{N_1}(\mathcal{H}\nabla_{Z_1}\omega_J QZ_2, \omega_J SX_1), \\ g_{N_1}(\mathcal{H}\nabla_{Z_1}\omega_J\phi_J Z_2, X_2) &= g_{N_1}(\mathcal{H}\nabla_{Z_1}\omega_J Z_2, C_J X_2) + g_{N_1}(\mathcal{T}_{Z_1}\omega_J Z_2, B_J X_2) \end{aligned}$$

for $Z_1, Z_2 \in \Gamma(D_2^J)$, $X_1 \in \Gamma(D^J \oplus D_1^J)$ and $X_2 \in \Gamma(\ker F_*)^\perp$.

(d)

$$\begin{aligned} g_{N_1}(\mathcal{T}_{Z_1}\omega_K\phi_K Z_2, X_1) &= g_{N_1}(\mathcal{T}_{Z_1}\omega_K QZ_2, RPX_1 + \phi_K SX_1) + g_{N_1}(\mathcal{H}\nabla_{Z_1}\omega_K QZ_2, \omega_K SX_1), \\ g_{N_1}(\mathcal{H}\nabla_{Z_1}\omega_K\phi_K Z_2, X_2) &= g_{N_1}(\mathcal{H}\nabla_{Z_1}\omega_K Z_2, C_K X_2) + g_{N_1}(\mathcal{T}_{Z_1}\omega_K Z_2, B_K X_2) \end{aligned}$$

for $Z_1, Z_2 \in \Gamma(D_2^K)$, $X_1 \in \Gamma(D^K \oplus D_1^K)$ and $X_2 \in \Gamma(\ker F_*)^\perp$.

Theorem 3.15. *For F the following conditions are equivalent:*

(a) F is a totally geodesic map.

(b)

$$\begin{aligned} & g_{N_1}(\mathcal{H}\nabla_{Z_1}\omega_I\phi_I QZ_2 + \mathcal{H}\nabla_{Z_1}\omega_I\phi_I SZ_2 - \cos^2\theta_I^1\nabla_{Z_1}QZ_2 - \cos^2\theta_I^2\nabla_{Z_1}SZ_2, U_1) \\ = \; & g_{N_1}(\mathcal{V}\nabla_{Z_1}IPZ_2 + \mathcal{T}_{Z_1}\omega_I QZ_2 + \mathcal{T}_{Z_1}\omega_I SZ_2, B_I U_1) \\ & + g_{N_1}(\mathcal{T}_{Z_1}IPZ_2 + \mathcal{H}\nabla_{Z_1}\omega_I QZ_2 + \mathcal{H}\nabla_{Z_1}\omega_I SZ_2, C_I U_1), \end{aligned}$$

$$\begin{aligned} & g_{N_1}(\mathcal{H}\nabla_{U_1}\omega_I\phi_I QZ_2 + \mathcal{H}\nabla_{U_1}\omega_I\phi_I SZ_2 - \cos^2\theta_I^1\nabla_{U_1}QZ_2 - \cos^2\theta_I^2\nabla_{U_1}SZ_2, U_2) \\ = \; & g_{N_1}(\mathcal{V}\nabla_{U_1}IPZ_2 + \mathcal{T}_{U_1}\omega_I QZ_2 + \mathcal{T}_{U_1}\omega_I SZ_2, B_I U_2) \\ & + g_{N_1}(\mathcal{T}_{U_1}IPZ_2 + \mathcal{H}\nabla_{U_1}\omega_I QZ_2 + \mathcal{H}\nabla_{U_1}\omega_I SZ_2, C_I U_2) \end{aligned}$$

for $Z_1, Z_2 \in \Gamma(\ker F_*)$ and $U_1, U_2 \in \Gamma(\ker F_*)^\perp$.

(c)

$$\begin{aligned} & g_{N_1}(\mathcal{H}\nabla_{Z_1}\omega_J\phi_J QZ_2 + \mathcal{H}\nabla_{Z_1}\omega_J\phi_J SZ_2 - \cos^2\theta_J^1\nabla_{Z_1}QZ_2 - \cos^2\theta_J^2\nabla_{Z_1}SZ_2, U_1) \\ = \; & g_{N_1}(\mathcal{V}\nabla_{Z_1}JPZ_2 + \mathcal{T}_{Z_1}\omega_J QZ_2 + \mathcal{T}_{Z_1}\omega_J SZ_2, B_J U_1) \\ & + g_{N_1}(\mathcal{T}_{Z_1}JPZ_2 + \mathcal{H}\nabla_{Z_1}\omega_J QZ_2 + \mathcal{H}\nabla_{Z_1}\omega_J SZ_2, C_J U_1), \end{aligned}$$

