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Existence and Ulam stability results of hybrid Langevin pantograph ψ-fractional coupled systems

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Abstract. This study presents results on the solutions of a coupled system of hybrid Langevin fractional pantograph differential equations involving ψ-Caputo type fractional derivatives within Banach spaces. We establish the uniqueness of solutions using Banach's fixed-point theorem and confirm their existence through Dhage's hybrid fixed-point theorem for the sum of three operators. Additionally, we investigate the stability of these solutions in both the Ulam-Hyers sense and its generalized form. The theoretical findings are further supported by several illustrative examples.

1. Introduction

Due to their extensive applications in modeling diverse scientific and technical phenomena, fractional-order differential equations have garnered significant interest from researchers. These equations are employed in the study of various fields, including blood flow dynamics, electrical circuits, biology, chemistry, physics, control theory, wave propagation, and signal and image processing, among others. For further details, see [1–3, 12–15, 34, 37, 38]. In recent years, researchers have introduced various fractional operators. In 2017, Ricardo Almeida expanded the field by introducing the ψ-Caputo operator, thereby adding to the existing repertoire, which includes the Caputo, Caputo-Hadamard, Caputo-Erdélyi-Kober, and Caputo-Katugampola operators, see [10, 25].

Nonlinear coupled systems involving fractional derivatives have become a significant focus in contemporary research due to their applications in various areas of applied mathematics. Consequently, numerous studies and books have explored the existence, stability, and uniqueness of solutions for various fractional differential equations and inclusions, employing different fractional derivatives and boundary conditions.

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For more comprehensive details, refer to sources such as [4, 5, 8, 11, 16, 33]. Additionally, dynamic systems are often studied as particular instances of fractional differential equations. Hybrid fractional differential equations, which involve partial derivatives of a hybrid function with nonlinear dependence, are also a prominent area of modern research. Recent advances in this field are detailed in several research papers, see [6, 17, 20, 27, 28].

In [29], the authors Matar *et al.* investigated the following nonlinear fractional differential hybrid system subject to periodic boundary conditions of the form

$$\begin{cases} {}^{C}D_{0^+}^{\alpha_1,\psi}\left(z(\tau)g_1(\tau)\right) = f_1(\tau, z(\tau), y(\tau)), \quad \tau \in [a_1, a_2], \\ {}^{C}D_{0^+}^{\alpha_2,\psi}\left(y(\tau)g_2(\tau)\right) = f_2(\tau, z(\tau), y(\tau)), \quad \tau \in [a_1, a_2], \\ z(a_1) = z(a_2),\ z'(a_1) = z'(a_2), \\ y(a_1) = y(a_2),\ y'(a_1) = y'(a_2), \end{cases}$$

where ${}^{C}D_{0^+}^{\alpha_1,\psi}$ and ${}^{C}D_{0^+}^{\alpha_2,\psi}$ are the ψ-Caputo fractional derivatives of order $\alpha_1, \alpha_2 \in (1,2)$, $(g_i)_{i=1,2} : [a,b] \to \mathbb{R} \setminus \{0\}$ and $(f_i)_{i=1,2} : [a_1, a_2] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions.

Paul Langevin introduced the classical Langevin equation in 1908 to model the dynamics of physical systems under fluctuating conditions [26]. Since then, researchers have explored various generalizations of this equation [21, 22].

In [32], Salem *et al.* study the existence and uniqueness of solutions to dual systems of nonlinear fractional Langevin differential equations of the Caputo type with boundary value conditions given as follows:

$$\begin{cases} {}^{C}D_{0^+}^{\alpha_1}\left({}^{C}D_{0^+}^{\beta_1} + \lambda_1\right) z(\tau) = f_1(\tau, z(\tau), y(\tau)),\ \tau \in [0,1],\ 0 < \beta_1 \le 1,\ 1 < \alpha_1 \le 2, \\ {}^{C}D_{0^+}^{\alpha_2}\left({}^{C}D_{0^+}^{\beta_2} + \lambda_2\right) y(\tau) = f_2(\tau, z(\tau), y(\tau)),\ \tau \in [0,1],\ 0 < \beta_2 \le 1,\ 1 < \alpha_2 \le 2, \\ z(0) = 0,\ {}^{C}D_{0^+}^{\beta_1} z(0) = \Gamma(\beta_1 + 1)\, {}_1^{\gamma}I_{0^+}^{\nu_1} z(\epsilon_1), \\ \sum_{i=1}^{m_1} \rho_{i_1} z(\varepsilon_{i_1}) = \mu_1 {}^{AB}I_{0^+}^{\nu_2} z(\epsilon_2), \\ y(0) = 0,\ {}^{C}D_{0^+}^{\beta_2} y(0) = \Gamma(\beta_2 + 1)\, {}_2^{\gamma}I_{0^+}^{\nu_3} y(\epsilon_3), \\ \sum_{i=1}^{m_2} \rho_{i_2} y(\varepsilon_{i_2}) = \mu_2 {}^{AB}I_{0^+}^{\nu_4} y(\epsilon_4), \end{cases}$$

where ${}^{C}D_{0^+}$ is the Caputo fractional derivative of order α_j and β_j for j = 1, 2. ${}^{AB}I_{0^+}$ and ${}^{\gamma}I_{0^+}$ are Atangana-Baleanu, and Katugampola fractional integrals, respectively. $\gamma_i > 0$ and $\Lambda_i, \mu_i \in \mathbb{R}$ for $= 1, 2$, $\nu_n \in \mathbb{R}$ for $n = 1, 2, 3, 4$. $\rho_{i_j} \in \mathbb{R}$ for $i = 1, ..., m_i$ and j = 1, 2. $0 < \epsilon_n < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \cdots < \varepsilon_{m_i}$ for $i = 1, 2$ and $n = 1, 2, 3, 4$. $f_1, f_1 : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions.

The pantograph equation is a versatile differential equation applied across diverse fields, including electrodynamics, astrophysics, and cellular growth modeling. This broad applicability has led to a surge of recent studies on the fractional order pantograph equation by various researchers, see [19, 24, 39].

Additionaly, I. Ahmad *et al.* [7] demonstrated the existence of solutions for a nonlinear coupled system of pantograph fractional differential equations with Caputo fractional derivatives of the form

$$\begin{cases} {}^{C}D_{0^+}^{\alpha_1} z(\tau) = f_1(\tau, z(\tau), z(\vartheta\tau), y(\tau)),\ \tau \in [0,1], \\ {}^{C}D_{0^+}^{\alpha_2} y(\tau) = f_2(\tau, z(\tau), y(\tau), y(\vartheta\tau)),\ \tau \in [0,1], \\ a_1 z(0) - b_1 z(\vartheta_1) - c_1 z(1) = g_1(z), \\ a_2 z(0) - b_2 z(\vartheta_2) - c_2 y(1) = g_2(y), \end{cases}$$

where ${}^{C}D_{0^+}^{\alpha_i}$ represent the Caputo derivatives of order $\alpha_i \in (0,1)$, $0 < \vartheta_i < 1$, $a_1 \neq b_i + c_i$ which $a_i\, b_i, c_i \in \mathbb{R}$, $f_i : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ and $g_i : C([0,1], \mathbb{R}) \to \mathbb{R}$, $i = 1, 2$.

In recent times, substantial attention has been directed towards investigating the Ulam stability of solutions for coupled systems of fractional differential equations. In [31], A. Salim *et al.* investigate the existence, uniqueness and Ulam stability of differential coupled system involving the Riesz-Caputo derivative with boundary conditions of the from

$$\begin{cases} {}_0^{RC}D_T^{\alpha_1} z(\tau) = f_1(\tau, z(\tau), y(\tau), {}_0^{RC}D_T^{\alpha_1} z(\tau), {}_0^{RC}D_T^{\alpha_2} y(\tau)), \ \tau \in [0, \mathcal{T}], \\ {}_0^{RC}D_T^{\alpha_2} y(\tau) = f_2(\tau, z(\tau), y(\tau), {}_0^{RC}D_T^{\alpha_1} z(\tau), {}_0^{RC}D_T^{\alpha_2} y(\tau)), \ \tau \in [0, \mathcal{T}], \\ \beta_1 z(0) + \beta_2 z(\mathcal{T}) = \beta_3, \\ \gamma_1 y(0) + \gamma_2 y(\mathcal{T}) = \gamma_3, \end{cases}$$

where ${}_0^{RC}D_{\mathcal{T}}^{\alpha_i}$ represent the Riesz-Caputo derivatives of order $\alpha_i \in (0,1)$ for $i = 1,2$, $\gamma_j, \beta_j \in \mathbb{R}$ for $j = 1,2,3$, $\gamma_1 + \gamma_2 \neq 0$ and $\beta_1 + \beta_2 \neq 0$. Also $f_1, f_2 : [0, \mathcal{T}] \times \mathbb{R}^4 \to \mathbb{R}$ are given continuous functions.

The study of Ulam stability in fractional differential equations introduces a novel approach for researchers, paving the way for exploring various topics in nonlinear analysis. Moreover, the stability analysis of fractional-order differential equations is more complex than that of classical differential equations, due to the nonlocal nature and weakly singular kernels of fractional derivatives. Consequently, Ulam-type stability issues have attracted considerable interest from numerous researchers, see [15, 18, 23, 30, 33, 36].

Leveraging insights from prior research, this study presents a novel examination of fractional hybrid differential systems that combine Langevin processes with pantograph arguments. The primary objective is to assess the existence, uniqueness, and Ulam-Hyers stability of solutions for these systems, which are governed by the ψ-Caputo derivative, as detailed below

$$\begin{cases} {}^{C}D_{a_1^+}^{\alpha_1,\psi}\left[{}^{C}D_{a_1^+}^{\beta_1,\psi}\left(\dfrac{z(\tau) - h_1(\tau, z(\tau))}{g_1(\tau, z(\tau))}\right) + \lambda_1 z(\tau)\right] = f_1(\tau, z(\tau), y(\xi\tau)), \quad \tau \in \mathcal{T} = [a_1, a_2], \\ {}^{C}D_{a_1^+}^{\alpha_2,\psi}\left[{}^{C}D_{a_1^+}^{\beta_2,\psi}\left(\dfrac{y(\tau) - h_2(\tau, y(\tau))}{g_2(\tau, y(\tau))}\right) + \lambda_2 y(\tau)\right] = f_2(\tau, y(\tau), z(\tilde{\xi}\tau)), \quad \tau \in \mathcal{T} = [a_1, a_2], \end{cases} \tag{1}$$

under the given boundary conditions

$$\begin{cases} \left(\dfrac{z(\tau) - h_1(\tau, z(\tau))}{g_1(\tau, z(\tau))}\right)\Bigg|_{\tau=a_1} = \vartheta_1, \ \left(\dfrac{z(\tau) - h_2(\tau, z(\tau))}{g_2(\tau, z(\tau))}\right)\Bigg|_{\tau=a_1} = \vartheta_2, \\ \left(\dfrac{z(\tau) - h_1(\tau, z(\tau))}{g_1(\tau, z(\tau))}\right)'\Bigg|_{\tau=a_1} = \left(\dfrac{z(\tau) - h_2(\tau, z(\tau))}{g_2(\tau, z(\tau))}\right)'\Bigg|_{\tau=a_1} = 0, \\ \left(\dfrac{z(\tau) - h_1(\tau, z(\tau))}{g_1(\tau, z(\tau))}\right)\Bigg|_{\tau=\epsilon_1} = \vartheta_3, \ \left(\dfrac{z(\tau) - h_2(\tau, z(\tau))}{g_2(\tau, z(\tau))}\right)\Bigg|_{\tau=\epsilon_2} = \vartheta_4, \ a_1 < \epsilon_1, \epsilon_2 < a_2, \end{cases} \tag{2}$$

where ${}^{C}D_{a_1^+}^{\alpha_i,\psi}$, ${}^{C}D_{a_1^+}^{\beta_i,\psi}$ are the ψ-Caputo fractional derivatives of order $\alpha_i \in (0,1]$, $\beta_i \in (1,2]$, for $i = 1,2$, $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$, $0 < a_1 < a_2$, $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \in \mathbb{R}$, ($\vartheta_1 \neq \vartheta_3$ and $\vartheta_2 \neq \vartheta_4$) and $0 < \xi, \tilde{\xi} < 1$. The given functions $f_j : \mathcal{T} \times \mathbb{R}^2 \to \mathbb{R}$, $h_j : \mathcal{T} \times \mathbb{R} \to \mathbb{R}$, and $g_j : \mathcal{T} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ are continuous with $j = 1,2$. It is noteworthy that this is the first recorded instance in the literature where a coupled fractional model simultaneously investigates both Langevin and pantograph systems.

The key novelties presented in this study are as follows:

- The coupled system (1)-(2) integrates both Langevin and pantograph arguments and involves diverse boundary conditions, providing a comprehensive extension of the arguments constructed in the literature.
- The deformation of two functions is used to express the coupled system (1)-(2) in hybrid mode, enabling the application of Dhage's hybrid fixed-point theorems.

– By consolidating numerous fractional derivatives into a single operator, the ψ-fractional operator facilitates the integration of classical results and the development of new applications.
– We further extend the findings in [32] by examining a coupled Langevin hybrid fractional system with multi-point conditions of orders $\beta_i \in (0,1]$ and $\alpha_i \in (1,2]$, for $i = 1,2$.
– The qualitative concepts of uniqueness, existence, and stability are examined for the first time in relation to the coupled system (1)-(2).

