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Long time asymptotics for American maximum option with a dividend-paying asset

Yongqing Xu[a]

[a]*College of Big data and Internet, Shenzhen Technology University, Shenzhen 518118, China*

Abstract. American maximum options provide minimum payoff protections for the investors when the asset's price falls and bring more profits when the asset's price rises. It is interesting to explore the properties of them. In this paper, we pay attention to the long time behaviors of two optimal boundaries and the price function of American maximum option with a dividend-paying asset. We provide their long time behaviors by analyzing several integral equations related to the transformed optimal boundaries and the scaled price function and provide rigorous proofs for all results. Numerical examples are carried out to demonstrate the price and boundaries of American maximum option.

1. Introduction

American maximum options are a type of special financial contract (position), which permits the holders (investors) buy the underlying asset at a prespecifc price or obtain some cash at another prespecific level before or at the maturity of the contract. Thus American maximum options hold minimum payoff feature and provide protections for the investors. An American maximum option has two optimal exercise boundaries which have interactions and observe specific equations. When the underlying asset's price falls to the lower boundary, the holder chooses to exercise the option obtaining some cash at predetermined amount; when the underlying asset's price rises to the upper boundary, the holder chooses to buy the asset at a fixed strike price. With the advantages, American maximum options have received more and more applications from the financial industries and the management of companies.

There are some literatures concerning American maximum options, we can refer [1]-[5] for them. The earliest literature concerning American maximum option should be attributed to Guo and Shepp[1], where they considered two types of perpetual American maximum options as optimal stopping problems and obtained the price expressions of the options. One maximum option is a choice between the asset and a fixed cash, and then owns risk aversion feature. Another maximum option is a new type of stock option for a company, where the company promises a guaranteed minimum as an extra incentive in case the market appreciation of the stock is low, thus making the option more attractive to the employee. Jiang[2] studied the valuation of American options on the maximum/minimum of two assets using the method

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Email address: xyq195752@sina.com; xyq195752@gmail.com (Yongqing Xu)
ORCID iD: https://orcid.org/0000-0002-5726-175X (Yongqing Xu)

of PDE. He discussed the effects of some factors on the valuation if one factor was increasing while the others kept unchanged. Besides, the properties of the optimal boundary the option were studied including monotonicity, convexity and asymptotic behavior. Xu and Wu[3] investigated the valuation of perpetual American maximum option on two assets under Markov-modulated dynamics and derived the pricing formulas by solving a series of variational inequalities, and further built optimal stopping time rule. In [4], Khizhnyak studied estimates for the value of perpetual American maximum option on two risky assets, and built upper and lower bounds by different methods respectively. In [5], Chen et al. studied the properties of American maximum option with finite maturity, including the existence of solution and the monotonicity and smoothness of free boundaries, etc.

It is an interesting problem to study the asymptotic behaviors of American options with far maturity when considering the price of perpetual American option instead of the one of American option with long maturity. Ahn et al.[6] provided a detailed proof of a long time asymptotic behavior of the optimal exercise boundary and the rescaled price function for an American put option without dividend-paying. Their proof needs the aid of the convexity of exercise boundary. Chen et al.[7] studied the long time behavior of the optimal exercise boundary and the rescaled value function for an American put option written on a dividend-paying asset. They provided a sharp estimation in the proof and classification for the leading order of the error according to the related parameter. In current literature including [6] and [7], the studies were limited to the situation concerning single exercise boundary of American put option. To the best of our knowledge, there are no papers about the long time approximations of American options with multiple exercise boundaries, which are often applied in the financial markets. Our study will be the latest progress about these problems. Motivated by [6] and [7], we shall extend the former studies of single boundary to the case of double exercise boundaries, that is to be specific, we shall investigate the long time behaviors of the two optimal stopping boundaries of American maximum option with a dividend-paying asset. The analysis is performed by deciding the leading order of the very term in the equations satisfied by the optimal boundaries. The asymptotic results for the optimal boundaries are classified by the related parameter. Then we set up the long time behavior of the rescaled value function by deciding the leading order of the difference of the two value functions.

The paper is structured as follows. In Section 2, we give a formulation of the model and problem. In section 3, we provide and prove the result of the long time behaviors of the exercise boundaries under the case with dividend. In section 4, we offer and prove the long time behaviors of the transformed price function of American maximum option. Some concluding remarks are provided in the end.

2. The Model and Problem Formulation

We consider a complete financial market composed of a risk-less money account and a risky stock, whose price at time t denoted by B_t and S_t. The stock price is a stochastic processes defined by a geometric Brownian motion (GBM) or log-normal process and the money account observes an ordinary differential equation as follows

$$dS_t = S_t[(r-q)dt + \sigma dW_t], \qquad dB_t = rB_t dt,$$

where the constants $r > 0, q \geq 0$ denote the risk free interest rate and the dividend rate, respectively. The notation $\sigma > 0$ is the volatility of the asset price. $\{W_t\}_{t\geq 0}$ is a one-dimensional standard Brownian motion or wiener process defined on a complete probability space $(\Omega, (\mathcal{F})_{t\geq 0}, \mathbb{P})$, where $\mathbb{P}$ is a risk-neutral probability measure and the natural filtration $(\mathcal{F})_{t\geq 0}$ is generated by $\{W_t\}_{t\geq 0}$. Denote by L and K predetermined price levels ($L > K$) and by T the maturity date of the option. When the maximum option is exercised, the payoff is $\max\{S_t - K, L - K\}$. Let $F(S,t)$ be the value function of American maximum option at time t with $S_t = S$, which satisfies the following variational inequality

$$\begin{cases} \max\{\overline{\mathcal{L}}F, \max(L-K, S-K) - F\} = 0; & \text{in} \quad (0,+\infty)\times(0,T] \\ F(S_T, T) = \max(L-K, (S-K)_+); & \text{on} \quad (0,+\infty)\times\{T\} \\ A(t) = \sup\{S>0 | F(S,t) = L-K; & \text{on} \quad (0,T] \\ B(t) = \inf\{S>0 | F(S,t) = S-K; & \text{on} \quad (0,T]. \end{cases} \tag{1}$$

where the operator is defined as $\overline{\mathcal{L}}F = F_t + \frac{\sigma^2}{2}S^2F_{SS} + (r-q)SF_S - rF$, which is the usual Black-Scholes' operator. $A(t)$ and $B(t)$ denote the lower and upper exercise boundaries, respectively. When the underlying price hits the lower boundary $A(t)$, the holder of the option chooses to exercise it and obtains the cash $L-K$; When the underlying price hits the upper boundary $B(t)$, the holder of the option chooses to exercise it and obtain the asset S_t at the strike price K. For the convenience of presentation, we define several dimensionless quantities: $x = \ln(\frac{S}{K}), \tau = \frac{\sigma^2}{2}(T-t), f(x,\tau) = \frac{F(S,t)}{K}, a(\tau) = \ln\frac{A(t)}{K}, b(\tau) = \ln\frac{B(t)}{K}, k = \frac{L}{K}, \rho = \frac{2r}{\sigma^2}, \ell = \frac{2q}{\sigma^2}, \alpha = \rho - \ell - 1, \lambda = \rho + \frac{\alpha^2}{4}, f_0(x) = \max(k-1, e^x-1), \mathcal{L}f := f_{xx} + \alpha f_x - \rho f$. With above notations, the variational inequality (1) can be changed into the following form

$$\begin{cases} \max\{\mathcal{L}f - f_\tau, f_0 - f\} = 0; & \text{in} \quad (-\infty,+\infty)\times(0,T), \\ f(x,0) = f_0(x); & \text{on} \quad (-\infty,+\infty)\times\{0\}, \\ a(\tau) = \sup\{x<0 \mid f(x,\tau) = k-1; & \text{on} \quad (0,+\infty), \\ b(\tau) = \inf\{x>0 \mid f(x,\tau) = e^x - 1; & \text{on} \quad (0,+\infty). \end{cases} \tag{2}$$

