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Rough ideal convergence in G2NS

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Abstract. In this paper, we demonstrate that the generalized 2-norm space (G2NS) cannot be metrized, even if it is not a regular space. Additionally, we introduce and explore some issues related to the sets of rough $\mathcal{I}$-limit points and $\mathcal{I}$-cluster points over G2NS, and we show how these sets can differ from the established fundamental results.

1. Introduction

Gähler [15] introduced the concept of distance between three points in space, calling it a 2-metric. He later developed the idea of 2-normed spaces [16] and explored the geometry of these spaces [17]. Afterward, Chaipunya et al. [9] introduced a new concept of a general distance between three arbitrary points, presenting $g-3ps$. They also studied various fixed point theorems in $g-3ps$ spaces and proved that the topology of a $g-3ps$ space is T_1-separable but not T_2-separable. Recently, Kundu et al. [20] proposed a general version of a normed linear space called a generalized 2-normed space (G2NS). They discussed the topology and demonstrated that the space G2NS is T_2-separable. Additionally, they defined a generalized 2-norm (G2NS) on $\mathbb{R}^2$ and proved that the topology of G2NS on $\mathbb{R}^2$ differs from its standard topology.

Definition 1.1. ([20]) Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A function $N : X \times X \to \mathbb{R}^+ \cup \{0\}$ is called a generalized 2-norm or G2N if the following conditions are met:

(GN1) $N(x, y) = 0$ iff $x = y = \theta$.

(GN2) $N(\lambda x, \lambda y) = |\lambda| N(x, y)$ for every $x, y \in X$ and $\lambda \in \mathbb{R}$ or $\mathbb{C}$.

(GN3) There exist $r, s > 0$ such that $N(x - z, y - z) < s$ for all $x, y, z \in X$ and $N(x, x), N(y, y), N(z, z) < r$.

The pair (X, N) is called a generalized 2-normed space or G2NS.

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Definition 1.2. ([20]) Let (X, N) be a *G2NS* and consider the family $\mathcal{B}_N$ of all open balls of (X, N) where,

$$\mathcal{B}_N = \{B_N(x, r) : x \in X,\ r > 0\} \text{ and } B_N(x, r) = \{y \in X : N(x - y, x - y) < r\}.$$

The topology of (X, N), denoted by τ_N, is the topology with $\mathcal{B}_N$ as subbase which is T_2-separable. Furthermore, a subset $A \subseteq X$ is bounded if $M(A) < \infty$, where

$$M(A) = \sup\{N(x - z, y - z) : x, y, z \in A\}$$

is the maximal perimeter of A and for all $x \in X$ and $r > 0$, we have

$$M(B_N(x, r)) = rM(B_N(0, 1)) < \infty.$$

Proposition 1.3 (([20])). *Suppose $\lambda : [0, 2\pi) \to \mathbb{R}$ is periodic with period π such that $\lambda([0, 2\pi)) \subseteq [m_*, m*]$ for some $m_*, m* > 0$. Consider $N : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined as follows:*

$$N(\mathbf{r}_1 e^{i\alpha_1}, \mathbf{r}_2 e^{i\alpha_2}) = \frac{\mathbf{r}_1 + \mathbf{r}_2}{2} \lambda\left(\frac{\alpha_1 + \alpha_2}{2}\right).$$

Then, $(\mathbb{R}^2, N)$ is a G2NS. Following a specific selection of the function $\lambda : [0, 2\pi) \to \mathbb{R}$ defined by

$$\lambda(\alpha) = \begin{cases} 1 & \text{if } \alpha \in \mathbb{Q} \cap [0, \pi) \\ \frac{1}{2} & \text{if } \alpha \in \mathbb{Q}^c \cap [0, \pi) \\ 1 & \text{if } \alpha \in \pi + \mathbb{Q} \cap [0, \pi) \\ \frac{1}{2} & \text{if } \alpha \in \pi + \mathbb{Q}^c \cap [0, \pi) \end{cases}$$

in the aforementioned observation, Kundu et. al. [20, Example 4.17] demonstrated that the topology on $\mathbb{R}^2$ caused by that G2NS is distinct from the usual topology on $\mathbb{R}^2$. For this particular G2N N one can obtain:

$$\begin{aligned} B_N(0, 1) = \{\mathbf{r}e^{i\alpha} : \mathbf{r}\lambda(\alpha) < 1\} &= \{\mathbf{r}e^{i\alpha} : \mathbf{r} < 1, \alpha \in \mathbb{Q} \cap [0, \pi)\} \\ &\cup \{\mathbf{r}e^{i\alpha} : \mathbf{r} < 2, \alpha \in \mathbb{Q}^c \cap [0, \pi)\} \\ &\cup \{\mathbf{r}e^{i\alpha} : \mathbf{r} < 1, \alpha \in \pi + \mathbb{Q} \cap [0, \pi)\} \\ &\cup \{\mathbf{r}e^{i\alpha} : \mathbf{r} < 2, \alpha \in \pi + \mathbb{Q}^c \cap [0, \pi)\}. \end{aligned}$$

Kundu et al. [20] have shown that the space G2NS is T_2-separable, and the topology of G2NS on $\mathbb{R}^2$ differs from its usual topology. This leads to several important questions: Is the generalized 2-normed space metrizable? Under which conditions can any norm be induced from a generalized 2-norm? Is the convergence in the topology induced by a generalized 2-norm on $\mathbb{R}^2$ the same as the ordinary convergence in $\mathbb{R}^2$? To address these questions, we will provide some relevant examples. In conclusion, we declare that the questions do not hold.

On the other hand, we will briefly discuss an important concept in set theory called the ideal of subsets of $\mathbb{N}$. A family $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is termed an ideal on $\mathbb{N}$ if it satisfies the following conditions: (i) $A \cup B \in \mathcal{I}$ whenever $A, B \in \mathcal{I}$, (ii) $B \in \mathcal{I}$ whenever $A \in \mathcal{I}$ and $B \subseteq A$. An ideal $\mathcal{I}$ is considered non-trivial if it is not empty and not equal to $\mathcal{P}(\mathbb{N})$ [19]. The concept of rough convergence was first introduced in the works of Phu [23, 24] and has since gained significant attention from mathematicians. Over time, this particular area has drawn noticeably more attention to numerous mathematicians [2, 5–7, 21, 25–27]. Now, let's review the concept of rough $\mathcal{I}$-convergence in normed spaces.

