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Approximation by bivariate Bernstein-Kantorovich operators that reproduce exponential functions

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Abstract. In this study, we construct bivariate Bernstein-Kantorovich polynomials which reproduce exponential function and investigate some approximation results for these operators. We use the test functions to prove Korovkin-type theorem. Then we obtain the rate of convergence by means of the modulus of continuity. We demonstrate the Voronovskaya-type theorem for the newly constructed operator. In the last section, we represent some illustrative graphics to show the convergence of the constructed operators and we give some numerical results.

1. Introduction

Bivariate form of Bernstein operators have been constructed in the literature. Some of these operators are given in [5, 14, 15, 17, 18, 21, 25]. Bivariate Bernstein polynomial of order n on the simplex $S \equiv \left\{(x, y) \in \mathbb{R}^2; x, y \geq 0, x + y \leq 1\right\}$ for $f \in C(S)$ and $n \in \mathbb{N}$ is given as follows:

$$B_n f(x, y) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} f\left(\frac{k}{n}, \frac{j}{n}\right) p_{n,k,j}(x, y), \tag{1}$$

where

$$p_{n,k,j}(x, y) = \binom{n}{k}\binom{n-k}{j} x^k y^j (1 - x - y)^{n-k-j}.$$

In [34], for $n \in \mathbb{N}$, $f \in L_1([0, 1] \times [0, 1])$ Pop and Farcas constructed two variable Bernstein-Kantorovich type operators
$K_n : L_1(S) \to C([0, 1] \times [0, 1])$. For any $(x, y) \in S$, these operators are defined as:

$$K_n(f; x, y) = (n + 1)^2 \sum_{k=0}^{n} \sum_{j=0}^{n-k} p_{n,k,j}(x, y) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} f(t, s)\, ds\, dt, \tag{2}$$

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where $k, j \geq 0$.

In [10, 11], Aral et al. give modification of exponential forms of Bernstein operators defined as

$$G_n f(x) = G_n(f; x) \quad = \quad \sum_{k=0}^{n} f\left(\frac{k}{n}\right) e^{-\alpha k/n} e^{-\alpha x} p_{n,k}(a_n(x)),\ x \in [0,1],\ n \in \mathbb{N},$$

where

$$p_{n,k}(a_n(x)) = \binom{n}{k}(a_n(x))^k (1 - a_n(x))^{n-k}$$

and

$$a_n(x) = \frac{e^{\alpha x/n} - 1}{e^{\alpha/n} - 1}. \tag{3}$$

They defined the relation of their operators to the classical Bernstein operators as

$$G_n f(x) = exp_\alpha(x) B_n\left(\frac{f}{exp_\alpha}; a_n(x)\right). \tag{4}$$

Here, the exponential function is defined as $exp_\alpha(x) = e^{\alpha x}$, for a real parameter $\alpha > 0$. The generalization given by Aral et al. [11] is a particular case of the modification introduced by Morigi and Neamtu in [33].

In 2021, Bozkurt et al. [16] give the bivariate Bernstein operators, which reproduce exponential functions for each integer n and $\alpha, \beta > 0$ in the domain $S_{\alpha,\beta} = \{(x, y) \in \mathbb{R}^2; x, y \geq 0, a_n(\alpha, x) + a_n(\beta, y) \leq 1\} \subset S$ as follows:

$$B_n^{\alpha,\beta} f(x, y) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} f\left(\frac{k}{n}, \frac{j}{n}\right) p_{n,k,j}(a_n(\alpha, x), a_n(\beta, y)), \tag{5}$$

where

$$p_{n,k,j}(a_n(\alpha, x), a_n(\beta, y)) = \binom{n}{k}\binom{n-k}{j} a_n(\alpha, x)^k a_n(\beta, y)^j (1 - a_n(\alpha, x) - a_n(\beta, y))^{n-k-j}.$$

Altomare et al. [7] showed a work of the approximation properties for a extensive class of Kantorovich-type operators. The Kantorovich-type version of the well-known linear positive operators studied by several mathematicians for example; in [35], the authors introduced a generalization of the Kantorovich-type Bernstein operators based on q-integers. In [4], Acu and Muraru introduced a bivariate generalization of the Bernstein Schurer–Kantorovich operators based on q-integers. In [1], the authors obtained Kantorovich modifications of (p, q)-Bernstein operators for bivariate functions using a new (p, q)-integral. In [6], the authors producted of a generalized λ-Bernstein-Kantorovich type operators. Also they comparisoned for the bivariate λ-Bernstein-Kantorovich type operators and the associated GBS operators by means of graphs and tables for certain functions by using some algorithms. In [12], Aslan gave some approximation properties of univariate and bivariate Kantorovich type new class Bernstein operators by means of shape parameter $\lambda \in [-1, 1]$. In [3], Acu, Aral and Raşa, studied Voronovskaya type results and convergence in variation for the exponential Bernstein Kantorovich operators. For more research see [2],[8],[13],[19],[20],[22],[24],[26],[27],[28],[29],[30],[31],[32],[36] and [37].

This paper consists of 6 sections. In Section 2, we give the definition of a new type generalized two variable Bernstein-Kantorovich operators and we obtain some auxiliary results such as calculation of moment and limits of central moments. In Section 3, we mention the rate of convergence with the help of modulus of continuity. In Section 4, we present Voronovskaya-type results. In Section 5, we illustrate numerical examples with graphics. In Section 6, we give conclusion.

2. Preliminary

In this article, we construct bivariate Bernstein-Kantorovich operators that reproduce exponential functions.