The structure of our paper is organized as follows: Section 2 introduces the fundamental concepts and definitions relevant to our study. Section 3 demonstrates the existence and uniqueness of solutions for the ψ-Caputo coupled system (1)-(2) through the application of fixed-point theorems. In Section 4, we investigate the stability of the coupled system within the framework of Ulam-Hyers stability and its generalizations. The final section presents illustrative examples that underscore the main findings of our research.

2. Preliminaries

In this section, we introduce some notations, definitions, and preliminary tools which are used throughout this paper.

Let $\mathcal{T} = [a_1, a_2]$. By $C(\mathcal{T}, \mathbb{R})$ we denote the Banach space of all continuous functions from $\mathcal{T} \longrightarrow \mathbb{R}$ with the norm

$$\|z\|_\infty = \sup\{|z(\tau)| : \tau \in \mathcal{T}\}.$$

By $C^n(\mathcal{T}, \mathbb{R})$, we denote the space of functions that are n-times continuously differentiable on $\mathcal{T}$.

Now, we consider the following Banach space

$$\Xi = \{(z, y) : z, y \in C(\mathcal{T}, \mathbb{R})\},$$

endowed with the norm

$$\|(z, y)\|_\Xi = \|z\| + \|y\|.$$

Let $\psi \in C^1(\mathcal{T}, \mathbb{R}^+)$ be an increasing differentiable function such that $\psi'(\tau) \neq 0$, for all $\tau \in \mathcal{T}$.

Definition 2.1 ([9]). *For $\alpha > 0$, the ψ-Riemann-Liouville fractional integral of order α for an integrable function $v : \mathcal{T} \mapsto \Xi$ is given by*

$$I_{a^+}^{\alpha,\psi} v(\tau) = \int_a^\tau \frac{\psi'(s)(\psi(\tau) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} v(s) ds, \tag{3}$$

where Γ is the classical Euler Gamma function.

Definition 2.2 ([9]). *Let $n-1 < \alpha < n$, $v : \mathcal{T} \mapsto \Xi$ be an integrable function and ψ be defined as in Definition 2.1. The ψ-Riemann Liouville fractional derivative of order α of a function v is defined by*

$$\begin{aligned} D_{a^+}^{\alpha,\psi} v(\tau) &= \left[\frac{1}{\psi'(\tau)} \frac{d}{d\tau}\right]^n I_{a^+}^{n-\alpha,\psi} v(\tau) \\ &= \left[\frac{1}{\psi'(\tau)} \frac{d}{d\tau}\right]^n \int_a^\tau \frac{\psi'(s)(\psi(\tau) - \psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)} v(s) ds, \end{aligned} \tag{4}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α.

Definition 2.3 ([9]). *Assume that $v \in C^n(\mathcal{T}, \Xi)$ and let ψ be defined as in Definition 2.1. ψ-Caputo fractional derivative of a function v of order $\alpha \in (n-1, n)$, is determined as*

$${}^C D_{a^+}^{\alpha,\psi} v(\tau) = I_{a^+}^{n-\alpha,\psi} v_\psi^{[n]}(\tau),$$

where $v_{\psi}^{[n]}(\tau) = \left[\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\right]^{n} v(\tau)$ *and* $n = [\alpha] + 1$ *for* $\alpha \notin \mathbb{N}$, $n = \alpha$ *for* $\alpha \in \mathbb{N}$. *By the definition, we have*

$$ {}^{C}D_{a^+}^{\alpha,\psi} v(\tau) = \begin{cases} \int_a^{\tau} \frac{\psi'(s)(\psi(\tau)-\psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)} v_{\psi}^{[n]}(s)ds, & n \notin \mathbb{N}, \\ v_{\psi}^{[n]}(\tau), & n \in \mathbb{N}, \end{cases} $$

where $\mathbb{N}$ *denotes the set of positive integers.*

Lemma 2.4 ([9]). *For* $\alpha > 0$, *we obtain*

I) ${}^{C}D_{a^+}^{\alpha,\psi} I_{a^+}^{\alpha,\psi} v(\tau) = v(\tau)$ *for all functions* $v \in C(\mathcal{T}, \Xi)$.

II) If $v \in C^n(\mathcal{T}, \Xi)$, *then* $I_{a^+}^{\alpha,\psi}\, {}^{C}D_{a^+}^{\alpha,\psi} v(\tau) = v(\tau) - \sum_{k=0}^{n-1} \frac{v_{\psi}^{[k]}(a)}{k!}[\psi(\tau) - \psi(a)]^k$.

Lemma 2.5 ([9]). *For the functions* $v,\ \psi \in C(\mathcal{T}, \mathbb{R})$ *and* $\alpha > 0$, *we have*

I) $I_{a^+}^{\alpha,\psi}(.)$ *is linear and bounded form* $C(\mathcal{T}, \Xi)$ *to* $C(\mathcal{T}, \Xi)$.

II) $I_{a^+}^{\alpha,\psi} v\,(a) = \lim_{\tau \to a^+} I_{a^+}^{\alpha,\psi} v(\tau) = 0$.

Lemma 2.6 ([9, 35]). *Let* $\alpha,\ \beta > 0$ *and* $v \in C(\mathcal{T}, \Xi)$. *Then for each* $\tau \in \mathcal{T}$, *we have*

(C1) $I_{a^+}^{\alpha,\psi}[\psi(\tau) - \psi(a)]^{\beta-1} = \dfrac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}[\psi(\tau) - \psi(a)]^{\alpha+\beta-1}$,

(C2) if $\beta > n \in \mathbb{N}$, *then* ${}^{C}D_{a^+}^{\alpha,\psi}[\psi(\tau) - \psi(a)]^{\beta-1} = \dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}[\psi(\tau) - \psi(a)]^{\beta-\alpha-1}$,

(C3) $\forall k \in \{0, 1, \cdots, n-1\}$, *n is a positive integer, then* ${}^{C}D_{a^+}^{\alpha,\psi}[\psi(\tau) - \psi(a)]^k = 0$,

(C4) $I_{a^+}^{\alpha,\psi} I_{a^+}^{\beta,\psi} v(\tau) = I_{a^+}^{\alpha+\beta,\psi} v(\tau)$,

(C5) for any constant ρ, *we always have* ${}^{C}D_{a^+}^{\alpha,\psi} \rho = 0$.

Here we recall some fixed point theorems to be used in the study.

Theorem 2.7 (Banach's fixed point theorem [13]). *Let X be a Banach space and* $\mathcal{A} : X \to X$ *be a contraction, i.e. there exists* $\Lambda \in [0, 1)$ *such that*

$$\|\mathcal{A}(z_1) - \mathcal{A}(z_2)\| \leq \Lambda \|z_1 - z_2\|,$$

for all $z_1, z_2 \in X$. *Then* $\mathcal{A}$ *has a unique fixed point.*

Theorem 2.8 (Dhage fixed point theorem [13]). *Let* $\mathbb{J}$ *be a closed, convex, bounded and nonempty subset of a Banach algebra* $(C(\mathcal{T}, \mathbb{R}), \|\cdot\|)$, *and let* $\mathcal{P}, \mathcal{Q} : C(\mathcal{T}, \mathbb{R}) \to C(\mathcal{T}, \mathbb{R})$ *and* $\mathcal{R} : \mathbb{J} \to C(\mathcal{T}, \mathbb{R})$ *be three operators such that*

1) $\mathcal{P}$ *and* $\mathcal{Q}$ *are Lipschitzian with Lipschitz constants* Θ_1 *and* Θ_2, *respectively,*

2) $\mathcal{R}$ *is compact and continuous,*

3) $z = \mathcal{P}z\mathcal{R}y + \mathcal{Q}z \Rightarrow z \in \mathbb{J}$ *for all* $y \in \mathbb{J}$,

4) $\Theta_1 \rho + \Theta_2 < 1$, *where* $\rho = \|\mathcal{R}(\mathbb{J})\| = \sup\{\|\mathcal{R}(y)\| : y \in \mathbb{J}\}$.

Then the operator equation $\mathcal{P}z\mathcal{R}z + \mathcal{Q}z = z$ *has a solution in* $\mathbb{J}$.

3. The existence of solutions

Let $\varpi : \mathcal{T} \to \mathbb{R}$ be a function such that $\mathcal{Y}(\cdot) \in C(\mathcal{T}, \mathbb{R})$, $\mathcal{G} \in C(\mathcal{T}, \mathbb{R}\backslash\{0\})$, and the function $\mathcal{H} \in C(\mathcal{T}, \mathbb{R})$. We consider the following linear fractional differential equation related to (1) .

$$ {}^{C}D^{\alpha,\psi}_{a_1^+}\left[{}^{C}D^{\beta,\psi}_{a_1^+}\left(\frac{\mathrm{x}(\tau)-\mathcal{H}(\tau)}{\mathcal{G}(\tau)}\right)+\lambda \mathrm{x}(\tau)\right]=\mathcal{Y}(\tau), \quad \tau\in[a_1,a_2], \tag{5} $$

where $\alpha \in (0, 1]$, $\beta \in (1, 2]$, $\lambda > 0$, with the boundary conditions

$$ \begin{cases} \left(\dfrac{\mathrm{x}(\tau)-\mathcal{H}(\tau)}{\mathcal{G}(\tau)}\right)\Bigg|_{\tau=a_1} = c, \\ \left(\dfrac{\mathrm{x}(\tau)-\mathcal{H}(\tau)}{\mathcal{G}(\tau)}\right)'\Bigg|_{\tau=a_1} = 0, \\ \left(\dfrac{\mathrm{x}(\tau)-\mathcal{H}(\tau)}{\mathcal{G}(\tau)}\right)\Bigg|_{\tau=\epsilon} = \ell, \ a_1<\epsilon<a_2\ , \end{cases} \tag{6} $$

where $\mathcal{G} \in C(\mathcal{T}, \mathbb{R}\backslash\{0\})$, $\mathcal{H} \in C(\mathcal{T}, \mathbb{R})$, $c, \ell \in \mathbb{R}$ with $c \neq \ell$ and ϵ, is pre-fixed point satisfying $a_1 < \epsilon < a_2$, $\mathcal{Y} \in C(\mathcal{T}, \mathbb{R})$. The following theorem shows that the problem (5)-(6) have a unique solution given by:

$$ \begin{aligned} \mathrm{x}(\tau) :=& \mathcal{G}(\tau)\Bigg[I^{\alpha+\beta,\psi}_{a_1^+}\mathcal{Y}(\tau)-\lambda I^{\beta,\psi}_{a_1^+}\mathrm{x}(\tau)+c-\ell \\ &+\frac{[\psi(\tau)-\psi(a_1)]^{\beta}}{[\psi(\epsilon)-\psi(a_1)]^{\beta}}\left(\lambda I^{\beta,\psi}_{a_1^+}\mathrm{x}(\epsilon)-I^{\alpha+\beta,\psi}_{a_1^+}\mathcal{Y}(\epsilon)+\ell-c\right)\Bigg]+\mathcal{H}(\tau). \end{aligned} \tag{7} $$

Theorem 3.1. *The function* x *satisfies problem* (5)-(6) *if and only if it satisfies* (7).

Proof. Assume that x satisfies the problem (5)-(6) and such that the function $\eta : \tau \longrightarrow \left(\frac{\mathrm{x}(\tau)-\mathcal{H}(\tau)}{\mathcal{G}(\tau)}\right) \in C(\mathcal{T}, \mathbb{R})$. We prove that z is a solution to the equation (7). Applying the fractional integral $I^{\alpha,\psi}_{a_1^+}$ to both sides of (5) and using Lemma 2.4, we have

$$ {}^{C}D^{\beta,\psi}_{a_1^+}\eta(\tau)+\lambda \mathrm{x}(\tau)=I^{\alpha,\psi}_{a_1^+}\mathcal{Y}(\tau)+\varsigma_0. \tag{8} $$

Now, applying $I^{\beta,\psi}_{a_1^+}$ to both sides of (8)

$$ \eta(\tau)=I^{\alpha+\beta,\psi}_{a_1^+}\mathcal{Y}(\tau)+\varsigma_0\frac{[\psi(\tau)-\psi(a_1)]^{\beta}}{\Gamma(\beta+1)}-\lambda I^{\beta,\psi}_{a_1^+}\mathrm{x}(\tau)+\varsigma_1+\varsigma_2[\psi(\tau)-\psi(a_1)], \tag{9} $$

Then,

$$ \frac{\mathrm{x}(\tau)-\mathcal{H}(\tau)}{\mathcal{G}(\tau)}=I^{\alpha+\beta,\psi}_{a_1^+}\mathcal{Y}(\tau)+\varsigma_0\frac{[\psi(\tau)-\psi(a_1)]^{\beta}}{\Gamma(\beta+1)}-\lambda I^{\beta,\psi}_{a_1^+}\mathrm{x}(\tau)+\varsigma_1+\varsigma_2[\psi(\tau)-\psi(a_1)], $$

which implies that

$$ \begin{aligned} \mathrm{x}(\tau) =& \mathcal{G}(\tau)\Bigg[I^{\alpha+\beta,\psi}_{a_1^+}\mathcal{Y}(\tau)+\varsigma_0\frac{[\psi(\tau)-\psi(a_1)]^{\beta}}{\Gamma(\beta+1)}-\lambda I^{\beta,\psi}_{a_1^+}\mathrm{x}(\tau) \\ &+\varsigma_1+\varsigma_2[\psi(\tau)-\psi(a_1)]+\mathcal{H}(\tau). \end{aligned} \tag{10} $$

such as $\varsigma_i \in \mathbb{R}$, with $i = 0, 1, 2$. Next, by the condition $\frac{\mathrm{x}(a_1)-\mathcal{H}(a_1)}{\mathcal{G}(a_1)} = c$ and $\left(\frac{\mathrm{x}(\tau)-\mathcal{H}(\tau)}{\mathcal{G}(\tau)}\right)'\Big|_{\tau=a_1} = 0$ gives

$$ \varsigma_1 := c \text{ and } \varsigma_2 := 0. \tag{11} $$

On the other hand by $\frac{x(\epsilon)-\mathcal{H}(\epsilon)}{\mathcal{G}(\epsilon)} = \ell$, we have

$$\varsigma_0 := \frac{\Gamma(\beta+1)}{[\psi(\epsilon)-\psi(a_1)]^\beta}\left(\lambda I^{\beta,\psi}_{a_1^+}x(\epsilon) - I^{\alpha+\beta,\psi}_{a_1^+}\mathcal{Y}(\epsilon) + \ell - c\right). \tag{12}$$

Substituting (11) and (12) into (10), we obtain (7).