The main targets are to explore the long time asymptotic behaviors of exercise boundaries $a(\tau), b(\tau)$ and value function $p(x,\tau)$ as $\tau \to \infty$. By basic properties of American maximum options in [5], we know

$$f(x,\tau) \nearrow \bar{f}(x), \quad a(\tau) \searrow \bar{a}, \quad b(\tau) \nearrow \bar{b}, \quad \text{as } \tau \nearrow \infty.$$

Where the notation $\bar{f}(x)$ denotes the rescaled price of perpetual American maximum option, i.e. $\bar{f}(x) := \lim_{\tau\to\infty} f(x,\tau)$. $\bar{a}$ and $\bar{b}$ represent the lower and upper exercise boundaries of perpetual American maximum option respectively, which are provided in [1]. According to the situation of the dividend, we classify two cases to display the expressions of the rescaled price of perpetual American maximum option as follows:

(1) Case 1: $q = 0$; The rescaled value function of perpetual American maximum option is given by

$$\bar{f}(x) = \begin{cases} k-1; & \text{as} \quad x < x^*, \\ (k-1)[\frac{\gamma_1}{\gamma_1-\gamma_0}e^{\gamma_0(x-\ln l)} - \frac{\gamma_0}{\gamma_1-\gamma_0}e^{\gamma_1(x-\ln l)}]; & \text{as} \quad x \geq x^*, \end{cases} \tag{3}$$

where x^* satisfies an equation for smooth boundary condition. Under this case, there is no upper exercise boundary ($\bar{b} = \infty$), moreover, $b(\tau) = \infty$. This is a degenerate situation. We refer Theorem 2 and Corollary 1 in [1] for the results.

(2) Case 2: $q > 0$; The rescaled value function of perpetual American maximum has the form

$$\bar{f}(x) = \begin{cases} k-1; & \text{as} \quad x \leq \bar{a}, \\ (k-1)[\frac{\gamma_1}{\gamma_1-\gamma_0}e^{\gamma_0(x-\bar{a})} + \frac{\gamma_0}{\gamma_0-\gamma_1}e^{\gamma_1(x-\bar{a})}]; & \text{as} \quad \bar{a} < x < \bar{b}, \\ e^x - 1; & \text{as} \quad x \geq \bar{b} \end{cases} \tag{4}$$

where the exercise boundaries are $\bar{a} = \ln\frac{A}{K}$ and $\bar{b} = \ln\frac{B}{K}$ with $A > L, B < L$ satisfying the equations

$$\frac{A-K}{L-K} = \frac{\gamma_1}{\gamma_1-\gamma_0}\theta^{\gamma_0} + \frac{\gamma_0}{\gamma_0-\gamma_1}\theta^{\gamma_1}$$

and

$$B = \frac{L-K}{\theta(\gamma_0-\gamma_1)}[\gamma_0\theta^{\gamma_1} - \gamma_1\theta^{\gamma_0}] + \frac{K}{\theta}$$

with $\theta = \frac{A}{B}$ being the unique real root bigger than 1 to the equation:

$$f(x) = \frac{1}{\gamma_0-\gamma_1}[\gamma_1(1-\gamma_0)x^{\gamma_0} - \gamma_0(1-\gamma_1)x^{\gamma_1}] - \frac{K}{L-K} = 0.$$

where $\gamma_0 > 1 > 0 > \gamma_1$ are two roots of the characteristic equation $\gamma^2 + \alpha\gamma - \rho = 0$. Besides, $a_0 := \lim_{\tau\to 0} a(\tau) = \ln\frac{L}{K}$ and $b_0 := \lim_{\tau\to 0} b(\tau) = \max\{\ln\frac{L}{K}, \ln\frac{\rho}{\ell}\}$. We refer Theorem 2 in [1] for (4) and Theorem 4 in [5] for a_0 and b_0, respectively.

We denote the basic solution of $\partial_\tau - \mathcal{L}$ by Γ with the following form

$$\Gamma(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} e^{-(x+\alpha\tau)^2/(4\tau)-\rho\tau}.$$

As $q = 0$, this is a trivial case with single boundary, we will not explore it. As $q > 0$, since $f_\tau - \mathcal{L}f = \rho(k-1)\chi_{(-\infty,a(\tau))}(x) + (\ell e^x - \rho)\chi_{(b(\tau),\infty)}(x)$ with χ_A being the indicator function of set A, we can write the solution of $f(x,\tau)$ under the case with dividend-paying

$$\begin{aligned} f(x,\tau) &= \int_{-\infty}^{\infty} f_0(y)\Gamma(x-y,\tau)dy + \int_0^\tau \int_{-\infty}^{a(u)} \rho(k-1)\Gamma(x-y,\tau-u)dydu \\ &\quad + \int_0^\tau \int_{b(u)}^{\infty} (\ell e^y - \rho)\Gamma(x-y,\tau-u)dydu. \end{aligned} \tag{5}$$

In next sections, we derive the long time asymptotic behaviors of the optimal exercise boundaries and the rescaled price function of American maximum option.

3. The long time behaviors of exercise boundaries

For the long time behaviors of the optimal exercise boundaries of American maximum option written on a dividend-paying asset, we have the following theorem.

Theorem 3.1. *There are two constants m_1 and m_2 such that for $\tau \to \infty$,*

$$a(\tau) = \bar{a} + \begin{cases} [m_1 + O(1)\tau^{-1}]\tau^{-\frac{1}{2}}e^{-\lambda\tau} & \text{if } \alpha \neq 0 \\ O(1)\tau^{-\frac{3}{2}}e^{-\lambda\tau} & \text{if } \alpha = 0. \end{cases}$$

and

$$b(\tau) = \bar{b} + \begin{cases} [m_2 + O(1)\tau^{-1}]\tau^{-\frac{1}{2}}e^{-\lambda\tau} & \text{if } \alpha \neq 0 \\ O(1)\tau^{-\frac{3}{2}}e^{-\lambda\tau} & \text{if } \alpha = 0. \end{cases}$$

where $O(1)$ is a generic bounded function for sufficiently large τ.