$$\begin{aligned} & g_{N_1}(\mathcal{H}\nabla_{U_1}\omega_J\phi_J QZ_2 + \mathcal{H}\nabla_{U_1}\omega_J\phi_J SZ_2 - \cos^2\theta_J^1\nabla_{U_1}QZ_2 - \cos^2\theta_J^2\nabla_{U_1}SZ_2, U_2) \\ = \; & g_{N_1}(\mathcal{V}\nabla_{U_1}JPZ_2 + \mathcal{T}_{U_1}\omega_J QZ_2 + \mathcal{T}_{U_1}\omega_J SZ_2, B_J U_2) \\ & + g_{N_1}(\mathcal{T}_{U_1}JPZ_2 + \mathcal{H}\nabla_{U_1}\omega_J QZ_2 + \mathcal{H}\nabla_{U_1}\omega_J SZ_2, C_J U_2), \end{aligned}$$

for $Z_1, Z_2 \in \Gamma(\ker F_*)$ and $U_1, U_2 \in \Gamma(\ker F_*)^\perp$.

(d)

$$\begin{aligned} & g_{N_1}(\mathcal{H}\nabla_{Z_1}\omega_K\phi_K QZ_2 + \mathcal{H}\nabla_{Z_1}\omega_K\phi_K SZ_2 - \cos^2\theta_K^1\nabla_{Z_1}QZ_2 - \cos^2\theta_K^2\nabla_{Z_1}SZ_2, U_1) \\ = \; & g_{N_1}(\mathcal{V}\nabla_{Z_1}KPZ_2 + \mathcal{T}_{Z_1}\omega_K QZ_2 + \mathcal{T}_{Z_1}\omega_K SZ_2, B_K U_1) \\ & + g_{N_1}(\mathcal{T}_{Z_1}KPZ_2 + \mathcal{H}\nabla_{Z_1}\omega_K QZ_2 + \mathcal{H}\nabla_{Z_1}\omega_K SZ_2, C_K U_1), \end{aligned}$$

$$\begin{aligned} & g_{N_1}(\mathcal{H}\nabla_{U_1}\omega_K\phi_K QZ_2 + \mathcal{H}\nabla_{U_1}\omega_K\phi_K SZ_2 - \cos^2\theta_K^1\nabla_{U_1}QZ_2 - \cos^2\theta_K^2\nabla_{U_1}SZ_2, U_2) \\ = \; & g_{N_1}(\mathcal{V}\nabla_{U_1}KPZ_2 + \mathcal{T}_{U_1}\omega_K QZ_2 + \mathcal{T}_{U_1}\omega_K SZ_2, B_K U_2) \\ & + g_{N_1}(\mathcal{T}_{U_1}KPZ_2 + \mathcal{H}\nabla_{U_1}\omega_K QZ_2 + \mathcal{H}\nabla_{U_1}\omega_K SZ_2, C_K U_2), \end{aligned}$$

for $Z_1, Z_2 \in \Gamma(\ker F_*)$ and $U_1, U_2 \in \Gamma(\ker F_*)^\perp$.

Proof. Since F is a Riemannian submersion, we have

$$(\nabla F_*)(U_1, U_2) = 0$$

for $Z, W \in \Gamma(\ker F_*)^\perp$.

For $Z_1, Z_2 \in \Gamma(\ker F_*)$ and $U_1, U_2 \in \Gamma(\ker F_*)^\perp$. Using equations (3), (4), (7), (10), (13), (14), (18) and Lemma 3.4, we have

