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Deferred logarithmic summability and its applications to Tauberian theory

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Abstract. In this paper, we define and introduce deferred logarithmic transformations, which are more general and effective than well-known transformations. Additionally, we provide several inclusion theorems and illustrative examples. We investigate the limiting behavior of deferred logarithmic moving averages of numerical sequences. We also present necessary and sufficient Tauberian conditions under which convergence of a sequence or its certain subsequences follows from its deferred logarithmic summability. As a result, the method and theory developed in this paper may contribute to obtaining more interesting and useful results concerning other advanced summability methods and to extending these findings to additional areas of research.

1. Introduction

In summability theory, despite the significant progress made with existing methods, there remains a persistent need for continued research and refinement. While the current transformations exhibit many advantageous properties, it has been observed that nearly all of them also present undesirable characteristics. For instance, the Cesàro transformation of any positive order increases ultimate bounds and oscillations of certain sequences of functions and does not always preserve some convergence types for sequences of functions such as continuous, uniform or mean-square. To overcome these limitations, Agnew [1] introduced the deferred Cesàro mean as a generalization of the Cesàro mean, which possesses useful properties not present in Cesàro's and other well-known transformations. In this pioneering study, Agnew also introduced a necessary and sufficient condition so that the deferred Cesàro transformation may include the Cesàro transformation. Since then, this topic has been extensively investigated by researchers from various perspectives. Moreover, deferred Cesàro means have been studied in various contexts, including paranormed spaces [9], sequence spaces of random variables [12] and fuzzy numbers [33]. The integration of deferred Cesàro means with the concept of statistical convergence has significantly expanded the scope of research, paving the way for further studies [8, 11, 13, 20]. In recent years, deferred Cesàro means have

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become a highly active area of research, due to their applications in summability and approximation theory. For more details, we refer to [3–5, 7, 10, 14–18, 25–27, 29, 32, 34].

In this paper, inspired by the above-mentioned studies, the logarithmic method will form the basis of our research. In the literature, there has been a surge of studies on Tauberian conditions for logarithmic summability methods, both in the classical sense and in the statistical sense. Additionally, researchers have expanded these studies to various sequence spaces. Here, we reference only a few of them [2, 6, 19, 21–24, 28, 31, 36].

Although the aforementioned studies are milestones in this field, our primary motivation for conducting this research is our shared goal with Agnew [1]. In this study, we introduce the deferred logarithmic transformation by modifying the logarithmic method, resulting in a more general and effective transformation compared to the logarithmic and other well-known methods in terms of its features.

We consider a sequence (u_n) of real or complex numbers. The deferred logarithmic means $D^{p,q}_{\ell_n}(u)$ of (u_n) is defined by

$$D^{p,q}_{\ell_n}(u) = \frac{1}{\ell^{p,q}_n}\sum_{k=p_n+1}^{q_n}\frac{u_k}{k}, \quad \text{where} \quad \ell^{p,q}_n = \sum_{k=p_n+1}^{q_n}\frac{1}{k} \sim \log\frac{q_n}{p_n}, \tag{1}$$

where (p_n) and (q_n) are the sequences of non-negative integers satisfying

$$p_n < q_n, \quad n = 1, 2, \ldots, \tag{2}$$

and

$$\lim_{n\to\infty} q_n = \infty. \tag{3}$$

It is important to observe that when $p_n = n-1$ and $q_n = n$, we obtain the identity transformation, and when $p_n = 0$ and $q_n = n$, the corresponding deferred logarithmic mean reduces to the logarithmic mean.

Since

$$\lim_{n\to\infty}\frac{\ell^{p,q}_n}{\log\dfrac{q_n}{p_n}} = 1,$$

the sequences $\{D_{\ell_n}\}$ and $\{{}^{*}D_{\ell_n}\}$ are equiconvergent with the same limit, where

$${}^{*}D^{p,q}_{\ell_n}(u) = \frac{1}{\log\dfrac{q_n}{p_n}}\sum_{k=p_n+1}^{q_n}\frac{u_k}{k},$$

in which case the logarithm is to the natural base e.

