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# A new criteria for two Weyl's type theorems

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**Abstract.** Let $\mathcal{B}(\mathcal{H})$ be the collection of all bounded linear operators on $\mathcal{H}$, where $\mathcal{H}$ is an infinite dimensional complex Hilbert space. For $T \in \mathcal{B}(\mathcal{H})$, we say property $(UW_{\Pi})$( property $(\omega)$) holds for $T$ if $\sigma_a(T)\backslash\sigma_{ea}(T) = \Pi(T)(\pi_{00}(T))$, where $\sigma_a(T)$ and $\sigma_{ea}(T)$ denote the approximate point spectrum and the essential approximate point spectrum of $T$, respectively, also $\Pi(T)$ and $\pi_{00}(T)$ severally denote the set of all poles and all finite dimensional isolated eigenvalues. In this note, we introduce a new judgement method for bounded linear operators and their function calculus satisfying property $(UW_{\Pi})$ and property $(\omega)$ together by the deformed property. Meanwhile, we investigate the relationships among property $(UW_{\Pi})$, property $(\omega)$ and hypercyclic property.

## 1. Introduction

Spectral theory has a vital role in functional analysis theory, while Weyl's theorem as a significant branch of it has appealed a great deal of attention since Weyl[18] found it in 1909. After that, plentiful results appeared based on the Weyl's theorem, such as the collection of operators fullfilling the Weyl's theorem was expanded (from [7, 8]) and the theorem were varied in many aspects ([4, 5, 14, 16]) which are called Weyl's type theorems. Property $(\omega)$ and property $(UW_{\Pi})$ as two Weyl's type theorems, which proposed by Rakočević[14] and Berkani[5] respectively, have catched numerous academicians([2, 9, 10, 15, 19]) eyes recently. We will go on this subject. Let us begin with some symbols and terminologies.

In this note, let $\mathcal{B}(\mathcal{H})$ denote the set of all continuous linear operators defined on $\mathcal{H}$, where $\mathcal{H}$ is a complex and separable Hilbert space. For $T \in \mathcal{B}(\mathcal{H})$, $N(T)$ $(R(T))$ stands for the kernel (range) of $T$ and $T^*$ denotes the adjoint operator, the nullity $n(T)$ and deficiency $d(T)$ are defined by $n(T) = \dim N(T)$ and $d(T) = \dim N(T^*)$ respectively. $T \in \mathcal{B}(\mathcal{H})$ is said to be a semi-Fredholm operator when $R(T)$ is closed and $\min\{n(T), d(T)\} < \infty$, in this case, the index of $T$ is defined by $\operatorname{ind}(T) = n(T) - d(T)$. We say $T$ is an upper semi-Fredholm if the index exists and $n(T)$ is finite. Analogically, $T$ is called a lower semi-Fredholm when the index exists and $d(T)$ is finite. Particularly, if $T$ is an upper semi-Fredholm operator with $n(T) = 0$, then we say $T$ is a bounded below operator. $T \in \mathcal{B}(\mathcal{H})$ is a Fredholm operator if $\operatorname{ind}(T)$ is finite. Especially, we say $T$ is a Weyl operator if $\operatorname{ind}(T) = 0$. Moreover, if $T$ is upper semi-Fredholm with $\operatorname{ind}(T) \leq 0$, then $T$ is called an upper semi-Weyl operator. The symbol $\operatorname{asc}(T)$ stands for the ascent of $T$ which is a smallest integer $n$ making $N(T^n) = N(T^{n+1})$. Similarly, the descent $\operatorname{des}(T)$ is also a smallest integer $n$ making $R(T^n) = R(T^{n+1})$.

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If for any $n \in \mathbb{N}$, $N(T^n) \subsetneq N(T^{n+1})$ is always true, then we write $\text{asc}(T) = \infty$ (resp. $\text{des}(T) = \infty$), where $\mathbb{N}$ is a collection of all nonnegative integers. If $\text{asc}(T) = \text{des}(T) < \infty$, then we say $T$ is a Drazin invertible operator. What's more, $T$ is said to be a Browder operator if $T$ is a Drazin invertible operator with $\text{ind}(T) < \infty$.

Throughout the paper, $\mathbb{C}$ denotes the set of complex numbers. For $T \in \mathcal{B}(\mathcal{H})$, $\rho(T)$, $\rho_a(T)$, $\rho_w(T)$, $\rho_e(T)$, $\rho_{SF_+}(T)$, $\rho_{SF_-}(T)$, $\rho_{SF}(T)$, $\rho_{ea}(T)$, $\rho_D(T)$ and $\rho_b(T)$, respectively, denote the collection of complex numbers $\lambda$ which make $T - \lambda I$ be a invertible operator, bounded below operator, Weyl operator, Fredholm operator, upper semi-Fredholm operator, lower semi-Fredholm operator, semi-Fredholm operator, upper semi-Weyl operator, Drazin invertible operator, and Browder operator. Meanwhile $\sigma(T) = \mathbb{C} \backslash \rho(T)$ denotes the spectrum of $T$. Similarly, $\sigma_*(T) = \mathbb{C} \setminus \rho_*(T)$ denotes the corresponding spectrum, where $* \in \{a, w, e, SF_+, SF_-, SF, ea, D, b\}$.

For a set $M \subseteq \mathbb{C}$, we write $\partial M$, $\text{int}M$, $\text{acc}M$ and $\text{iso}M$ as the set of boundary points, interior points, accumulation points and isolated points of $M$, respectively. Simultaneously, $\overline{M}$ is denoted as the closure of $M$. Let $\sigma_0(T) = \sigma(T) \setminus \sigma_b(T)$. Set $\sigma_d(T) = \{\lambda \in \mathbb{C} : R(T - \lambda I) \text{ is not closed }\}$ and $\rho_d(T) = \mathbb{C} \setminus \sigma_d(T)$.

For $T \in \mathcal{B}(\mathcal{H})$, property $(UW_\Pi)$ holds for operator $T$ (write $T \in (UW_\Pi)$) if

$$\sigma_a(T) \backslash \sigma_{ea}(T) = \Pi(T),$$

and $T$ satisfies property $(\omega)$ (write $T \in (\omega)$) if

$$\sigma_a(T) \backslash \sigma_{ea}(T) = \pi_{00}(T),$$

where $\Pi(T) = \sigma(T) \backslash \sigma_D(T)$ and $\pi_{00}(T) = \{\lambda \in \text{iso}\sigma(T) : 0 < n(T - \lambda I) < \infty\}$.

In paper [6], the author gave the definition of consistency in Fredholm and index property. Next, we will provide a deformation property based on above property. And we can get the following Lemma 1.2 that lists the judgement of operators having the deformed property by the similar approaches from Theorem 3.2 in [6].