Definition 2.1. *Let $S_{\alpha,\beta} = \{(x,y) \in \mathbb{R}^2; x, y \geq 0, a_n(\alpha,x) + a_n(\beta,y) \leq 1\} \subset S$ for each $n \in \mathbb{N}$ and $\alpha, \beta > 0$. We define bivariate Bernstein-Kantorovich operators for the functions $f \in C(S_{\alpha,\beta})$ as follows:*

$$\widetilde{K}_n^{\alpha,\beta}(f;x,y) = a'_{n+1}(\alpha,x)a'_{n+1}(\beta,y)(n+1)^2 e^{\alpha x+\beta y}\sum_{k=0}^{n}\sum_{j=0}^{n-k} p_{n,k,j}(a_{n+1}(\alpha,x), a_{n+1}(\beta,y)) \times \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} e^{-\alpha t-\beta s} f(t,s)\,ds\,dt, \tag{6}$$

where

$$p_{n,k,j}(a_{n+1}(\alpha,x), a_{n+1}(\beta,y)) = \binom{n}{k}\binom{n-k}{j} a_{n+1}(\alpha,x)^k a_{n+1}(\beta,y)^j (1 - a_{n+1}(\alpha,x) - a_{n+1}(\beta,y))^{n-k-j},$$

$$a_{n+1}(\alpha,x) = \frac{e^{\alpha x/n+1}-1}{e^{\alpha/n+1}-1},\quad a_{n+1}(\beta,y) = \frac{e^{\beta y/n+1}-1}{e^{\beta/n+1}-1}$$

and

$$a'_{n+1}(\alpha,x) = \frac{\alpha e^{\alpha x/n+1}}{(n+1)(e^{\alpha/n+1}-1)},\quad a'_{n+1}(\beta,y) = \frac{\beta e^{\beta y/n+1}}{(n+1)(e^{\beta/n+1}-1)}.$$

$\alpha, \beta > 0$ are real parameters and $exp_{i,j}^{\alpha,\beta}$ represent the exponential function defined by $exp_{i,j}^{\alpha,\beta}(t,s) := e^{i\alpha t + j\beta s}$ for $0 \leq i, j \leq 4$.

Lemma 2.2. *Let $n \in \mathbb{N}$ and $(x,y) \in S_{\alpha,\beta}$. Then the following equalities hold:*

$$\begin{aligned}
\widetilde{K}_n^{\alpha,\beta}(1;x,y) &= e^{\frac{(\alpha x+\beta y)(n+2)-\alpha-\beta}{n+1}}(e^{\frac{-\alpha}{n+1}}(1-e^{\frac{\alpha x}{n+1}}) + e^{\frac{-\beta}{n+1}}(1-e^{\frac{\beta y}{n+1}}+e^{\frac{\beta}{n+1}}))^n,\\
\widetilde{K}_n^{\alpha,\beta}(e^{\alpha t};x,y) &= \frac{\alpha e^{\frac{(\alpha x+\beta y)(n+2)-\beta}{n+1}}}{(e^{\frac{\alpha}{n+1}}-1)(n+1)}\left(e^{\frac{-\beta}{n+1}}(1-e^{\frac{\beta y}{n+1}}+e^{\frac{\beta}{n+1}})\right)^n,\\
\widetilde{K}_n^{\alpha,\beta}(e^{\beta s};x,y) &= \frac{\beta e^{\frac{(\alpha x+\beta y)(n+2)-\alpha}{n+1}}}{(e^{\frac{\beta}{n+1}}-1)(n+1)}\left(e^{\frac{-\alpha}{n+1}}(1-e^{\frac{\alpha x}{n+1}}+e^{\frac{\alpha}{n+1}})\right)^n,\\
\widetilde{K}_n^{\alpha,\beta}(e^{2\alpha t};x,y) &= e^{\frac{(\alpha x+\beta y)(n+2)-\beta}{n+1}}\left(e^{\frac{\alpha x}{n+1}} - e^{\frac{-\beta}{n+1}}(e^{\frac{\beta y}{n+1}}+1)\right)^n,\\
\widetilde{K}_n^{\alpha,\beta}(e^{2\beta s};x,y) &= e^{\frac{(\alpha x+\beta y)(n+2)-\alpha}{n+1}}\left(e^{\frac{\beta y}{n+1}} - e^{\frac{-\alpha}{n+1}}(e^{\frac{\alpha x}{n+1}}+1)\right)^n,\\
\widetilde{K}_n^{\alpha,\beta}(e^{3\alpha t};x,y) &= \frac{1}{2}\left(e^{\frac{\alpha}{n+1}}+1\right)e^{\frac{\beta y-\beta}{n+1}} + e^{\frac{\alpha x(2+n)}{n+1}}\left(e^{\frac{\alpha x}{n+1}}(e^{\frac{\alpha}{n+1}}+1) - e^{\frac{-\beta}{n+1}}(e^{\frac{\beta y}{n+1}}-1) - e^{\frac{\alpha}{n+1}}\right)^n,\\
\widetilde{K}_n^{\alpha,\beta}(e^{4\alpha t};x,y) &= \frac{1}{3}\left(1+e^{\frac{\alpha}{n+1}}+e^{\frac{2\alpha}{n+1}}\right)e^{\frac{(\alpha x+\beta y)(n+2)}{n+1}}\left(e^{\frac{\alpha x}{n+1}}(e^{\frac{2\alpha}{n+1}}+e^{\frac{\alpha}{n+1}}+1) - e^{\frac{\alpha}{n+1}}(e^{\frac{\alpha}{n+1}}+1) - e^{\frac{-\beta}{n+1}}(e^{\frac{\beta y}{n+1}}-1)\right)^n,\\
\widetilde{K}_n^{\alpha,\beta}(e^{\alpha t+\beta s};x,y) &= \frac{\alpha\beta e^{\frac{(\alpha x+\beta y)(n+2)}{n+1}}}{(e^{\frac{\alpha}{n+1}}-1)(e^{\frac{\beta}{n+1}}-1)}(n+1)^2.
\end{aligned}$$