Conversely, assume x satisfies the equation (7) such that the function $\eta : \tau \longrightarrow \left(\frac{x(\tau)-\mathcal{H}(\tau)}{\mathcal{G}(\tau)}\right) \in C(\mathcal{T}, \mathbb{R})$. Applying operator ${}^{C}D^{\beta,\psi}_{a^+}$ on both sides of (7), and since $\mathcal{G}(\tau) \neq 0$ for all $\tau \in \mathcal{T}$, then, from Lemma 2.4 and Lemma 2.6, we obtain

$${}^{C}D^{\beta,\psi}_{a^+}\eta(\tau) = I^{\alpha,\psi}_{a_1^+}\mathcal{Y}(\tau) - \lambda x(\tau) + \frac{\Gamma(\beta+1)}{[\psi(\epsilon)-\psi(a_1)]^\beta}\left(\lambda I^{\beta,\psi}_{a_1^+}x(\epsilon) - I^{\alpha+\beta,\psi}_{a_1^+}\mathcal{Y}(\epsilon) + \ell - c\right). \tag{13}$$

Reapplying, ${}^{C}D^{\alpha,\psi}_{a^+}$ to the above equation, we obtain

$${}^{C}D^{\alpha,\psi}_{a_1^+}\left[{}^{C}D^{\beta,\psi}_{a_1^+}\left(\frac{x(\tau)-\mathcal{H}(\tau)}{\mathcal{G}(\tau)}\right) + \lambda x(\tau)\right] = \mathcal{Y}(\tau), \tag{14}$$

Taking the limit $\tau \to a_1$ of (7) we obtain

$$\frac{x(a_1)-\mathcal{H}(a_1)}{\mathcal{G}(a_1)} = c, \tag{15}$$

Substituting $\tau = \epsilon$ into (7), we have

$$\frac{z(\epsilon_1)-\mathcal{H}(\epsilon)}{\mathcal{G}(\epsilon)} = \ell.$$

Now, applying $D^1_{a_1^+}$ to both sides of (7) gives

$$\begin{aligned}\left(\frac{x(\tau)-\mathcal{H}(\tau)}{\mathcal{G}(\tau)}\right)' &= I^{\alpha+\beta-1,\psi}_{a_1^+}\mathcal{Y}(\tau) - \lambda I^{\beta-1,\psi}_{a_1^+}x(\tau) \\ &\quad + \frac{\Gamma(\beta+1)}{[\psi(\epsilon)-\psi(a_1)]^\beta}\left(\lambda I^{\beta,\psi}_{a_1^+}x(\epsilon) - I^{\alpha+\beta,\psi}_{a_1^+}\mathcal{Y}(\epsilon) + \ell - c\right) I^{\beta-1,\psi}_{a_1^+}1.\end{aligned} \tag{16}$$

Taking the limit $\tau \to a_1$ of (16) we have

$$\left.\left(\frac{x(\tau)-\mathcal{H}(\tau)}{\mathcal{G}(\tau)}\right)'\right|_{\tau=a_1} = 0. \tag{17}$$

This shows that the boundary conditions (6) are satisfied. □

Next, we present the solution for the coupled system (1)-(2).

Definition 3.2. *A function* $x \in C(\mathcal{T}, \mathbb{R})$ *that satisfies the equations* (5) *and* (6) *on* $\mathcal{T}$ *is considered a solution to the fractional problem* (5)-(6).

Lemma 3.3. *Let* $\alpha_i \in (0, 1]$, $\beta_i \in (1, 2]$, *for* $i = 1, 2$, $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$, $0 < a_1 < a_2$, $a_1 < \epsilon_1, \epsilon_2 < a_2$, $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \in \mathbb{R}$, ($\vartheta_1 \neq \vartheta_3$ *and* $\vartheta_2 \neq \vartheta_4$) *and* $0 < \xi, \tilde{\xi} < 1$, *let* $f_j : \mathcal{T} \times \mathbb{R}^2 \to \mathbb{R}$, $h_j : \mathcal{T} \times \mathbb{R} \to \mathbb{R}$, *and* $g_j : \mathcal{T} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ *are continuous functions with* $j = 1, 2$. *If the function* $\tau \longrightarrow \left(\frac{z(\tau)-h(\tau,z(\tau))}{g(\tau,z(\tau))}\right) \in C(\mathcal{T}, \mathbb{R})$ *and similarly, the function* $\tau \longrightarrow \left(\frac{y(\tau)-h(\tau,y(\tau))}{g(\tau,y(\tau))}\right) \in C(\mathcal{T}, \mathbb{R})$, *then* $(z, y) \in \Xi$ *satisfies the coupled system* (1)-(2) *if and only if* (z, y) *is the fixed point of the operator* $\mathcal{S} : \Xi \longrightarrow \Xi$ *defined by*

$$\mathcal{S}(z, y)(\tau) := (\mathcal{S}_1(z, y)(\tau), \mathcal{S}_2(z, y)(\tau)),$$

such as

$$\mathcal{S}_1(z,y)(\tau) := g_1(\tau, z(\tau))\left[\tilde{\Psi}(z,y)(\tau) - \frac{(\psi(\tau)-\psi(a_1))^{\beta_1}}{(\psi(\epsilon_1)-\psi(a_1))^{\beta_1}}\tilde{\Psi}(z,y)(\epsilon_1)\right] + h_1(\tau, z(\tau)), \tag{18}$$

and

$$\mathcal{S}_2(z,y)(\tau) := g_2(\tau, y(\tau))\left[\tilde{\Phi}(z,y)(\tau) - \frac{(\psi(\tau)-\psi(a_1))^{\beta_2}}{(\psi(\epsilon_2)-\psi(a_1))^{\beta_2}}\tilde{\Phi}(z,y)(\epsilon_2)\right] + h_2(\tau, y(\tau)), \tag{19}$$

where

$$\begin{aligned}\tilde{\Psi}(z,y)(\tau) := &\int_{a_1}^{\tau} \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\alpha_1+\beta_1-1}}{\Gamma(\alpha+\beta)} f_1(s, z(s), y(\xi s))ds \\ &- \lambda_1 \int_{a_1}^{\tau} \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\beta_1-1}}{\Gamma(\beta_1)} z(s)ds + \vartheta_1 - \vartheta_3,\end{aligned}$$

and

$$\begin{aligned}\tilde{\Phi}(z,y)(\tau) := &\int_{a_1}^{\tau} \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_2+\beta_2)} f_2(s, y(s), z(\tilde{\xi} s))ds \\ &- \lambda_2 \int_{a_1}^{\tau} \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\beta_2-1}}{\Gamma(\beta_2)} y(s)ds + \vartheta_2 - \vartheta_4.\end{aligned}$$

Given that the functions g_i and h_i are continuous and $f_i(\tau,\cdot,\cdot) \in C(\mathcal{T}\times\mathbb{R}^2,\mathbb{R})$, for $i = 1,2$, it follows that $\mathcal{S}(z,y) \in \Xi$.

The next result relies on the application of the Banach fixed-point theorem. Furthermore, we assume the following conditions for this outcome.

($C1$) The functions $(f_i)_{i=1,2} : \mathcal{T}\times\mathbb{R}^2 \to \mathbb{R}$, $(h_i)_{i=1,2} : \mathcal{T}\times\mathbb{R} \to \mathbb{R}$, and $(g_i)_{i=1,2} : \mathcal{T}\times\mathbb{R} \to \mathbb{R}\setminus\{0\}$ are continuous.

($C2$) There exist positive functions $p_i, q_i, r_i \in C(\mathcal{T},\mathbb{R}^+)$ such that

$$\begin{aligned}|f_i(\tau,z,y) - f_i(\tau,\bar{z},\bar{y})| &\le p_i(\tau)\left(|z-\bar{z}| + |y-\bar{y}|\right), \\ |g_i(\tau,z) - g_i(\tau,\bar{z})| &\le q_i(\tau)|z-\bar{z}|,\end{aligned}$$

and

$$|h_i(\tau,z) - h_i(\tau,\bar{z})| \le r_i(\tau)|z-\bar{z}|,$$

for $i = 1,2$, for all $\tau \in \mathcal{T}$ and $z, y, \bar{z}, \bar{y} \in \mathbb{R}$,
where

$$p_i^* = \sup_{\tau\in\mathcal{T}} p_i(\tau),\ q_i^* = \sup_{\tau\in\mathcal{T}} q_i(\tau) \text{ and } r_i^* = \sup_{\tau\in\mathcal{T}} r_i(\tau),\ i = 1,2.$$

($C3$) There exist positive constants $\mathcal{L}_i$ and $\mathcal{M}_i$, where $|f_i(\tau,\cdot,\cdot)| < \mathcal{L}_i$ and $|g_i(\tau,\cdot)| < \mathcal{M}_i$, for all $\tau \in \mathcal{T}$ and $\cdot \in \mathbb{R}$, where $i = 1,2$.

For the sake of clarity, we denote

$$\begin{cases} \Delta_1 = \dfrac{[(\psi(a_2)-\psi(a_1))^{\alpha_1+\beta_1} + (\psi(a_2)-\psi(a_1))^{\beta_1}(\psi(\epsilon_1)-\psi(a_1))^{\alpha_1}]}{\Gamma(\alpha_1+\beta_1+1)}, \\ \Delta_2 = \frac{2|\lambda_1|(\psi(a_2)-\psi(a_1))^{\beta_1}}{\Gamma(\beta_1+1)}, \\ \Delta_3 = (|\vartheta_1|+|\vartheta_3|)\left(\dfrac{[\psi(a_2)-\psi(a_1)]^{\beta_1}}{[\psi(\epsilon_1)-\psi(a_1)]^{\beta_1}}+1\right), \\ \nabla_1 = \dfrac{[(\psi(a_2)-\psi(a_1))^{\alpha_2+\beta_2} + (\psi(a_2)-\psi(a_1))^{\beta_2}(\psi(\epsilon_2)-\psi(a_1))^{\alpha_2}]}{\Gamma(\alpha_2+\beta_2+1)}, \\ \nabla_2 = \frac{2|\lambda_2|(\psi(a_2)-\psi(a_1))^{\beta_2}}{\Gamma(\beta_2+1)}, \\ \nabla_3 = (|\vartheta_2|+|\vartheta_4|)\left(\dfrac{[\psi(a_2)-\psi(a_1)]^{\beta_2}}{[\psi(\epsilon_2)-\psi(a_1)]^{\beta_2}}+1\right), \\ \varpi_1 = (2p_1^*\mathcal{M}_1 + q_1^*\mathcal{L}_1)\Delta_1 + (\mathcal{M}_1+q_1^*)\Delta_2 + q_1^*\Delta_3 + r_1^*, \\ \varpi_2 = (2p_2^*\mathcal{M}_2 + q_2^*\mathcal{L}_2)\nabla_1 + (\mathcal{M}_2+q_2^*)\nabla_2 + q_2^*\nabla_3 + r_2^*. \end{cases}$$

Theorem 3.4. *Suppose that* $(C1)-(C3)$ *holds. If*

$$\sum_{i=1}^{2} \varpi_i < 1, \tag{20}$$

then the coupled system (1)-(2) has a unique solution on $\mathcal{T} = [a_1, a_2]$.