Proof. To facilitate the analysis, we need to derive the expression of the derivative of $f(x,\tau)$ about τ. To this end, we use the technique of integration by parts to rewrite the form of f in (5). We firstly define a function

$$g(x,\tau) := \int_{-\infty}^{\infty} f_0(y)\Gamma_\tau(x-y,\tau)dy. \tag{6}$$

Actually, the basic calculus theorem and the property of Dirac function will disclose that

$$\int_0^\tau \int_{-\infty}^{\infty} f_0(y)\Gamma_\tau(x-y,u)dydu = \int_{-\infty}^{\infty} f_0(y)\Gamma(x-y,\tau)dy - f_0(x). \tag{7}$$

That will generate the following equality

$$\int_{-\infty}^{\infty} f_0(y)\Gamma(x-y,\tau)dy = f_0(x) + \int_0^\tau g(x,u)du. \tag{8}$$

By virtue of the basic fact $\Gamma_\tau = \Gamma_{xx} + \alpha\Gamma_x - \rho\Gamma$, we change the form of g as follows:

$$g(x,\tau) = \int_{-\infty}^{\infty} f_0(y)(\Gamma_{xx} + \alpha\Gamma_x - \rho\Gamma)(x-y,\tau)dy. \tag{9}$$

So we can divide the integral into three terms. The first term in (9) can be computed as follows:

$$
\begin{aligned}
I_1 &:= \int_{-\infty}^{\infty} f_0(y)\Gamma_{xx}(x-y,\tau)dy = -\int_{-\infty}^{\infty} f_0(y)d\Gamma_x(x-y,\tau) \\
&= -f_0(y)\Gamma_x(x-y,\tau)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \Gamma_x(x-y,\tau)f_0'(y)dy \\
&= -\Gamma(x-y,\tau)f_0'(y)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \Gamma(x-y,\tau)f_0''(y)dy \\
&= k\Gamma(x-\ln k,\tau) + \int_{\ln k}^{\infty} e^y\Gamma(x-y,\tau)dy. \qquad (10)
\end{aligned}
$$

where $f_0(y) = (k-1)\chi_{(-\infty,\ln k)}(y) + (e^y-1)\chi_{[\ln k,\infty)}(y)$ is the payoff function. The first term of the right side in the third equality is zero since the higher order appears in the exponential function in the derivative of Γ. The fifth equality uses the fact $f_0'(y) = e^y\chi_{[\ln k,\infty)}(y)$.

Similar to above computations, the second term in (9) can be computed as follows:

$$
\begin{aligned}
I_2 &:= \int_{-\infty}^{\infty} \alpha f_0(y)\Gamma_x(x-y,\tau)dy = -\int_{-\infty}^{\infty} \alpha f_0(y)d\Gamma(x-y,\tau) \\
&= -\alpha f_0(y)\Gamma(x-y,\tau)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \alpha f_0'(y)\Gamma(x-y,\tau)dy \\
&= \int_{\ln k}^{\infty} \alpha e^y\Gamma(x-y,\tau)dy. \qquad (11)
\end{aligned}
$$

The third term in (9) can be computed as follows:

$$
I_3 := -\rho\int_{-\infty}^{\infty} f_0(y)\Gamma(x-y,\tau)dy = -\rho(k-1)\int_{-\infty}^{\ln k}\Gamma(x-y,\tau)dy - \rho\int_{\ln k}^{\infty}(e^y-1)\Gamma(x-y,\tau)dy. \qquad (12)
$$

Uniting above expressions in (10–12), we obtain the following result for (9)

$$
g(x,\tau) = k\Gamma(x-\ln k,\tau) + \int_{\ln k}^{\infty}(\rho-\ell e^y)\Gamma(x-y,\tau)dy - \int_{-\infty}^{\ln k}\rho(k-1)\Gamma(x-y,\tau)dy. \qquad (13)
$$

Thus, using the solution formula in (5) plus the expressions in (8) and (13), we can rewrite the form of $f(x,\tau)$ as follows:

$$
\begin{aligned}
f(x,\tau) &= f_0(x) + \int_0^{\tau} k\Gamma(x-\ln k,u)du + \int_0^{\tau}\int_{\ln k}^{\infty}(\rho-\ell e^y)\Gamma(x-y,u)dydu \\
&\quad - \int_0^{\tau}\int_{-\infty}^{\ln k}\rho(k-1)\Gamma(x-y,u)dydu + \int_0^{\tau}\int_{-\infty}^{a(\tau-u)}\rho(k-1)\Gamma(x-y,u)dydu \\
&\quad + \int_0^{\tau}\int_{b(\tau-u)}^{\infty}(\ell e^y-\rho)\Gamma(x-y,u)dydu. \qquad (14)
\end{aligned}
$$

Note that $a_0 = \ln k$ and $b_0 = \max\{\ln k, \ln\frac{\rho}{\ell}\}$, we differentiate f with respect to τ and x successively, and then acquire easily the two derivatives after some algebraic operations

$$
\begin{aligned}
f_\tau(x,\tau) &= \int_{\ln k}^{b_0}(\rho-\ell e^y)\Gamma(x-y,\tau)dy + \rho(k-1)\int_0^{\tau}\Gamma(x-a(u),\tau-u)\dot{a}(u)du \\
&\quad - \int_0^{\tau}(\ell e^{b(u)}-\rho)\Gamma(x-b(u),\tau-u)\dot{b}(u)du + k\Gamma(x-\ln k,\tau) \qquad (15)
\end{aligned}
$$

and

$$f_{\tau x}(x,\tau) = \int_{\ln k}^{b_0}(\rho-\ell e^y)\Gamma_x(x-y,\tau)dy+\rho(k-1)\int_0^\tau \Gamma_x(x-a(u),\tau-u)\dot a(u)du \\ -\int_0^\tau(\ell e^{b(u)}-\rho)\Gamma_x(x-b(u),\tau-u)\dot b(u)du+k\Gamma_x(x-\ln k,\tau). \tag{16}$$

According to the boundary conditions of American maximum option at $x=a(\tau)$ and $x=b(\tau)$: $f_\tau(a(\tau),\tau)=0, f_\tau(b(\tau),\tau)=0$ and $f_{\tau x}(a(\tau)\pm,\tau)=-(\frac{1}{2}\pm\frac{1}{2})\rho(k-1)\dot a(\tau)$, $f_{\tau x}(b(\tau)\pm,\tau)=-(\frac{1}{2}\mp\frac{1}{2})(\ell e^{b(\tau)}-\rho)\dot b(\tau)$, we can write out the following two equations about the boundaries. The first two equations are satisfied by $a(\tau)$ as follows:

$$\int_{\ln k}^{b_0}(\rho-\ell e^y)\Gamma(a(\tau)-y,\tau)dy+\rho(k-1)\int_0^\tau \Gamma(a(\tau)-a(u),\tau-u)\dot a(u)du \\ -\int_0^\tau(\ell e^{b(u)}-\rho)\Gamma(a(\tau)-b(u),\tau-u)\dot b(u)du+k\Gamma(a(\tau)-\ln k,\tau)=0. \tag{17}$$

and

$$2\int_{\ln k}^{b_0}(\rho-\ell e^y)\Gamma_x(a(\tau)-y,\tau)dy+2\rho(k-1)\int_0^\tau \Gamma_x(a(\tau)-a(u),\tau-u)\dot a(u)du \\ -2\int_0^\tau(\ell e^{b(u)}-\rho)\Gamma_x(a(\tau)-b(u),\tau-u)\dot b(u)du+2k\Gamma_x(a(\tau)-\ln k,\tau)=-\rho(k-1)\dot a(\tau). \tag{18}$$