Proof. By taking $f(t,s)=1$ in (6), we obtain

$$\begin{aligned}\widetilde{K}_n^{\alpha,\beta}(1;x,y) &= \frac{\alpha\beta}{(n+1)^2}\frac{e^{\frac{\alpha x}{n+1}}}{e^{\frac{\alpha}{n+1}}-1}\frac{e^{\frac{\beta y}{n+1}}}{e^{\frac{\beta}{n+1}}-1}e^{\alpha x+\beta y}(n+1)^2\sum_{k=0}^{n}\sum_{j=0}^{n-k}p_{n,k,j}(a_{n+1}(\alpha,x),a_{n+1}(\beta,y))\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}}e^{-\alpha t-\beta s}\,ds\,dt\\ &= e^{\frac{(\alpha x+\beta y)(n+2)-\alpha-\beta}{n+1}}(e^{\frac{-\alpha}{n+1}}(1-e^{\frac{\alpha x}{n+1}})+e^{\frac{-\beta}{n+1}}(1-e^{\frac{\beta y}{n+1}}+e^{\frac{\beta}{n+1}}))^n.\end{aligned}$$

By taking $f(t,s)=e^{\alpha t}$ in (6), we achieve

$$\begin{aligned}\widetilde{K}_n^{\alpha,\beta}(e^{\alpha t};x,y) &= \frac{\alpha\beta}{(n+1)^2}\frac{e^{\frac{\alpha x}{n+1}}}{e^{\frac{\alpha}{n+1}}-1}\frac{e^{\frac{\beta y}{n+1}}}{e^{\frac{\beta}{n+1}}-1}e^{\alpha x+\beta y}(n+1)^2\sum_{k=0}^{n}\sum_{j=0}^{n-k}p_{n,k,j}(a_{n+1}(\alpha,x),a_{n+1}(\beta,y))\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}}e^{-\beta s}\,ds\,dt\\ &= \frac{\alpha e^{\frac{(\alpha x+\beta y)(n+2)-\beta}{n+1}}}{(e^{\frac{\alpha}{n+1}}-1)(n+1)}\left(e^{\frac{-\beta}{n+1}}(1-e^{\frac{\beta y}{n+1}}+e^{\frac{\beta}{n+1}})\right)^n.\end{aligned}$$

By taking $f(t,s)=e^{\beta s}$ in (6), we get

$$\begin{aligned}\widetilde{K}_n^{\alpha,\beta}(e^{\beta s};x,y) &= \frac{\alpha\beta}{(n+1)^2}\frac{e^{\frac{\alpha x}{n+1}}}{e^{\frac{\alpha}{n+1}}-1}\frac{e^{\frac{\beta y}{n+1}}}{e^{\frac{\beta}{n+1}}-1}e^{\alpha x+\beta y}(n+1)^2\sum_{k=0}^{n}\sum_{j=0}^{n-k}p_{n,k,j}(a_{n+1}(\alpha,x),a_{n+1}(\beta,y))\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}}e^{-\alpha t}\,ds\,dt\\ &= \frac{\beta e^{\frac{(\alpha x+\beta y)(n+2)-\alpha}{n+1}}}{(e^{\frac{\beta}{n+1}}-1)(n+1)}\left(e^{\frac{-\alpha}{n+1}}(1-e^{\frac{\alpha x}{n+1}}+e^{\frac{\alpha}{n+1}})\right)^n.\end{aligned}$$

By taking $f(t,s)=e^{2\alpha t}$ in (6), we have

$$\begin{aligned}\widetilde{K}_n^{\alpha,\beta}(e^{2\alpha t};x,y) &= \frac{\alpha\beta}{(n+1)^2}\frac{e^{\frac{\alpha x}{n+1}}}{e^{\frac{\alpha}{n+1}}-1}\frac{e^{\frac{\beta y}{n+1}}}{e^{\frac{\beta}{n+1}}-1}e^{\alpha x+\beta y}(n+1)^2\sum_{k=0}^{n}\sum_{j=0}^{n-k}p_{n,k,j}(a_{n+1}(\alpha,x),a_{n+1}(\beta,y))\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}}e^{\alpha t-\beta s}\,ds\,dt\\ &= e^{\frac{(\alpha x+\beta y)(n+2)-\beta}{n+1}}\left(e^{\frac{\alpha x}{n+1}}-e^{\frac{-\beta}{n+1}}(e^{\frac{\beta y}{n+1}}+1)\right)^n.\end{aligned}$$

By taking $f(t,s)=e^{2\beta s}$ in (6), we obtain

$$\begin{aligned}\widetilde{K}_n^{\alpha,\beta}(e^{2\beta s};x,y) &= \frac{\alpha\beta}{(n+1)^2}\frac{e^{\frac{\alpha x}{n+1}}}{e^{\frac{\alpha}{n+1}}-1}\frac{e^{\frac{\beta y}{n+1}}}{e^{\frac{\beta}{n+1}}-1}e^{\alpha x+\beta y}(n+1)^2\sum_{k=0}^{n}\sum_{j=0}^{n-k}p_{n,k,j}(a_{n+1}(\alpha,x),a_{n+1}(\beta,y))\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}}e^{-\alpha t+\beta s}\,ds\,dt\\ &= e^{\frac{(\alpha x+\beta y)(n+2)-\alpha}{n+1}}\left(e^{\frac{\beta y}{n+1}}-e^{\frac{-\alpha}{n+1}}(e^{\frac{\alpha x}{n+1}}+1)\right)^n.\end{aligned}$$

Other results can be obtained in a similar way.
□

Theorem 2.3. *Let $\alpha,\beta\in(0,\infty)$. Then we have*

$$\lim_{n\to\infty}\widetilde{K}_n^{\alpha,\beta}\left(exp_{i,j}^{\alpha,\beta};x,y\right)=exp_{i,j}^{\alpha,\beta}(x,y)$$

for $(i,j)\in\{(0,0),(1,0),(0,1),(2,0),(0,2)\}$.