We say that (u_n) is deferred logarithmic summable of first order, briefly: summable $(D_{\ell_n}, 1)$, if there exists some $L \in \mathbb{C}$ such that

$$\lim_{n\to\infty} D^{p,q}_{\ell_n}(u) = L. \tag{4}$$

In this paper, innovating the definition of a properly deferred sequence given by [1], we provide several inclusion theorems and illustrative examples. Additionally, we investigate the limiting behavior of deferred logarithmic moving averages of numerical sequences. In the main result of this paper, we present converse conclusions, namely Tauberian theorems, for the deferred logarithmic method, whose regularity we have established. Motivated by the definitions of slowly decreasing and slowly oscillating sequences with respect to logarithmic summability [24], we define the concepts of slow decrease and slow oscillation for a sequence of real and complex numbers with respect to deferred logarithmic summability, respectively. As a result, we obtain necessary and/or sufficient conditions under which convergence of a sequence (u_n) or its certain subsequences follows from summability by deferred logarithmic means.

2. Inclusions

Throughout this paper, we shall proceed under the assumption that conditions (2) and (3) hold. Any additional restrictions on the sequences of non-negative integers (p_n) and (q_n), if necessary, will be specified in the relevant theorems.

Remark 2.1. We recall that for the sequence (u_n) of real or complex numbers, the sequence (σ_n) of its arithmetic (also called $(C,1)$) averages is defined by

$$\sigma_n(u) = \frac{1}{n}\sum_{k=1}^{n} u_k, \quad n = 1, 2 \ldots, \tag{5}$$

while the sequence (τ_n) of its logarithmic (also called $(\ell,1)$) averages is defined by

$$\tau_n(u) = \frac{1}{\ell_n}\sum_{k=1}^{n} \frac{u_k}{k}, \quad \text{where} \quad \ell_n = \sum_{k=1}^{n}\frac{1}{k} \sim \log n, \quad n = 1, 2 \ldots. \tag{6}$$

Moreover, if a sequence is $(C,1)$ summable to L, then it is $(\ell,1)$ summable to L (see, e.g., [35]). However, the converse of this statement is not true in general.

Theorem 2.2. *Every sequence that is $(C,1)$ summable to a finite limit L is also $(D_{\ell_n},1)$ summable to the same value.*

Proof. For the reader's convenience, we provide a sketch of the proof of this claim. To begin with, we express u_n in terms of σ_n as follows:

$$u_n = n\sigma_n - (n-1)\sigma_{n-1}.$$

Accordingly, we can state the following,

$$\begin{aligned} D_{\ell_n}^{p,q}(u) &= \frac{1}{\ell_n^{p,q}}\sum_{k=p_n+1}^{q_n}\frac{1}{k}(k\sigma_k - (k-1)\sigma_{k-1}) = \frac{1}{\ell_n^{p,q}}\sum_{k=p_n+1}^{q_n}\left(\sigma_k - \sigma_{k-1} + \frac{1}{k}\sigma_{k-1}\right) \\ &= \frac{\sigma_{q_n} - \sigma_{p_n}}{\ell_n^{p,q}} + \frac{1}{\ell_n^{p,q}}\sum_{k=p_n+1}^{q_n}\frac{\sigma_{k-1}}{k} \to 0 + L, \quad n \to \infty. \end{aligned}$$

So, $u_n \to L \ (C,1) \implies u_n \to L \ (D_{\ell_n},1), \ n \to \infty.$ $\square$

The converse of Theorem 2.2 is not always true, as shown in the following example.

Example 2.3. Let $p_n = 2n-1$ and $q_n = 4n-1$. The sequence $(u_n) = ((-1)^n n)$ is $(D_{\ell_n},1)$ summable, but not $(C,1)$ summable.