The upper boundary conditions generate the following two equations satisfied by $b(\tau)$

$$\int_{\ln k}^{b_0}(\rho-\ell e^y)\Gamma(b(\tau)-y,\tau)dy+\rho(k-1)\int_0^\tau \Gamma(b(\tau)-a(u),\tau-u)\dot a(u)du \\ -\int_0^\tau(\ell e^{b(u)}-\rho)\Gamma(b(\tau)-b(u),\tau-u)\dot b(u)du+k\Gamma(b(\tau)-\ln k,\tau)=0. \tag{19}$$

and

$$2\int_{\ln k}^{b_0}(\rho-\ell e^y)\Gamma_x(b(\tau)-y,\tau)dy+2\rho(k-1)\int_0^\tau \Gamma_x(b(\tau)-a(u),\tau-u)\dot a(u)du \\ -2\int_0^\tau(\ell e^{b(u)}-\rho)\Gamma_x(b(\tau)-b(u),\tau-u)\dot b(u)du+2k\Gamma_x(b(\tau)-\ln k,\tau)=-(\ell e^{b(\tau)}-\rho)\dot b(\tau). \tag{20}$$

We firstly consider the behavior of lower boundary $a(\tau)$. For this aim, we observe asymptotic value of the first and last items in (17) as follows. Direct computations and calculus theorem disclose that, as $\tau\to\infty$

$$\int_{\ln k}^{b_0}(\rho-\ell e^y)\Gamma(a(\tau)-y,\tau)dy=[c_a+O(1)\tau^{-1}]\tau^{-\frac{1}{2}}e^{-\lambda\tau} \tag{21}$$

where the coefficient is defined as $c_a=\frac{\rho}{\sqrt{\pi}\alpha}[e^{\frac{\alpha}{2}(b_0-\bar a)}-e^{\frac{\alpha}{2}(\ln k-\bar a)}]-\frac{\ell e^{\bar a}}{\sqrt{\pi}}\frac{1}{\alpha+2}[e^{(b_0-\bar a)\frac{\alpha+2}{2}}-e^{(\ln k-\bar a)\frac{\alpha+2}{2}}]$ and the notation $O(1)$ denotes a generic bounded function as $\tau\to\infty$. We write out the specific coefficient of the leading term with order $\tau^{-\frac{1}{2}}e^{-\lambda\tau}$ and keep the term with order $\tau^{-\frac{3}{2}}e^{-\lambda\tau}$ multiplied by $O(1)$. Actually we use the method of completing square to obtain the following equality

$$\int_{\ln k}^{b_0}e^y\Gamma(a(\tau)-y,\tau)dy=\frac{e^{-\rho\tau}}{\sqrt{\pi}}e^{a(\tau)+(\alpha+1)\tau}\int_{\frac{\ln k-a(\tau)-(\alpha+2)\tau}{2\sqrt{\tau}}}^{\frac{b_0-a(\tau)-(\alpha+2)\tau}{2\sqrt{\tau}}}e^{-u^2}du.$$

and a proper change of variable can lead to

$$\int_{\ln k}^{b_0} \Gamma(a(\tau) - y, \tau)dy = \frac{e^{-\rho\tau}}{\sqrt{\pi}} \int_{\frac{\ln k - a(\tau) - \alpha\tau}{2\sqrt{\tau}}}^{\frac{b_0 - a(\tau) - \alpha\tau}{2\sqrt{\tau}}} e^{-u^2} du.$$

The integrals in the right sides of above two equalities can be simplified by the following result: as $t \to \infty$,

$$\int_{at^{-1}-ct}^{bt^{-1}-ct} e^{-u^2} du = [\frac{e^{2bc} - e^{2ac}}{2c} + O(1)t^{-2}]t^{-1}e^{-c^2t^2},$$

if the parameter c equals zero, we should understand the value of $\frac{e^{2bc}-e^{2ac}}{2c}$ by its limits $b - a$. This formula can be proven by L'hospital rule. The last term in (17) is showed as: with $\tau \to \infty$,

$$k\Gamma(a(\tau) - \ln k, \tau) = [c_1 + O(1)\tau^{-1}]\tau^{-\frac{1}{2}}e^{-\lambda\tau}, \tag{22}$$

with $c_1 = \frac{k}{\sqrt{4\pi}} e^{\frac{\alpha(\ln k - \bar{a})}{2}}$. Combining (21) and (22), equality (17) tells us that

$$\begin{aligned} &\rho(k-1)\int_0^\tau \Gamma(a(\tau) - a(u), \tau - u)\dot{a}(u)du - \int_0^\tau (\ell e^{b(u)} - \rho)\Gamma(a(\tau) - b(u), \tau - u)\dot{b}(u)du \\ &= -[c_1 + c_a + O(1)\tau^{-1}]\tau^{-\frac{1}{2}}e^{-\lambda\tau}. \end{aligned} \tag{23}$$

To deal with equality (18), we need use the relationship equality $\Gamma_x(x, \tau) = -\frac{(x+\alpha\tau)}{2\tau}\Gamma(x, \tau)$. We handle the first term in the left side of equality (18). Direct computations can produce the result

$$2\int_{\ln k}^{b_0} \rho\Gamma_x(a(\tau) - y, \tau)dy = \frac{\rho e^{-\rho\tau}}{\sqrt{\pi}\sqrt{\tau}}\Big[e^{-\frac{(\ln k - a(\tau) - \alpha\tau)^2}{4\tau}} - e^{-\frac{(b_0 - a(\tau) - \alpha\tau)^2}{4\tau}}\Big] = [c_2 + O(1)\tau^{-1}]\tau^{-\frac{1}{2}}e^{-\lambda\tau} \tag{24}$$

where $c_2 = \frac{\rho}{\sqrt{\pi}}[e^{\frac{(\ln k - \bar{a})\alpha}{4}} - e^{\frac{(b_0 - \bar{a})\alpha}{4}}]$. Letting $u = y - a(\tau) - (\alpha+2)\tau$, $u_2 = b_0 - a(\tau) - (\alpha+2)\tau$ and $u_1 = \ln k - a(\tau) - (\alpha+2)\tau$, we can compute the following integral

$$\begin{aligned} -2\int_{\ln k}^{b_0} \ell e^y \Gamma_x(a(\tau) - y, \tau)dy &= \frac{-\ell e^{-\rho\tau}}{\sqrt{\pi\tau}} e^{a(\tau)+(\alpha+1)\tau} \int_{u_1}^{u_2} \frac{u}{2\tau} e^{-\frac{u^2}{4\tau}} du - 2\int_{\ln k}^{b_0} \ell e^y \Gamma(a(\tau) - y, \tau)dy \\ &= [c_3 + O(1)\tau^{-1}]\tau^{-\frac{1}{2}}e^{-\lambda\tau} \end{aligned} \tag{25}$$

where the amount is defined by $c_3 = (\frac{\ell e^{\bar{a}}}{\sqrt{\pi}} - \frac{\ell e^{\bar{a}}}{\sqrt{\pi}}\frac{2}{\alpha+2})[e^{(b_0-\bar{a})\frac{\alpha+2}{2}} - e^{(\ln k - \bar{a})\frac{\alpha+2}{2}}]$. The second and third terms in the left side of (18) can be computed as follows:

$$\begin{aligned} &2\rho(k-1)\int_0^\tau \Gamma_x(a(\tau) - a(u), \tau - u)\dot{a}(u)du - 2\int_0^\tau (\ell e^{b(u)} - \rho)\Gamma_x(a(\tau) - b(u), \tau - u)\dot{b}(u)du \\ &= \begin{cases} (c_1\bar{a} + c_a\bar{a} + O(1)\tau^{-1})\tau^{-\frac{3}{2}}e^{-\lambda\tau}; & \alpha = 0 \\ (c_1\alpha + c_a\alpha + O(1)\tau^{-1})\tau^{-\frac{1}{2}}e^{-\lambda\tau}; & \alpha \neq 0. \end{cases} \end{aligned} \tag{26}$$