Proof. Hereby, by choosing test function $exp_{i,j}^{\alpha,\beta}(t,s)=e^{i\alpha t+j\beta s}$ for $(i,j)\in\{(0,0),(1,0),(0,1)\}$ we obtain

$$\lim_{n\to\infty}\widetilde{K}_n^{\alpha,\beta}(1;x,y)=1, \tag{7}$$

$$\lim_{n\to\infty}\widetilde{K}_n^{\alpha,\beta}(e^{\alpha t};x,y)=e^{\alpha x}, \tag{8}$$

$$\lim_{n\to\infty} \widetilde{K}_n^{\alpha,\beta}(e^{\beta s}; x, y) = e^{\beta y}. \tag{9}$$

For $(i, j) = (2, 0)$ and $(i, j) = (0, 2)$, we get

$$\lim_{n\to\infty} \widetilde{K}_n^{\alpha,\beta}(e^{2\alpha t} + e^{2\beta s}; x, y) = e^{2\alpha x} + e^{2\beta y}. \tag{10}$$

□

Theorem 2.4. *Let $\alpha, \beta \in (0, \infty)$ and $f \in C(S_{\alpha,\beta})$. $\widetilde{K}_n^{\alpha,\beta}(f; x, y)$ converge uniformly to f on $C(S_{\alpha,\beta})$.*

Proof. *According to Korovkin theorem [23], from (7), (8), (9) and (10) ,*

$$\lim_{n\to\infty} \|\widetilde{K}_n^{\alpha,\beta}\left(exp_{i,j}^{\alpha,\beta}; x, y\right) - exp_{i,j}^{\alpha,\beta}(x, y)\|_{C(S_{\alpha,\beta})} = 0$$

where $(i, j) \in \{(0, 0), (1, 0), (0, 1), (2, 0), (0, 2)\}$. We obtain the desired result. □

Lemma 2.5. *For any $(x, y) \in S_{\alpha,\beta}$, we obtain the limits of the central moments as follows:*

$$\begin{aligned}
\lim_{n\to\infty} n(\widetilde{K}_n^{\alpha,\beta}(1; x, y) - 1) &= \alpha^2(-1 + x)x + \beta(-1 + (2 + \beta)y - \beta y^2) - \alpha(1 + x(-2 + \beta y)), && (11)\\
\lim_{n\to\infty} n\widetilde{K}_n^{\alpha,\beta}((e^{\alpha t} - e^{\alpha x}); x, y) &= \frac{1}{2}\alpha e^{\alpha x}(1 + 2\alpha x^2 - 2x(1 + \alpha - \beta y)), && (12)\\
\lim_{n\to\infty} n\widetilde{K}_n^{\alpha,\beta}((e^{\beta s} - e^{\beta y}); x, y) &= \frac{1}{2}\beta e^{\beta y}(1 + 2\beta y^2 - 2y(1 + \beta - \alpha x)), && (13)\\
\lim_{n\to\infty} n\widetilde{K}_n^{\alpha,\beta}((e^{\alpha t} - e^{\alpha x})^2; x, y) &= -\alpha^2 e^{2\alpha x}(-1 + x)x, && (14)\\
\lim_{n\to\infty} n\widetilde{K}_n^{\alpha,\beta}((e^{\beta s} - e^{\beta y})^2; x, y) &= -\beta^2 e^{2\beta y}(-1 + y)y, && (15)\\
\lim_{n\to\infty} n^2\widetilde{K}_n^{\alpha,\beta}((e^{\alpha t} - e^{\alpha x})^4; x, y) &= 3\alpha^4 e^{4\alpha x}(-1 + x)^2 x^2, && (16)\\
\lim_{n\to\infty} n^2\widetilde{K}_n^{\alpha,\beta}((e^{\beta s} - e^{\beta y})^4; x, y) &= 3\beta^4 e^{4\beta y}(-1 + y)^2 y^2, && (17)\\
\lim_{n\to\infty} n(\widetilde{K}_n^{\alpha,\beta}((e^{\alpha t} - e^{\alpha x})(e^{\beta s} - e^{\beta y}); x, y)) &= -\alpha\beta xy e^{\alpha x+\beta y}. && (18)
\end{aligned}$$

3. Rate of convergence

The modulus of continuity $\omega(f, \delta)$ for two-dimensional functions as follows [9]:

$$\omega(f, \delta) = \sup\{|f(x, y) - f(t, s)| : (t, s) \in S, \sqrt{(t - x)^2 + (s - y)^2} \leq \delta\}.$$

Theorem 3.1. *Let $f \in C(S_{\alpha,\beta})$. Then following inequality holds*

$$|\widetilde{K}_n^{\alpha,\beta}(f; x, y) - f(x, y)| \leq \left(1 + \frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)\omega\left(f, \delta\right),$$

where

$$\begin{aligned}
\delta^2 \;=\; & e^{\frac{(\alpha x+\beta y)(n+2)-\beta}{n+1}}\left(e^{\frac{\alpha x}{n+1}} - e^{\frac{-\beta}{n+1}}\left(e^{\frac{\beta y}{n+1}} - 1\right)\right)^n + e^{\frac{(\alpha x+\beta y)(2+n)-\alpha-\beta}{n+1}}\left(e^{-\frac{\alpha}{n+1}}(1 - e^{\frac{\alpha x}{n+1}}) + e^{\frac{-\beta}{n+1}}(1 - e^{\frac{\beta y}{n+1}}) + 1\right)^n (e^{2\alpha x} + e^{2\beta y})\\
& + e^{\frac{(2+n)(\alpha x+\beta y)-\alpha}{n+1}}\left(e^{\frac{-\alpha}{n+1}}(1 - e^{\frac{\alpha x}{n+1}}) + e^{\frac{\beta y}{n+1}}\right)^n - \frac{2}{n+1}\left[\frac{\alpha e^{\frac{\alpha x(3+2n)+\beta y(n+2)-\beta}{n+1}}}{e^{\frac{\alpha}{n+1}} - 1}\left(e^{\frac{-\beta}{n+1}}(1 - e^{\frac{\beta y}{n+1}} + e^{\frac{\beta}{n+1}})\right)^n\right.\\
& \left. + \frac{\beta e^{\frac{\alpha x(2+n)+\beta y(3+2n)-\alpha}{n+1}}}{e^{\frac{\beta}{n+1}} - 1}\left(e^{\frac{-\alpha}{n+1}}(1 - e^{\frac{\alpha x}{n+1}} + e^{\frac{\alpha}{n+1}})\right)^n\right].
\end{aligned}$$