Definition 1.4. ([11, 22]) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a normed space X is said to be rough $\mathcal{I}$-convergent to x_* with degree of roughness r (≥ 0), if for each $\varepsilon > 0$, $\{n \in \mathbb{N} : \|x_n - x_*\| \geq r + \varepsilon\} \in \mathcal{I}$. Observe that $r = 0$ in the aforementioned definition corresponds to the definition of $\mathcal{I}$-convergence of sequences [19]. In this scenario, the definition of (classical) rough convergence is obtained when one set $\mathcal{I}$ =Fin (the set of finite subsets of $\mathbb{N}$).

We could follow references [1, 3, 4, 12–14, 18] related to rough convergence, rough statistical convergence and rough ideal convergence.

Our main goal is to explore the concept of a rough version of ideal convergence in G2NS. This concept naturally encompasses rough ideal convergence in normed spaces by considering the generalized 2-norm N appropriately. Additionally, we aim to provide a more general characterization of the rough limit set $\mathcal{I}\text{-LIM}^r_N x_i$ for any G2NS. Furthermore, the article thoroughly examines some related findings from [6, 7, 22–24]. Due to certain properties of G2NS, the results we have obtained differ from those presented in previous literature.

2. Main results

To address the previous question, we present an example that demonstrates the difference between convergence in the topology induced by the G2NS on $\mathbb{R}^2$ and the standard convergence in $\mathbb{R}^2$.

Example 2.1. First, we define a real-valued function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} \frac{1}{2}, & x \in \mathbb{Q}, \\ \frac{1}{4}, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

Next, we define the generalized 2-norm $N : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^+ \cup \{0\}$ in the following manner:

$$N((x_1, x_2), (y_1, y_2)) = \begin{cases} (|x_1| + |x_2| + |y_1| + |y_2|) \cdot \frac{1}{2}, & \text{if } x_1 = 0 \\ (|x_1| + |x_2| + |y_1| + |y_2|) \cdot f(\frac{x_2}{x_1}), & \text{if } x_1 \neq 0. \end{cases}$$

(GN1) Let $N(\mathbf{x},\mathbf{y}) = 0$ where $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2)$. This ensures that $x_1 = 0$, otherwise $N(\mathbf{x},\mathbf{y})$ would never zero. Therefore, we have

$$N((x_1, x_2), (y_1, y_2)) = 0 \Leftrightarrow (|x_2| + |y_1| + |y_2|) \cdot \frac{1}{2} = 0 \Leftrightarrow \mathbf{x} = \mathbf{y} = 0.$$

(GN2) Let $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$. Then, it is evident that

$$N(\lambda\mathbf{x}, \lambda\mathbf{y}) = |\lambda| N(\mathbf{x}, \mathbf{y}), \text{ for every } \lambda \in \mathbb{R}.$$

(GN3) Let $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2), \mathbf{z} = (z_1, z_2) \in \mathbb{R}^2$ and let r be a positive real number such that

$$N(\mathbf{x},\mathbf{x}), N(\mathbf{y},\mathbf{y}), N(\mathbf{z},\mathbf{z}) < r \text{ and } s = 8r.$$

Then, it follows that

$$\begin{aligned} N(\mathbf{x\text{-}z}, \mathbf{y\text{-}z}) &= \begin{cases} (|x_1 - z_1| + |x_2 - z_2| + |y_1 - z_1| + |y_2 - z_2|) \cdot f(\frac{x_2 - z_2}{x_1 - z_1}) & \text{when } x_1 \neq z_1 \\ (|x_2 - z_2| + |y_1 - z_1| + |y_2 - z_2|) \cdot \frac{1}{2} & \text{when } x_1 = z_1 \end{cases} \\ &\leq \frac{1}{2}(|x_1 - z_1| + |x_2 - z_2| + |y_1 - z_1| + |y_2 - z_2|) \\ &\leq \frac{1}{2}\{(|x_1| + |x_2|) + (|y_1| + |y_2|) + 2(|z_1| + |z_2|)\} < s \end{aligned}$$

Thus, we can conclude that $(\mathbb{R}^2, N)$ is a generalized 2-normed space.

Observe now that we can write $B_N(0, 1) = A \cup B$, where $A = \{(x, y) \in \mathbb{R}^2 : |x| + |y| < 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 : 1 \leq |x| + |y| < 2 \text{ and } y/x \in \mathbb{R} - \mathbb{Q}\}$.

Consider the topological space $(\mathbb{R}^2, \tau_N)$ induced by the G2NS $(\mathbb{R}^2, N)$. The sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{u_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ are set up as follows: for each $n \in \mathbb{N}$, we have

$$x_n = \left(1 + \frac{1}{2n}\right)^n \text{ and } u_n = \frac{1}{2(4n-1)(4n-3)}.$$

Let us set $y_n = \sum_{k=1}^{n} u_k$. Then, it is obvious that both $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}$ are increasing and $(\sqrt{e}, \frac{\pi}{16})$ is the usual limit of $\{(x_n, y_n)\}_{n\in\mathbb{N}}$. Now observe that

$$\begin{aligned} &\left(\sqrt{e}, \frac{\pi}{16}\right) \in \{(x, y) \in \mathbb{R}^2 : 1 \le |x| + |y| < 2 \text{ and } y/x \in \mathbb{R} - \mathbb{Q}\} \subseteq B_N(0, 1) \\ \implies &\left(\sqrt{e}, \frac{\pi}{16}\right) \in B_N(0, 1), \text{ whereas } (x_n, y_n) \notin B_N(0, 1) \quad \text{for any } n \in \mathbb{N}. \end{aligned}$$

This assures that $(x_n, y_n) \nrightarrow \left(\sqrt{e}, \frac{\pi}{16}\right)$ in $(\mathbb{R}^2, \tau_N)$. We assert that $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ has no limit in $(\mathbb{R}^2, \tau_N)$. Assume, on the contrary, $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ converges to (x_*, y_*) in the topological space $(\mathbb{R}^2, \tau_N)$. Since τ_N is finer than the usual topology on $\mathbb{R}^2$, it follows that $(x_*, y_*) = (\sqrt{e}, \frac{\pi}{16})$. Which is a contradiction, as we already obtain $(x_n, y_n) \nrightarrow \left(\sqrt{e}, \frac{\pi}{16}\right)$ in $(\mathbb{R}^2, \tau_N)$. The assertion therefore meets.