$$\begin{aligned}
& g_{N_2}((\nabla F_*)(Z_1, Z_2), F_*(U_1)) \\
= \;& -g_{N_1}(\nabla_{Z_1} Z_2, U_1), \\
= \;& -g_{N_1}(\nabla_{Z_1} RZ_2, RU_1), \\
= \;& -g_{N_1}(\nabla_{Z_1} RPZ_2, RU_1) - g_{N_1}(\nabla_{Z_1} RQZ_2, RU_1) - g_{N_1}(\nabla_{Z_1} RSZ_2, JU_1), \\
= \;& -g_{N_1}(\nabla_{Z_1} RPZ_2, RU_1) - g_{N_1}(\nabla_{Z_1} \phi_R QZ_2, RU_1) - g_{N_1}(\nabla_{Z_1} \phi_R SZ_2, RU_1) \\
& -g_{N_1}(\nabla_{Z_1} \omega_R QZ_2, RU_1) - g_{N_1}(\nabla_{Z_1} \omega_R SZ_2, RU_1), \\
= \;& -g_{N_1}(\mathcal{V}\nabla_{Z_1} RPZ_2 + \mathcal{T}_{Z_1} \omega_R QZ_2 + \mathcal{T}_{Z_1} \omega_R SZ_2, B_R U_1) \\
& -g_{N_1}(\mathcal{T}_{Z_1} RPZ_2 + \mathcal{H}\nabla_{Z_1} \omega_R QZ_2 + \mathcal{H}\nabla_{Z_1} \omega_R SZ_2, C_R U_1) \\
& -g_{N_1}(\cos^2\theta_R^1 \nabla_{Z_1} QZ_2 + \cos^2\theta_R^2 \nabla_{Z_1} SZ_2 - \mathcal{H}\nabla_{Z_1} \omega_R \phi_R QZ_2 - \mathcal{H}\nabla_{Z_1} \omega_R \phi_R SZ_2, U_1).
\end{aligned}$$

Next, using equations (5), (6), (7), (10), (13), (14), (18) and lemma 3.4, we have

$$\begin{aligned}
& g_{N_2}((\nabla F_*)(U_1, Z_2), F_*(U_2)) \\
= \;& -g_{N_1}(\nabla_{U_1} Z_2, U_2), \\
= \;& -g_{N_1}(\nabla_{U_1} RZ_2, RU_2), \\
= \;& -g_{N_1}(\nabla_{U_1} RPZ_2, RU_2) - g_{N_1}(\nabla_{U_1} QZ_2, RU_2) - g_{N_1}(\nabla_{U_1} RSZ_2, RU_2), \\
= \;& -g_{N_1}(\nabla_{U_1} RPZ_2, RU_2) - g_{N_1}(\nabla_{U_1} \phi_R QZ_2, RU_2) - g_{N_1}(\nabla_{U_1} \phi_R SZ_2, RU_2) \\
& -g_{N_1}(\nabla_{U_1} \omega_R QZ_2, RU_2) - g_{N_1}(\nabla_{U_1} \omega_R SZ_2, RU_2), \\
= \;& -g_{N_1}(\mathcal{V}\nabla_{U_1} RPZ_2 + \mathcal{A}_{U_1} \omega_R QZ_2 + \mathcal{A}_{U_1} \omega_R SZ_2, B_R U_2) \\
& -g_{N_1}(\mathcal{A}_{U_1} RPZ_2 + \mathcal{H}\nabla_{U_1} \omega_R QZ_2 + \mathcal{H}\nabla_{U_1} \omega_R SZ_2, C_R U_2) \\
& -g_{N_1}(\cos^2\theta_R^1 \nabla_{U_1} QZ_2 + \cos^2\theta_R^2 \nabla_{U_1} SZ_2 - \mathcal{H}\nabla_{U_1} \omega_R \phi_R QZ_2 - \mathcal{H}\nabla_{U_1} \omega_R \phi_R SZ_2, U_2).
\end{aligned}$$

□

4. Example

Note that given an Euclidean space R^{4s} with coordinates $(x_1, x_2, ..., x_{4s})$, we can canonically choose complex structures I, J, K on R^{4s} as follows:

$$\begin{aligned}
I(\frac{\partial}{\partial x_{4r+1}}) &= \frac{\partial}{\partial x_{4r+2}}, I(\frac{\partial}{\partial x_{4r+2}}) = -\frac{\partial}{\partial x_{4r+1}}, I(\frac{\partial}{\partial x_{4r+3}}) = \frac{\partial}{\partial x_{4r+4}}, \\
I(\frac{\partial}{\partial x_{4r+4}}) &= -\frac{\partial}{\partial x_{4r+3}}, J(\frac{\partial}{\partial x_{4r+1}}) = \frac{\partial}{\partial x_{4r+3}}, J(\frac{\partial}{\partial x_{4r+2}}) = -\frac{\partial}{\partial x_{4r+4}}, \\
J(\frac{\partial}{\partial x_{4r+3}}) &= -\frac{\partial}{\partial x_{4r+1}}, J(\frac{\partial}{\partial x_{4r+4}}) = \frac{\partial}{\partial x_{4r+2}}, K(\frac{\partial}{\partial x_{4r+1}}) = \frac{\partial}{\partial x_{4r+4}}, \\
K(\frac{\partial}{\partial x_{4r+2}}) &= \frac{\partial}{\partial x_{4r+3}}, K(\frac{\partial}{\partial x_{4r+3}}) = -\frac{\partial}{\partial x_{4r+2}}, K(\frac{\partial}{\partial x_{4r+4}}) = -\frac{\partial}{\partial x_{4r+1}},
\end{aligned}$$

for $r \in \{0, 1, 2, ..., ..., s-1\}$.