Proof. Let $(z,y), (\bar{z},\bar{y}) \in \Xi$ and $\tau \in \mathcal{T}$. By $(C2)$, we get

$$\begin{aligned} &|\tilde{\Psi}(z,y)(\tau) - \tilde{\Psi}(\bar{z},\bar{y})(\tau)| \\ &\le \int_{a_1}^{\tau} \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)} |f_1(s,z(s),y(\xi_1 s)) - f_1(s,\bar{z}(s),\bar{y}(\xi_1 s))| ds \\ &\quad + |\lambda_1| \int_{a_1}^{\tau} \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\beta_1-1}}{\Gamma(\beta_1)} |z(s)-\bar{z}(s)| ds \\ &\le \left(\frac{2p_1^*(\psi(\tau)-\psi(a_1))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{|\lambda_1|(\psi(\tau)-\psi(a_1))^{\beta_1}}{\Gamma(\beta_1+1)} \right) \|z-\bar{z}, y-\bar{y}\|_{\Xi}, \end{aligned}$$

and by condition $(C3)$, we obtain

$$|\tilde{\Psi}(z,y)(\tau)| \le \frac{\mathcal{L}_1(\psi(\tau)-\psi(a_1))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{|\lambda_1|(\psi(\tau)-\psi(a_1))^{\beta_1}}{\Gamma(\beta_1+1)} + |\vartheta_1| + |\vartheta_3|.$$

In the same way, we obtain

$$\begin{aligned} &|\tilde{\Phi}(z,y)(\tau) - \tilde{\Phi}(\bar{z},\bar{y})(\tau)| \\ &\le \int_{a_1}^{\tau} \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_2+\beta_2)} |f_2(s,y(s),z(\xi_2 s)) - f_2(s,\bar{y}(s),\bar{z}(\xi_2 s))| ds \\ &\quad + |\lambda_2| \int_{a_1}^{\tau} \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\beta_2-1}}{\Gamma(\beta_2)} |y(s)-\bar{y}(s)| ds \\ &\le \left(\frac{2p_2^*(\psi(\tau)-\psi(a_1))^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{|\lambda_2|(\psi(\tau)-\psi(a_1))^{\beta_2}}{\Gamma(\beta_2+1)} \right) \|z-\bar{z}, y-\bar{y}\|_{\Xi}, \end{aligned}$$

and by condition $(C3)$, we get

$$|\tilde{\Phi}(z,y)(\tau)| \le \frac{\mathcal{L}_2(\psi(\tau)-\psi(a_1))^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{|\lambda_2|(\psi(\tau)-\psi(a_1))^{\beta_1}}{\Gamma(\beta_2+1)} + |\vartheta_2| + |\vartheta_4|.$$

By applying the triangle inequality, we obtain

$$\begin{aligned}
&|\mathcal{S}_1(z,y)(\tau)-\mathcal{S}_1(\bar{z},\bar{y})(\tau)| \\
&\leq |g_1(\tau,z(\tau))\tilde{\Psi}(z,y)(\tau)-g_1(\tau,w(\tau))\tilde{\Psi}(\bar{z},\bar{y})(\tau)| \\
&\quad+\frac{(\psi(\tau)-\psi(a_1))^{\beta_1}}{(\psi(\epsilon_1)-\psi(a_1))^{\beta_1}}|g_1(\tau,z(\tau))\tilde{\Psi}(z,y)(\epsilon_1)-g_1(\tau,\bar{z}(\tau))\tilde{\Psi}(\bar{z},\bar{y})(\epsilon_1)| \\
&\quad+|h_1(\tau,z(\tau))-h_1(\tau,\bar{z}(\tau))|.
\end{aligned}$$

We conclude that

$$\begin{aligned}
&|\mathcal{S}_1(z,y)(\tau)-\mathcal{S}_1(\bar{z},\bar{y})(\tau)| \\
&\leq \mathcal{M}_1\left(\frac{2p_1^*(\psi(\tau)-\psi(a_1))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}+\frac{|\lambda_1|(\psi(\tau)-\psi(a_1))^{\beta_1}}{\Gamma(\beta_1+1)}\right)\|z-\bar{z},y-\bar{y}\|_\Xi \\
&\quad+q_1^*\left(\frac{\mathcal{L}_1(\psi(\tau)-\psi(a_1))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}+\frac{|\lambda_1|(\psi(\tau)-\psi(a_1))^{\beta_1}}{\Gamma(\beta_1+1)}\right)\|z-\bar{z},y-\bar{y}\|_\Xi \\
&\quad+\mathcal{M}_1(\psi(\tau)-\psi(a_1))^{\beta_1}\left(\frac{2p_1^*(\psi(\epsilon_1)-\psi(a_1))^{\alpha_1}}{\Gamma(\alpha_1+\beta_1+1)}+\frac{|\lambda_1|}{\Gamma(\beta_1+1)}\right)\|z-\bar{z},y-\bar{y}\|_\Xi \\
&\quad+q_1^*(\psi(\tau)-\psi(a_1))^{\beta_1}\left(\frac{\mathcal{L}_1(\psi(\epsilon_1)-\psi(a_1))^{\alpha_1}}{\Gamma(\alpha_1+\beta_1+1)}+\frac{|\lambda_1|}{\Gamma(\beta_1+1)}\right)\|z-\bar{z},y-\bar{y}\|_\Xi \\
&\quad+\left[q_1^*(|\vartheta_1|+|\vartheta_3|)\left(\frac{[\psi(\tau)-\psi(a_1)]^{\beta_1}}{[\psi(\epsilon_1)-\psi(a_1)]^{\beta_1}}+1\right)+r_1^*\right]\|z-\bar{z},y-\bar{y}\|_\Xi.
\end{aligned}$$

After taking the supremum over $\mathcal{T}$ and simplifying, we get

$$\|\mathcal{S}_1(z,y)-\mathcal{S}_1(\bar{z},\bar{y})\|_\Xi \leq \varpi_1\|z-\bar{z},y-\bar{y}\|_\Xi. \tag{21}$$

Similarly, we obtain

$$\|\mathcal{S}_2(z,y)-\mathcal{S}_2(\bar{z},\bar{y})\|_\Xi \leq \varpi_2\|z-\bar{z},y-\bar{y}\|_\Xi. \tag{22}$$

It follows from (21) and (22) that

$$\|\mathcal{S}(x,y)-\mathcal{S}(\bar{x},\bar{y})\|_\Xi \leq (\varpi_1+\varpi_2)\|x-\tilde{x},y-\tilde{y}\|_\Xi.$$

Since $\sum_{i=1}^{2}\varpi_i<1$, $\mathcal{S}$ is a contraction operator. Consequently by utilizing Banach's fixed point theorem, the coupled system (1)-(2) has a unique solution. In the following result, we establish the existence of solutions for the hybrid coupled Langevin fractional pantograph system (1)-(2). □

This is achieved by utilizing a theorem derived from Dhage's fixed-point theorem.

Theorem 3.5. *Assume that the following hypotheses hold.*

(CD1) *The functions* $f_1, f_2 : \mathcal{T}\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ *are continuous on* $\mathcal{T}$.

(CD2) *The functions* $g_i : \mathcal{T}\times\mathbb{R} \longrightarrow \mathbb{R}\setminus\{0\}$ *and* $h_i : \mathcal{T}\times\mathbb{R} \longrightarrow \mathbb{R}$ *are continuous. Moreover, there exist positive continuous functions* $q_i, r_i \in C(\mathcal{T},[0,\infty))$ *such that*

$$|g_i(\tau,z)-g_i(\tau,\bar{z})| \leq q_i(\tau)|z-\bar{z}|,$$

and

$$|h_i(\tau,z)-h_i(\tau,\bar{z})| \leq r_i(\tau)|z-\bar{z}|,$$

for $i=1,2$, *and for any* $z,\bar{z}\in\mathbb{R}$ *and* $\tau\in\mathcal{T}$.

($\mathcal{CD}3$) There exist functions $p_1, p_2, p_3, \eta_1, \eta_2, \eta_3 \in C(\mathcal{T}, \mathbb{R}^+)$ *such that*

$$|f_1(\tau, z, \bar{z})| < p_1(\tau) + p_2(\tau)|z| + p_3(\tau)|\bar{z}|,$$

and

$$|f_2(\tau, z, \bar{z})| < \eta_1(\tau) + \eta_2(\tau)|z| + \eta_3(\tau)|\bar{z}|,$$

for all $\tau \in \mathcal{T}$ *and* $z, \bar{z} \in \mathbb{R}$.

($\mathcal{CD}4$) There exists a number $\gamma > 0$ *such that*

$$\frac{g_1^* \mathcal{N}_\gamma + g_2^* \mathcal{E}_\gamma + h_1^* + h_2^*}{1 - [q_1^* \mathcal{N}_\gamma + q_2^* \mathcal{E}_\gamma + r_1^* + r_2^*]} \leq \gamma,$$

and

$$q_1^* \mathcal{N}_\gamma + q_2^* \mathcal{E}_\gamma + r_1^* + r_2^* < 1, \tag{23}$$

where

$$q_j^* = \sup_{\tau \in \mathcal{T}} q_j(\tau), r_j^* = \sup_{\tau \in \mathcal{T}} r_j(\tau),\ g_j^* = \sup_{\tau \in \mathcal{T}} |g_j(\tau, 0)|,\ h_j^* = \sup_{\tau \in \mathcal{T}} |h_j(\tau, 0)|,\ j = 1, 2,$$
$$p_i^* = \sup_{\tau \in \mathcal{T}} p_i(\tau),\ \eta_i^* = \sup_{\tau \in \mathcal{T}} \eta_i(\tau), i = 1, 2, 3,$$

and

$$\mathcal{N}_\gamma = \Delta_1 P_1^* + \Big[\Delta_1 (P_2^* + P_3^*) + \Delta_2\Big]\gamma + \Delta_3,$$
$$\mathcal{E}_\gamma = \nabla_1 \eta_1^* + \Big[\nabla_1 (\eta_2^* + \eta_3^*) + \nabla_2\Big]\gamma + \nabla_3.$$

Then the coupled system (1)-(2) *has at least one mild solution on* $\mathcal{T}$.

Proof. We define a subset $\mathcal{D}_\gamma$ of $C(\mathcal{T}, \mathbb{R})$.

$$\mathcal{D}_\gamma = \{(z, y) \in \Xi\ :\ \|z, y\|_\Xi \leq \gamma\}.$$

Next, consider the operator $\mathcal{S}_1$ and $\mathcal{S}_2$ defined in (18) and (19), respectively. Additionally, introduce the operators $\mathcal{P}, \mathcal{Q}, \mathcal{U}, \mathcal{V} : C(\mathcal{T}, \mathbb{R})^2 \longrightarrow C(\mathcal{T}, \mathbb{R})$ by

$$\begin{cases} \mathcal{P}(z, y)(\tau) = g_1(\tau, z(\tau)),\ \tau \in \mathcal{T}, \\ \mathcal{Q}(z, y)(\tau) = h_1(\tau, z(\tau)),\ \tau \in \mathcal{T}, \\ \mathcal{U}(z, y)(\tau) = g_2(\tau, y(\tau)),\ \tau \in \mathcal{T}, \\ \mathcal{V}(z, y)(\tau) = h_2(\tau, y(\tau)),\ \tau \in \mathcal{T}, \end{cases}$$

and $\mathcal{R}, \mathcal{K} : \mathcal{D}_\gamma \longrightarrow C(\mathcal{T}, \mathbb{R})$ by

$$\begin{cases} \mathcal{R}(z, y)(\tau) = \tilde{\Psi}(z, y)(\tau) - \frac{(\psi(\tau) - \psi(a_1))^{\beta_1}}{(\psi(\epsilon_1) - \psi(a_1))^{\beta_1}} \tilde{\Psi}(z, y)(\epsilon_1),\ \tau \in \mathcal{T}, \\ \mathcal{K}(z, y)(\tau) = \tilde{\Phi}(z, y)(\tau) - \frac{(\psi(\tau) - \psi(a_1))^{\beta_2}}{(\psi(\epsilon_2) - \psi(a_1))^{\beta_2}} \tilde{\Phi}(z, y)(\epsilon_2),\ \tau \in \mathcal{T}. \end{cases}$$

Then,

$$\mathcal{S}_1(z, y) = \mathcal{P}(z, y)\mathcal{R}(z, y) + \mathcal{Q}(z, y),$$

and

$$\mathcal{S}_2(z, y) = \mathcal{U}(z, y)\mathcal{K}(z, y) + \mathcal{V}(z, y).$$

Step 1: Firstly, we show that $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{U}$ and $\mathcal{V}$ are Lipschitzian on $C(\mathcal{T},\mathbb{R})^2$. Let $(z,y),(\bar{z},\bar{y}) \in C(\mathcal{T},\mathbb{R})^2$. Then by $(\mathcal{CD}2)$ we have

$$\begin{aligned}|\mathcal{P}(z,y)(\tau) - \mathcal{P}(\bar{z},\bar{y})(\tau)| &= |g_1(\tau,z(\tau)) - g_1(\tau,\bar{z}(\tau))| \\ &\leq q_1(\tau)|z(\tau) - \bar{z}(\tau)|.\end{aligned}$$

Then for each $\tau \in [a_1, a_2]$ we obtain

$$\|\mathcal{P}(z,y) - \mathcal{P}(\bar{z},\bar{y})\| \leq q_1^*\|(z,y),(\bar{z},\bar{y})\|_\Xi. \tag{24}$$

As before, we have

$$\|\mathcal{U}(z,y) - \mathcal{U}(\bar{z},\bar{y})\| \leq q_2^*\|(z,y),(\bar{z},\bar{y})\|_\Xi, \tag{25}$$

and for each $\tau \in [a_1, a_2]$ we get

$$\begin{aligned}|\mathcal{Q}(z,y)(\tau) - \mathcal{Q}(\bar{z},\bar{y})(\tau)| &= |h_1(\tau,z(\tau)) - h_1(\tau,\bar{z}(\tau))| \\ &\leq r_1(\tau)|z(\tau) - \bar{z}(\tau)|.\end{aligned}$$

Then,

$$\|\mathcal{Q}(z,y) - \mathcal{Q}(\bar{z},\bar{y})\| \leq r_1^*\|(z,y),(\bar{z},\bar{y})\|_\Xi, \tag{26}$$

and

$$\|\mathcal{V}(z,y) - \mathcal{V}(\bar{z},\bar{y})\| \leq r_2^*\|(z,y),(\bar{z},\bar{y})\|_\Xi. \tag{27}$$

Therefore $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{U}$ and $\mathcal{V}$ are Lipschitzian on $C(\mathcal{T},\mathbb{R})^2$ with Lipschitz constants q_i^* and r_i^*, for $i = 1, 2$.