Indeed, one can check easily that

$$\begin{aligned} D_{\ell_n}^{p,q}(u) &= \frac{1}{\ell_n^{p,q}}\sum_{k=p_n+1}^{q_n}\frac{u_k}{k} \\ &= \frac{1}{\ell_n^{p,q}}\sum_{k=2n}^{4n-1}(-1)^k \to 0, \quad n \to \infty. \end{aligned}$$

So, we say that (u_n) is deferred logarithmic summable to 0.

On the other hand, we know that if $\lim_{n\to\infty}\sigma_n$ exists, then $u_n = o(n), n \to \infty$. From this fact, since $\lim_{n\to\infty}\frac{u_n}{n} = \lim_{n\to\infty}(-1)^n$ does not exist, we can conclude that the limit of (σ_n) does not exist. Hence, (u_n) is not $(C,1)$ summable.

As is well-known, the (C, 1) summability method is regular, meaning that every convergent sequence is Cesàro summable to the same limit.

Corollary 2.4. *The deferred logarithmic summability method is regular under conditions (2) and (3), meaning that if conditions (2) and (3) are satisfied and a sequence (u_n) of real or complex numbers converges to a finite limit L, then $(D_{\ell_n}, 1)$ also converges to the same limit L.*

However, it is shown by the following example that convergence does not follow from deferred logarithmic summability in general.

Example 2.5. Consider the divergent sequence $(u_n) = ((-1)^n)$.
It is clear that $\sigma_n \to 0, \quad n \to \infty$. Hence, $D^{p,q}_{\ell_n} \to 0, \quad n \to \infty$.

The converse cases hold only under a suitable condition. Such a condition is called a Tauberian condition, and the resulting theorem is called a Tauberian theorem. It is important to note that these theorems are referred to as 'Tauberian' in honor of A. Tauber, who was the first to prove one of the simplest theorems of this kind. Next, in the main result of this paper, we present Tauberian theorems for the deferred logarithmic summability of sequences.

Motivated by the definition of a properly deferred sequence [1], we say that $(D_{\ell_n}, 1)$ is properly deferred if $\left(\dfrac{\ell_{p_n}}{\ell^{p,q}_n}\right)$ is bounded.

Theorem 2.6. *If $(D_{\ell_n}, 1)$ is proper, then a complex sequence that is $(\ell, 1)$ summable to a finite limit L is also $(D_{\ell_n}, 1)$ summable to the same value.*

Proof. We outline the proof of this claim. First, using (6), we obtain

$$\ell_n \tau_n = \sum_{k=1}^{n} \frac{u_k}{k}. \tag{7}$$

Accordingly, based on (1) and (7), we can state

$$\begin{aligned} D^{p,q}_{\ell_n}(u) &= \frac{1}{\ell^{p,q}_n}\left(\ell_{q_n}\tau_{q_n} - \ell_{p_n}\tau_{p_n}\right) \\ &= \tau_{q_n} + \left(\frac{\ell_{q_n}}{\ell^{p,q}_n} - 1\right)\tau_{q_n} - \frac{\ell_{p_n}}{\ell^{p,q}_n}\tau_{p_n} \\ &= \tau_{q_n} + \frac{\ell_{p_n}}{\ell^{p,q}_n}(\tau_{q_n} - \tau_{p_n}) \to L + 0, \quad n \to \infty. \end{aligned}$$

Since $(D_{\ell_n}, 1)$ is proper, then $u_n \to L\ (\ell, 1) \implies u_n \to L\ (D_{\ell_n}, 1),\ n \to \infty$. □

However, the converse of Theorem 2.6 is not true in general as the following example shows.