Proof. From the definition of the modulus of continuity, we have

$$|\widetilde{K}_n^{\alpha,\beta}(f;x,y)-f(x,y)| \leq \left(1+\frac{\widetilde{K}_n^{\alpha,\beta}\left((t-x)^2+(s-y)^2;x,y\right)}{\delta^2}\right)\omega(f,\delta).$$

By using the Mean Value Theorem, we get

$$|\widetilde{K}_n^{\alpha,\beta}(f;x,y)-f(x,y)| \quad \leq \quad \left\{1+\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)\frac{\widetilde{K}_n^{\alpha,\beta}\left((e^{\alpha t}-e^{\alpha x})^2+(e^{\beta s}-e^{\beta y})^2;x,y\right)}{\delta^2}\right\}\omega(f,\delta).$$

Here, if we choose

$$\delta^2 \quad = \quad \widetilde{K}_n^{\alpha,\beta}\left((e^{\alpha t}-e^{\alpha x})^2+(e^{\beta s}-e^{\beta y})^2;x,y\right),$$

we can write

$$|\widetilde{K}_n^{\alpha,\beta}(f;x,y)-f(x,y)| \leq \left(1+\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)\omega\left(f,\delta\right),$$

where

$$\begin{aligned}
\delta^2 \quad = \quad & e^{\frac{(\alpha x+\beta y)(n+2)-\beta}{n+1}}\left(e^{\frac{\alpha x}{n+1}}-e^{\frac{-\beta}{n+1}}\left(e^{\frac{\beta y}{n+1}}-1\right)\right)^n+e^{\frac{(\alpha x+\beta y)(2+n)-\alpha-\beta}{n+1}}\left(e^{-\frac{\alpha}{n+1}}(1-e^{\frac{\alpha x}{n+1}})+e^{\frac{-\beta}{n+1}}(1-e^{\frac{\beta y}{n+1}})+1\right)^n(e^{2\alpha x}+e^{2\beta y}) \\
& +e^{\frac{(2+n)(\alpha x+\beta y)-\alpha}{n+1}}\left(e^{\frac{-\alpha}{n+1}}(1-e^{\frac{\alpha x}{n+1}})+e^{\frac{\beta y}{n+1}}\right)^n-\frac{2}{n+1}\left[\frac{\alpha e^{\frac{\alpha x(3+2n)+\beta y(n+2)-\beta}{n+1}}}{e^{\frac{\alpha}{n+1}}-1}\left(e^{\frac{-\beta}{n+1}}(1-e^{\frac{\beta y}{n+1}}+e^{\frac{\beta}{n+1}})\right)^n\right. \\
& \left.+\frac{\beta e^{\frac{\alpha x(2+n)+\beta y(3+2n)-\alpha}{n+1}}}{e^{\frac{\beta}{n+1}}-1}\left(e^{\frac{-\alpha}{n+1}}(1-e^{\frac{\alpha x}{n+1}}+e^{\frac{\alpha}{n+1}})\right)^n\right].
\end{aligned}$$

□

4. Voronovskaya-Type Theorem

In this section, we mention a Voronovskaya-type asymptotic theorem for the $\widetilde{K}_n^{\alpha,\beta}(f;x,y)$. The inverse of the exponential function for first variable t is $\log_\alpha^\beta$ and the inverse of the exponential function for second variable s is $\log_\beta^\alpha$.

From Taylor's expansion, for $(x,y)\in\delta_{\alpha,\beta}$ we have

$$\begin{aligned}
f(t,s) \quad = \quad & f(x,y)+(e^{\alpha t}-e^{\alpha x})\left[\frac{\partial}{\partial x}f(\log_\alpha^\beta,.)\right]\Bigg|_{(e^{\alpha x},e^{\beta y})}+(e^{\beta s}-e^{\beta y})\left[\frac{\partial}{\partial y}f(.,\log_\beta^\alpha)\right]\Bigg|_{(e^{\alpha x},e^{\beta y})} \\
& +\frac{1}{2}\left\{(e^{\alpha t}-e^{\alpha x})^2\left[\frac{\partial^2}{\partial x^2}f(\log_\alpha^\beta,.)\right]\Bigg|_{(e^{\alpha x},e^{\beta y})}+2(e^{\alpha t}-e^{\alpha x})(e^{\beta s}-e^{\beta y})\left[\frac{\partial^2}{\partial y\partial x}f(\log_\alpha^\beta,\log_\beta^\alpha)\right]\Bigg|_{(e^{\alpha x},e^{\beta y})}\right. \\
& \left.+(e^{\beta s}-e^{\beta y})^2\left[\frac{\partial^2}{\partial y^2}f(.,\log_\beta^\alpha)\right]\Bigg|_{(e^{\alpha x},e^{\beta y})}\right\}+R(f,t,s;x,y)\{(e^{\alpha t}-e^{\alpha x})^2+(e^{\beta s}-e^{\beta y})^2\} \qquad (19)
\end{aligned}$$

where $R(f,t,s;x,y)\to 0$, as $(t,s)\to(x,y)$.