**Definition 1.1.** *Let $T \in \mathcal{B}(\mathcal{H})$. If for every $S \in \mathcal{B}(\mathcal{H})$, one of the situations occurs:*

(1) *Neither $TS$ nor $ST$ are Fredholm operators;*

(2) *$TS$ and $ST$ both are Fredholm operators and $ind(TS) = ind(ST) \leq ind(S)$,*

then we say $T$ has the property of consistency in Fredholm and nonpositive index, and $T$ is said to a $CFI_-$ operator.

**Lemma 1.2.** *Let $T \in \mathcal{B}(\mathcal{H})$, then $T$ is a $CFI_-$ operator if and only if one of the following conditions holds:*

(1) *$T$ is a Fredholm operator and $ind(T) \leq 0$;*

(2) *$R(T)$ is not closed;*

(3) *$R(T)$ is closed and $n(T) = d(T) = \infty$.*

Then, the new spectrum is defined by

$$\sigma_1(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a } CFI_- \text{ operator}\},$$

and $\rho_1(T) = \mathbb{C} \setminus \sigma_1(T)$. Hence by above Lemma 1.2, we can prove that $\sigma_1(T) \subseteq \sigma(T)$ is an open set and $\text{iso}\sigma(T) \subseteq \text{int}\rho_1(T)$ easily.

Let us see some examples in the beginning.

**Example 1.3.** Let $T \in \mathcal{B}(\ell^2)$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (0, x_2, 0, x_4, \cdots),$$

then we see that $\sigma_a(T) = \sigma_{ea}(T) = \{0, 1\}$, $\pi_{00}(T) = \emptyset$, $\Pi(T) = \{0, 1\}$, namely $T \in (\omega)$. But $T \notin (UW_\Pi)$.

Hence, property $(\omega)$ holds does not imply property $(UW_\Pi)$ holds.

**Example 1.4.** Let $T \in \mathcal{B}(\ell^2)$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (0, 0, \frac{x_2}{2}, \frac{x_3}{3} \cdots),$$

then we get that $\sigma_{ea}(T) = \sigma_a(T) = \{0\}$, $\pi_{00}(T) = \{0\}$, $\Pi(T) = \emptyset$. Thus $T$ satisfies property $(UW_\Pi)$. However $T$ does not satisfy property $(\omega)$.

Hence, property $(UW_\Pi)$ holds does not imply property $(\omega)$ holds as well.

**Example 1.5.** Let $A, B \in \mathcal{B}(\ell^2)$ and $T \in \mathcal{B}(\ell^2 \oplus \ell^2)$ be defined by

$$A(x_1, x_2, x_3, \cdots) = (0, x_2, x_3, x_4, \cdots),\ B(x_1, x_2, x_3, \cdots) = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3} \cdots).$$

Make $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, then we have $\sigma_a(T) = \sigma_{ea}(T) = \{0, 1\}$, $\pi_{00}(T) = \{0\}$, $\Pi(T) = \{1\}$. Thus $T \notin (\omega)$ and $T \notin (UW_\Pi)$.

Hence, we can find $T \in \mathcal{B}(\mathcal{H})$ which both property $(\omega)$ and property $(UW_\Pi)$ don't hold.

From above examples, we can conclude that there is no association between property $(\omega)$ and property $(UW_\Pi)$. Consequently, our purpose in section 2 is to describe the situation that bounded linear operators obeying property $(\omega)$ and property $(UW_\Pi)$ together. Moreover, in section 3 we also give the sufficient and necessary conditions in terms of the issue that property $(\omega)$ and property $(UW_\Pi)$ holds for functions of bounded linear operators. In the last section 4, we will explore the associations among hypercylic property and these two Weyl's type theorems.

## 2. A new judgement of property $(\omega)$ and property $(UW_\Pi)$

For convenience, in this paper $T \in [(UW_\Pi) \cap (\omega)]$ is a notation that $T$ satisfies both property $(\omega)$ and property $(UW_\Pi)$.

**Theorem 2.1.** *let $T \in \mathcal{B}(\mathcal{H})$, then the following statements are equivalent.*

(1) $T \in [(UW_\Pi) \cap (\omega)]$.

(2) $\sigma_b(T) = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T - \lambda I) < \infty\}] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup [acc\sigma(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathbb{C} : des(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$.

*Proof.* (1) $\Rightarrow$ (2). First, take arbitarily $\lambda_0$ which does not belong to the right side of the equation, without loss of generality, assume that $\lambda_0 \in \sigma(T)$, then $n(T - \lambda_0 I) > 0$.

**Claim**. $d(T - \lambda_0 I) < \infty$. If not, then $d(T - \lambda_0 I) = \infty$, we will consider two cases.

**Case 1** Assume that $n(T - \lambda_0 I) < \infty$. If $\lambda_0 \notin \mathrm{acc}\sigma(T)$, then $\lambda_0 \in \pi_{00}(T)$, according to property $(\omega)$, we know $T - \lambda_0 I$ is a Browder operator; if $\lambda_0 \notin \sigma_{SF_+}(T)$, then $\lambda_0 \in \sigma_a(T) \backslash \sigma_{ea}(T)$, $T - \lambda_0 I$ also is a Browder operator by condition (1).

**Case 2** Assume that $\mathrm{des}(T - \lambda_0 I) < \infty$, then $n(T - \lambda_0 I) \geq d(T - \lambda_0 I)$( [1, Theorem 1.22]). If $\lambda_0 \notin \mathrm{acc}\sigma(T)$, we get that $\lambda_0 \in \Pi(T)$, then $\lambda_0 \notin \sigma_b(T)$ from $T \in (UW_\Pi)$; if $\lambda_0 \notin \sigma_{SF_+}(T)$, then $T - \lambda_0 I$ is a Fredholm operator.

Whenever case 1 or case 2 happens, it always creates a contradiction with $d(T - \lambda_0 I) = \infty$. Hence, it follows that $\lambda_0 \notin \sigma_1(T)$. By the definition of $\rho_1(T)$ and $d(T - \lambda_0 I) < \infty$, we get $\lambda_0 \notin \sigma_b(T)$ easily. And the opposite inclusion is obivious.

(2) $\Rightarrow$ (1). It is clear that $[\sigma_a(T) \backslash \sigma_{ea}(T) \cup \pi_{00}(T) \cup \Pi(T)] \cap \{[\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T - \lambda I) < \infty\}] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup [\mathrm{acc}\sigma(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathbb{C} : \mathrm{des}(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]\} = \emptyset$, so $\sigma_a(T) \backslash \sigma_{ea}(T) \cup \pi_{00}(T) \cup \Pi(T) \subseteq \sigma_0(T)$, which means that $T \in [(UW_\Pi) \cap (\omega)]$. $\square$

**Remark 2.2.** (*i*) Assumes $T \in [(UW_\Pi) \cap (\omega)]$, then every part of the decomposition of $\sigma_b(T)$ in Theorem 2.1 can not be avoided. The following instances can account for it:

**Example 2.3.** Let $T \in \mathcal{B}(\ell^2)$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (x_2, x_3, x_4, \cdots),$$

then $T \in [(UW_\Pi) \cap (\omega)]$ and $\sigma_b(T) = \mathbb{D}$. But $\{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup [\mathrm{acc}\sigma(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathbb{C} : \mathrm{des}(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}] = \mathbb{T}$, where $\mathbb{D}$ denotes the unit disk and $\mathbb{T}$ is the unit circle. Hence $\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T - \lambda I) < \infty\}$ can not be avoided.