Example 2.7. Let $p_n = 2n$ and $q_n = 4n^2$. Consider the sequence (u_n) defined by

$$u_n = \begin{cases} \frac{n(n+1)}{2}, & n \text{ is odd}, \\ -\frac{n^2}{2}, & n \text{ is even}. \end{cases}$$

It is easy to verify that $u_n \to 0\ (D_{\ell_n}, 1),\ n \to \infty$.
On the other hand, we have

$$\tau_n(u) = \begin{cases} \frac{n+1}{2\ell_n}, & n \text{ is odd}, \\ 0, & n \text{ is even}. \end{cases}$$

Clearly, (u_n) is not $(\ell, 1)$ summable.

When examining the implication relations among the aforementioned summability methods, we obtain the following corollary.

Corollary 2.8. *If $(D_{\ell_n}, 1)$ is proper, then summability $(D_{\ell_n}, 1)$ is more effective than summability $(\ell, 1)$. Consequently, it is also more effective than summability $(C, 1)$.*

3. Auxiliary results

rWe need the following lemma which is essential in the proofs of our main results.

Lemma 3.1. *(i) If $\lambda > 1$ and n is large enough in the sense that $[q_n^\lambda] > q_n$, then,*

$$u_{q_n} - D_{\ell_n}^{p,q}(u) = \frac{\ell_n^{p,[q^\lambda]}}{\ell_n^{q,[q^\lambda]}}\left(D_{\ell_n}^{p,[q^\lambda]}(u) - D_{\ell_n}^{p,q}(u)\right) - \frac{1}{\ell_n^{q,[q^\lambda]}}\sum_{k=q_n+1}^{[q_n^\lambda]}\left(\frac{u_k - u_{q_n}}{k}\right). \tag{8}$$

(ii) If $0 < \lambda < 1$ and n is large enough in the sense that $q_n > [q_n^\lambda]$, then,

$$u_{q_n} - D_{\ell_n}^{[p^\lambda],[q^\lambda]}(u) = \frac{\ell_n^{[p^\lambda],q}}{\ell_n^{[q^\lambda],q}}\left(D_{\ell_n}^{[p^\lambda],q}(u) - D_{\ell_n}^{[p^\lambda],[q^\lambda]}(u)\right) + \frac{1}{\ell_n^{[q^\lambda],q}}\sum_{k=[q_n^\lambda]+1}^{q_n}\left(\frac{u_{q_n} - u_k}{k}\right), \tag{9}$$

where $[q_n^\lambda]$ denotes the integer part of q_n^λ.

Proof. (i) By definition,

$$\begin{aligned} D_{\ell_n}^{p,[q^\lambda]}(u) &= \frac{1}{\ell_n^{p,[q^\lambda]}}\sum_{k=p_n+1}^{[q_n^\lambda]}\frac{u_k}{k} \\ &= \frac{1}{\ell_n^{p,[q^\lambda]}}\sum_{k=p_n+1}^{q_n}\frac{u_k}{k} + \frac{1}{\ell_n^{p,[q^\lambda]}}\sum_{k=q_n+1}^{[q_n^\lambda]}\frac{u_k}{k} \\ &= \frac{\ell_n^{p,q}}{\ell_n^{p,[q^\lambda]}}D_{\ell_n}^{p,q}(u) + \frac{1}{\ell_n^{p,[q^\lambda]}}\sum_{k=q_n+1}^{[q_n^\lambda]}\frac{u_k}{k}. \end{aligned}$$

Therefore,

$$\frac{\ell_n^{p,[q^\lambda]}}{\ell_n^{q,[q^\lambda]}}\left(D_{\ell_n}^{p,[q^\lambda]}(u) - D_{\ell_n}^{p,q}(u)\right) - \frac{1}{\ell_n^{q,[q^\lambda]}}\sum_{k=q_n+1}^{[q_n^\lambda]}\frac{u_k}{k} = -D_{\ell_n}^{p,q}(u),$$

that is equivalent to (8).

(ii) The proof of (9) is similar. □

Furthermore, we determine the limiting behavior of deferred logarithmic moving averages of a sequence real or complex numbers.