Step 2: We demonstrate that the operators $\mathcal{R}$ and $\mathcal{K}$ are completely continuous on $\mathcal{D}_\gamma$. To achieve this, we first establish that the operators $\mathcal{R}$ and $\mathcal{K}$ are continuous on $C(\mathcal{T},\mathbb{R})$. Let $\{z_n, y_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}_\gamma$ that converges to a point $(z,y) \in \mathcal{D}_\gamma$. Then, we have

$$\begin{aligned}&\lim_{n\to+\infty} \mathcal{R}(z_n,y_n)(\tau) \\ &= \lim_{n\to+\infty}\left\{\tilde{\Psi}(z_n,y_n)(\tau) - \frac{(\psi(\tau)-\psi(a_1))^{\beta_1}}{(\psi(\epsilon_1)-\psi(a_1))^{\beta_1}}\tilde{\Psi}(z_n,y_n)(\epsilon_1)\right\} \\ &= \lim_{n\to+\infty}\left\{\int_{a_1}^{\tau}\frac{\psi'(s)(\psi(\tau)-\psi(s))^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)}f_1(s,z_n(s),y_n(\xi_1 s))ds\right. \\ &- \lambda_1\int_{a_1}^{\tau}\frac{\psi'(s)(\psi(\tau)-\psi(s))^{\beta_1-1}}{\Gamma(\beta_1)}z_n(s)ds + \vartheta_1 - \vartheta_3 \\ &- \frac{(\psi(\tau)-\psi(a_1))^{\beta_1}}{(\psi(\epsilon_1)-\psi(a_1))^{\beta_1}}\left(\int_{a_1}^{\epsilon_1}\frac{\psi'(s)(\psi(\epsilon_1)-\psi(s))^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)}f_1(s,z_n(s),y_n(\xi_1 s))ds\right. \\ &\left.\left.-\lambda_1\int_{a_1}^{\epsilon_1}\frac{\psi'(s)(\psi(\epsilon_1)-\psi(s))^{\beta_1-1}}{\Gamma(\beta_1)}z_n(s)ds + \vartheta_1 - \vartheta_3\right)\right\}.\end{aligned}$$

By the continuity of the function f_1, we may obtain

$$\lim_{n\to+\infty}\mathcal{R}(z_n,y_n)(\tau) = \mathcal{R}(z,y)(\tau).$$

Hence,

$$\|\mathcal{R}(z_n,y_n) - \mathcal{R}(z,y)\|_\infty \longrightarrow 0, \text{ as } n \to \infty,$$

for all $\tau \in \mathcal{T}$. Similarly, we also have

$$\begin{aligned}
&\lim_{n\to+\infty} \mathcal{K}(z_n, y_n)(\tau) \\
&= \lim_{n\to+\infty} \left\{ \tilde{\Phi}(z_n, y_n)(\tau) - \frac{(\psi(\tau)-\psi(a_1))^{\beta_2}}{(\psi(\epsilon_2)-\psi(a_1))^{\beta_2}} \tilde{\Phi}(z_n, y_n)(\epsilon_2) \right\} \\
&= \lim_{n\to+\infty} \left\{ \int_{a_1}^{\tau} \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_2+\beta_2)} f_2(s, y_n(s), z_n(\xi_2 s)) ds \right. \\
&\quad - \lambda_2 \int_{a_1}^{\tau} \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\beta_2-1}}{\Gamma(\beta_2)} y_n(s) ds + \vartheta_2 - \vartheta_4 \\
&\quad - \frac{(\psi(\tau)-\psi(a_1))^{\beta_2}}{(\psi(\epsilon_2)-\psi(a_1))^{\beta_2}} \left(\int_{a_1}^{\epsilon_2} \frac{\psi'(s)(\psi(\epsilon_2)-\psi(s))^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_2+\beta_2)} f_2(s, y_n(s), z_n(\xi_2 s)) ds \right. \\
&\quad \left.\left. -\lambda_2 \int_{a_1}^{\epsilon_2} \frac{\psi'(s)(\psi(\epsilon_2)-\psi(s))^{\beta_2-1}}{\Gamma(\beta_2)} z_n(s) ds + \vartheta_2 - \vartheta_4 \right) \right\}.
\end{aligned}$$

By the continuity of the function f_2, we may obtain

$$\lim_{n\to+\infty} \mathcal{K}(z_n, y_n)(\tau) = \mathcal{K}(z, y)(\tau).$$

Hence,

$$\|\mathcal{K}(z_n, y_n) - \mathcal{K}(z, y)\|_\infty \longrightarrow 0, \text{ as } n \to \infty,$$

for all $\tau \in \mathcal{T}$. This shows that $\mathcal{R}$ and $\mathcal{K}$ are continuous operators on $\mathcal{D}_\gamma$.

Next, we prove that the sets $\mathcal{R}(\mathcal{D}_\gamma)$ and $\mathcal{K}(\mathcal{D}_\gamma)$ are uniformly bounded in $\mathcal{D}_\gamma$. For any $(z, y) \in \mathcal{D}_\gamma$ and $\tau \in \mathcal{T}$, we have

$$|\tilde{\Psi}(z, y)(\tau)| \le \frac{(\psi(\tau)-\psi(a_1))^{\alpha_1+\beta_1} \left(P_1^* + (P_2^* + P_3^*)\gamma\right)}{\Gamma(\alpha_1+\beta_1+1)} + \frac{|\lambda_1|(\psi(\tau)-\psi(a_1))^{\beta_1}\gamma}{\Gamma(\beta_1+1)} + |\vartheta_1| + |\vartheta_3|,$$

and

$$|\tilde{\Phi}(z, y)(\tau)| \le \frac{(\psi(\tau)-\psi(a_1))^{\alpha_2+\beta_2} \left(\mathfrak{y}_1^* + (\mathfrak{y}_2^* + \mathfrak{y}_3^*)\gamma\right)}{\Gamma(\alpha_2+\beta_2+1)} + \frac{|\lambda_2|(\psi(\tau)-\psi(a_1))^{\beta_2}\gamma}{\Gamma(\beta_2+1)} + |\vartheta_2| + |\vartheta_4|.$$

Therefore,

$$\begin{aligned}
&|\mathcal{R}(z, y)(\tau)| \\
&\le |\tilde{\Psi}(z, y)(\tau)| + \frac{(\psi(\tau)-\psi(a_1))^{\beta_1}}{(\psi(\epsilon_1)-\psi(a_1))^{\beta_1}} |\tilde{\Psi}(z, y)(\epsilon_1)| \\
&\le \frac{(\psi(\tau)-\psi(a_1))^{\alpha_1+\beta_1} \left(P_1^* + (P_2^* + P_3^*)\gamma\right)}{\Gamma(\alpha_1+\beta_1+1)} + \frac{2|\lambda_1|(\psi(\tau)-\psi(a_1))^{\beta_1}\gamma}{\Gamma(\beta_1+1)} \\
&\quad + \frac{(\psi(\tau)-\psi(a_1))^{\beta_1}(\psi(\epsilon_1)-\psi(a_1))^{\alpha_1} \left(P_1^* + (P_2^* + P_3^*)\gamma\right)}{\Gamma(\alpha_1+\beta_1+1)} \\
&\quad + (|\vartheta_1| + |\vartheta_3|) \left(\frac{[\psi(a_2)-\psi(a_1)]^{\beta_1}}{[\psi(\epsilon_1)-\psi(a_1)]^{\beta_1}} + 1 \right) \qquad (28) \\
&\le \mathcal{N}_\gamma.
\end{aligned}$$

Then, we obtain

$$\|\mathcal{R}(z,y)\|_\infty \le \infty,$$

for all $(z,y) \in \mathcal{D}_\gamma$. We also have

$$\begin{aligned}
&|\mathcal{K}(z,y)(\tau)| \\
&\le |\tilde{\Phi}(z,y)(\tau)| + \frac{(\psi(\tau)-\psi(a_1))^{\beta_2}}{(\psi(\epsilon_2)-\psi(a_1))^{\beta_2}}|\tilde{\Phi}(z,y)(\epsilon_2)| \\
&\le \frac{(\psi(\tau)-\psi(a_1))^{\alpha_2+\beta_2}\left(\mathfrak{y}_1^* + (\mathfrak{y}_2^*+\mathfrak{y}_3^*)\gamma\right)}{\Gamma(\alpha_2+\beta_2+1)} + \frac{2|\lambda_2|(\psi(\tau)-\psi(a_1))^{\beta_2}\gamma}{\Gamma(\beta_2+1)} \\
&\quad + \frac{(\psi(\tau)-\psi(a_1))^{\beta_2}(\psi(\epsilon_2)-\psi(a_1))^{\alpha_2}\left(\mathfrak{y}_1^* + (\mathfrak{y}_2^*+\mathfrak{y}_3^*)\gamma\right)}{\Gamma(\alpha_2+\beta_2+1)} \\
&\quad + (|\vartheta_2|+|\vartheta_4|)\left(\frac{[\psi(a_2)-\psi(a_1)]^{\beta_2}}{[\psi(\epsilon_2)-\psi(a_1)]^{\beta_2}}+1\right) \\
&\le \mathcal{E}_\gamma.
\end{aligned} \tag{29}$$

Then, we get

$$\|\mathcal{K}(z,y)\|_\infty \le \infty,$$

for all $(z,y) \in \mathcal{D}_\gamma$.

This prove that the sets $\mathcal{R}(\mathcal{D}_\gamma)$ and $\mathcal{K}(\mathcal{D}_\gamma)$ are uniformly bounded in $\mathcal{D}_\gamma$.

On the other hand, we demonstrate that $\mathcal{R}(\mathcal{D}_\gamma)$ and $\mathcal{K}(\mathcal{D}_\gamma)$ are equicontinuous sets in $\mathcal{D}_\gamma$. We take $\tau_1,\tau_2 \in [a_1,a_2]$ with $\tau_1 < \tau_2$ and $(z,y) \in \mathcal{D}_\gamma$. Then,

$$\begin{aligned}
&|\mathcal{R}(z,y)(\tau_2) - \mathcal{R}(z,y)(\tau_1)| \\
&\le \int_{a_1}^{\tau_1} \frac{\psi'(s)[(\psi(\tau_2)-\psi(s))^{\alpha_1+\beta_1-1} - (\psi(\tau_1)-\psi(s))^{\alpha_1+\beta_1-1}]}{\Gamma(\alpha_1+\beta_1)}|f_1(s,z(s),y(\xi_1 s))|ds \\
&\quad + \int_{\tau_1}^{\tau_2} \frac{\psi'(s)(\psi(\tau_2)-\psi(s))^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)}|f_1(s,z(s),y(\xi_1 s))|ds \\
&\quad + \int_{a_1}^{\tau_1} \frac{|\lambda_1|\psi'(s)[(\psi(\tau_2)-\psi(s))^{\beta_1-1} - (\psi(\tau_1)-\psi(s))^{\beta_1-1}]}{\Gamma(\beta_1)}|z(s)|ds \\
&\quad + \int_{\tau_1}^{\tau_2} \frac{|\lambda_1|\psi'(s)(\psi(\tau_2)-\psi(s))^{\beta_1-1}}{\Gamma(\beta_1)}|z(s)|ds \\
&\quad + \frac{[(\psi(\tau_2)-\psi(a_1))^{\beta_1} - (\psi(\tau_1)-\psi(a_1))^{\beta_1}]}{(\psi(\epsilon_1)-\psi(a_1))^{\beta_1}} \\
&\quad \times\Bigg(\int_{a_1}^{\epsilon_1} \frac{\psi'(s)(\psi(\epsilon_1)-\psi(s))^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)}|f_1(s,z(s),y(\xi_1 s))|ds \\
&\quad + |\lambda_1|\int_{a_1}^{\epsilon_1} \frac{\psi'(s)(\psi(\epsilon_1)-\psi(s))^{\beta_1-1}}{\Gamma(\beta_1)}|z(s)|ds + |\vartheta_1| + |\vartheta_3|\Bigg).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&|\mathcal{R}(z,y)(\tau_2)-\mathcal{R}(z,y)(\tau_1)|\\
&\le |\tilde{\Psi}(z,y)(\tau_2)-\tilde{\Psi}(z,y)(\tau_1)| + \frac{[(\psi(\tau_2)-\psi(a_1))^{\beta_1}-(\psi(\tau_1)-\psi(a_1))^{\beta_1}]}{(\psi(\epsilon_1)-\psi(a_1))^{\beta_1}}|\tilde{\Psi}(z,y)(\epsilon_1)|\\
&\le \frac{[(\psi(\tau_1)-\psi(a_1))^{\alpha_1+\beta_1}-(\psi(\tau_2)-\psi(a_1))^{\alpha_1+\beta_1}]}{\Gamma(\alpha_1+\beta_1+1)}\left[P_1^*+(P_2^*+P_3^*)\gamma\right]\\
&\quad+\left(\frac{(\psi(\tau_2)-\psi(\tau_1))^{\beta_1}-(\psi(\tau_2)-\psi(a_1))^{\beta_1}+(\psi(\tau_1)-\psi(a_1))^{\beta_1}}{\Gamma(\beta_1+1)}\right)|\lambda_1|\gamma\\
&\quad+[(\psi(\tau_1)-\psi(a_1))^{\beta_1}-(\psi(\tau_2)-\psi(a_1))^{\beta_1}]\left(\frac{[\psi(\epsilon_1)-\psi(a_1)]^{\alpha_1}}{\Gamma(\alpha_1+\beta_1+1)}+\frac{|\lambda_1|}{\Gamma(\beta_1+1)}+|\vartheta_1|+|\vartheta_3|\right).
\end{aligned}$$