**Example 2.4.** Let $T \in \mathcal{B}(\ell^2)$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (0, x_1, 0, x_2, \cdots),$$

then $T \in [(UW_\Pi) \cap (\omega)]$ and $\sigma_b(T) = \mathbb{D}$. But $[\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T - \lambda I) < \infty\}] \cup [\mathrm{acc}\sigma(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathbb{C} : \mathrm{des}(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}] = \mathbb{T}$. Hence $\{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$ can not be avoided.

**Example 2.5.** Let $T \in \mathcal{B}(\ell^2)$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (0, \frac{x_2}{2}, \frac{x_3}{3}, \cdots),$$

then $T \in [(UW_\Pi) \cap (\omega)]$ and $\sigma_b(T) = \{0\}$. But $[\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T - \lambda I) < \infty\}] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup [\{\lambda \in \mathbb{C} : \mathrm{des}(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}] = \emptyset$. Hence $\mathrm{acc}\sigma(T) \cap \sigma_{SF_+}(T)$ can not be avoided.

**Example 2.6.** Let $A, B \in \mathcal{B}(\ell^2)$ and $T \in \mathcal{B}(\ell^2 \oplus \ell^2)$ be defined by

$$A(x_1, x_2, x_3, \cdots) = (0, x_1, \frac{x_2}{2}, \cdots),\ B(x_1, x_2, x_3, \cdots) = (0, x_1, 0, \frac{x_3}{3}, \cdots).$$

Make $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, then $T \in [(UW_\Pi) \cap (\omega)]$ and $\sigma_b(T) = \{0\}$. But $[\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T - \lambda I) < \infty\}] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup [\mathrm{acc}\sigma(T) \cap \sigma_{SF_+}(T)] = \emptyset$. Hence $\{\lambda \in \mathbb{C} : \mathrm{des}(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}$ can not be avoided.

(*ii*) From the proof of Theorem 2.1, it is obvious that $T \in [(UW_\Pi) \cap (\omega)] \iff \sigma_b(T) = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T - \lambda I) < \infty\}] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup [\mathrm{acc}\sigma(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathrm{iso}\sigma(T) : \mathrm{des}(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$.

By the Theorem 2.1 and consistency in Fredholm and nonpositive index property, it is easy to prove that $\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T - \lambda I) < \infty\} \subseteq \sigma_1(T) \cap \mathrm{int}\sigma_{ea}(T) \subseteq \sigma_1(T) \cap \mathrm{acc}\sigma_{ea}(T) \subseteq \sigma_1(T) \cap \sigma_{ea}(T)$, hence we have the following result.

**Corollary 2.7.** *Let $T \in \mathcal{B}(\mathcal{H})$, then the following statements are equivalent.*

(1) $T \in [(UW_\Pi) \cap (\omega)]$.

(2) $\sigma_b(T) = [\sigma_1(T) \cap int\sigma_{ea}(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup [acc\sigma(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathbb{C} : des(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$.

(3) $\sigma_b(T) = [\sigma_1(T) \cap acc\sigma_{ea}(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup [acc\sigma(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathbb{C} : des(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$.

(4) $\sigma_b(T) = [\sigma_1(T) \cap \sigma_{ea}(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup [acc\sigma(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathbb{C} : des(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$.

**Corollary 2.8.** *Let $T \in \mathcal{B}(\mathcal{H})$, then the following statements are equivalent.*

(1) $T \in [(UW_\Pi) \cap (\omega)]$.

(2) $\sigma_b(T) = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T - \lambda I) < \infty\}] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup [acc\sigma(T) \cap \sigma_{SF}(T)] \cup [\{\lambda \in \mathbb{C} : des(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$.

(3) $\sigma_b(T) = [\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup [acc\sigma(T) \cap \sigma_{SF}(T)] \cup [\rho_{SF_+}(T) \cap acc\sigma_{ea}(T)] \cup [\{\lambda \in \mathbb{C} : des(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$.

(4) $\sigma_b(T) = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup [acc\sigma(T) \cap \sigma_{SF}(T)] \cup [\rho_{SF_+}(T) \cap acc\sigma_{ea}(T)] \cup [\{\lambda \in \mathbb{C} : des(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$.

*Proof.* (1) $\Rightarrow$ (2). By the Theorem 2.1, we get that $\sigma_b(T) = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T-\lambda I) < \infty\}] \cup \{\lambda \in \sigma(T) : n(T-\lambda I) = 0\} \cup [\mathrm{acc}\sigma(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathbb{C} : \mathrm{des}(T-\lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\}]$. To begin with, we have $\mathrm{acc}\sigma(T) \cap \sigma_{SF_+}(T) = [\mathrm{acc}\sigma(T) \cap \sigma_d(T)] \cup [\mathrm{acc}\sigma(T) \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\}]$. And it is easy to get that $\mathrm{acc}\sigma(T) \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \subseteq [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T-\lambda I) < \infty\}] \cup [\mathrm{acc}\sigma(T) \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = d(T-\lambda I) = \infty\}]$, thus " $\subseteq$ " is proved combining with $\sigma_{SF}(T) = \{\lambda \in \mathbb{C} : n(T-\lambda I) = d(T-\lambda I) = \infty\} \cup \sigma_d(T)$. The clusion "$\supseteq$ " is evident. Hence, (2) is right.

By the definition of $\sigma_1(T)$, we can rapidly get that $\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T-\lambda I) < \infty\} \subseteq [\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup [\rho_{SF_+}(T) \cap \mathrm{acc}\sigma_{ea}(T)]$ and $[\sigma_1(T) \cap \sigma_{SF_+}(T)] = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\}]$, therefore (2) $\Rightarrow$ (3) $\Rightarrow$ (4) is valid.

While for (4) $\Rightarrow$ (1), the method is same as Theorem 2.1. $\square$

Since $\sigma_1(T)$ is an open set, then $\sigma_1(T) \neq \overline{\sigma_1(T)}$. Because of $[\sigma_a(T) \backslash \sigma_{ea}(T) \cup \pi_{00}(T) \cup \Pi(T)] \cap [\overline{\sigma_1(T)} \cap \{\lambda \in \mathbb{C} : d(T-\lambda I) < \infty\}] = \emptyset$, then we gain the following corollary from the Theorem 2.1, Corollary 2.7 and Corollary 2.8.