The subsequent example depicts that the topological space $(\mathbb{R}^2, \tau_N)$ defined in Example 2.1 is not regular.

Example 2.2. Consider the closed set $K = \mathbb{R}^2 \setminus B_N(0, 1)$ in $(\mathbb{R}^2, \tau_N)$ and $p = (\sqrt{e}, \frac{\pi}{16}) \notin K$. Consider an open set V_* in $(\mathbb{R}^2, \tau_N)$ such that $p \in V_* \subseteq B_N(0, 1)$. Then, for sufficiently small $\varepsilon > 0$, we can write

$$\left\{(x, y) \in \mathbb{R}^2 : \sqrt{(x - \sqrt{e})^2 + \left(y - \frac{\pi}{16}\right)^2} < \varepsilon\right\} \cap \left\{(x, y) \in \mathbb{R}^2 : \frac{y}{x} \in \mathbb{R} \setminus \mathbb{Q}\right\} \subseteq V_*.$$

We set $p_n = (x_n, y_n)$, where $x_n = \left(1 + \frac{1}{2n}\right)^n$ and $y_n = \displaystyle\sum_{k=1}^{n} \frac{1}{2(4k-1)(4k-3)}$, for each $n \in \mathbb{N}$. Note that $\{p_n\}_{n\in\mathbb{N}}$ converges to p usually. So there exists $n_\varepsilon \in \mathbb{N}$ such that for each $n \ge n_\varepsilon$,

$$p_n \in B_\varepsilon(p), \text{ where } B_r(p) \text{ denotes the usual-open ball with center } p \text{ and radius } r.$$

For each $n \in \mathbb{N}$, let us choose $V_n \in \tau_N$ such that $p_n = (x_n, y_n) \in V_n$. Since $\{\frac{y_n}{x_n}\}_{n\in\mathbb{N}} \subseteq \mathbb{Q}$, therefore p_n must belong to the concentrate part of V_n, for each $n \in \mathbb{N}$. This ensures that for each $n \in \mathbb{N}$, there exists $\delta_n > 0$ such that $B_{\delta_n}(p_n) \subseteq V_n$. Thus, for each $n \ge n_\varepsilon$, we have

$$\begin{aligned} &B_{\delta_n}(p_n) \cap B_\varepsilon(p) \neq \emptyset \\ \implies &B_{\delta_n}(p_n) \cap (B_\varepsilon(p) \cap \{(x, y) : y/x \in \mathbb{R} \setminus \mathbb{Q}\}) \neq \emptyset, \\ \implies &V_n \cap (B_\varepsilon(p) \cap \{(x, y) : y/x \in \mathbb{R} \setminus \mathbb{Q}\}) \neq \emptyset \Rightarrow V_n \cap V_* \neq \emptyset. \end{aligned}$$

Therefore, it follows that every open cover of K is also an open cover of $\{p_n : n \in \mathbb{N}\}$ and every open cover of $\{p_n : n \in \mathbb{N}\}$ intersects every open set V_* containing p. This ensures that the point p and the closed set K cannot be strongly separated by open sets. Thus, the topological space $(\mathbb{R}^2, \tau_N)$ is not regular. Consequently, it is not metrizable.

Remark 2.3. Does every generalized 2-norm ($G2N$) on a vector space X be obtained from a norm $||.||$ on X? The answer is yes. Setting $N(\mathbf{x}, \mathbf{y}) = ||\mathbf{x}|| + ||\mathbf{y}||$, for all $\mathbf{x}, \mathbf{y} \in X$.

A generalized 2-norm ($G2N$) on a vector space X defines a norm $||.||$ on X which is given by $||\mathbf{x}|| = N(\mathbf{x}, 0)$, for all $\mathbf{x} \in X$, provided $G2N$ satisfies the following property

$$N(\mathbf{x} + \mathbf{y}, 0) \le N(\mathbf{x}, 0) + N(\mathbf{y}, 0) \text{ for all } \mathbf{x}, \mathbf{y} \in X.$$

The following example shows that any norm cannot be directly induced from the G2N by placing one coordinate 0, i.e., $\|\mathbf{x}\| = N(\mathbf{x}, 0)$.

Example 2.4. Let us consider $X = \ell^{\frac{1}{2}}$ the space of sequences of complex numbers $\{\xi_n\}_{n\in\mathbb{N}}$ satisfying $\sum_{n=1}^{\infty}|\xi_n|^{\frac{1}{2}} < \infty$.

Let us define $N : \ell^{\frac{1}{2}} \times \ell^{\frac{1}{2}} \to \mathbb{R}_{\geq}0$ by $N(\mathbf{x},\mathbf{y}) = \left(\sum_{j=1}^{\infty}|\xi_j|^{\frac{1}{2}}\right)^2 + \left(\sum_{j=1}^{\infty}|\eta_j|^{\frac{1}{2}}\right)^2$, where $\mathbf{x} = \{\xi_n,\}_{n\in\mathbb{N}}, \mathbf{y} = \{\eta_n\}_{n\in\mathbb{N}} \in \ell^{\frac{1}{2}}$.

(GN1) Now $N(\mathbf{x},\mathbf{y}) = 0 \iff \left(\sum_{j=1}^{\infty}|\xi_j|^{\frac{1}{2}}\right)^2 + \left(\sum_{j=1}^{\infty}|\eta_j|^{\frac{1}{2}}\right)^2 = 0 \iff \xi_j = 0 = \eta_j$ for all $j \in \mathbb{N} \iff \mathbf{x} = \mathbf{0} = \mathbf{y}$.