Hence,

$$|\mathcal{R}(z,y)(\tau_2)-\mathcal{R}(z,y)(\tau_1)| \longrightarrow 0, \text{ as } \tau_1\to\tau_2.$$

In addition, we have

$$\begin{aligned}
&|\mathcal{K}(z,y)(\tau_2)-\mathcal{K}(z,y)(\tau_1)|\\
&\le |\tilde{\Phi}(z,y)(\tau_2)-\tilde{\Phi}(z,y)(\tau_1)| + \frac{[(\psi(\tau_2)-\psi(a_1))^{\beta_2}-(\psi(\tau_1)-\psi(a_1))^{\beta_2}]}{(\psi(\epsilon_2)-\psi(a_1))^{\beta_2}}|\tilde{\Phi}(z,y)(\epsilon_2)|\\
&\le \frac{[(\psi(\tau_1)-\psi(a_1))^{\alpha_2+\beta_2}-(\psi(\tau_2)-\psi(a_1))^{\alpha_2+\beta_2}]}{\Gamma(\alpha_2+\beta_2+1)}\left[P_1^*+(P_2^*+P_3^*)\gamma\right]\\
&\quad+\left(\frac{(\psi(\tau_2)-\psi(\tau_1))^{\beta_2}-(\psi(\tau_2)-\psi(a_1))^{\beta_2}+(\psi(\tau_1)-\psi(a_1))^{\beta_2}}{\Gamma(\beta_2+1)}\right)|\lambda_2|\gamma\\
&\quad+[(\psi(\tau_1)-\psi(a_1))^{\beta_2}-(\psi(\tau_2)-\psi(a_1))^{\beta_2}]\left(\frac{[\psi(\epsilon_2)-\psi(a_1)]^{\alpha_2}}{\Gamma(\alpha_2+\beta_2+1)}+\frac{|\lambda_1|}{\Gamma(\beta_2+1)}+|\vartheta_2|+|\vartheta_4|\right).
\end{aligned}$$

Consequently,

$$|\mathcal{K}(z,y)(\tau_2)-\mathcal{K}(z,y)(\tau_1)| \longrightarrow 0, \text{ as } \tau_1\to\tau_2.$$

Thus, $\mathcal{R}(\mathcal{D}_\gamma)$ and $\mathcal{K}(\mathcal{D}_\gamma)$ are equicontinuous on $\mathcal{T}$. Hence, by the Arzelà-Ascoli theorem, $\mathcal{R}$ and $\mathcal{K}$ are completely continuous on $\mathcal{D}_\gamma$.

Step 3: We show that condition (3) of Theorem 2.8 is satisfied. For $(z,y)\in C(\mathcal{T},\mathbb{R})$ and $(\bar{z},\bar{y})\in\mathcal{D}_\gamma$, where $z=\mathcal{P}z\mathcal{R}\bar{z}+\mathcal{Q}z$ and $y=\mathcal{U}y\mathcal{K}\bar{y}+\mathcal{V}y$, we get

$$\begin{aligned}
|z(\tau)| &= |\mathcal{P}z(\tau)\mathcal{R}\bar{z}(\tau)+\mathcal{Q}z(\tau)|\\
&\le [|g_1(\tau,z(\tau))+g_1(\tau,0)|+|g_1(\tau,0)|]\mathcal{N}_\gamma+|h_1(\tau,z(\tau))+h_1(\tau,0)|+|h_1(\tau,0)|\\
&\le [q_1^*\|z\|+g_1^*]\mathcal{N}_\gamma+r_1^*\|z\|+h_1^*,
\end{aligned}$$

and

$$\begin{aligned}
|y(\tau)| &= |\mathcal{U}y(\tau)\mathcal{K}\bar{y}(\tau)+\mathcal{V}y(\tau)|\\
&\le [|g_2(\tau,y(\tau))+g_2(\tau,0)|+|g_2(\tau,0)|]\mathcal{E}_\gamma+|h_2(\tau,y(\tau))+h_2(\tau,0)|+|h_2(\tau,0)|\\
&\le [q_2^*\|y\|+g_2^*]\mathcal{E}_\gamma+r_2^*\|y\|+h_2^*,
\end{aligned}$$

which implies that

$$\|(z,y)\|_\Xi \le \frac{g_1^*\mathcal{N}_\gamma+g_2^*\mathcal{E}_\gamma+h_1^*+h_2^*}{1-[q_1^*\mathcal{N}_\gamma+q_2^*\mathcal{E}_\gamma+r_1^*+r_2^*]}\le\gamma.$$

This shows that condition (3) of Theorem 2.8 is satisfied.

Step 4: Finally, we have

$$\rho = \|(\mathcal{R}(\mathcal{D}_\gamma), \mathcal{K}(\mathcal{D}_\gamma))\|_\Xi = \sup\left\{\|\mathcal{R}(z,y)\| + \|\mathcal{K}(z,y)\| : (z,y) \in \mathcal{D}_\gamma\right\} \leq \mathcal{N}_\gamma + \mathcal{E}_\gamma.$$

From the above estimate we obtain

$$q_1^*\mathcal{N}_\gamma + q_2^*\mathcal{E}_\gamma + r_1^* + r_2^* < 1.$$

Hence, all the conditions of Theorem 2.8 are satisfied, and therefore, the operator equation $z = \mathcal{P}z\mathcal{R}z + \mathcal{Q}z$ and $y = \mathcal{U}y\mathcal{K}y + \mathcal{V}y$ has a solution in $\mathcal{D}_\gamma$. Consequently, the coupled system (1)-(2) has a solution on $\mathcal{T}$. □

4. Ulam-type stability

This section discusses the Ulam stability of the coupled system (1)-(2). Let $(z,y) \in \Xi$ and $\varrho_1, \varrho_2 > 0$, and $\mathcal{X}_1, \mathcal{X}_2 : [a_1, a_2] \longrightarrow \mathbb{R}^+$ be a continuous function. We consider the following inequalities:

$$\begin{cases} |{}^C D_{a_1^+}^{\alpha_1,\psi}\left[{}^C D_{a_1^+}^{\beta_1,\psi}\left(\frac{z(\tau)-h_1(\tau,z(\tau))}{g_1(\tau,z(\tau))}\right) + \lambda_1 z(\tau)\right] - f_1(\tau, z(\tau), y(\xi_1\tau))| < \varrho, \quad \tau \in \mathcal{T}, \\ |{}^C D_{a_1^+}^{\alpha_2,\psi}\left[{}^C D_{a_1^+}^{\beta_2,\psi}\left(\frac{y(\tau)-h_2(\tau,y(\tau))}{g_2(\tau,y(\tau))}\right) + \lambda_2 y(\tau)\right] - f_2(\tau, y(\tau), z(\xi_2\tau))| < \varrho, \quad \tau \in \mathcal{T}, \end{cases} \tag{30}$$

and

$$\begin{cases} |{}^C D_{a_1^+}^{\alpha_1,\psi}\left[{}^C D_{a_1^+}^{\beta_1,\psi}\left(\frac{z(\tau)-h_1(\tau,z(\tau))}{g_1(\tau,z(\tau))}\right) + \lambda_1 z(\tau)\right] - f_1(\tau, z(\tau), y(\xi_1\tau))| < \varrho\mathcal{X}_1(\tau), \quad \tau \in \mathcal{T}, \\ |{}^C D_{a_1^+}^{\alpha_2,\psi}\left[{}^C D_{a_1^+}^{\beta_2,\psi}\left(\frac{y(\tau)-h_2(\tau,y(\tau))}{g_2(\tau,y(\tau))}\right) + \lambda_2 y(\tau)\right] - f_2(\tau, y(\tau), z(\xi_2\tau))| < \varrho\mathcal{X}_2(\tau), \quad \tau \in \mathcal{T}, \end{cases} \tag{31}$$

Definition 4.1 ([15, 33]). *The coupled system* (1)-(2) *is Ulam-Hyers stable if there exists a real number* $\Theta = \Theta_1 + \Theta_2 > 0$ *with* $\Theta_1, \Theta_2 > 0$ *such that for each* $\varrho > 0$ *and for each solution* $(z,y) \in \Xi$ *to the previous inequality* (30)*, there exists a solution* $(\bar{z}, \bar{y}) \in \Xi$ *of the coupled system* (1)-(2) *with*

$$\|(z,y) - (\bar{z},\bar{y})\|_\Xi \leq \varrho\Theta.$$

Definition 4.2 ([15, 33]). *The coupled system* (1)-(2) *is generalized Ulam-Hyers stable if there exists* $\mathcal{J} \in C(\mathbb{R}^+, \mathbb{R}^+)$*, such that* $\mathcal{J}(0) = 0$ *for any* $\varrho > 0$*, and for each solution* $(z,y) \in \Xi$ *to the inequality* (30)*, there exists a solution* $(\bar{z}, \bar{y}) \in \Xi$ *of the coupled system* (1)-(2) *with*

$$\|(z,y) - (\bar{z},\bar{y})\|_\Xi \leq \mathcal{J}(\varrho).$$

Remark 4.3. *It is clear that:*

1. *Definition 4.1⇒ Definition 4.2.*

A function $(z,y) \in \Xi$ *is a solution of the inequality* (30) *if and only if there exist a function* $\mathcal{W}_i \in C(\mathcal{T}, \mathbb{R})$ *such that for all* $i = 1,2$ *we have*

i. $|\mathcal{W}_i(\tau)| \leq \varrho$*, for all* $\tau \in \mathcal{T}$*,*

ii. ${}^C D_{a_1^+}^{\alpha_1,\psi}\left[{}^C D_{a_1^+}^{\beta_1,\psi}\left(\dfrac{z(\tau) - h_1(\tau, z(\tau))}{g_1(\tau, z(\tau))}\right) + \lambda_1 z(\tau)\right] - f_1(\tau, z(\tau), y(\xi_1\tau)) = \mathcal{W}_1(\tau)$*, for all* $\tau \in \mathcal{T}$*,*

iii. ${}^C D_{a_1^+}^{\alpha_2,\psi}\left[{}^C D_{a_1^+}^{\beta_2,\psi}\left(\dfrac{y(\tau) - h_2(\tau, y(\tau))}{g_2(\tau, y(\tau))}\right) + \lambda_2 y(\tau)\right] - f_2(\tau, y(\tau), z(\xi_2\tau)) = \mathcal{W}_2(\tau)$*, for all* $\tau \in \mathcal{T}$*.*

We now discuss the Ulam stability of the solution to the problem (1)-(2).

Theorem 4.4. *Assume that* $(C1)$, $(C2)$ *and* $(C3)$ *are satisfied. then the coupled system (1)-(2) is Ulam-Hyers stable and hence generalizes Ulam-Hyers stable under the condition* (20).

Proof. Assume that $\varrho > 0$ and that $(z, y) \in \Xi$ fulfills the inequality (30), and let $(\bar{z}, \bar{y}) \in \Xi$ be the sole solution of the coupled system (1)-(2). Since $(z, y) \in \Xi$ satisfes the inequality (30), then it follows from Remark 4.3 that

$$\begin{cases} {}^{C}D^{\alpha_1,\psi}_{a_1^+}\left[{}^{C}D^{\beta_1,\psi}_{a_1^+}\left(\dfrac{z(\tau)-h_1(\tau,z(\tau))}{g_1(\tau,z(\tau))}\right)+\lambda_1 z(\tau)\right]-f_1(\tau,z(\tau),y(\xi\tau))=\mathcal{W}_1(\tau), \quad \tau\in\mathcal{T}, \\ {}^{C}D^{\alpha_2,\psi}_{a_1^+}\left[{}^{C}D^{\beta_2,\psi}_{a_1^+}\left(\dfrac{y(\tau)-h_2(\tau,y(\tau))}{g_2(\tau,y(\tau))}\right)+\lambda_2 y(\tau)\right]-f_2(\tau,y(\tau),z(\tilde{\xi}\tau))=\mathcal{W}_2(\tau), \quad \tau\in\mathcal{T}, \end{cases}$$

under the given boundary conditions

$$\begin{cases} \left(\dfrac{z(\tau)-h_1(\tau,z(\tau))}{g_1(\tau,z(\tau))}\right)\Bigg|_{\tau=a_1}=\vartheta_1,\ \left(\dfrac{z(\tau)-h_2(\tau,z(\tau))}{g_2(\tau,z(\tau))}\right)\Bigg|_{\tau=a_1}=\vartheta_2, \\ \left(\dfrac{z(\tau)-h_1(\tau,z(\tau))}{g_1(\tau,z(\tau))}\right)'\Bigg|_{\tau=a_1}=\left(\dfrac{z(\tau)-h_2(\tau,z(\tau))}{g_2(\tau,z(\tau))}\right)'\Bigg|_{\tau=a_1}=0, \\ \left(\dfrac{z(\tau)-h_1(\tau,z(\tau))}{g_1(\tau,z(\tau))}\right)\Bigg|_{\tau=\epsilon_1}=\vartheta_3,\ \left(\dfrac{z(\tau)-h_2(\tau,z(\tau))}{g_2(\tau,z(\tau))}\right)\Bigg|_{\tau=\epsilon_2}=\vartheta_4,\ a_1<\epsilon_1,\epsilon_2<a_2. \end{cases}$$