**Corollary 2.9.** *Let $T \in \mathcal{B}(\mathcal{H})$. Then the following statements are equivalent.*

(1) $T \in [(UW_\Pi) \cap (\omega)]$.

(2) $\sigma_b(T) = [\overline{\sigma_1(T)} \cap \{\lambda \in \mathbb{C} : d(T-\lambda I) < \infty\}] \cup \{\lambda \in \sigma(T) : n(T-\lambda I) = 0\} \cup [acc\sigma(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathbb{C} : des(T-\lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\}]$.

(3) $\sigma_b(T) = [\overline{\sigma_1(T)} \cap \sigma_{ea}(T)] \cup \{\lambda \in \sigma(T) : n(T-\lambda I) = 0\} \cup [acc\sigma(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathbb{C} : des(T-\lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\}]$.

(4) $\sigma_b(T) = [\overline{\sigma_1(T)} \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\}] \cup \{\lambda \in \sigma(T) : n(T-\lambda I) = 0\} \cup [acc\sigma(T) \cap \sigma_{SF}(T)] \cup [\rho_{SF_+}(T) \cap acc\sigma_{ea}(T)] \cup [\{\lambda \in \mathbb{C} : des(T-\lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\}]$.

## 3. Property ($\omega$) and property ($UW_\Pi$) for operator functions

Let us see some examples to explain there is no direct connection between operators and their functions satisfying both property ($UW_\Pi$) and property ($\omega$) together.

**Example 3.1.** Let $A, B, C, D \in \mathcal{B}(\ell^2)$ and $T \in \mathcal{B}(\ell^2 \oplus \ell^2 \oplus \ell^2 \oplus \ell^2)$ be defined by

$$A(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots),\ B(x_1, x_2, \cdots) = (x_1, 0, \cdots),$$

$$C(x_1, x_2, \cdots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots),\ D(x_1, x_2, \cdots) = (0, x_1, \frac{x_2}{2}, \cdots).$$

Make $T = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B+I & 0 & 0 \\ 0 & 0 & C+3I & 0 \\ 0 & 0 & 0 & D-4I \end{pmatrix}$, then $\sigma_a(T) = \mathbb{T} \cup \{-4, 2, 3, 4, \frac{7}{2}, \frac{10}{3}, \cdots\}$, $\sigma_{ea}(T) = \mathbb{T} \cup \{-4, 3\}$, $\Pi(T) = \pi_{00}(T) = \{2, 4, \frac{7}{2}, \frac{10}{3}, \cdots\}$, which means that $T \in [(UW_\Pi) \cap (\omega)]$.

Let $p(T) = (T-2I)T$, $q(T) = (T-4I)(T+4I)$. By calculations, for polynomial $p(T)$ we can gain that $0 \in \sigma_a(p(T)) \backslash \sigma_{ea}(p(T))$, but $0 \notin \pi_{00}(p(T)) \cup \Pi(p(T))$. And for polynomial $q(T)$ we get that $0 \in \pi_{00}(q(T))$, but $0 \notin \sigma_a(q(T)) \backslash \sigma_{ea}(q(T))$.

Therefore, $T$ both satisfies property ($\omega$) and property ($UW_\Pi$) does not entail $p(T) \in [(UW_\Pi) \cap (\omega)]$ for arbitrary polynomial $p$.

**Example 3.2.** Let $A, B, C \in \mathcal{B}(\ell^2)$ and $T \in \mathcal{B}(\ell^2 \oplus \ell^2 \oplus \ell^2)$ be defined by

$$A(x_1, x_2, \cdots) = (x_1, 0, x_3, \cdots),\ B(x_1, x_2, \cdots) = (0, x_1, \frac{x_2}{2}, \cdots),\ C(x_1, x_2, \cdots) = (0, x_1, 0, x_2, \cdots).$$

Make $T = \begin{pmatrix} \frac{5}{4}A - \frac{1}{4} & 0 & 0 \\ 0 & B-I & 0 \\ 0 & 0 & C+I \end{pmatrix}$, then we have $\sigma_a(T) = \sigma_{ea}(T) = \{\lambda \in \mathbb{C} : |\lambda - 1| = 1\} \cup \{-1, -\frac{1}{4}\}$, $\pi_{00}(T) = \{-1\}$, $\Pi(T) = \{-\frac{1}{4}\}$, that is $T \notin (\omega)$ and $T \notin (UW_\Pi)$.

Let $p(T) = T^2$, then $\sigma_a(T^2) = \sigma_{ea}(T^2) = \{re^{i\theta} : r = 2(1 + \cos\theta)\} \cup \{1, \frac{1}{16}\}$, $\pi_{00}(T^2) = \Pi(T^2) = \emptyset$, so $T^2 \in [(UW_\Pi) \cap (\omega)]$.

Therefore, $p(T) \in [(UW_\Pi) \cap (\omega)]$ for some polynomial $p$ does not entail $T \in [(UW_\Pi) \cap (\omega)]$.

Hence we will characterize the case that property $(UW_\Pi)$ and property $(\omega)$ hold for function calculus through the new spectrum in the following. We have a fact that if for arbitrary polynomial $p$, $p(T)$ obeys property $(UW_\Pi)$ or property $(\omega)$, then for any $\lambda,\ \mu \in \rho_{SF_+}(T)$, $\text{ind}(T - \lambda I)\text{ind}(T - \mu I) \geq 0$.

**Theorem 3.3.** *Let $T \in \mathcal{B}(\mathcal{H})$, then for each polynomial $p$, $p(T) \in [(UW_\Pi) \cap (\omega)]$ if and only if the following assertions hold:*

*(1) $T \in [(UW_\Pi) \cap (\omega)]$;*

*(2) For any $\lambda,\ \mu \in \rho_{SF_+}(T)$, $ind(T - \lambda I)ind(T - \mu I) \geq 0$;*

*(3) If $\sigma_0(T) \neq \emptyset$, then $\sigma_b(T) = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T - \lambda I) < \infty\}] \cup [acc\sigma(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathbb{C} : des(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$.*

*Proof.* Necessity. We only need to prove (3). Let $B^o(\lambda_0;\epsilon) = \{\lambda \in \mathbb{C} : 0 < |\lambda - \lambda_0| < \epsilon\}$. Since $T \in [(UW_\Pi) \cap (\omega)]$, then by Theorem 2.1 we have $\sigma_b(T) = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T - \lambda I) < \infty\}] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup [\text{acc}\sigma(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathbb{C} : \text{des}(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$. Among that we know $\{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} = \{\lambda \in \text{iso}\sigma(T) : n(T - \lambda I) = 0\} \cup \{\lambda \in \text{acc}\sigma(T) : n(T - \lambda I) = 0\}$.