Then we easily check that (I, J, K) is a hyperkähler structure on R^{4s}, where $\langle, \rangle$ denotes the Euclidean metric on R^{4s}. Throughout this section, we will use these notations.

Example 4.1. *Define a map* $F : R^{16} \to R^8$ *by*

$$F(x_1, x_2, ..., x_{16}) = (x_1, x_2, \frac{x_3 - x_5}{\sqrt{2}}, x_4, \frac{x_{10} + x_{13}}{\sqrt{2}}, x_{14}, x_{15}, x_{16}).$$

Then the map F is an almost h-qbs submersion such that

$$\ker F_* = \left\langle \begin{array}{c} \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5}), \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8}, \\ \frac{\partial}{\partial x_9}, \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_{10}} - \frac{\partial}{\partial x_{13}}), \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} \end{array} \right\rangle,$$

$$(\ker F_*)^\perp = \left\langle \begin{array}{c} \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_4}, \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x3} - \frac{\partial}{\partial x_5}), \\ \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_{10}} + \frac{\partial}{\partial x_{13}}), \frac{\partial}{\partial x_{14}}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_{16}} \end{array} \right\rangle,$$

$$\begin{aligned}
D^I &= \left\langle \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} \right\rangle, D_1^I = \left\langle \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x3} - \frac{\partial}{\partial x_5}), \frac{\partial}{\partial x_6} \right\rangle, \\
D_2^I &= \left\langle \frac{\partial}{\partial x_9}, \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_{10}} + \frac{\partial}{\partial x_{13}}) \right\rangle, D^J = \left\langle \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{11}} \right\rangle, \\
D_1^J &= \left\langle \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x3} - \frac{\partial}{\partial x_5}), \frac{\partial}{\partial x_7} \right\rangle, D_2^J = \left\langle \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_{10}} + \frac{\partial}{\partial x_{13}}), \frac{\partial}{\partial x_{12}} \right\rangle, \\
D^K &= \left\langle \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{12}} \right\rangle, D_1^K = \left\langle \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x3} - \frac{\partial}{\partial x_5}), \frac{\partial}{\partial x_8} \right\rangle, \\
D_2^K &= \left\langle \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_{10}} + \frac{\partial}{\partial x_{13}}), \frac{\partial}{\partial x_{11}} \right\rangle,
\end{aligned}$$

with the almost h-qbs angles $\{\theta_I^1 = \theta_J^1 = \theta_K^1 = \frac{\pi}{4}, \theta_I^2 = \theta_J^2 = \theta_K^2 = \frac{\pi}{4}\}$.

Example 4.2. *Define a map* $F : R^{16} \to R^8$ *by*

$$F(x_1, x_2, ..., x_{16}) = (\cos\alpha x_1 - \sin\alpha x_5, x_2, x_3, x_4, \sin\beta x_{11} + \cos\beta x_{13}, x_{14}, x_{15}, x_{16}).$$

Then the map F is an almost h-qbs submersion such that

$$\ker F_* = \left\langle \begin{array}{c} (\sin\alpha\frac{\partial}{\partial x_1} + \cos\alpha\frac{\partial}{\partial x_5}), \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8}, \\ \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, (\cos\beta\frac{\partial}{\partial x_{11}} - \sin\beta\frac{\partial}{\partial x_{13}}), \frac{\partial}{\partial x_{12}} \end{array} \right\rangle,$$

$$(\ker F_*)^\perp = \left\langle \begin{array}{c} \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, (\cos\alpha\frac{\partial}{\partial x_1} - \sin\alpha\frac{\partial}{\partial x_5}), \\ (\sin\beta\frac{\partial}{\partial x_{11}} + \cos\beta\frac{\partial}{\partial x_{13}}), \frac{\partial}{\partial x_{14}}, \frac{\partial}{\partial x_{15}}, \frac{\partial}{\partial x_{16}} \end{array} \right\rangle,$$