Theorem 3.2. *If a sequence (u_n) of real or complex numbers is deferred logarithmic summable to a finite limit L, then*

$$\lim_{n\to\infty}\frac{1}{\ell_n^{q,[q^\lambda]}}\sum_{k=q_n+1}^{[q_n^\lambda]}\frac{u_k}{k} = L, \tag{10}$$

for each $\lambda > 1$*, and*

$$\lim_{n\to\infty} \frac{1}{\ell_n^{[q^\lambda],q}} \sum_{k=[q_n^\lambda]+1}^{q_n} \frac{u_k}{k} = L, \tag{11}$$

for each $0 < \lambda < 1$*.*

Proof. If $\lambda > 1$ and n is large enough such that $[q_n^\lambda] > q_n$, then from Lemma 3.1, we have

$$\frac{1}{\ell_n^{q,[q^\lambda]}} \sum_{k=q_n+1}^{[q_n^\lambda]} \frac{u_k}{k} = D_{\ell_n}^{p,q}(u) + \frac{\ell_n^{p,[q^\lambda]}}{\ell_n^{q,[q^\lambda]}} \left(D_{\ell_n}^{p,[q^\lambda]}(u) - D_{\ell_n}^{p,q}(u) \right).$$

It is obvious that for all $\lambda > 1$ and sufficiently large n,

$$1 < \frac{\ell_n^{p,[q^\lambda]}}{\ell_n^{q,[q^\lambda]}} < \frac{3\lambda - 1}{\lambda - 1}. \tag{12}$$

Thus (10) is obtained from (12) and the assumed convergence of $(D_{\ell_n}^{p,q}(u))$.

If $0 < \lambda < 1$ and n is large enough such that $q_n > [q_n^\lambda]$, then from Lemma 3.1, we have

$$\frac{1}{\ell_n^{[q^\lambda],q}} \sum_{k=[q_n^\lambda]+1}^{q_n} \frac{u_k}{k} = D_{\ell_n}^{[p^\lambda],[q^\lambda]}(u) + \frac{\ell_n^{[p^\lambda],q}}{\ell_n^{[q^\lambda],q}} \left(D_{\ell_n}^{[p^\lambda],q}(u) - D_{\ell_n}^{[p^\lambda],[q^\lambda]}(u) \right).$$

It is clear that for all $0 < \lambda < 1$ and sufficiently large n,

$$1 < \frac{\ell_n^{[p^\lambda],q}}{\ell_n^{[q^\lambda],q}} < \frac{3 - \lambda}{1 - \lambda}. \tag{13}$$

Thus (11) is obtained from (13) and the assumed convergence of $(D_{\ell_n}^{p,q}(u))$. □

4. Main results

For real sequences, we prove the following one-sided theorems. First, we give necessary and sufficient Tauberian conditions under which convergence of a certain subsequence of a sequence of real numbers follows from its deferred logarithmic summability.

Theorem 4.1. *If a sequence* (u_n) *of real numbers is deferred logarithmic summable to a finite limit L, then*

$$\lim_{n\to\infty} u_{q_n} = L \tag{14}$$

if and only if

$$\limsup_{\lambda\downarrow 1} \liminf_{n\to\infty} \frac{1}{\ell_n^{q,[q^\lambda]}} \sum_{k=q_n+1}^{[q_n^\lambda]} \left(\frac{u_k - u_{q_n}}{k} \right) \geq 0 \tag{15}$$

and

$$\limsup_{\lambda\uparrow 1} \liminf_{n\to\infty} \frac{1}{\ell_n^{[q^\lambda],q}} \sum_{k=[q_n^\lambda]+1}^{q_n} \left(\frac{u_{q_n} - u_k}{k} \right) \geq 0, \tag{16}$$

in which case we necessarily have

$$\lim_{n\to\infty} \frac{1}{\ell_n^{q,[q^\lambda]}} \sum_{k=q_n+1}^{[q_n^\lambda]} \left(\frac{u_k - u_{q_n}}{k}\right) = 0 \tag{17}$$

for all $\lambda > 1$*, and*

$$\lim_{n\to\infty} \frac{1}{\ell_n^{[q^\lambda],q}} \sum_{k=[q_n^\lambda]+1}^{q_n} \left(\frac{u_{q_n} - u_k}{k}\right) = 0 \tag{18}$$

for all $0 < \lambda < 1$.