The last term in the left side of equality (18) is showed as

$$2k\Gamma_x(a(\tau) - \ln k, \tau) = [-\alpha c_1 + O(1)\tau^{-1}]\tau^{-\frac{1}{2}}e^{-\lambda\tau}, \text{ as } \alpha \neq 0. \tag{27}$$

and

$$2k\Gamma_x(a(\tau) - \ln k, \tau) = [(\ln k - \bar{a})c_1 + O(1)\tau^{-1}]\tau^{-\frac{3}{2}}e^{-\lambda\tau}, \text{ as } \alpha = 0. \tag{28}$$

Combing the results (21–28), equality (18) can be simplified into the following forms. As $\alpha = 0$, the coefficients become as $c_1 = \frac{k}{\sqrt{4\pi}}$, $c_2 = 0$ and $c_3 = 0$. Thus equality (18) will become as

$$-\rho(k-1)\dot{a}(\tau) = O(1)\tau^{-\frac{3}{2}}e^{-\lambda\tau}, \text{ and furthermore, } \dot{a}(\tau) = O(1)\tau^{-\frac{3}{2}}e^{-\lambda\tau}.$$

Integration from τ to ∞, we can get that, for $\alpha = 0$,

$$a(\tau) = \bar{a} + O(1)\tau^{-\frac{3}{2}} e^{-\lambda\tau}. \tag{29}$$

As $\alpha \neq 0$, equality (18) will become as

$$-\rho(k-1)\dot{a}(\tau) = [c_2 + c_3 + \alpha c_a + O(1)\tau^{-1}]\tau^{-\frac{1}{2}} e^{-\lambda\tau}. \tag{30}$$

Integration from τ to ∞, we obtain the following result

$$a(\tau) = \bar{a} + [\bar{c}_2 + \bar{c}_3 + \alpha\bar{c}_a + O(1)\tau^{-1}]\tau^{-\frac{1}{2}} e^{-\lambda\tau}. \tag{31}$$

where the coefficients $\bar{c}_2, \bar{c}_3, \bar{c}_a$ can be written accordingly after integration by the equality (30) .

Next, we investigate the behavior of upper boundary $b(\tau)$. To this end, we deal with the terms in (19). By some computations similar to (21), we can obtain the following result for the first term in (19)

$$\int_{\ln k}^{b_0} (\rho - \ell e^{y})\Gamma(b(\tau) - y, \tau)dy = [c_b + O(1)\tau^{-1}]\tau^{-\frac{1}{2}} e^{-\lambda\tau} \tag{32}$$

where the coefficient is defined as $c_b = \frac{\rho}{\sqrt{\pi}\alpha}[e^{\frac{\alpha}{2}(b_0-\bar{b})} - e^{\frac{\alpha}{2}(\ln k-\bar{b})}] - \frac{\ell e^{\bar{b}}}{\sqrt{\pi}}\frac{1}{\alpha+2}[e^{(b_0-\bar{b})\frac{\alpha+2}{2}} - e^{(\ln k-\bar{b})\frac{\alpha+2}{2}}]$. The last term in (19) is showed as

$$k\Gamma(b(\tau) - \ln k, \tau) = [c_4 + O(1)\tau^{-1}]\tau^{-\frac{1}{2}} e^{-\lambda\tau}, \text{ with } c_4 = \frac{k}{\sqrt{4\pi}} e^{\frac{\alpha(\ln k-\bar{b})}{2}}. \tag{33}$$

Combining (32) and (33), the sum of the middle two terms in (19) equals to

$$\begin{aligned}\rho(k-1)\int_0^{\tau} \Gamma(b(\tau) - a(u), \tau - u)\dot{a}(u)du - \int_0^{\tau} (\ell e^{b(u)} - \rho)\Gamma(b(\tau) - b(u), \tau - u)\dot{b}(u)du \\ = -[c_4 + c_b + O(1)\tau^{-1}]\tau^{-\frac{1}{2}} e^{-\lambda\tau}.\end{aligned} \tag{34}$$

Similar to (24), we handle the first term in the left side of equality (18). Direct computations can demonstrate the result

$$2\int_{\ln k}^{b_0} \rho\Gamma_x(b(\tau) - y, \tau)dy = \frac{\rho e^{-\rho\tau}}{\sqrt{\pi}\sqrt{\tau}}\Big[e^{-\frac{(\ln k-b(\tau)-\alpha\tau)^2}{4\tau}} - e^{-\frac{(b_0-b(\tau)-\alpha\tau)^2}{4\tau}}\Big] = [c_5 + O(1)\tau^{-1}]\tau^{-\frac{1}{2}} e^{-\lambda\tau} \tag{35}$$

where $c_5 = \frac{\rho}{\sqrt{\pi}}[e^{\frac{(\ln k-\bar{b})\alpha}{4}} - e^{\frac{(b_0-\bar{b})\alpha}{4}}]$. Letting $v = y - b(\tau) - (\alpha+2)\tau$, $v_2 = b_0 - b(\tau) - (\alpha+2)\tau$ and $v_1 = \ln k - b(\tau) - (\alpha+2)\tau$, we can compute the integral in first term of (20)

$$\begin{aligned}-2\int_{\ln k}^{b_0} \ell e^{y}\Gamma_x(b(\tau) - y, \tau)dy = \frac{-\ell e^{-\rho\tau}}{\sqrt{\pi\tau}} e^{b(\tau)+(\alpha+1)\tau} \int_{v_1}^{v_2} \frac{v}{2\tau} e^{-\frac{v^2}{4\tau}} dv - 2\int_{\ln k}^{b_0} \ell e^{y}\Gamma(b(\tau) - y, \tau)dy \\ = [c_6 + O(1)\tau^{-1}]\tau^{-\frac{1}{2}} e^{-\lambda\tau}\end{aligned} \tag{36}$$

where the notation is defined by $c_6 = (\frac{\ell e^{\bar{b}}}{\sqrt{\pi}} - \frac{\ell e^{\bar{b}}}{\sqrt{\pi}}\frac{2}{\alpha+2})[e^{(b_0-\bar{b})\frac{\alpha+2}{2}} - e^{(\ln k-\bar{b})\frac{\alpha+2}{2}}]$. The middle two terms in the left side of (20) can be computed as follows:

$$\begin{aligned}&2\rho(k-1)\int_0^{\tau} \Gamma_x(b(\tau) - a(u), \tau - u)\dot{a}(u)du - 2\int_0^{\tau} (\ell e^{b(u)} - \rho)\Gamma_x(b(\tau) - b(u), \tau - u)\dot{b}(u)du \\ &= \begin{cases} (c_4\bar{b} + c_b\bar{b} + O(1)\tau^{-1})\tau^{-\frac{3}{2}} e^{-\lambda\tau}; & \alpha = 0 \\ (c_4\alpha + c_b\alpha + O(1)\tau^{-1})\tau^{-\frac{1}{2}} e^{-\lambda\tau}; & \alpha \neq 0. \end{cases}\end{aligned} \tag{37}$$