**Claim 1**. $\{\lambda \in \text{iso}\sigma(T) : n(T - \lambda I) = 0\} = \emptyset$.

If not, take $\mu \in \{\lambda \in \text{iso}\sigma(T) : n(T - \lambda I) = 0\}$ and $\lambda \in \sigma_0(T)$, then put $p(T) = (T - \lambda I)(T - \mu I)$. Since $\lambda,\ \mu \in \text{iso}\sigma(T)$, then there exist $B^o(\lambda;\delta_1)$ and $B^o(\mu;\delta_2)$ such that $\forall t \in B^o(\lambda;\delta_1) \cup B^o(\mu;\delta_2)$, we have $T - tI$ is invertible. Let $\delta = \min\{\delta_1,\ \delta_2\}$ and for all $\lambda_0 \in B^o(0;\delta^2)$, set $p(T) - \lambda_0 I = (T - \lambda' I)(T - \mu' I)$ , then we claim $\{\lambda', \mu'\} \subseteq B^o(\lambda;\delta_1) \cup B^o(\mu;\delta_2)$, if not we obtain $|\lambda_0| = |p(\lambda')| = |p(\mu')| > \delta^2$, namely $\lambda_0 \notin B^o(0;\delta^2)$. So it implies $0 \in \text{iso}\sigma(p(T))$. Combining with $0 < n(T - \lambda I) < \infty$, it follows that $0 \in \pi_{00}(p(T))$. From the condition, we have $\mu \notin \sigma_b(T)$, a contradiction.

**Claim 2**. $\sigma_D(T) = \sigma_{ea}(T)$.

In fact, because of $T \in (UW_\Pi)$, then " $\supseteq$ " is evident. For the converse, take $\lambda_0 \notin \sigma_{ea}(T)$ arbitrarily, then $\lambda_0 \in \rho_a(T) \cup [\sigma_a(T)\backslash\sigma_{ea}(T)]$. Then it must have $\lambda_0 \in \sigma_a(T)\backslash\sigma_{ea}(T)$. If not, then $\lambda_0 \in \rho_a(T)$. Take $\mu_0 \in \sigma_0(T)$ and let $p(T) = (T - \lambda_0 I)(T - \mu_0 I)$, it follows that $0 \notin \sigma_b(p(T))$. Hence $\lambda_0 \notin \sigma_b(T)$, contradicted with $\lambda_0 \in \rho_a(T)$. Thus $\lambda_0 \notin \sigma_D(T)$ easily.

By claim 2, we have $\{\lambda \in \text{acc}\sigma(T) : n(T - \lambda I) = 0\} \cap \rho_d(T) = \emptyset$. Then it follows that $\{\lambda \in \text{acc}\sigma(T) : n(T - \lambda I) = 0\} \subseteq \text{acc}\sigma(T) \cap \sigma_{SF_+}(T)$.

Therefore, the condition (3) holds due to claim 1 and claim 2.

Sufficiency. There exists two cases.

**Case 1** Assume that $\sigma_0(T) = \emptyset$. Combining with $T \in [(UW_\Pi) \cap (\omega)]$, it follows that $\sigma_{ea}(T) = \sigma_a(T)$ and $\pi_{00}(T) = \Pi(T) = \emptyset$. Since for any $\lambda,\ \mu \in \rho_{SF_+}(T)$, $\text{ind}(T - \lambda I)\text{ind}(T - \mu I) \geq 0$, then $\sigma_{ea}(T)$ satisfies spectrum mapping theorem( [17, Theorem 2]). Hence, for any polynomial $p$, we have $\sigma_{ea}(p(T)) = p(\sigma_{ea}(T)) = p(\sigma_a(T)) = \sigma_a(p(T))$. Because $\Pi(p(T)) \subseteq p(\Pi(T))$ and $\pi_{00}(p(T)) \subseteq p(\pi_{00}(T))$, then $\pi_{00}(p(T)) = \Pi(p(T)) = \emptyset$. Thus $p(T) \in [(UW_\Pi) \cap (\omega)]$.

**Case 2** Assume that $\sigma_0(T) \neq \emptyset$. From the condition (3), it entails that $\sigma_{ea}(T) = \sigma_b(T)$ and $\{\lambda \in \text{iso}\sigma(T) : n(T - \lambda I) < \infty\} = \sigma_0(T)$. Take $\lambda \in \sigma_a(p(T))\backslash\sigma_{ea}(p(T))$ and suppose that

$$p(T) - \lambda I = a(T - \lambda_1 I)^{n_1}(T - \lambda_2 I)^{n_2} \cdots (T - \lambda_t I)^{n_t},$$

$\lambda_i \neq \lambda_j$ if $i \neq j$, where $i,\ j = 1, 2, \cdots, t$. Then $\lambda_i \notin \sigma_{ea}(T)$ where $i = 1, 2, \cdots, t$, it follows that $\lambda_i \notin \sigma_b(T)$. Thus $\sigma_a(p(T))\backslash\sigma_{ea}(p(T)) \subseteq \sigma_0(p(T))$. Take $\lambda \in \pi_{00}(p(T))$ and $p(T)$ has above decomposition. Without loss of generality, let $\lambda_i \in \sigma(T)$ where $i = 1, 2, \cdots, t$, then we can gain that $\lambda_i \in \text{iso}\sigma(T)$. From $n(p(T) - \lambda I) < \infty$, then $n(T - \lambda_i I) < \infty$. Hence $\lambda_i \in \{\lambda \in \text{iso}\sigma(T) : n(T - \lambda I) < \infty\}$, it follows that $\lambda_i \in \sigma_0(T)$. Thus $\pi_{00}(p(T)) \subseteq \sigma_0(p(T))$. Because $\sigma_D(T)$ satisfies spectrum mapping theorem and $\sigma_{ea}(T) = \sigma_b(T) = \sigma_D(T)$, we can show that $\Pi(p(T)) \subseteq \sigma_0(p(T))$.

Therefore, property $(UW_\Pi)$ and property $(\omega)$ hold for $p(T)$ for each polynomial $p$. $\square$

From the proof in Theorem 3.3, we can see the fact:

**Corollary 3.4.** *Let $T \in \mathcal{B}(\mathcal{H})$,*

(1) *If $\sigma_0(T) = \emptyset$, then $p(T) \in [(UW_\Pi) \cap (\omega)]$ for any polynomial $p$ if and only if $T \in [(UW_\Pi) \cap (\omega)]$ and for any $\lambda,\ \mu \in \rho_{SF_+}(T)$, $ind(T - \lambda I)ind(T - \mu I) \geq 0$;*

(2) *If $\sigma_0(T) \neq \emptyset$, then for any polynomial $p$, $p(T) \in [(UW_\Pi) \cap (\omega)]$ if and only if $\sigma_b(T) = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T - \lambda I) < \infty\}] \cup [acc\sigma(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathbb{C} : des(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$.*

In the sequel, we continue to explore the sufficient and necessary conditions of operator functions obeying both property $(UW_\Pi)$ and property $(\omega)$.