$$\begin{aligned}
D^I &= \left\langle \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}} \right\rangle, D_1^I = \left\langle (\sin\alpha\frac{\partial}{\partial x_1} + \cos\alpha\frac{\partial}{\partial x_5}), \frac{\partial}{\partial x_6} \right\rangle, \\
D_2^I &= \left\langle (\cos\beta\frac{\partial}{\partial x_{11}} - \sin\beta\frac{\partial}{\partial x_{13}}), \frac{\partial}{\partial x_{12}} \right\rangle, D^J = \left\langle \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{12}} \right\rangle, \\
D_1^J &= \left\langle (\sin\alpha\frac{\partial}{\partial x_1} + \cos\alpha\frac{\partial}{\partial x_5}), \frac{\partial}{\partial x_7} \right\rangle, D_2^J = \left\langle \frac{\partial}{\partial x_9}, (\cos\beta\frac{\partial}{\partial x_{11}} - \sin\beta\frac{\partial}{\partial x_{13}}) \right\rangle, \\
D^K &= \left\langle \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{12}} \right\rangle, D_1^K = \left\langle (\sin\alpha\frac{\partial}{\partial x_1} + \cos\alpha\frac{\partial}{\partial x_5}), \frac{\partial}{\partial x_8} \right\rangle, \\
D_2^K &= \left\langle \frac{\partial}{\partial x_{10}}, (\cos\beta\frac{\partial}{\partial x_{11}} - \sin\beta\frac{\partial}{\partial x_{13}}) \right\rangle,
\end{aligned}$$

with the almost h-qbs angles $\{\theta_I^1 = \theta_J^1 = \theta_K^1 = \alpha, \theta_I^2 = \theta_J^2 = \theta_K^2 = \beta\}$.

Example 4.3. *Define a map $F : R^{16} \to R^8$ by*

$$F(x_1, x_2, ..., x_{16}) = (\frac{x_3 + x_5}{\sqrt{2}}, x_6, x_7, x_8, \frac{\sqrt{3}x_9 - x_{16}}{2}, x_{10}, x_{11}, x_{12}).$$

Then the map F is an almost h-qbs submersion such that

$$\ker F_* = \left\langle \begin{array}{c} \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_4}, \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5}), \\ \frac{1}{2}(\frac{\partial}{\partial x_9} + \sqrt{3}\frac{\partial}{\partial x_{16}}), \frac{\partial}{\partial x_{13}}, \frac{\partial}{\partial x_{14}}, \frac{\partial}{\partial x_{15}} \end{array} \right\rangle,$$

$$(\ker F_*)^\perp = \left\langle \begin{array}{c} \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x3} + \frac{\partial}{\partial x_5}), \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8}, \\ \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}}, \frac{1}{2}(\sqrt{3}\frac{\partial}{\partial x_9} - \frac{\partial}{\partial x_{16}}) \end{array} \right\rangle,$$

$$\begin{aligned} D^I &= \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_{13}}, \frac{\partial}{\partial x_{14}} \right\rangle, D_1^I = \left\langle \frac{\partial}{\partial x_4}, \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5}) \right\rangle, \\ D_2^I &= \left\langle \frac{1}{2}(\frac{\partial}{\partial x_9} + \sqrt{3}\frac{\partial}{\partial x_{16}}), \frac{\partial}{\partial x_{15}} \right\rangle, D^J = \left\langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_{13}}, \frac{\partial}{\partial x_{15}} \right\rangle, \\ D_1^J &= \left\langle \frac{\partial}{\partial x_1}, \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5}) \right\rangle, D_2^J = \left\langle \frac{1}{2}(\frac{\partial}{\partial x_9} + \sqrt{3}\frac{\partial}{\partial x_{16}}), \frac{\partial}{\partial x_{14}} \right\rangle, \\ D^K &= \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_{14}}, \frac{\partial}{\partial x_{15}} \right\rangle, D_1^K = \left\langle \frac{\partial}{\partial x_2}, \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5}) \right\rangle, \\ D_2^K &= \left\langle \frac{1}{2}(\frac{\partial}{\partial x_9} + \sqrt{3}\frac{\partial}{\partial x_{16}}), \frac{\partial}{\partial x_{13}} \right\rangle, \end{aligned}$$

with the almost h-qbs angles $\theta_I^1 = \theta_J^1 = \theta_K^1 = \frac{\pi}{4}$ and $\theta_I^2 = \theta_J^2 = \theta_K^2 = \frac{\pi}{6}$.

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