By applying the operator $\widetilde{K}_n^{\alpha,\beta}(.;x,y)$ to both sides of (19), we write

$$\begin{aligned}
\widetilde{K}_n^{\alpha,\beta}(f;x,y) &= f(x,y)\widetilde{K}_n^{\alpha,\beta}(1;x,y) + \widetilde{K}_n^{\alpha,\beta}((e^{\alpha t}-e^{\alpha x});x,y)\left[\frac{\partial}{\partial x}f(\log_\alpha^\beta,.)\right]\Big|_{(e^{\alpha x},e^{\beta y})} \\
&+\widetilde{K}_n^{\alpha,\beta}((e^{\beta s}-e^{\beta y});x,y)\left[\frac{\partial}{\partial y}f(.,\log_\beta^\alpha)\right]\Big|_{(e^{\alpha x},e^{\beta y})} \\
&+\frac{1}{2}\Bigg\{\widetilde{K}_n^{\alpha,\beta}((e^{\alpha t}-e^{\alpha x})^2;x,y)\left[\frac{\partial^2}{\partial x^2}f(\log_\alpha^\beta,.)\right]\Big|_{(e^{\alpha x},e^{\beta y})} \\
&+2\,\widetilde{K}_n^{\alpha,\beta}((e^{\alpha t}-e^{\alpha x})(e^{\beta s}-e^{\beta y});x,y)\left[\frac{\partial^2}{\partial y\partial x}f(\log_\alpha^\beta,\log_\beta^\alpha)\right]\Big|_{(e^{\alpha x},e^{\beta y})} \\
&+\widetilde{K}_n^{\alpha,\beta}((e^{\beta s}-e^{\beta y})^2;x,y)\left[\frac{\partial^2}{\partial y^2}f(.,\log_\beta^\alpha)\right]\Big|_{(e^{\alpha x},e^{\beta y})}\Bigg\} \\
&+\widetilde{K}_n^{\alpha,\beta}(R(f,t,s;x,y)((e^{\alpha t}-e^{\alpha x})^2+(e^{\beta s}-e^{\beta y})^2);x,y).
\end{aligned}$$

$$\begin{aligned}
\widetilde{K}_n^{\alpha,\beta}(f;x,y)-f(x,y) &= f(x,y)(\widetilde{K}_n^{\alpha,\beta}(1;x,y)-1) + \widetilde{K}_n^{\alpha,\beta}((e^{\alpha t}-e^{\alpha x});x,y)\left[\frac{\partial}{\partial x}f(\log_\alpha^\beta,.)\right]\Big|_{(e^{\alpha x},e^{\beta y})} \\
&+\widetilde{K}_n^{\alpha,\beta}((e^{\beta s}-e^{\beta y});x,y)\left[\frac{\partial}{\partial y}f(.,\log_\beta^\alpha)\right]\Big|_{(e^{\alpha x},e^{\beta y})} \\
&+\frac{1}{2}\Bigg\{\widetilde{K}_n^{\alpha,\beta}((e^{\alpha t}-e^{\alpha x})^2;x,y)\left[\frac{\partial^2}{\partial x^2}f(\log_\alpha^\beta,.)\right]\Big|_{(e^{\alpha x},e^{\beta y})} \\
&+2\,\widetilde{K}_n^{\alpha,\beta}((e^{\alpha t}-e^{\alpha x})(e^{\beta s}-e^{\beta y});x,y)\left[\frac{\partial^2}{\partial y\partial x}f(\log_\alpha^\beta,\log_\beta^\alpha)\right]\Big|_{(e^{\alpha x},e^{\beta y})} \\
&+\widetilde{K}_n^{\alpha,\beta}((e^{\beta s}-e^{\beta y})^2;x,y)\left[\frac{\partial^2}{\partial y^2}f(.,\log_\beta^\alpha)\right]\Big|_{(e^{\alpha x},e^{\beta y})}\Bigg\} \\
&+\widetilde{K}_n^{\alpha,\beta}(R(f,t,s;x,y)((e^{\alpha t}-e^{\alpha x})^2+(e^{\beta s}-e^{\beta y})^2);x,y). \qquad (20)
\end{aligned}$$

By using the following equalities in (20),

$$\begin{aligned}
\left[\frac{\partial f(\log_\alpha^\beta,.)}{\partial x}\right]\Big|_{(e^{\alpha x},e^{\beta y})} &= \frac{1}{\alpha}e^{-\alpha x}\frac{\partial f(x,y)}{\partial x}, \\
\left[\frac{\partial^2 f(\log_\alpha^\beta,.)}{\partial x^2}\right]\Big|_{(e^{\alpha x},e^{\beta y})} &= e^{-2\alpha x}\left(\frac{1}{\alpha^2}\frac{\partial^2 f(x,y)}{\partial x^2}-\frac{1}{\alpha}\frac{\partial f(x,y)}{\partial x}\right), \\
\left[\frac{\partial^2 f(\log_\alpha^\beta,\log_\beta^\alpha)}{\partial y\partial x}\right]\Big|_{(e^{\alpha x},e^{\beta y})} &= \frac{1}{\alpha}\frac{1}{\beta}e^{-(\alpha x+\beta y)}\frac{\partial f(x,y)}{\partial y\partial x}, \qquad (21)
\end{aligned}$$

we obtain

$$\begin{aligned}\widetilde{K}_n^{\alpha,\beta}(f;x,y)-f(x,y) &= f(x,y)(\widetilde{K}_n^{\alpha,\beta}(1;x,y)-1)+\frac{e^{-\alpha x}}{\alpha}\frac{\partial f(x,y)}{\partial x}\widetilde{K}_n^{\alpha,\beta}((e^{\alpha t}-e^{\alpha x});x,y)\\
&+\frac{e^{-\beta y}}{\beta}\frac{\partial f(x,y)}{\partial y}\widetilde{K}_n^{\alpha,\beta}((e^{\beta s}-e^{\beta y});x,y)\\
&+\frac{1}{2}\left\{e^{-2\alpha x}\left(\frac{1}{\alpha^2}\frac{\partial^2 f(x,y)}{\partial x^2}-\frac{1}{\alpha}\frac{\partial f(x,y)}{\partial x}\right)\widetilde{K}_n^{\alpha,\beta}((e^{\alpha t}-e^{\alpha x})^2;x,y)\right.\\
&+\frac{2}{\alpha\beta}e^{-(\alpha x+\beta y)}\frac{\partial f(x,y)}{\partial y\partial x}\widetilde{K}_n^{\alpha,\beta}((e^{\alpha t}-e^{\alpha x})(e^{\beta s}-e^{\beta y});x,y)\\
&\left.+e^{-2\beta y}\left(\frac{1}{\beta^2}\frac{\partial^2 f(x,y)}{\partial y^2}-\frac{1}{\beta}\frac{\partial f(x,y)}{\partial y}\right)\widetilde{K}_n^{\alpha,\beta}((e^{\beta s}-e^{\beta y})^2;x,y)\right\}\\
&+\widetilde{K}_n^{\alpha,\beta}(R(f,t,s;x,y)((e^{\alpha t}-e^{\alpha x})^2+(e^{\beta s}-e^{\beta y})^2);x,y)\Big\}. \qquad (22)\end{aligned}$$