**Lemma 3.5.** *Let $T \in \mathcal{B}(\mathcal{H})$. Then $\sigma_0(T) = \emptyset$ and for any polynomial $p$, $p(T) \in [(UW_\Pi) \cap (\omega)]$ if and only if one of the following assertions holds:*

(1) $\sigma(T) = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T - \lambda I) < \infty\}] \cup [acc\sigma(T) \cap \sigma_{SF_+}(T)] \cup \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\} \cup [\{\lambda \in \mathbb{C} : des(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$.

(2) $\sigma(T) = [\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup [acc\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup [\{\lambda \in \mathbb{C} : des(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = d(T - \lambda I) = \infty\}] \cup [acc\{\lambda \in \mathbb{C} : n(T - \lambda I) > d(T - \lambda I)\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$.

*Proof.* If (1)(or (2)) holds, we can prove that $\sigma_0(T) = \emptyset$ and for any $\lambda \in \rho_{SF_+}(T)$ $ind(T - \lambda I) \geq 0$ (or $ind(T - \lambda I) \leq 0$). Using Theorem 2.1 and Corollary 3.1, it is easy to prove that the sufficiency of this corollary. Next we will show the necessity.

Since $T \in [(UW_\Pi) \cap (\omega)]$ and $\sigma_0(T) = \emptyset$, it follows that $\sigma(T) = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T - \lambda I) < \infty\}] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup [\mathrm{acc}\sigma(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathbb{C} : \mathrm{des}(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$ by Theorem 2.1. Accordingly for any $\lambda,\ \mu \in \rho_{SF_+}(T)$, $\mathrm{ind}(T - \lambda I)\mathrm{ind}(T - \mu I) \geq 0$, then there are two situations.

**Case 1** Suppose that $\lambda \in \rho_{SF_+}(T)$, $\mathrm{ind}(T - \lambda I) \geq 0$, then $\sigma_a(T) = \sigma(T)$, it entails that (1) is correct.

**Case 2** Suppose that $\lambda \in \rho_{SF_+}(T)$, $\mathrm{ind}(T - \lambda I) \leq 0$, then it follows that $[\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T - \lambda I) < \infty\}] = [\sigma_1(T) \cap \sigma_{SF_+}(T)]$ and $[\{\lambda \in \mathbb{C} : \mathrm{des}(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}] \cup [\mathrm{acc}\sigma(T) \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}] \subseteq [\{\lambda \in \mathbb{C} : \mathrm{des}(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = d(T - \lambda I) = \infty\}] \cup [\mathrm{acc}\{\lambda \in \mathbb{C} : n(T - \lambda I) > d(T - \lambda I)\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$. Hence $\sigma(T) \subseteq [\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup [\mathrm{acc}\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup [\{\lambda \in \mathbb{C} : \mathrm{des}(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = d(T - \lambda I) = \infty\}] \cup [\mathrm{acc}\{\lambda \in \mathbb{C} : n(T - \lambda I) > d(T - \lambda I)\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$. The inclusion " $\supseteq$ " is obvious. Thus the condition (2) is proved. □

According to the Corollary 3.4 and Lemma 3.5, we can get:

**Theorem 3.6.** *Let $T \in \mathcal{B}(\mathcal{H})$, then for any polynomial $p$, $p(T) \in [(UW_\Pi) \cap (\omega)]$ if and only if one of the following assertions holds:*

(1) $\sigma(T) = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T - \lambda I) < \infty\}] \cup [acc\sigma(T) \cap \sigma_{SF_+}(T)] \cup \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\} \cup [\{\lambda \in \mathbb{C} : des(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$.

(2) $\sigma(T) = [\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup [acc\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup [\{\lambda \in \mathbb{C} : des(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = d(T - \lambda I) = \infty\}] \cup [acc\{\lambda \in \mathbb{C} : n(T - \lambda I) > d(T - \lambda I)\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$.

(3) $\sigma_b(T) = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T - \lambda I) < \infty\}] \cup [acc\sigma(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathbb{C} : des(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$.

We claim that: If for any polynomial $p$, $p(T) \in (\omega)$ and $T \in (UW_\Pi)$, then $p(T) \in (UW_\Pi)$ for arbitrary polynomial $p$.

In fact, since for any polynomial $p$, $p(T) \in (\omega)$, then we can get $\sigma_a(p(T)) \backslash \sigma_{ea}(p(T)) \subseteq \Pi(p(T))$. For the opposition, $\forall \lambda \in \Pi(p(T))$ and $p(T)$ has the same decomposition like Theorem 3.3, it follows that $\lambda_i \notin \sigma_D(T)$. Also because of $T \in (UW_\Pi)$, then we have $\lambda_i \notin \sigma_b(T)$. It implies that $\Pi(p(T)) \subseteq \sigma_0(p(T))$. Hence $p(T) \in (UW_\Pi)$.

According to Theorem 3.6, we can gain a conclusion as follows.

**Corollary 3.7.** *Let $T \in \mathcal{B}(\mathcal{H})$, then for any polynomial $p$, $p(T) \in [(UW_\Pi) \cap (\omega)]$ if and only if $T \in (UW_\Pi)$ and one of the following assertions holds:*

(1) $\sigma(T) = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T - \lambda I) < \infty\}] \cup [acc\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\} \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}$.

(2) $\sigma(T) = [\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup [acc\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma(T) : n(T-\lambda I) = 0\} \cup \{\lambda \in \mathbb{C} : n(T-\lambda I) = d(T-\lambda I) = \infty\}]$.
(3) $\sigma_b(T) = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T-\lambda I) < \infty\}] \cup [acc\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\}$.

**Corollary 3.8.** *Let $T \in \mathcal{B}(\mathcal{H})$, then for any polynomial $p$, $p(T) \in [(UW_\Pi) \cap (\omega)]$ if and only if $T \in (\omega)$ and one of the following assertions holds:*
(1) $\sigma(T) = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T-\lambda I) < \infty\}] \cup [acc\sigma(T) \cap \sigma_{SF_+}(T)] \cup \{\lambda \in \sigma_a(T) : n(T-\lambda I) = 0\} \cup [\{\lambda \in \mathbb{C} : des(T-\lambda I) = \infty\} \cap \sigma_a(T)]$.
(2) $\sigma(T) = [\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup [acc\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma(T) : n(T-\lambda I) = 0\} \cup [\{\lambda \in \mathbb{C} : des(T-\lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : d(T-\lambda I) = \infty\}] \cup [acc\{\lambda \in \mathbb{C} : n(T-\lambda I) > d(T-\lambda I)\} \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\}]$.
(3) $\sigma_b(T) = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T-\lambda I) < \infty\}] \cup [acc\sigma(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathbb{C} : des(T-\lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) > 0\}]$.