(GN2) Let $\alpha \in \mathbb{C}$ and $\mathbf{x} = \{\xi_n,\}_{n\in\mathbb{N}}, \mathbf{y} = \{\eta_n\}_{n\in\mathbb{N}} \in \ell^{\frac{1}{2}}$. Then, we have

$$N(\alpha\mathbf{x},\alpha\mathbf{y}) = \left(\sum_{j=1}^{\infty}|\alpha\xi_j|^{\frac{1}{2}}\right)^2 + \left(\sum_{j=1}^{\infty}|\alpha\eta_j|^{\frac{1}{2}}\right)^2 = |\alpha|N(\mathbf{x},\mathbf{y}).$$

(GN3) Finally, assume that $\mathbf{x} = \{\xi_n\}_{n\in\mathbb{N}}, \mathbf{y} = \{\eta_n\}_{n\in\mathbb{N}}, \mathbf{z} = \{\zeta_n\}_{n\in\mathbb{N}} \in \ell^{\frac{1}{2}}$ and r be a positive number such that $N(\mathbf{x},\mathbf{x}), N(\mathbf{y},\mathbf{y}), N(\mathbf{z},\mathbf{z}) < r$, and set $s = 4r$. Observe that

$$N(\mathbf{x},\mathbf{x}) < r \Rightarrow \sum_{j=1}^{\infty}|\xi_j|^{\frac{1}{2}} < \sqrt{\frac{r}{2}}\ .$$

Similarly, we obtain

$$\sum_{j=1}^{\infty}|\eta_j|^{\frac{1}{2}} < \sqrt{\frac{r}{2}} \text{ and } \sum_{j=1}^{\infty}|\zeta_j|^{\frac{1}{2}} < \sqrt{\frac{r}{2}}\ .$$

Note that for any $a, b \in \mathbb{C}$, we have $|a-b|^{\frac{1}{2}} \leq |a|^{\frac{1}{2}} + |b|^{\frac{1}{2}}$. So we obtain

$$\sum_{i=1}^{\infty}|\xi_i - \zeta_i|^{\frac{1}{2}} \leq \sum_{i=1}^{\infty}|\xi_i|^{\frac{1}{2}} + \sum_{i=1}^{\infty}|\zeta_i|^{\frac{1}{2}} = \sqrt{2r} \text{ and } \sum_{i=1}^{\infty}|\eta_i - \zeta_i|^{\frac{1}{2}} < \sqrt{2r}\ .$$

Therefore, it follows that

$$N(\mathbf{x}-\mathbf{z},\mathbf{y}-\mathbf{z}) = \left(\sum_{j=1}^{\infty}|\xi_j - \zeta_j|^{\frac{1}{2}}\right)^2 + \left(\sum_{j=1}^{\infty}|\eta_j - \zeta_j|^{\frac{1}{2}}\right)^2 \leq 4r = s.$$

Thus, N is a generalized 2-norm on $\ell^{\frac{1}{2}}$. Since ℓ^p is not a normed space for $0 < p < 1$, we conclude that $N(\mathbf{x}, 0)$ or $N(0, \mathbf{x})$ cannot be a norm on X.

In our study of G2NS, we have introduced the concepts of rough $\mathcal{I}$-convergence and $\mathcal{I}$-cluster points for a sequence. To illustrate this, we offer a significant example (see Example 2.6) demonstrating that the set of rough ideal convergent sequences is distinct from the set of ideal convergent sequences in G2NS. Additionally, we have defined the rough ideal limit set and established a connection between ideal cluster points and rough ideal limit points of a G2NS-valued sequence.

Definition 2.5. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a G2NS, (X, N) is considered rough $\mathcal{I}$-convergent to x_* with a roughness degree $r \geq 0$, denoted by $x_n \xrightarrow[N]{(\mathcal{I},r)} x_*$, if for every $\varepsilon > 0$,

$$\{n \in \mathbb{N} : N(x_n - x_*, x_n - x_*) \geq r + \varepsilon\} \in \mathcal{I}.$$

The non-negative real number r is referred to as the degree of roughness. The collection

$$\mathcal{I} - LIM_N^r x_i = \left\{x_* \in X : x_n \xrightarrow[N]{(\mathcal{I},r)} x_*\right\}$$

is designated as the r-limit set of $\{x_n\}_{n\in\mathbb{N}}$. We say that a sequence $\{x_n\}_{n\in\mathbb{N}}$ is rough $\mathcal{I}$-convergent if there exists $r \geq 0$ for which $\mathcal{I} - LIM_N^r x_i \neq \emptyset$. In this case, the convergence associated with the ideal Fin is known as rough convergence.

Let us demonstrate the novelty of the concept with a non-trivial example. Consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ with values in G2NS that does not $\mathcal{I}$-converge to any point, but

$$\mathcal{I} - \mathrm{LIM}_N^r x_i \neq \emptyset, \text{ for some } r > 0.$$

Example 2.6. Let us consider the G2NS as in Example 1.3 and $\mathbb{G} = \{g : \mathbb{N} \to [0, \infty) : g(n) \to \infty$ and $n/g(n) \nrightarrow 0\}$. We also consider an ideal [8] of the form

$$\mathcal{Z}_g(f) = \{A \subset \mathbb{N} : d_g^f(A) = 0\}, \text{ where } d_g^f(A) = \lim_{n\to\infty} \frac{f(|A \cap \{1, 2, ..., n\}|)}{f(g(n))},$$

where f is an unbounded modulus function and $g \in \mathbb{G}$ such that $f(n)/f(g(n)) \nrightarrow 0$ as $n \to \infty$.