Using Lemma 3.3 once more, we have

$$z(\tau) := g_1(\tau,z(\tau))\left[\tilde{\Psi}(z,y)(\tau)-\frac{(\psi(\tau)-\psi(a_1))^{\beta_1}}{(\psi(\epsilon_1)-\psi(a_1))^{\beta_1}}\tilde{\Psi}(z,y)(\epsilon_1)\right]+h_1(\tau,z(\tau)),$$

and

$$y(\tau) := g_2(\tau,y(\tau))\left[\tilde{\Phi}(z,y)(\tau)-\frac{(\psi(\tau)-\psi(a_1))^{\beta_2}}{(\psi(\epsilon_2)-\psi(a_1))^{\beta_2}}\tilde{\Phi}(z,y)(\epsilon_2)\right]+h_2(\tau,y(\tau)),$$

where

$$\tilde{\Psi}(z,y)(\tau) := I^{\alpha_1+\beta_1,\psi}_{a_1^+}[f_1(\tau,z(\tau),y(\xi\tau))+\mathcal{W}_1(\tau)]-\lambda_1 I^{\alpha_1,\psi}_{a_1^+}z(\tau)+\vartheta_1-\vartheta_3,$$

and

$$\tilde{\Phi}(z,y)(\tau) := I^{\alpha_2+\beta_2,\psi}_{a_1^+}[f_2(\tau,y(\tau),z(\tilde{\xi}\tau))+\mathcal{W}_2(\tau)]-\lambda_2 I^{\alpha_2,\psi}_{a_1^+}y(\tau)+\vartheta_2-\vartheta_4.$$

Moreover, using part (*i*) of Remark 4.3 and (*C*2), we can obtain the following formula for each $\tau \in \mathcal{T}$.

$$\begin{aligned} &|\tilde{\Psi}(z,y)(\tau)-\tilde{\Psi}(\bar{z},\bar{y})(\tau)| \\ &\le I^{\alpha_1+\beta_1,\psi}_{a_1^+}|f_1(\tau,z(\tau),y(\xi\tau))-f_1(\tau,\bar{z}(\tau),\bar{y}(\xi\tau))|+\lambda_1 I^{\alpha_1,\psi}_{a_1^+}|z(\tau)-\bar{z}(\tau)|+I^{\alpha_1+\beta_1,\psi}_{a_1^+}|\mathcal{W}_1(\tau)| \\ &\le\left(\frac{2p_1^*(\psi(\tau)-\psi(a_1))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}+\frac{|\lambda_1|(\psi(\tau)-\psi(a_1))^{\beta_1}}{\Gamma(\beta_1+1)}\right)\|(z,y)-(\bar{z},\bar{y})\|_\Xi \\ &\quad+\frac{\varrho(\psi(\tau)-\psi(a_1))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}, \end{aligned} \tag{32}$$

and

$$\begin{aligned} &|\tilde{\Phi}(z,y)(\tau)-\tilde{\Phi}(\bar{z},\bar{y})(\tau)| \\ &\le I^{\alpha_2+\beta_2,\psi}_{a_1^+}|f_2(\tau,y(\tau),z(\xi\tau))-f_2(\tau,\bar{y}(\tau),\bar{z}(\tilde{\xi}\tau))|+\lambda_2 I^{\alpha_2,\psi}_{a_1^+}|y(\tau)-\bar{y}(\tau)|+I^{\alpha_2+\beta_2,\psi}_{a_1^+}|\mathcal{W}_2(\tau)| \\ &\le\left(\frac{2p_2^*(\psi(\tau)-\psi(a_1))^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}+\frac{|\lambda_2|(\psi(\tau)-\psi(a_1))^{\beta_2}}{\Gamma(\beta_2+1)}\right)\|(z,y)-(\bar{z},\bar{y})\|_\Xi \\ &\quad+\frac{\varrho(\psi(\tau)-\psi(a_1))^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}, \end{aligned} \tag{33}$$

in addition to,

$$\begin{cases} |\tilde{\Psi}(z,y)(\tau)| & \leq & \frac{(\mathcal{L}_1+\varrho)(\psi(\tau)-\psi(a_1))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{|\lambda_1|(\psi(\tau)-\psi(a_1))^{\beta_1}}{\Gamma(\beta_1+1)} + |\vartheta_1| + |\vartheta_3|, \\ |\tilde{\Psi}(\bar{z},\bar{y})(\tau)| & \leq & \frac{\mathcal{L}_1(\psi(\tau)-\psi(a_1))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{|\lambda_1|(\psi(\tau)-\psi(a_1))^{\beta_1}}{\Gamma(\beta_1+1)} + |\vartheta_1| + |\vartheta_3|, \\ |\tilde{\Phi}(z,y)(\tau)| & \leq & \frac{(\mathcal{L}_2+\varrho)(\psi(\tau)-\psi(a_1))^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{|\lambda_2|(\psi(\tau)-\psi(a_1))^{\beta_2}}{\Gamma(\beta_2+1)} + |\vartheta_2| + |\vartheta_4|, \\ |\tilde{\Phi}(\bar{z},\bar{y})(\tau)| & \leq & \frac{\mathcal{L}_2(\psi(\tau)-\psi(a_1))^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{|\lambda_2|(\psi(\tau)-\psi(a_1))^{\beta_2}}{\Gamma(\beta_2+1)} + |\vartheta_2| + |\vartheta_4|. \end{cases} \tag{34}$$

Applying the triangle inequality to (32)-(34), we have

$$\begin{aligned} &|z(\tau) - \bar{z}(\tau)| \\ &\leq |g_1(\tau, z(\tau))\tilde{\Psi}(z,y)(\tau) - g_1(\tau, \tilde{z}(\tau))\tilde{\Psi}(\bar{z},\bar{y})(\tau)| \\ &\quad + \frac{(\psi(\tau) - \psi(a_1))^{\beta_1}}{(\psi(\epsilon_1) - \psi(a_1))^{\beta_1}} |g_1(\tau, z(\tau))\tilde{\Psi}(z,y)(\epsilon_1) - g_1(\tau, \bar{z}(\tau))\tilde{\Psi}(\bar{z},\bar{y})(\epsilon_1)| + |h_1(\tau, z(\tau)) - h_1(\tau, \bar{z}(\tau))|, \end{aligned}$$

so that we obtain

$$\|z - \bar{z}\|_\infty \leq \varpi_1 \|(z,y) - (\bar{z},\bar{y})\|_\Xi + \mathcal{M}_1 \Delta_1 \varrho. \tag{35}$$

On the other hand, we have

$$\|y - \bar{y}\|_\infty \leq \varpi_2 \|(z,y) - (\bar{z},\bar{y})\|_\Xi + \mathcal{M}_2 \nabla_1 \varrho. \tag{36}$$

Combining the two last inequalities (35) and (36), we get

$$\|(z,y) - (\bar{z},\bar{y})\|_\Xi \leq \left[1 - \sum_{i=1}^{2} \varpi_i\right]^{-1} (\mathcal{M}_1\Delta_1 + \mathcal{M}_2\nabla_2)\, \varrho. \tag{37}$$

Let is put $\Theta = \left[1 - \sum_{i=1}^{2} \varpi_i\right]^{-1} (\mathcal{M}_1\Delta_1 + \mathcal{M}_2\nabla_2)$. Taking into account $\sum_{i=1}^{2} \varpi_i < 1$, we notice that $\Theta > 0$. Thus, we have

$$\|(z,y) - (\bar{z},\bar{y})\|_\Xi \leq \varrho\Theta.$$

Consequently, the coupled system (1)-(2) is stable in the sense of Ulam-Hyers. This completes the proof using Ulam-Hyers definition. □

Theorem 4.5. *Suppose the conditions of Theorem 4.4 hold. If there exists $\mathcal{J} \in C(\mathbb{R}^+, \mathbb{R}^+)$, such that $\mathcal{J}(0) = 0$ with $\varrho > 0$, then the coupled system* (1)-(2) *is generalized Ulam-Hyers stable.*

Proof. For $\mathcal{J}(\varrho) = \Theta\varrho$; $\mathcal{J}(0) = 0$, we prove that the solution to the coupled system (1)-(2) is also generalized Ulam-Hyers stable. □

5. Examples

Example 5.1. *Consider the coupled system*

$$\begin{cases} {}^C D_{1^+}^{0.45, \sqrt{\tau}} \left[{}^C D_{1^+}^{1.65, \sqrt{\tau}} \left(\dfrac{z(\tau) - h_1(\tau, z(\tau))}{g_1(\tau, z(\tau))} \right) + \frac{25}{100} z(\tau) \right] = f_1(\tau, z(\tau), y(\frac{1}{7}\tau)), \quad \tau \in \mathcal{T}, \\ {}^C D_{1^+}^{0.70, \sqrt{\tau}} \left[{}^C D_{1^+}^{1.75, \sqrt{\tau}} \left(\dfrac{y(\tau) - h_2(\tau, y(\tau))}{g_2(\tau, y(\tau))} \right) + 2y(\tau) \right] = f_2(\tau, y(\tau), z(\frac{4}{10}\tau)), \quad \tau \in \mathcal{T}, \end{cases} \tag{38}$$

under the boundary conditions

$$\begin{cases} \left(\dfrac{z(\tau)-h_1(\tau,z(\tau))}{g_1(\tau,z(\tau))}\right)(1)=\frac{15}{100}, \left(\dfrac{y(\tau)-h_2(\tau,y(\tau))}{g_2(\tau,y(\tau))}\right)(1)=\frac{30}{100}, \\ \left(\dfrac{z(\tau)-h_1(\tau,z(\tau))}{g_1(\tau,z(\tau))}\right)'\Big|_{\tau=1}=\left(\dfrac{y(\tau)-h_2(\tau,y(\tau))}{g_2(\tau,y(\tau))}\right)'\Big|_{\tau=1}=0, \\ \left(\dfrac{z(\tau)-h_1(\tau,z(\tau))}{g_1(\tau,z(\tau))}\right)(\frac{7}{6})=\frac{-15}{100}, \left(\dfrac{y(\tau)-h_2(\tau,y(\tau))}{g_2(\tau,y(\tau))}\right)(\frac{6}{5})=\frac{-30}{100}, 1<\frac{7}{6},\frac{6}{5}<2\,, \end{cases} \tag{39}$$

where $\mathcal{T}=[1,2]$, $a_1=1$, $a_2=2$ *and*

$$\begin{cases} g_1(\tau,z(t))=\dfrac{(\tau-\frac{1}{2})(|z(\tau)|+1)}{\pi e^{-\tau+3}},\ \tau\in\mathcal{T},\ z\in C(\mathcal{T},\mathbb{R}), \\ g_2(\tau,y(t))=\dfrac{\cos(\tau)(|y(\tau)|+0.01)}{\pi^2+100\tau^2},\ \tau\in\mathcal{T},\ y\in C(\mathcal{T},\mathbb{R}), \\ h_1(\tau,z(\tau))=\dfrac{\sqrt{\tau-\frac{1}{2}z(\tau)}}{12\pi e\sqrt{6-\tau}}+\dfrac{1}{e^{2+\tau}+32\pi},\ \tau\in\mathcal{T},\ z\in C(\mathcal{T},\mathbb{R}), \\ h_2(\tau,y(\tau))=\dfrac{\cos(2\tau)\,y(\tau)}{12\pi(\tau+1)}+\dfrac{e^2}{32\pi},\ \tau\in\mathcal{T},\ y\in C(\mathcal{T},\mathbb{R}), \\ f_1(\tau,z(\tau),y(\frac{\tau}{7}))=\dfrac{\sqrt{\tau-\frac{1}{2}}\left(\sin(\tau)y(\dfrac{1}{7}\tau)+\cos(\tau)z(\tau)-1\right)}{55e^{-\tau+3}(1+|z(\tau)|\sqrt{\tau-\frac{1}{2}})},\ \tau\in\mathcal{T},\ z,y\in C(\mathcal{T},\mathbb{R}), \\ f_2(\tau,y(\tau),z(\frac{4\tau}{10}))=\dfrac{\sin(\tau)\left(\sin(\tau)z(\dfrac{4}{10}\tau)+y(\tau)-0.02\right)}{(55+e^{-\tau+3})(1+|y(\tau)|)},\ \tau\in\mathcal{T},\ z,y\in C(\mathcal{T},\mathbb{R}). \end{cases}$$

Clearly, the continuous functions $f_i\in C(\mathcal{T}\times\mathbb{R}^2,\mathbb{R})$, $h_i\in C(\mathcal{T}\times\mathbb{R},\mathbb{R})$ *and* $g_i\in C(\mathcal{T}\times\mathbb{R},\mathbb{R}\setminus\{0\})$, *for* $i=1,2$. *Hence the condition* $(C1)$ *is satisfied.*
For each $z,\bar{z},y,\bar{y}\in\mathbb{R}$ *and* $\tau\in\mathcal{T}$, *we have*

$$\begin{cases} |f_1(\tau,z,y)-f_1(\tau,\bar{z},\bar{y})|\le\dfrac{\sqrt{\tau-\frac{1}{2}}}{55e^{-\tau+3}}\left(|z-\bar{z}|+|\,y-\bar{y}|\right), \\ |f_2(\tau,z,y)-f_2(\tau,\bar{z},\bar{y})|\le\dfrac{1}{55+e^{-\tau+3}}\left(|z-\bar{z}|+|\,y-\bar{y}|\right), \\ |g_1(\tau,z)-g_1(\tau,\bar{z})|\le\dfrac{(\tau-\frac{1}{2})}{\pi e^{-\tau+3}}\,|z-\bar{z}|, \\ |g_2(\tau,y)-g_2(\tau,\bar{y})|\le\dfrac{1}{\pi^2+100\tau^2}\left|y-\bar{y}\right|, \\ |h_1(\tau,z)-h_1(\tau,\bar{z})|\le\dfrac{\sqrt{\tau-\frac{1}{2}}}{12\pi e\sqrt{6-\tau}}\,|z-\bar{z}|, \\ |h_2(\tau,y)-h_2(\tau,\bar{y})|\le\dfrac{1}{12\pi(\tau+1)}\left|y-\bar{y}\right|. \end{cases}$$