*Proof.* First, it is evident to show the necessity due to Theorem 3.6. Next, we finish the proof of sufficiency.

If the condition (1) or (2) holds, we have that $\sigma_0(T) = \emptyset$ and for any $\lambda,\ \mu \in \rho_{SF_+}(T)$, $\text{ind}(T-\lambda I)\text{ind}(T-\mu I) \geq 0$. Also, we obtain that $\Pi(T) = \emptyset$. It follows $T \in (UW_\Pi)$ from $\sigma_0(T) = \emptyset$ and $T \in (\omega)$. Thus, we show that $p(T) \in [(UW_\Pi) \cap (\omega)]$ for every polynomial $p$. If the condition (3) holds, it implies that $\Pi(T) \subseteq \sigma_0(T)$ and $\text{iso}\sigma(T) \subseteq \{\lambda \in \mathbb{C} : n(T-\lambda I) > 0\}$. Moreover, combining with $T \in (\omega)$ we can get $\sigma_{ea}(T) = \sigma_b(T)$ and $\sigma_0(T) = \{\lambda \in \text{iso}\sigma(T) : n(T-\lambda I) > 0\}$. Then we can prove for any polynomial $p$, $p(T) \in [(UW_\Pi) \cap (\omega)]$ from the proof in Theorem 3.3. □

## 4. Hypercyclic property and Weyl's type theorems

For $h \in \mathcal{H}$, the orbit of $h$ under the bounded linear operator $T$ is defined as $\text{Orb}(T, h) = \{T^n h : n \in \mathbb{N}\}$. A element $h$ is said to be hypercyclic regarding to $T$ if $\text{Orb}(T, h)$ is dense in $\mathcal{H}$. We denote $HC(\mathcal{H})$ the collection of all bounded linear operators which have hypercyclic vectors. In 1974, Hilden and Wallen put forward hypercyclic property in [13]. Later, many authors have greatly interests on it and have obtained a lot of outcomes in ([3, 11, 12]). Then we will use the new tool to research it sequentially.

**Lemma 4.1.** [12] *Let $T \in \mathcal{B}(\mathcal{H})$, then $T \in \overline{HC(\mathcal{H})}$ if and only if the following assertions hold:*
(1) *$\sigma_w(T) \cup \mathbb{T}$ is a connected set;*
(2) *$\sigma_0(T) = \emptyset$;*
(3) *$\rho_{SF}^-(T) = \{\lambda \in \rho_{SF}(T) : ind(T-\lambda I) < 0\} = \emptyset$.*

In the following , we can take some examples to illustrate there is no relevance among property $(UW_\Pi)$, property $(\omega)$ and hypercyclic property.

**Example 4.2.** Let $A, B, C \in \mathcal{B}(\ell^2)$ and $T \in \mathcal{B}(\ell^2 \oplus \ell^2 \oplus \ell^2)$ be defined by

$$A(x_1, x_2, \cdots) = (x_1, 0, \cdots),\ B(x_1, x_2, \cdots) = (0, x_1, \frac{x_2}{2}, \cdots),\ C(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots).$$

Put $T = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C-2I \end{pmatrix}$, then we can get $T \in [(UW_\Pi) \cap (\omega)]$ but $T \notin \overline{HC(\mathcal{H})}$ due to Lemma 4.1. That is $T \in [(UW_\Pi) \cap (\omega)] \nRightarrow T \in \overline{HC(\mathcal{H})}$.

**Example 4.3.** Let $A, B, C, D \in \mathcal{B}(\ell^2)$ and $T \in \mathcal{B}(\ell^2 \oplus \ell^2 \oplus \ell^2 \oplus \ell^2)$ be defined by

$$A(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots),\ B(x_1, x_2, \cdots) = (x_2, x_3, \cdots),$$
$$C(x_1, x_2, \cdots) = (0, 0, \frac{x_2}{2}, \frac{x_3}{3}, \cdots),\ D(x_1, x_2, \cdots) = (0, x_2, x_3, \cdots).$$

Put $T = \begin{pmatrix} A+iI & 0 & 0 & 0 \\ 0 & B+iI & 0 & 0 \\ 0 & 0 & C-I & 0 \\ 0 & 0 & 0 & 2D-I \end{pmatrix}$, then we can get $T \in \overline{HC(\mathcal{H})}$. However $T \notin (\omega)$ and $T \notin (UW_\Pi)$. That is $T \in \overline{HC(\mathcal{H})} \nRightarrow T \in [(UW_\Pi) \cap (\omega)]$.

**Theorem 4.4.** *let $T \in \mathcal{B}(\mathcal{H})$, then $T \in \overline{HC(\mathcal{H})}$ and $T \in [(UW_{\Pi}) \cap (\omega)]$ if and only if $\sigma(T) = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T - \lambda I) < \infty\}] \cup \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\} \cup [acc\sigma(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathbb{C} : des(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$ and $\sigma(T) \cup \mathbb{T}$ is connected.*

*Proof.* For the necessity, suppose that $T \in \overline{HC(\mathcal{H})}$, then $\sigma_a(T) = \sigma_b(T) = \sigma(T)$. Thus $\sigma(T) = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T - \lambda I) < \infty\}] \cup \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\} \cup [\text{acc}\sigma(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathbb{C} : \text{des}(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$ according to the Theorem 2.1. Meanwhile, it follows that $\sigma_b(T) = \sigma_w(T)$ from $T \in [(UW_{\Pi}) \cap (\omega)]$. Hence, $\sigma(T) \cup \mathbb{T}$ is connected by Lemma 4.1.

For the sufficiency, we can obtain that $\rho_{SF}^-(T) = \sigma_0(T) = \emptyset$ and $\sigma_w(T) = \sigma(T)$ when the equation holds, thus $T$ satisfies all three properties due to Theorem 2.1 and Lemma 4.1. □

**Remark 4.5.** (*i*) There also exists some instances which can explain the four compositions of $\sigma(T)$ are independent when $T \in \overline{HC(\mathcal{H})}$ and $T \in [(UW_{\Pi}) \cap (\omega)]$.

(*ii*) By the definition, it is clear that if $T$ obeys all three properties, then $\sigma_1(T) = \{\lambda \in \mathbb{C} : n(T - \lambda I) > d(n - \lambda I)\}$ and $\rho_1(T) = \rho(T) \cup \{\lambda \in \rho_d(T) : n(T - \lambda I) = d(T - \lambda I) = \infty\} \cup \sigma_d(T)$.

(*iii*) Suppose that $T \in \overline{HC(\mathcal{H})}$ and $T \in [(UW_{\Pi}) \cap (\omega)]$, then we obtain $\sigma(T) = \sigma_1(T) \cup \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\} \cup [\text{acc}\sigma(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathbb{C} : \text{des}(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$. But there is an operator $T(x_1, x_1, \cdots) = (0, x_1, x_2, \cdots)$ which can declare hypercyclic property doesn't hold when $\sigma(T)$ has above construction and $\sigma(T) \cup \mathbb{T}$ is connected.