Consider the unbounded modulus funtion $f(x) = \sqrt{x}, x \in [0, \infty)$, and the weight $g(n) = \sqrt[4]{n}, n \in \mathbb{N}$. Since $\lim\limits_{n\to\infty} \dfrac{\sqrt{n^{1/5}}}{\sqrt{n^{1/4}}} = 0$, it follows that $A \in \mathcal{Z}_g(f)$, where $A = \{n^5 : n \in \mathbb{N}\}$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^2$ is now set up as follows:

$$x_n = \begin{cases} \mathbf{r}_n \exp(i\theta_n) & \text{if } n \in \mathbb{N} \setminus A \\ ((-1)^n, n!) & \text{if } n \in A \end{cases}$$

where the sequences $\{\theta_n\}_{n\in\mathbb{N}\setminus A}, \{\mathbf{r}_n\}_{n\in\mathbb{N}\setminus A}$ in $\mathbb{R}$ are defined, respectively, as follows:

$$\theta_n = \begin{cases} \frac{\pi}{2} + \frac{1}{n} & \text{if } n \in \{2t : t \in \mathbb{N}\} \setminus A \\ \pi - \frac{1}{n} & \text{if } n \in \{2t-1 : t \in \mathbb{N}\} \setminus A \end{cases} \text{ and } \mathbf{r}_n = 3 - \frac{1}{n} \text{ for } n \in \mathbb{N} \setminus A.$$

It is evident that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is not $\mathcal{Z}_g(f)$-convergent to any point in $(\mathbb{R}^2, N)$, since

$$\{2t : t \in \mathbb{N}\} \setminus A, \{2t-1 : t \in \mathbb{N}\} \setminus A \notin \mathcal{Z}_g(f).$$

Let us now show that $x_n \xrightarrow[N]{(\mathcal{Z}_g(f),r)} x_*$, where $r = 1.5$ and $x_* = 0$. Observe that for each $\varepsilon > 0$, we can write

$$B_N(0, 1.5 + \varepsilon) \supseteq \{\mathbf{r}e^{i\theta} : \mathbf{r} < 1.5 + \varepsilon, \theta \in \mathbb{Q} \cap [0, \pi)\} \cup \{\mathbf{r}e^{i\theta} : r < 3 + 2\varepsilon, \theta \in \mathbb{Q}^c \cap [0, \pi)\}.$$

Therefore, we have

$$\{n \in \mathbb{N} : N(x_n - x_*, x_n - x_*) \geq r + \varepsilon\} \subseteq A.$$

Since $A \in \mathcal{Z}_g(f)$, we have

$$\{n \in \mathbb{N} : N(x_n - x_*, x_n - x_*) \geq r + \varepsilon\} \in \mathcal{Z}_g(f).$$

Thus, we can conclude that $x_n \xrightarrow[N]{(\mathcal{Z}_g(f),1.5)} 0$ in $(\mathbb{R}^2, N)$.

Definition 2.7. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a *G2NS* is called $\mathcal{I}$-bounded if there exists $M > 0$ such that

$$\{n \in \mathbb{N} : N(x_n, x_n) \geq M\} \in \mathcal{I}.$$

Our initial finding indicates that a G2NS-valued sequence is rough $\mathcal{I}$-convergent if and only if it is $\mathcal{I}$-bounded.

Theorem 2.8. *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a G2NS is rough $\mathcal{I}$-convergent if and only if it is $\mathcal{I}$-bounded.*

Proof. First assume that $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-bounded. Then, there exists a positive $M > 0$ such that

$$\{n \in \mathbb{N} : N(x_n, x_n) \geq M\} \in \mathcal{I}.$$

Therefore, it follows that $x_n \xrightarrow[N]{(\mathcal{I},M)} 0$.

Conversely, suppose that $x_n \xrightarrow[N]{(\mathcal{I},r)} x_*$. Then,

$$A = \{n \in \mathbb{N} : N(x_n - x_*, x_n - x_*) > r + 1\} \in \mathcal{I}.$$

Let us pick arbitrary $j \in \mathbb{N} \setminus A$. Then, we have $N(x_j - x_*, x_j - x_*) \leq r + 1$. We set $t = N(-x_*, -x_*)$. By (GN3), there exists $s > 0$ such that $N(x_j, x_j) \leq s$. This ensures that

$$\{n \in \mathbb{N} : N(x_n, x_n) > s\} \subseteq A.$$

Consequently, we have

$$\{n \in \mathbb{N} : N(x_n, x_n) > s\} \in \mathcal{I}.$$

Hence, we deduce that $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-bounded. □

The following result shows that the maximum perimeter of the set $\mathcal{I} - \mathrm{LIM}_N^r x_i$ is finite.

Theorem 2.9. *Suppose $\{x_n\}_{n\in\mathbb{N}}$ is a $\mathcal{I}$-bounded sequence in a G2NS (X, N). Then, the limit set $\mathcal{I} - \mathrm{LIM}_N^r x_i$ is bounded.*

Proof. Let us fix $x_* \in \mathcal{I} - \mathrm{LIM}_N^r x_i$. Now consider arbitrary $y_* \in \mathcal{I} - \mathrm{LIM}_N^r x_i$. Then, we have $A, B \in \mathcal{I}$, where

$$A = \{n \in \mathbb{N} : N(x_n - x_*, x_n - x_*) > r + 1\},$$

$$B = \{n \in \mathbb{N} : N(x_n - y_*, x_n - y_*) > r + 1\}.$$

Note that $\mathbb{N} \setminus (A \cup B) \neq \emptyset$, since $\mathbb{N} \setminus (A \cup B) \in \mathcal{F}(\mathcal{I})$ (where $\mathcal{F}(\mathcal{I})$ is the filter associated to the ideal $\mathcal{I}$). Let us choose any $j \in \mathbb{N} \setminus (A \cup B)$. Then, we have

$$N(x_j - x_*, x_j - x_*) \leq r + 1 \text{ and } N(x_j - y_*, x_j - y_*) \leq r + 1.$$

Now from (GN3), it follows that there exists $s > 0$ such that

$$N(y_* - x_*, y_* - x_*) < s.$$

This implies that $y_* \in B_N(x_*, s)$. Since y_* was arbitrary, we have $\mathcal{I} - \mathrm{LIM}_N^r x_i \subseteq B_N(x_*, s)$. Therefore, we have

$$\begin{aligned} M(\mathcal{I} - \mathrm{LIM}_N^r x_i) &\leq M(B_N(x_*, s)) \\ &= sM(B_N(0, 1)) < \infty. \end{aligned}$$