By taking limit of (22), we achieve

$$\begin{aligned}\lim_{n\to\infty} n(\widetilde{K}_n^{\alpha,\beta}(f;x,y)-f(x,y)) &= f(x,y)\lim_{n\to\infty} n(\widetilde{K}_n^{\alpha,\beta}(1;x,y)-1)\\
&+\frac{e^{-\alpha x}}{\alpha}\frac{\partial f(x,y)}{\partial x}\lim_{n\to\infty} n(\widetilde{K}_n^{\alpha,\beta}((e^{\alpha t}-e^{\alpha x});x,y))\\
&+\frac{e^{-\beta y}}{\beta}\frac{\partial f(x,y)}{\partial y}\lim_{n\to\infty} n(\widetilde{K}_n^{\alpha,\beta}((e^{\beta s}-e^{\beta y});x,y))\\
&+\frac{1}{2}\left\{e^{-2\alpha x}\left(\frac{1}{\alpha^2}\frac{\partial^2 f(x,y)}{\partial x^2}-\frac{1}{\alpha}\frac{\partial f(x,y)}{\partial x}\right)\right.\\
&\times\lim_{n\to\infty} n(\widetilde{K}_n^{\alpha,\beta}((e^{\alpha t}-e^{\alpha x})^2;x,y))\\
&+\frac{2}{\alpha\beta}e^{-(\alpha x+\beta y)}\frac{\partial f(x,y)}{\partial y\partial x}\lim_{n\to\infty} n(\widetilde{K}_n^{\alpha,\beta}((e^{\alpha t}-e^{\alpha x})(e^{\beta s}-e^{\beta y});x,y))\\
&\left.+e^{-2\beta y}\left(\frac{1}{\beta^2}\frac{\partial^2 f(x,y)}{\partial y^2}-\frac{1}{\beta}\frac{\partial f(x,y)}{\partial y}\right)\lim_{n\to\infty} n(\widetilde{K}_n^{\alpha,\beta}((e^{\beta s}-e^{\beta y})^2;x,y))\right\}\\
&+\lim_{n\to\infty} n(\widetilde{K}_n^{\alpha,\beta}(R(f,t,s;x,y)((e^{\alpha t}-e^{\alpha x})^2+(e^{\beta s}-e^{\beta y})^2);x,y))). \qquad (23)\end{aligned}$$

By using equations (11)-(18), we obtain

$$\begin{aligned}\lim_{n\to\infty} n(\widetilde{K}_n^{\alpha,\beta}(f;x,y)-f(x,y)) &= f(x,y)\alpha^2(-1+x)x+\beta(-1+(2+\beta)y-\beta y^2)-\alpha(1+x(-2+\beta y))\\
&+\frac{\partial f(x,y)}{\partial x}\frac{1}{2}\left(1+2\alpha x^2-2x(1+\alpha-\beta y)\right)\\
&+\frac{\partial f(x,y)}{\partial y}\frac{1}{2}\left(1+2\beta y^2-2y(1+\beta-\alpha x)\right)\\
&+\frac{1}{2}\left\{\left(-\frac{\partial^2 f(x,y)}{\partial x^2}+\alpha\frac{\partial f(x,y)}{\partial x}\right)x(x-1)-\frac{\partial f(x,y)}{\partial y\partial x}2xy\right.\\
&\left.+\left(-\frac{\partial^2 f(x,y)}{\partial y^2}+\beta\frac{\partial f(x,y)}{\partial y}\right)y(y-1)\right\}\\
&+\lim_{n\to\infty} n(\widetilde{K}_n^{\alpha,\beta}(R(f,t,s;x,y)((e^{\alpha t}-e^{\alpha x})^2+(e^{\beta s}-e^{\beta y})^2);x,y))). \qquad (24)\end{aligned}$$

When we use the Cauchy-Schwarz inequality for

$$n(\widetilde{K}_n^{\alpha,\beta}(R(f,t,s;x,y)((e^{\alpha t}-e^{\alpha x})^2+(e^{\beta s}-e^{\beta y})^2);x,y))),$$

we obtain

$$\begin{aligned} n\widetilde{K}_n^{\alpha,\beta}(R(t,s;x,y)((e^{\alpha t}-e^{\alpha x})^2+(e^{\beta s}-e^{\beta y})^2);x,y) &= n\widetilde{K}_n^{\alpha,\beta}(R(t,s;x,y)((e^{\alpha t}-e^{\alpha x})^2;x,y)\\ &\quad +n\widetilde{K}_n^{\alpha,\beta}(R(t,s;x,y)(e^{\beta s}-e^{\beta y})^2);x,y)\\ &\leq \sqrt{\widetilde{K}_n^{\alpha,\beta}(R^2(t,s;x,y);x,y)}\Bigg(\sqrt{n^2\widetilde{K}_n^{\alpha,\beta}((e^{\alpha t}-e^{\alpha x})^4;x,y)}\\ &\quad +\sqrt{n^2\widetilde{K}_n^{\alpha,\beta}((e^{\beta s}-e^{\beta y})^4;x,y)}\Bigg). \end{aligned}$$