The last term in the left side of equality (20) is showed as

$$2k\Gamma_x(b(\tau) - \ln k, \tau) = [-\alpha c_4 + O(1)\tau^{-1}]\tau^{-\frac{1}{2}}e^{-\lambda\tau}, \text{ as } \alpha \neq 0. \tag{38}$$

and

$$2k\Gamma_x(b(\tau) - \ln k, \tau) = [(\ln k - \bar{b})c_4 + O(1)\tau^{-1}]\tau^{-\frac{3}{2}}e^{-\lambda\tau}, \text{ as } \alpha = 0. \tag{39}$$

Combing the results (32–39), equality (20) can be simplified into the following forms. As $\alpha = 0$, the coefficients become as $c_4 = \frac{k}{\sqrt{4\pi}}$, $c_5 = 0$ and $c_6 = 0$. Thus equality (20) will become as

$$-(\ell e^{b(\tau)} - \rho)\dot{b}(\tau) = O(1)\tau^{-\frac{3}{2}}e^{-\lambda\tau},$$

Since $\lim_{\tau\to\infty}(\ell e^{b(\tau)} - \rho) = \ell e^{\bar{b}} - \rho$, we furthermore obtain that $\dot{b}(\tau) = O(1)\tau^{-\frac{3}{2}}e^{-\lambda\tau}$. Integration from τ to ∞, we can get that, for $\alpha = 0$,

$$b(\tau) = \bar{b} + O(1)\tau^{-\frac{3}{2}}e^{-\lambda\tau}. \tag{40}$$

As $\alpha \neq 0$, equality (20) will become as

$$-(\ell e^{b(\tau)} - \rho)\dot{b}(\tau) = [c_5 + c_6 + \alpha c_b + O(1)\tau^{-1}]\tau^{-\frac{1}{2}}e^{-\lambda\tau}. \tag{41}$$

Since $\lim_{\tau\to\infty}(\ell e^{b(\tau)} - \rho) = \ell e^{\bar{b}} - \rho$, equality (41) reduces to

$$\dot{b}(\tau) = [c_5' + c_6' + \alpha c_b' + O(1)\tau^{-1}]\tau^{-\frac{1}{2}}e^{-\lambda\tau}. \tag{42}$$

where the coefficients c_5', c_6', c_b' are given by c_5, c_6, c_b dividing by $\rho - \ell e^{\bar{b}}$. Integration from τ to ∞, we obtain the following result

$$b(\tau) = \bar{b} + [\bar{c}_7 + \bar{c}_8 + \alpha\bar{c}_b + O(1)\tau^{-1}]\tau^{-\frac{1}{2}}e^{-\lambda\tau}. \tag{43}$$

where the coefficients $\bar{c}_5, \bar{c}_6, \bar{c}_b$ can be written accordingly after integration by the equality (42). Therefore, we finish the proof of Theorem 3.1.

4. Long time asymptotic behavior of the value function

In this section, we will prove the following theorem for the long time asymptotic behavior of the rescaled value function.

Theorem 4.1. *There exists a constant ϑ such that for $\tau \to \infty$*

$$0 \leq \bar{f}(x) - f(x, \tau) \leq \vartheta\tau^{-\frac{1}{2}}e^{-\lambda\tau}, \ \ \forall x \in (-\infty, \infty).$$

or in brief

$$\bar{f}(x) - f(x, \tau) = O(1)\tau^{-\frac{1}{2}}e^{-\lambda\tau}, \ \ \forall x \in (-\infty, \infty).$$

Proof. Let us recall that the rescaled value function of perpetual American maximum has the form

$$\bar{f}(x) = \begin{cases} k - 1; & \text{as } \ x \leq \bar{a}, \\ (k-1)[\frac{\gamma_1}{\gamma_1 - \gamma_0}e^{\gamma_0(x-\bar{a})} + \frac{\gamma_0}{\gamma_0 - \gamma_1}e^{\gamma_1(x-\bar{a})}]; & \text{as } \ \bar{a} < x < \bar{b}, \\ e^x - 1; & \text{as } \ x \geq \bar{b}. \end{cases}$$

To facilitate the analysis, we define a function $h(x,\tau) = \bar{f}(x) - f(x,\tau)$, which is a piecewise function with the following form:

$$h(x,\tau) = \begin{cases} 0; & \text{as} \quad x \le \bar{a} \\ (k-1)[\frac{\gamma_1}{\gamma_1-\gamma_0}e^{\gamma_0(x-\bar{a})} + \frac{\gamma_0}{\gamma_0-\gamma_1}e^{\gamma_1(x-\bar{a})}] - (k-1); & \text{as} \quad \bar{a} < x \le a(\tau) \\ (k-1)[\frac{\gamma_1}{\gamma_1-\gamma_0}e^{\gamma_0(x-\bar{a})} + \frac{\gamma_0}{\gamma_0-\gamma_1}e^{\gamma_1(x-\bar{a})}] - f(x,\tau); & \text{as} \quad a(\tau) < x < b(\tau) \\ (k-1)[\frac{\gamma_1}{\gamma_1-\gamma_0}e^{\gamma_0(x-\bar{a})} + \frac{\gamma_0}{\gamma_0-\gamma_1}e^{\gamma_1(x-\bar{a})}] - (e^x-1); & \text{as} \quad b(\tau) \le x < \bar{b} \\ 0; & \text{as} \quad x \ge \bar{b}. \end{cases} \tag{44}$$

Note that, for $a(\tau) < x < b(\tau)$, $f(x,\tau)$ satisfies the equation $f_\tau - \mathcal{L}f = 0$, and for $\bar{a} < x < \bar{b}$, $\bar{f}(x)$ satisfies the equation $\bar{f}_\tau - \mathcal{L}\bar{f} = 0$. Thus, A direct computation will demonstrate us that

$$h_\tau - \mathcal{L}h = \begin{cases} 0; & \text{as} \quad x \le \bar{a} \\ -\rho(k-1); & \text{as} \quad \bar{a} < x \le a(\tau) \\ 0; & \text{as} \quad a(\tau) < x < b(\tau) \\ \rho - \ell e^x; & \text{as} \quad b(\tau) < x < \bar{b} \\ 0; & \text{as} \quad x \ge \bar{b}. \end{cases} \tag{45}$$

and

$$h(x,0) = \begin{cases} 0; & \text{as} \quad x \le \bar{a} \\ (k-1)[\frac{\gamma_1}{\gamma_1-\gamma_0}e^{\gamma_0(x-\bar{a})} + \frac{\gamma_0}{\gamma_0-\gamma_1}e^{\gamma_1(x-\bar{a})}] - (k-1); & \text{as} \quad \bar{a} < x \le a_0 \\ (k-1)[\frac{\gamma_1}{\gamma_1-\gamma_0}e^{\gamma_0(x-\bar{a})} + \frac{\gamma_0}{\gamma_0-\gamma_1}e^{\gamma_1(x-\bar{a})}] - (e^x-1); & \text{as} \quad a_0 \le x < \bar{b} \\ 0; & \text{as} \quad x \ge \bar{b}. \end{cases} \tag{46}$$

where $a_0 := \lim\limits_{\tau\to 0} a(\tau) = \ln k\ (>0)$ and $b_0 := \lim\limits_{\tau\to 0} b(\tau) = \max\{\ln k, \ln\frac{\rho}{\ell}\}$. Since $x > a_0$, it leads to $f_0(x) = e^x - 1$, and we have incorporated the forms of $h(x,0)$ in the intervals (a_0, b_0) and $[b_0, \bar{b})$. By virtue of Green's Theorem, $h(x,\tau)$ admits the following integral representation