Thus, we can conclude that $\mathcal{I} - \mathrm{LIM}_N^r x_i$ is bounded in (X, N). □

Theorem 2.10. *Suppose $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are sequences in (X, N) such that $x_n \xrightarrow[N]{(\mathcal{I},r_1)} x_*$ and for each $n \in \mathbb{N}$, we have $y_n \in B_N(x_n, r_2)$, where $r_1, r_2 \geq 0$. Then, there exists $s > 0$ such that $y_n \xrightarrow[N]{(\mathcal{I},s)} x_*$.*

Proof. Let us fix $\varepsilon_* > 0$ and set $t = \max\{r_1 + \varepsilon_*, r_2\}$. Since $x_n \xrightarrow[N]{(\mathcal{I},r_1)} x_*$, it follows that

$$A = \{n \in \mathbb{N} : N(x_n - x_*, x_n - x_*) > r_1 + \varepsilon_*\} \in \mathcal{I}.$$

Then, for any $j \in \mathbb{N} \setminus A$, we have

$$N(x_j - x_*, x_j - x_*) \leq r_1 + \varepsilon_* \leq t \text{ and } N(x_j - y_j, x_j - y_j) < r_2 \leq t \text{ (since } y_j \in B_N(x_j, r_2)).$$

By (GN3), there exists $s = s(t) > 0$ such that $N(y_j - x_*, y_j - x_*) < s$. Pick $\varepsilon > 0$ be arbitrary. Then, we have

$$\{n \in \mathbb{N} : N(y_n - x_*, y_n - x_*) \geq s + \varepsilon\} \subseteq A.$$

Thus, for any $\varepsilon > 0$, we have

$$\{n \in \mathbb{N} : N(y_n - x_*, y_n - x_*) \geq s + \varepsilon\} \in \mathcal{I}.$$

Therefore, we can deduce that $y_n \xrightarrow[N]{(\mathcal{I},s)} x_*$. □

Theorem 2.11. *If the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are rough ideal convergent to the same point x_* in (X, N), then there exists $A \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus A \in \mathcal{I}$ and the maximal perimeter of the set $\{x_n - y_n : n \in A\}$ is bounded by some constant multiple of $M(B_N(0, 1))$.*

Proof. Note that the given hypothesis ensures that there exist $r_1, r_2 \geq 0$ such that $M_1, M_2 \in \mathcal{I}$, and for each $j \in \mathbb{N} \setminus (M_1 \cup M_2)$ we have

$$N(x_j - x_*, x_j - x_*) \leq r_1 + \varepsilon_0 \text{ and } N(y_j - x_*, y_j - x_*) \leq r_2 + \varepsilon_0,$$

where ε_0 is a fixed positive number. Let us take $A = \mathbb{N} \setminus (M_1 \cup M_2)$. Then, it is easy to observe that $\mathbb{N} \setminus A = M_1 \cup M_2 \in \mathcal{I}$. Let us now set $t = \max\{r_1 + \varepsilon_0, r_2 + \varepsilon_0\}$. Thus, from (GN3), there exists $s = s(t) > 0$ such that $N(x_j - y_j, x_j - y_j) < s$ i.e., $x_j \in B_N(y_j, s)$ for each $j \in A$. This ensures that $\{x_n - y_n : n \in A\} \subseteq sB_N(0, 1)$. Consequently we have $M(\{x_n - y_n : n \in A\}) \leq sM(B_N(0, 1))$. □

A connection between ideal convergence and rough ideal convergence is illustrated in the following result.

Theorem 2.12. *If $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent to x in (X, N), then for any $r \geq 0$, there exists $s > 0$ such that $\mathcal{I}-\mathrm{LIM}_N^r x_i \subseteq B_N(x, s)$ and $x \in \bigcup_{s>0} \bigcap_{x_* \in \mathcal{I}-\mathrm{LIM}_N^r x_i} B_N(x_*, s)$.*

Proof. Since $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent to x, for any $r \geq 0$, we have $x_n \xrightarrow[N]{(\mathcal{I},r)} x$. Therefore, Theorem 2.9 entails that there exists $s > 0$ such that for any $y_* \in \mathcal{I} - \mathrm{LIM}_N^r x_i$, we have

$$x \in B_N(y_*, s). \tag{1}$$

This gives $y_* \in B_N(x, s)$ for every $y_* \in \mathcal{I} - \mathrm{LIM}_N^r x_i$ i.e., $\mathcal{I} - \mathrm{LIM}_N^r x_i \subseteq B_N(x, s)$. Now from Equation 1, it follows that

$$\begin{aligned} & x \in B_N(x_*, s) \text{ for all } x_* \in \mathcal{I} - \mathrm{LIM}_N^r x_i \text{ and for some } s > 0 \\ \Rightarrow\ & x \in \bigcap_{x_* \in \mathcal{I}-\mathrm{LIM}_N^r x_i} B_N(x_*, s) \text{ for some } s > 0 \\ \Rightarrow\ & x \in \bigcup_{s>0} \bigcap_{x_* \in \mathcal{I}-\mathrm{LIM}_N^r x_i} B_N(x_*, s). \end{aligned}$$

Hence, our assertion follows. □

Theorem 2.13. *For any $r, \sigma \geq 0$, there exists $s = s(r, \sigma) > 0$ such that*

$$\mathcal{I} - \text{LIM}^r_N x_i + B_N(0, \sigma) \subseteq \mathcal{I} - \text{LIM}^s_N x_i.$$

Proof. We set $t = \max\{r + 1, \sigma\}$. Let $x_* = y + z$, where $y \in \mathcal{I} - \text{LIM}^r_N x_i$ and $z \in B_N(0, \sigma)$. Therefore, we have

$$A = \{n \in \mathbb{N} : N(x_n - y, x_n - y) > t\} \in \mathcal{I} \text{ and } N(z, z) < \sigma < t.$$

Now by (GN3), there exists $s > 0$ such that

$$\{n \in \mathbb{N} : N(x_n - y - z, x_n - y - z) > s + \varepsilon\} \subseteq A, \text{ for each } \varepsilon > 0.$$