Hence condition $(C2)$ *is satisfied with*

$$p_1(\tau) = \frac{\sqrt{\tau - \frac{3}{2}}}{55e^{-\tau+3}}, \quad p_2(\tau) = \frac{1}{55 + e^{-\tau+3}},$$
$$q_1(\tau) = \frac{(\tau - \frac{1}{2})}{\pi e^{-\tau+3}}, \quad q_2(\tau) = \frac{1}{\pi^2 + 100\tau^2},$$
$$r_1(\tau) = \frac{\sqrt{\tau - \frac{1}{2}}}{12\pi e\sqrt{6-\tau}}, \; r_2(\tau) = \frac{1}{12\pi(\tau+1)},$$

so we have

$$p_1^* = \frac{\sqrt{3}}{55e\sqrt{2}}, \quad p_2^* = \frac{1}{55 + e^2},$$
$$q_1^* = \frac{3}{2\pi e}, \quad q_2^* = \frac{1}{\pi^2 + 100},$$
$$r_1^* = \frac{\sqrt{3}}{24\pi e\sqrt{2}}, \quad r_2^* = \frac{1}{24\pi},$$

and so the condition $(C3)$ *is satisfied with*

$$\mathcal{M}_1 = \frac{1}{55e}, \; \mathcal{L}_1 = \frac{3}{2\pi e}, \; \mathcal{M}_2 = \frac{1}{55}, \; \mathcal{L}_2 = \frac{1}{100}.$$

We can show that

$$\sum_{i=1}^{2} \varpi_i = 0.97329 < 1.$$

Thus, all the conditions of Theorem 3.4 are satisfied. Hence, the coupled system (38)-(39) *has a unique solution on* $[1,2]$.

Let $\varrho = \frac{2}{3} > 0$, *as illustrated Theorem 4.4 and by (30). If* $z, y \in C([1,2],\mathbb{R})$ *complies with*

$$\left|{}^{C}D_{1^+}^{0.45,\sqrt{\tau}}\left[{}^{C}D_{1^+}^{1.65,\sqrt{\tau}}\left(\frac{z(\tau) - h_1(\tau, z(\tau))}{g_1(\tau, z(\tau))}\right) + \tfrac{25}{100}z(\tau)\right] - f_1(\tau, z(\tau), y(\tfrac{1}{7}\tau))\right| < \frac{2}{3},$$

and

$$\left|{}^{C}D_{1^+}^{0.70,\sqrt{\tau}}\left[{}^{C}D_{1^+}^{1.75,\sqrt{\tau}}\left(\frac{y(\tau) - h_2(\tau, y(\tau))}{g_2(\tau, y(\tau))}\right) + 2y(\tau)\right] - f_2(\tau, y(\tau), z(\tfrac{4}{10}\tau))\right| < \frac{2}{3},$$

there exists a solution $\bar{z}, \bar{y} \in C([1,2],\mathbb{R})$ *of the coupled system* (38)-(39) *with*

$$\|(z, y) - (\bar{z}, \bar{y})\|_{\Xi} \le \frac{2}{3}\Theta,$$

where

$$\Theta = \left[1 - \sum_{i=1}^{2}\varpi_i\right]^{-1}(\mathcal{M}_1\Delta_1 + \mathcal{M}_2\nabla_2) = 0.38176 > 0.$$

Consequently, the coupled system (38)-(39) *is Ulam-Hyers stable on* $[1,2]$. *Finally, we assume that* $\varrho = 0$, *we obtain* $\mathcal{J}(0) = 0$. *Hence, the problem* (38)-(39) *is generalized Ulam-Hyers stable.*

Example 5.2. *Consider the coupled system*

$$\begin{cases} {}^{C}D_{1^+}^{0.45,\frac{\tau^2}{2}}\left[{}^{C}D_{1^+}^{1.65,\frac{\tau^2}{2}}\left(\dfrac{z(\tau)-h_1(\tau,z(\tau))}{g_1(\tau,z(\tau))}\right)+\frac{25}{100}z(\tau)\right]=f_1(\tau,z(\tau),y(\tfrac{1}{5}\tau)), \quad \tau\in\mathcal{T}, \\ {}^{C}D_{1^+}^{0.75,\frac{\tau^2}{2}}\left[{}^{C}D_{1^+}^{1.25,\frac{\tau^2}{2}}\left(\dfrac{z(\tau)-h_2(\tau,y(\tau))}{g_2(\tau,y(\tau))}\right)+\frac{4}{100}y(\tau)\right]=f_2(\tau,y(\tau),z(\tfrac{1}{5}\tau)), \quad \tau\in\mathcal{T}, \end{cases} \tag{40}$$

under the boundary conditions

$$\begin{cases} \left(\dfrac{z(\tau)-h_1(\tau,z(\tau))}{g_1(\tau,z(\tau))}\right)\Bigg|_{\tau=1}=\frac{2}{10}, \ \left(\dfrac{y(\tau)-h_2(\tau,y(\tau))}{g_2(\tau,y(\tau))}\right)\Bigg|_{\tau=1}=\frac{35}{100}, \\ \left(\dfrac{z(\tau)-h_1(\tau,z(\tau))}{g_1(\tau,z(\tau))}\right)'\Bigg|_{\tau=1}=\left(\dfrac{y(\tau)-h_2(\tau,y(\tau))}{g_2(\tau,y(\tau))}\right)'\Bigg|_{\tau=1}=0, \\ \left(\dfrac{z(\tau)-h_1(\tau,z(\tau))}{g_1(\tau,z(\tau))}\right)\Bigg|_{\tau=\frac{3}{2}}=\frac{-1}{10}, \ \left(\dfrac{y(\tau)-h_2(\tau,y(\tau))}{g_2(\tau,y(\tau))}\right)\Bigg|_{\tau=\frac{5}{4}}=\frac{-25}{100}, \ 1<\frac{3}{2},\frac{5}{4}<2\,, \end{cases} \tag{41}$$

where $\mathcal{T}=[1,2]$, $a_1=1$, $a_2=2$ *and*

$$\begin{cases} g_1(\tau,z(\tau))=\dfrac{\sqrt{(\tau-\frac{1}{2})}}{\pi e^{-\tau+3}}(|\cos(\tau)||z(\tau)|+2), \ \tau\in\mathcal{T}, \ z\in C(\mathcal{T},\mathbb{R}), \\ g_2(\tau,y(\tau))=\dfrac{|\sin(\tau)|}{24^{\tau}}(|y(\tau)|+|\cos(\tau)|+1), \ \tau\in\mathcal{T}, \ y\in C(\mathcal{T},\mathbb{R}), \\ h_1(\tau,z(\tau))=\dfrac{\sqrt{\tau-1}|\sin(\tau)|z(\tau)}{33\pi e}+\dfrac{16\pi}{35e^{2+\tau}}, \ \tau\in\mathcal{T}, \ z\in C(\mathcal{T},\mathbb{R}), \\ h_2(\tau,y(\tau))=\dfrac{|\cos^2(\tau)|}{73e^{2+\tau}}(|y(\tau)|+|\tan^{-1}(\tau)|+2\pi), \ \tau\in\mathcal{T}, \ y\in C(\mathcal{T},\mathbb{R}), \\ f_1(\tau,z(\tau),y(\tfrac{\tau}{5}))=\dfrac{\sqrt{\tau-\frac{1}{2}}|\sin(\tau)|\left(z(\dfrac{1}{5}\tau)+\cos(\tau)z(\tau)+1\right)}{70e^{-\tau+3}(2+|z(\tau)|+|z(\frac{\tau}{5})|)}, \ \tau\in\mathcal{T}, \ z,y\in C(\mathcal{T},\mathbb{R}), \\ f_2(\tau,y(\tau),z(\tfrac{\tau}{5}))=\dfrac{|\sin^2(\tau)|}{35e^{\tau+3}}\left(\dfrac{\cos(\tau)z(\tau)}{\pi+|z(\tau)|}+\dfrac{z(\dfrac{1}{5}\tau)}{2\pi+|z(\dfrac{1}{5}\tau)|}\right)+\dfrac{e^{\pi}}{35^{\tau^2}}, \ \tau\in\mathcal{T}, \ z,y\in C(\mathcal{T},\mathbb{R}). \end{cases}$$

Clearly, the functions $f_1, f_2 \in C(\mathcal{T}\times\mathbb{R}^2,\mathbb{R})$ *are continuous. Hence the condition* $(\mathcal{CD}1)$ *is satisfied. For each* $z,\bar{z},y,\bar{y}\in\mathbb{R}$ *and* $\tau\in\mathcal{T}$, *we have*

$$\begin{cases} |g_1(\tau,z)-g_1(\tau,\bar{z})|\leq\dfrac{\sqrt{(\tau-\frac{1}{2})}}{\pi e^{-\tau+3}}\,|z-\bar{z}|\,, \\ |g_2(\tau,y)-g_2(\tau,\bar{y})|\leq\dfrac{|\sin(\tau)|}{24^{\tau}}\,\left|y-\bar{y}\right|, \\ |h_1(\tau,z)-h_1(\tau,\bar{z})|\leq\dfrac{\sqrt{\tau-1}}{33\pi e}\,|z-\bar{z}|\,, \\ |h_2(\tau,y)-h_2(\tau,\bar{y})|\leq\dfrac{|\cos^2(\tau)|}{73e^{2+\tau}}\,\left|y-\bar{y}\right|. \end{cases}$$

Hence, the condition $(\mathcal{CD}2)$ *is satisfied with*

$$\begin{cases} q_1(\tau) = \dfrac{\sqrt{(\tau - \frac{1}{2})}}{\pi e^{-\tau+3}}, \\ r_1(\tau) = \dfrac{\sqrt{\tau - 1}}{33\pi e}, \\ q_2(\tau) = \dfrac{|\sin(\tau)|}{24^{\tau}}, \\ r_2(\tau) = \dfrac{|\cos^2(\tau)|}{73e^{2+\tau}}, \end{cases}$$

so we have

$$q_1^* = \frac{\sqrt{3}}{\pi e\sqrt{2}},\ r_1^* = \frac{1}{33\pi e},\ q_2^* = \frac{1}{24},$$
$$r_2^* = \frac{1}{73e^3}.$$

Let z, y ∈ $\mathbb{R}$. *Then*

$$|f_1(\tau, z, y)| \leq \frac{\sqrt{\tau - \frac{1}{2}}}{70e^{-\tau+3}}(|y| + |z| + 1),\ \tau \in \mathcal{T},$$
$$|f_2(\tau, z, y)| \leq \frac{|\sin^2(\tau)|}{35e^{\tau+3}}(|y| + |z|) + |\tan^{-1}(\tau)| + e^{\pi},\ \tau \in \mathcal{T},$$

and so, the condition $(\mathcal{CD}3)$ *is satisfied with*

$$p_1(\tau) = p_2(\tau) = p_3(\tau) = \frac{\sqrt{\tau - \frac{1}{2}}}{70e^{-\tau+3}},$$
$$p_1^* = p_2^* = p_3^* = \frac{\sqrt{3}}{70e\sqrt{2}},$$
$$\mathfrak{y}_1(\tau) = \frac{e^{\pi}}{35^{\tau^2}},\quad \mathfrak{y}_2(\tau) = \mathfrak{y}_3(\tau) = \frac{|\sin^2(\tau)|}{35e^{\tau+3}},$$
$$\mathfrak{y}_1^* = \frac{e^{\pi}}{35},\quad \mathfrak{y}_2^* = \mathfrak{y}_3^* = \frac{1}{35e^4}.$$

Set

$$g_1^* = \frac{\sqrt{6}}{\pi e},\ h_1^* = \frac{16\pi}{35e^3},$$
$$g_2^* = \frac{1}{12},\ h_2^* = \frac{5\pi}{70e^4}.$$

In addition, condition $(\mathcal{CD}4)$ *and* (23) *of Theorem 3.5 are satisfied if we take*

$$0.91882 \leq \gamma \leq 11.983.$$

By Theorem 3.5, the coupled system (40)-(41) *has at least one solution in* $\mathcal{T}$.

6. Conclusion

In this paper, we employed the ψ-Caputo derivative to analyze solutions of a novel class of hybrid Langevin-pantograph fractional coupled systems. Our study primarily focused on the existence, uniqueness, and stability of these solutions under non-local and two-point boundary conditions. This work represents the first exploration of the combined Langevin and pantograph differential systems within a fractional framework.

The hybrid formulation of the coupled system allowed us to apply Dhage's hybrid fixed-point theorem, specifically for the sum of three operators, alongside with Banach's fixed-point inequalities. We also examined the system's stability through the Ulam-Hyers stability model and its generalized variants.

To substantiate our theoretical findings, we provided practical examples, contributing new insights and broadening the scope of previous research in this domain.

Looking ahead, we recommend further research on similar problems that incorporate generalized fractional derivatives within the context of impulsive systems. Additionally, future studies could explore alternative fractional models, including those utilizing multipoint boundary conditions.

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