$$\begin{aligned} h(x,\tau) &= \int_{-\infty}^{\infty} h(y,0)\Gamma(x-y,\tau)dy - \int_0^\tau \int_{\bar{a}}^{a(u)} \rho(k-1)\Gamma(x-y,\tau-u)dydu \\ &\quad + \int_0^\tau \int_{b(u)}^{\bar{b}} (\rho - \ell e^y)\Gamma(x-y,\tau-u)dydu \\ &= (k-1)\int_{\bar{a}}^{a_0} [\frac{\gamma_1}{\gamma_1-\gamma_0}e^{\gamma_0(y-\bar{a})} + \frac{\gamma_0}{\gamma_0-\gamma_1}e^{\gamma_1(y-\bar{a})} - 1]\Gamma(x-y,\tau)dy \\ &\quad + \int_{a_0}^{\bar{b}} \Big((k-1)[\frac{\gamma_1}{\gamma_1-\gamma_0}e^{\gamma_0(y-\bar{a})} + \frac{\gamma_0}{\gamma_0-\gamma_1}e^{\gamma_1(y-\bar{a})}] - (e^y-1)\Big)\Gamma(x-y,\tau)dy \\ &\quad - \int_0^\tau \int_{\bar{a}}^{a(u)} \rho(k-1)\Gamma(x-y,\tau-u)dydu + \int_0^\tau \int_{b(u)}^{\bar{b}} (\rho - \ell e^y)\Gamma(x-y,\tau-u)dydu. \end{aligned} \tag{47}$$

Since $b(u) \ge b_0$ for $u \ge 0$ and $b_0 = \max\{\ln k, \ln\frac{\rho}{\ell}\}$, it will lead to $\ell e^y \ge \max\{k\ell, \rho\}$ and $\rho - \ell e^y \le \min\{\rho - k\ell, 0\} \le 0$ for $y \ge b(u)$. Besides, plus $-\rho(k-1) < 0$, we derive that

$$\begin{aligned} h(x,\tau) &\le (k-1)\int_{\bar{a}}^{a_0} [\frac{\gamma_1}{\gamma_1-\gamma_0}e^{\gamma_0(y-\bar{a})} + \frac{\gamma_0}{\gamma_0-\gamma_1}e^{\gamma_1(y-\bar{a})} - 1]\Gamma(x-y,\tau)dy \\ &\quad + \int_{a_0}^{\bar{b}} \Big((k-1)[\frac{\gamma_1}{\gamma_1-\gamma_0}e^{\gamma_0(y-\bar{a})} + \frac{\gamma_0}{\gamma_0-\gamma_1}e^{\gamma_1(y-\bar{a})}] - (e^y-1)\Big)\Gamma(x-y,\tau)dy \\ &\le (k-1)\int_{\bar{a}}^{\bar{b}} [\frac{\gamma_1}{\gamma_1-\gamma_0}e^{\gamma_0(y-\bar{a})} + \frac{\gamma_0}{\gamma_0-\gamma_1}e^{\gamma_1(y-\bar{a})} - 1]\Gamma(x-y,\tau)dy \end{aligned}$$

$$\begin{aligned}
&\leq (k-1)e^{-\lambda\tau}\frac{1}{\sqrt{4\pi\tau}}\int_{\bar{a}}^{\bar{b}}[\frac{\gamma_1}{\gamma_1-\gamma_0}e^{\gamma_0(y-\bar{a})}+\frac{\gamma_0}{\gamma_0-\gamma_1}e^{\gamma_1(y-\bar{a})}-1]e^{-\frac{(x-y)\alpha}{2}}dy \\
&\leq (k-1)e^{-\lambda\tau}\frac{1}{\sqrt{4\pi\tau}}e^{\frac{(\bar{b}-\bar{a})|\alpha|}{2}}\int_{\bar{a}}^{\bar{b}}[\frac{\gamma_1}{\gamma_1-\gamma_0}e^{\gamma_0(y-\bar{a})}+\frac{\gamma_0}{\gamma_0-\gamma_1}e^{\gamma_1(y-\bar{a})}-1]dy \\
&= \vartheta\tau^{-\frac{1}{2}}e^{-\lambda\tau}. \qquad (48)
\end{aligned}$$

where the second inequality comes from the fact that $e^y - 1 \geq k-1$ for $y \geq a_0 = \ln k$. The third inequality is true because of the basic fact

$$\Gamma(x,\tau) = \frac{1}{\sqrt{4\pi\tau}}e^{-\frac{x^2}{4\tau}}e^{-\frac{\alpha x}{2}}e^{-\lambda\tau} \leq \frac{1}{\sqrt{4\pi\tau}}e^{-\frac{\alpha x}{2}}e^{-\lambda\tau}.$$

Because the function $h(x,\tau)$ equals zero for $x \geq \bar{b}$ and $x \leq \bar{a}$. The fourth inequality is true since we have the result $|x-y| \leq \bar{b}-\bar{a}$ for $\bar{a} < x < \bar{b}$ and $\bar{a} \leq y \leq \bar{b}$. The coefficient is defined as $\vartheta = \frac{k-1}{\sqrt{4\pi}}e^{\frac{(\bar{b}-\bar{a})|\alpha|}{2}}\Big[\frac{\gamma_1}{\gamma_0(\gamma_1-\gamma_0)}(e^{\gamma_0(\bar{b}-\bar{a})}-1)+\frac{\gamma_0}{\gamma_1(\gamma_0-\gamma_1)}(e^{\gamma_1(\bar{b}-\bar{a})}-1)-(\bar{b}-\bar{a})\Big]$. From above result (48), we have already obtained that

$$0 \leq \bar{f}(x) - f(x,\tau) \leq \vartheta\tau^{-\frac{1}{2}}e^{-\lambda\tau}, \ \ \forall x \in (-\infty,\infty), \qquad (49)$$

that is to say

$$\bar{f}(x) - f(x,\tau) = O(1)\tau^{-\frac{1}{2}}e^{-\lambda\tau}, \ \ \forall x \in (-\infty,\infty). \qquad (50)$$

Therefore, we finish the proof of Theorem 4.1.

5. Numerical Examples

In this section, we consider the numerical method to the price function and optimal exercise boundaries of the American maximum option using finite difference. For the problem 2, we rewrite it as follows:

$$\begin{cases} \max\{\partial_{xx}f + \alpha\partial_x f - \rho f - \partial_\tau f, f_0 - f\} = 0; & \text{in} \quad (-\infty,+\infty)\times(0,T), \\ f(x,0) = f_0(x) := \max\{k-1, e^x - 1\}; & \text{on} \quad (-\infty,+\infty)\times\{0\}. \end{cases} \qquad (51)$$

For fixed mesh size $\Delta x, \Delta\tau > 0$, we denote by $f_j^n = f(j\Delta x, n\Delta\tau)$ the numerical solution of the price at point $(j\Delta x, n\Delta\tau)$, we adopt difference quotient to replace the derivatives and obtain the following difference equation:

$$\begin{cases} \max\Big\{\frac{f_{j+1}^{n-1}-2f_j^{n-1}+f_{j-1}^{n-1}}{(\Delta x)^2} + \alpha\frac{f_{j+1}^{n-1}-f_{j-1}^{n-1}}{2\Delta x} - \rho f_j^n - \frac{f_j^n - f_j^{n-1}}{\Delta\tau}, \max\{k-1, e^{j\Delta x}-1\} - f_j^n\Big\} = 0, \\ f_j^0 = \max\{k-1, e^{j\Delta x}-1\}. \end{cases} \qquad (52)$$