This implies $x_* \in \mathcal{I} - \text{LIM}^s_N x_i$. Consequently, $\mathcal{I} - \text{LIM}^r_N x_i + B_N(0, \sigma) \subseteq \mathcal{I} - \text{LIM}^s_N x_i$. $\square$

Theorem 2.14. *Suppose $\{x_n\}_{n\in\mathbb{N}}$ is a sequence taking values in a G2NS (X, N). Then, for any $r \geq 0$, there exists $s = s(r) > 0$ such that $\mathcal{I} - \text{LIM}^r_N x_i \subseteq int(\mathcal{I} - \text{LIM}^s_N x_i)$.*

Proof. If $\mathcal{I} - \text{LIM}^r_N x_i = \emptyset$, then we are done. So we assume that $\mathcal{I} - \text{LIM}^r_N x_i \neq \emptyset$ and take arbitrary $x_* \in \mathcal{I} - \text{LIM}^r_N x_i$. Therefore, we have

$$A = \{n \in \mathbb{N} : N(x_n - x_*, x_n - x_*) > r + 1\} \in \mathcal{I}.$$

Pick arbitrary $y \in B_N(x_*, r)$. Again using (GN3), we can find an $s = s(r) > 0$ such that for each $\varepsilon > 0$, we have

$$\{n \in \mathbb{N} : N(x_n - y, x_n - y) > s + \varepsilon\} \subseteq A.$$

This ensures that $y \in \mathcal{I} - \text{LIM}^s_N x_i$. Therefore, we have

$$B_N(x_*, r) \subseteq \mathcal{I} - \text{LIM}^s_N x_i.$$

Since x_* was arbitrary, therefore we deduce that $\mathcal{I} - \text{LIM}^r_N x_i \subseteq int(\mathcal{I} - \text{LIM}^s_N x_i)$. $\square$

Definition 2.15. Let $x = \{x_n\}_{n\in\mathbb{N}}$ be a sequence G2NS (X, N). Then, $\gamma \in X$ is an $\mathcal{I}$-cluster point of x, if for each $\varepsilon > 0$,

$$\{k \in \mathbb{N} : N(x_k - \gamma, x_k - \gamma) < \varepsilon\} \notin \mathcal{I}.$$

The set $\Gamma^{\mathcal{I}}_{x,N}$ denotes the assortment of $\mathcal{I}$-cluster points of x in (X, N).

Example 2.16. Consider the sequence as in Example 2.6. Since $d^f_g(\{2t : t \in \mathbb{N}\} \setminus A) \neq 0$ and $d^f_g(\{2t - 1 : t \in \mathbb{N}\} \setminus A) \neq 0$, for each $\varepsilon > 0$, it follows that

$$\left\{n \in \mathbb{N} : x_n \in B_N((3, \frac{\pi}{2}), \varepsilon)\right\}, \{n \in \mathbb{N} : x_n \in B_N((3, \pi), \varepsilon)\} \notin \mathcal{Z}_g(f).$$

It is easy to realize that the sequence $\{x_n\}_{n\in\mathbb{N}}$ has no other $\mathcal{I}$-cluster point. Thus, we have

$$\Gamma^{\mathcal{I}}_{x,N} = \{(3, \frac{\pi}{2}), (3, \pi)\}.$$

The subsequent results are pertained with $\mathcal{I}$-cluster points and the rough $\mathcal{I}$-limit set of a G2NS valued sequence.

Theorem 2.17. *For any sequence $x = \{x_n\}_{n\in\mathbb{N}}$ in (X, N), we have*

$$\mathcal{I} - \text{LIM}^r_N x_i \subseteq \bigcap_{\gamma \in \Gamma^{\mathcal{I}}_{x,N}} B_N(\gamma, s), \textit{ for some } s = s(r) > 0.$$

Proof. Let $x_* \in \mathcal{I} - \mathrm{LIM}_N^r x_i$ be arbitrary. Let us fix $t > r$. Then,

$$A = \{n \in \mathbb{N} : N(x_n - x_*, x_n - x_*) \geq t\} \in \mathcal{I}.$$

Now for any $\gamma \in \Gamma_{x,N}^{\mathcal{I}}$, we have

$$B = \{n \in \mathbb{N} : N(x_n - x_*, x_n - x_*) < t\} \notin \mathcal{I}.$$

Note that $(\mathbb{N} \setminus A) \cap B \neq \emptyset$, since $(\mathbb{N} \setminus A) \cap B \notin \mathcal{I}$. Let us pick any $j \in (\mathbb{N} \setminus A) \cap B$. Thus, we have

$$N(x_j - x_*, x_j - x_*) < t \text{ and } N(x_j - \gamma, x_j - \gamma) < t.$$

Therefore, (GN3) entails that there exists $s > 0$ such that $N(x_* - \gamma, x_* - \gamma) < s$. Since $x_* \in \mathcal{I}\text{-}LIM_N^r x_i$ was chosen arbitrarily, we have

$$\mathcal{I} - \mathrm{LIM}_N^r x_i \subseteq B_N(\gamma, s)$$

for each $\gamma \in \Gamma_{x,N}^{\mathcal{I}}$. Thus, we can conclude that

$$\mathcal{I} - \mathrm{LIM}_N^r x_i \subseteq \bigcap_{\gamma \in \Gamma_{x,N}^{\mathcal{I}}} B_N(\gamma, s).$$

Hence the results. □

Definition 2.18. A G2NS (X, N) is said to have property **P** if for each $x \in X$ there exists $\delta > 0$ such that $B_N(x, \delta) \subseteq B_N(y, s)$ whenever $x \in B_N(y, s)$ for some $y \in X$ and $s > 0$.