Since $R(t,s;x,y)\to 0$ as $(t,s)\to(x,y)$,

$$\lim_{n\to\infty}\widetilde{K}_n^{\alpha,\beta}(R(t,s;x,y);x,y)=0$$

is verified uniformly in $C(S_{\alpha,\beta})$. By using (16) and (17), we achieve the desired result

$$\begin{aligned} \lim_{n\to\infty} n(\widetilde{K}_n^{\alpha,\beta}(f;x,y)-f(x,y)) &= f(x,y)\alpha^2(-1+x)x+\beta(-1+(2+\beta)y-\beta y^2)-\alpha(1+x(-2+\beta y))\\ &\quad +\frac{\partial f(x,y)}{\partial x}\frac{1}{2}\left(1+2\alpha x^2-2x(1+\alpha-\beta y)\right)\\ &\quad +\frac{\partial f(x,y)}{\partial y}\frac{1}{2}\left(1+2\beta y^2-2y(1+\beta-\alpha x)\right)\\ &\quad +\frac{1}{2}\left\{\left(-\frac{\partial^2 f(x,y)}{\partial x^2}+\alpha\frac{\partial f(x,y)}{\partial x}\right)x(x-1)-\frac{\partial f(x,y)}{\partial y\partial x}2xy\right.\\ &\quad \left.+\left(-\frac{\partial^2 f(x,y)}{\partial y^2}+\beta\frac{\partial f(x,y)}{\partial y}\right)y(y-1)\right\}. \end{aligned}$$

5. Graphical and Numerical Analysis

In this section, we give graphical and numerical analysis of $\widetilde{K}_n^{\alpha,\beta}(f;x,y)$ operators which illustrate us the modelling of approximation for the function f.

Example 5.1. *Let $f(x,y)=\frac{sin(x)cos(y)}{x+y}$ for $x,y\in[0.1,0.9]$. We show the graphs of $\widetilde{K}_n^{\alpha,\beta}(f;x,y)$ operators for fixed $\beta=2$, $n=20$ and the various values of $\alpha\in\{0.5,0.75,1\}$.*

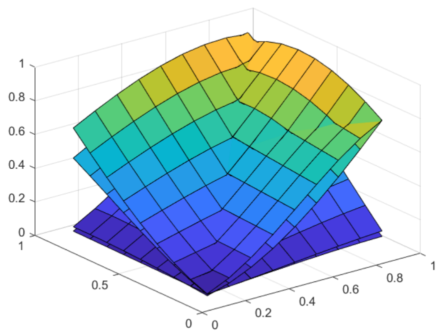

Figure 1: The graphs of $\widetilde{K}_n^{\alpha,\beta}(f;x,y)$ operators for $\beta=2$ and for the various values of α.

We calculate the maximum errors of $\|\widetilde{K}_n^{\alpha,\beta}(f;x,y)-f(x,y)\|$ for the function $f(x,y)=\frac{sin(x)cos(y)}{x+y}$ by choosing $x=y\in[0.1,0.9]$, step size $h=0.1$ in Table 1 and Table 2.

α	β	$\|\widetilde{K}_5^{\alpha,\beta}(f)-f\|$	$\|\widetilde{K}_8^{\alpha,\beta}(f)-f\|$	$\|\widetilde{K}_{10}^{\alpha,\beta}(f)-f\|$	$\|\widetilde{K}_{12}^{\alpha,\beta}(f)-f\|$	$\|\widetilde{K}_{15}^{\alpha,\beta}(f)-f\|$
0.5	2	0.7896	0.6460	0.6130	0.5964	0.5786
0.75	2	0.7747	0.6369	0.6050	0.5895	0.5738
1	2	0.7600	0.6291	0.5985	0.5838	0.5780

Table 1: Error table for fixed $\beta=2$ with different values of α.

β	α	$\|\widetilde{K}_5^{\alpha,\beta}(f)-f\|$	$\|\widetilde{K}_8^{\alpha,\beta}(f)-f\|$	$\|\widetilde{K}_{10}^{\alpha,\beta}(f)-f\|$	$\|\widetilde{K}_{12}^{\alpha,\beta}(f)-f\|$	$\|\widetilde{K}_{15}^{\alpha,\beta}(f)-f\|$
0.5	5	0.5443	0.3445	0.2868	0.2547	0.2285
1	5	0.5674	0.4295	0.4122	0.4086	0.4066
1.5	5	0.7364	0.5997	0.5778	0.5663	0.5571

Table 2: Error table for fixed $\alpha=5$ with different values of β.

Example 5.2. *Let* $f(x,y)=\frac{cos(x+y)}{x+y}$. *We give the graphs for* $\widetilde{K}_{20}^{0.01,0.2}(f;x,y)$, $\widetilde{K}_{20}^{0.5,0.3}(f;x,y)$ *and* $\widetilde{K}_{20}^{1,0.4}(f;x,y)$.

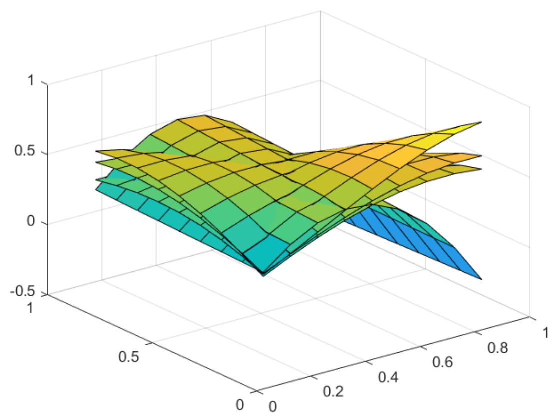

Figure 2: The graphs of $\widetilde{K}_n^{\alpha,\beta}(f;x,y)$ operators for $n=20$ and different values of α,β.

6. Conclusion

In this work, we construct the exponential bivariate Bernstein-Kantorovich operators. Then we calculate the rate of convergence with modulus of continuity for the functions define on $C(S_{\alpha,\beta})$. Also, we give the Voronovskaya-type asymptotic theorem. Finally, the error tables of the exponential bivariate Bernstein-Kantorovich operators are given for different value of n,α and β.

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