We select a proper pair of $(\Delta x, \Delta\tau)$ such that $\frac{2\Delta\tau}{(\Delta x)^2} = 1$ and obtain the following expressions

$$\begin{cases} f_j^n = \max\Big\{\frac{1}{1+\rho\Delta\tau}[(\frac{1}{2}+\frac{\alpha}{4}\sqrt{2\Delta\tau})f_{j+1}^{n-1} + (\frac{1}{2}-\frac{\alpha}{4}\sqrt{2\Delta\tau})f_{j-1}^{n-1}], k-1, e^{j\Delta x}-1\Big\}, \\ f_j^0 = \max\{k-1, e^{j\Delta x}-1\}. \end{cases} \qquad (53)$$

For the convenience of formulation, we need to define some new notations $\theta = e^{\sqrt{2\Delta\tau}}, d = \theta^{-1}, \beta = e^{\rho\Delta\tau}, p = \frac{\beta e^{-\ell\Delta\tau}-d}{\theta-d}$. Due to asymptotic expansion, as $\Delta\tau \to 0^+$, we obtain that

$$\frac{1}{2} + \frac{\alpha}{4}\sqrt{2\Delta\tau} = p + o(\sqrt{\Delta\tau}), \qquad \frac{1}{1+\rho\Delta\tau} = \frac{1}{\beta} + O(\Delta\tau^2).$$

Discarding the higher order term of $\sqrt{\Delta\tau}$, we can acquire

$$\begin{cases} f_j^n = \max\left\{\frac{1}{\beta}[pf_{j+1}^{n-1} + (1-p)f_{j-1}^{n-1}], k-1, \theta^j - 1\right\}, \\ f_j^0 = \max\{k-1, \theta^j - 1\}. \end{cases} \tag{54}$$

In the following parts, we consider numerical examples where the asset price model consists of the parameters $r = 0.04$, $q = 0.04$, $\sigma = 0.2$, and the option has the strike price $K = 40$, $L = 60$. To study the long time behavior of maximum option, we let the time-to- maturity be $\tau_0 = T - t = 2, 4, 6, 8, 10$ years and the asset's price staying in the interval $[0, 100]$. Using the method of finite difference, we will compute the prices and exercise boundaries of American maximum option under the different maturities, then compare them with the ones of perpetual American maximum option. Figure 1 gives a demonstration of the value functions of American maximum option under several different maturities. With increase of the maturities, it is apparent that the finite-time value functions will approach the perpetual value function. Figure 2 provides an exhibition of the change of two exercise boundaries with time and comparison with the ones of perpetual option. We can see that the lower boundary converges to the one of perpetual maximum option faster. Figure 3 offers a show of the maximum value difference of perpetual American maximum option and finite-time ones.

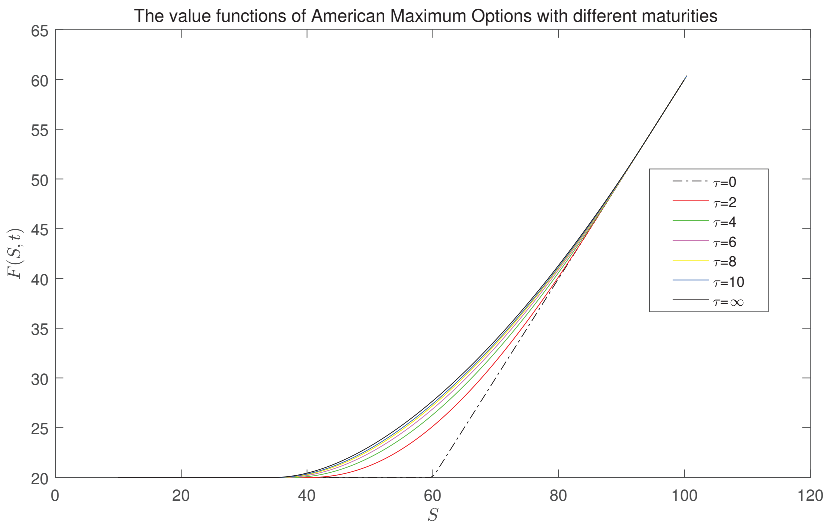

Figure 1: American Maximum Options with Different Maturities

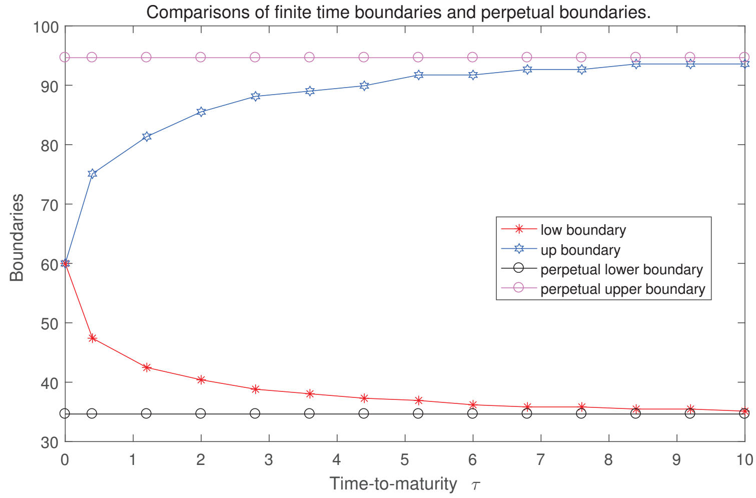

Figure 2: Boundaries of American Maximum options with Different Maturities

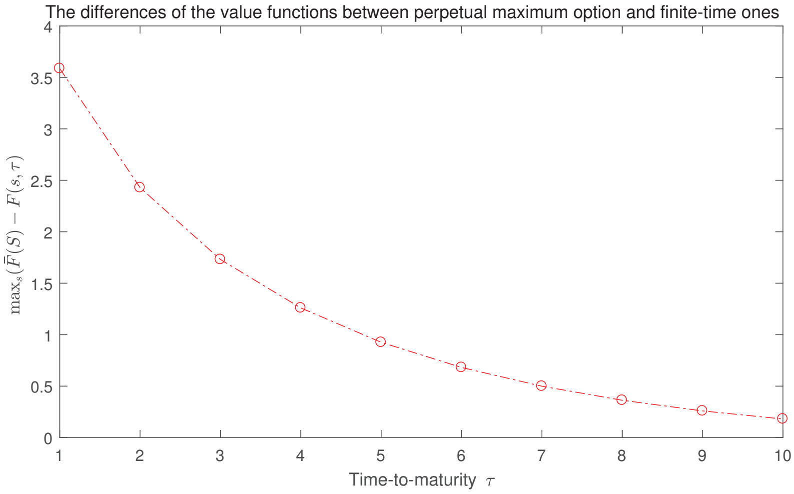

Figure 3: The difference of perpetual and finite-time American Maximum options

6. Conclusions

In this paper, we have investigated the long time asymptotic behaviors of the exercise boundaries and the scaled value function of American maximum option. For different relationships among related parameters, the exercise boundaries exhibit different long time asymptotic behaviors. Theses asymptotic results are useful for the approximation of American maximum option with long maturity. With above results, the error analysis can be performed when we use perpetual American maximum option to take place of the one with long maturity.

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