Theorem 2.19. *If the G2NS (X, N) has the property **P**, then $\Gamma_{x,N}^{\mathcal{I}}$ is a closed set.*

Proof. Assume $y \in cl(\Gamma_{x,N}^{\mathcal{I}})$ ($cl(A)$ denotes closure of A) and pick $\varepsilon > 0$. Evidently, we have

$$\Gamma_{x,N}^{\mathcal{I}} \cap B_N(y, \varepsilon) \neq \emptyset,$$

so choose $z \in \Gamma_{x,N}^{\mathcal{I}} \cap B_N(y, \varepsilon)$. By virtue of property **P**, there exists $\delta > 0$ such that $B_N(z, \delta) \subseteq B_N(y, \varepsilon)$. Therefore, we have

$$\{n \in \mathbb{N} : N(x_n - z, x_n - z) < \delta\} \subseteq \{n \in \mathbb{N} : N(x_n - y, x_n - y) < \varepsilon\}.$$

Since $\{n \in \mathbb{N} : N(x_n - z, x_n - z) < \delta\} \notin \mathcal{I}$, we deduce that $y \in \Gamma_{x,N}^{\mathcal{I}}$. □

3. Concluding remarks

In any normed space $(X, \|.\|)$, the space $(X, N_{\|.\|})$ is a G2NS, where $N_{\|.\|}(x, y) = \|x\| + \|y\|$, for all $x, y \in X$. We refer to this as the induced G2N on X by the norm $\|.\|$. It's easy to understand that $(X, N_{\|.\|})$ has the property **P**, but the G2NS given in Example 2.1 doesn't have the property **P**. This is because for each $(x, y) \in B_N(0, 1)$ with $\frac{y}{x} \in \mathbb{R} \setminus \mathbb{Q}$ and $1 \leq |x| + |y| < 2$, there doesn't exist any $t > 0$ such that $B_N((x, y), t) \subseteq B_N(0, 1)$. For this induced G2NS, we can assign carefully chosen values of 's' in terms of the degree of roughness 'r' (e.g. $s = r$ or $2r$) in the results obtained in the last section of this article. Several studies have already examined the cases when the specific G2NS $(X, N_{\|.\|})$ is taken into consideration (visit [5–7, 10, 23, 24]).

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References

[1] M. Arslan, E. Dündar, *Rough convergence in 2-normed spaces*, Bull. Math. Anal. Appl. **10** (2018), 1–9.
[2] M. Arslan, E. Dündar, *On rough convergence in 2-normed spaces and some properties*, Filomat **33** (2019), 5077–5086.
[3] M. Arslan, E. Dündar, *Rough statistical convergence in 2-normed spaces*, Honam Math. J. **43** (2021), 417–431.
[4] M. Arslan, E. Dündar, *Rough statistical cluster points in 2-normed spaces*, Thai J. Math. **20** (2022), 1419–1429.
[5] S. Aytar, *The rough limit set and the core of a real sequence*, Numer. Funct. Anal. Optim. **29** (2008), 283–290.
[6] S. Aytar, *Rough statistical convergence*, Numer. Funct. Anal. Optim. **29** (2008), 291–303.
[7] S. Aytar, *Rough statistical cluster points*, Filomat **31** (2017), 5295–5304.
[8] K. Bose, P. Das, A. Kwela, *Generating new ideals using weighted density via modulus functions*, Indag. Math. **29** (2018), 1196–1209.
[9] P. Chaipunya, P. Kumam, *On the distance between three arbitrary points*, J. Funct. Spaces. **2013** (2013).
[10] E. Dündar, *On rough $\mathcal{I}_2$-convergence of double sequences*, Numer. Funct. Anal. Optim. **37** (2016), 480–491.
[11] E. Dündar, C. Çakan, *Rough $\mathcal{I}$-convergence*, Demonstr. Math. **47** (2014), 638–651.
[12] E. Dündar, C. Çakan, *Rough convergence of double sequences*, Gulf J. Math. **2** (2014), 45–51.
[13] E. Dündar, U. Ulusu, *On rough $\mathcal{I}$-convergence and $\mathcal{I}$-Cauchy sequence for functions defined on amenable semigroups*, Univ. J. Math. Appl. **6** (2023), 86–90.
[14] E. Dündar, U. Ulusu, *Rough statistical convergent functions defined on amenable semigroups*, Proc. Nat. Acad. Sci. India Sect. A **94** (2024), 317–323.
[15] S. Gähler, *2-metrische Räume und ihre topologische struktur*, Math. Nachr. **26** (1963), 115–148.
[16] S. Gähler, *Lineare 2-normierte Räume*, Math. Nachr. **28** (1964), 1–43.
[17] S. Gähler, *Zur geometric 2-metriche raume*, Rev. Roumaine Math. Pures Appl. **40** (1966), 664–669.
[18] Ö. Kişi, E. Dündar, *Rough $\Delta\mathcal{I}$-statistical convergence*, J. Appl. Math. Inform. **40** (2022), 619–632.
[19] P. Kostyrko, T. Šalát, W. Wilczyński, *$\mathcal{I}$-convergence*, Real Anal. Exchange. **26** (2000), 669–685.
[20] A. Kundu, T. Bag, Sk. Nazmul, *A new generalization of normed linear space*, Topol. Appl. **256** (2019), 159–176.
[21] M. C. Listán-García, F. Rambla-Barreno, *Rough convergence and Chebyshev centers in Banach spaces*, Numer. Funct. Anal. Optim. **35** (2014), 432–442.
[22] S. K. Pal, D. Chandra, S. Dutta, *Rough ideal convergence*, Hacet. J. Math. Stat. **42** (2013), 633–640.
[23] H. X. Phu, *Rough convergence in normed linear spaces*, Numer. Funct. Anal. Optim. **22** (2001), 199–222.
[24] H. X. Phu, *Rough convergence in infinite dimensional normed space*, Numer. Funct. Anal. Optim. **24** (2003), 285–301.
[25] M. H. M. Rashid, *Rough statistical convergence and rough ideal convergence in random 2-normed spaces*, Filomat **38** (2024), 979–996.
[26] E. Savaş, M. Et, *On $(\Delta_\lambda^m, \mathcal{I})$-statistical convergence of order α*, Period. Math. Hungar. **71** (2015), 135–145.
[27] B. C. Tripathy, *On statistically convergent and statistically bounded sequences*, Bull. Malays. Math. Soc. (Second Series) **20** (1997), 31–33.