Since $[\sigma_a(T) \backslash \sigma_{ea}(T) \cup \pi_{00}(T) \cup \Pi(T) \cup \rho_{SF}^-(T)] \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) > d(T - \lambda I)\} = \emptyset$, then it follows a conclusion from (*i*) and (*ii*) in Remark 4.5.

**Corollary 4.6.** *let $T \in \mathcal{B}(\mathcal{H})$, then $T \in \overline{HC(\mathcal{H})}$ and $T \in [(UW_{\Pi}) \cap (\omega)]$ if and only if the following conditions hold:*

(1) *$\sigma(T) \cup \mathbb{T}$ is connected;*

(2) *$\sigma_1(T) = \{\lambda \in \mathbb{C} : n(T - \lambda I) > d(T - \lambda I)\}$;*

(3) *$\sigma(T) = \sigma_1(T) \cup \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\} \cup [acc\sigma(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathbb{C} : des(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$.*

Subsequently, we continue to explore the equivalent depictions of bounded linear operators meeting all three properties. By the means of Corollary 2.7 and Corollary 2.9, there is a corollary.

**Corollary 4.7.** *let $T \in \mathcal{B}(\mathcal{H})$, then the following statements are equivalent.*

(1) *$T \in \overline{HC(\mathcal{H})}$ and $T \in [(UW_{\Pi}) \cap (\omega)]$.*

(2) *$\sigma(T) = [\overline{\sigma_1(T)} \cap \{\lambda \in \mathbb{C} : d(T - \lambda I) < 0\}] \cup \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\} \cup [acc\sigma(T) \cap \sigma_{SF}(T)] \cup [\{\lambda \in \mathbb{C} : des(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$ and $\sigma(T) \cup \mathbb{T}$ is connected.*

(3) *$\sigma(T) = [\sigma_1(T) \cap \sigma_{ea}(T)] \cup \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\} \cup [acc\sigma(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathbb{C} : des(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$ and $\sigma(T) \cup \mathbb{T}$ is connected.*

In section 2, we give some equivalent characterizations that $T$ both obeys property $(\omega)$ and property $(UW_{\Pi})$. And we note that suppose $T \in \overline{HC(\mathcal{H})}$, then $T \in [(UW_{\Pi}) \cap (\omega)]$ if and only if $\sigma_{ea}(T) = \sigma_a(T)$ and $\pi_{00}(T) = \Pi(T) = \emptyset$. So we will make out above mentioned case through $\sigma_b(T)$ and $\sigma_1(T)$ when $T \in \overline{HC(\mathcal{H})}$ in the following. A fact is that if $T \in \overline{HC(\mathcal{H})}$, then $\sigma_1(T) = \sigma_1(T) \cap \{\lambda \in \mathbb{C} : d(T - \lambda I) < \infty\}$. Hence, a result will be seen through this fact and Theorem 4.4.

**Corollary 4.8.** *Let $T \in \mathcal{B}(\mathcal{H})$, suppose that $T \in \overline{HC(\mathcal{H})}$, then $T \in [(UW_{\Pi}) \cap (\omega)]$ if and only if $\sigma(T) = \sigma_1(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup [acc\sigma(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathbb{C} : des(T - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}]$.*

From the Theorem 3.6 and Theorem 4.4, we can get a corollary regarding to function calculus as follows.

**Corollary 4.9.** *Let $T \in \mathcal{B}(\mathcal{H})$, suppose that $T \in \overline{HC(\mathcal{H})}$, then the following assertions are equivalent.*

(1) *$T \in [(UW_{\Pi}) \cap (\omega)]$.*

(2) *For any polynomial $p$, $p(T)$ satisfies both property $(UW_{\Pi})$ and property $(\omega)$.*

## References


[1] P. Aiena, *Fredholm and Local Spectral Theory II, with Application to Weyl-type Theorems*, Springer Lecture Notes of Math 2235, 2018.
[2] P. Aiena, M. Kachad, *Property $(UW_{\Pi})$ under perturbations*, Ann. Func. Anal. **11**(2020), 29-46.
[3] M. Ansari, K. Hedayatian, B. Khani-Robati, *Strong hypercyclicity of Banach space operators*, J. Korean Math. Soc. **58**(2021): 91-107.
[4] E. Aponte, J. Sanabria, L. Vásquez, *Perturbation theory for property $(VE)$ and tensor product*, Mathematics, **9**(2021), 2775.
[5] M. Berkani, M. Kachad, *New Browder and Weyl type theorems*, Bull. Korean. Math. Soc. **52**(2015), 439-452.
[6] X. Cao, *Weyl spectrum of the products of operators*, J. Korean Math. Soc. **45**(2008), 771-780.
[7] M. Chō, M. Itoh, S. Ōshiro, *Weyl's theorem holds for p-hyponormal operators*, Glasgow Math. J. **39**(1997), 217-220.
[8] L. A. Coburn, *Weyl's theorem for nonnormal operators*, Michigan Math. J. **13**(1966), 285-288.
[9] L. Dai, X. H. Cao, Q. Guo, *Property $(\omega)$ and the Single-valued Extension Property*, Acta Math. Sin. Engl. Ser. **37**(2021), 1254-1266.
[10] A. Gupta, A. Kumar, *Property $(UW_{\Pi})$ and Perturbations*, Mediterr. J. Math. **16**(2019), 1-12.
[11] K. Hedayatian, M. Namegoshayfard, *Commutant hypercyclicity of Hilbert space operators*, Filomat, **37**(2023), 4857-4868.
[12] D. A. Herrero, *Limits of hypercyclic and supercyclic operators*, J. Funct. Anal. **99**(1991), 179-190.
[13] H. M. Hilden, L. J. Wallen, *Some cyclic and non-cyclic vectors of certain operators*, Indiana Univ. Math. J. **23**(1974), 557-565.
[14] V. Rakočević, *On a class of operators*, Mat. Vesnik. **37**(1985), 423-426.
[15] M. H. M. Rashid, *Upper triangular operator matrices, SVEP, and property $(\omega)$*, Acta Math. Vietnam. **44**(2019), 993-1004.
[16] M. H. M. Rashid, M. Cho, *The stability of property (gt) under perturbation and tensor product*, Oper. Matrices. **17**(2023), 259-277.
[17] C. Schmoeger, *The spectral mapping theorem for the essential approximate point spectrum*, Colloq. Math. **74**(1997), 167-176.
[18] H. Weyl, *Über beschränkte quadratische formen, deren differenz vollstetig ist*, Rend. Circ. Mat. Palermo, **27** (1909), 373-392.
[19] L. L. Yang, X. H. Cao, Property $(\omega)$ and its compact perturbations, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A. Mat. RACSAM. **115**(2021), 1-11.