We can reformulate conditions (15) and (16) as follows: To every $\epsilon > 0$ and $\lambda_1 > 1$, there exist $n_1 = n_1(\epsilon) > 0$ and $\lambda = \lambda(\epsilon)$ with $1 < \lambda < \lambda_1$ such that for every $n \geq n_1$ we have

$$\lim_{n\to\infty} \frac{1}{\ell_n^{q,[q^\lambda]}} \sum_{k=q_n+1}^{[q_n^\lambda]} \left(\frac{u_k - u_{q_n}}{k}\right) \geq -\epsilon,$$

and for another $1 < \lambda < \lambda_1$ we have

$$\lim_{n\to\infty} \frac{1}{\ell_n^{[q^{\lambda^{-1}}],q}} \sum_{k=[q_n^{\lambda^{-1}}]+1}^{q_n} \left(\frac{u_{q_n} - u_k}{k}\right) \geq -\epsilon.$$

Remark 4.2. The symmetric counterparts of conditions (15) and (16) are the following:

$$\liminf_{\lambda\downarrow 1} \limsup_{n\to\infty} \frac{1}{\ell_n^{q,[q^\lambda]}} \sum_{k=q_n+1}^{[q_n^\lambda]} \left(\frac{u_k - u_{q_n}}{k}\right) \leq 0 \tag{19}$$

and

$$\liminf_{\lambda\uparrow 1} \limsup_{n\to\infty} \frac{1}{\ell_n^{[q^\lambda],q}} \sum_{k=[q_n^\lambda]+1}^{q_n} \left(\frac{u_{q_n} - u_k}{k}\right) \leq 0, \tag{20}$$

respectively.

One can modify the proof of Theorem 4.1 so that the conclusion is valid if conditions (15) and (16) are replaced by (19) and (20), respectively.

The concept of slow decrease with respect to logarithmic summability for a sequence was defined by Kwee [21], similarly to Schmidt's concept of slow decrease in the case of Cesàro summability [30]. Kwee's definition is equivalent to Móricz's definition of slow decrease with respect to logarithmic summability (see, e.g., [24]).

Motivated by the definition of slow decrease with respect to logarithmic summability (see, e.g., [24]), we say that a real sequence (u_n) is slowly decreasing with respect to summability $(D_{\ell_n}, 1)$ if

$$\lim_{\lambda\downarrow 1} \liminf_{n\to\infty} \min_{q_n < k \leq [q_n^\lambda]} (u_k - u_{q_n}) \geq 0. \tag{21}$$

Expressed in terms of ϵ and λ, this can be formulated as: For given $\epsilon > 0$, there exist $n_1 = n_1(\epsilon) > 0$ and $\lambda = \lambda(\epsilon) > 1$ such that

$$u_k - u_{q_n} \geq -\epsilon \quad \text{whenever} \quad n \geq n_1 \text{ and } q_n < k \leq [q_n^\lambda].$$

An equivalent reformulation of (21) is the following:

$$\lim_{\lambda\uparrow 1}\liminf_{n\to\infty}\min_{[q_n^\lambda]<k\le q_n}(u_{q_n}-u_k)\ge 0. \tag{22}$$

Note that conditions (15) and (16) follow from conditions (21) and (22), respectively.

Next, we show that condition of being slowly decreasing with respect to summability $(D_{\ell_n},1)$ is sufficient for a deferred logarithmic summable sequence to be convergent.

Theorem 4.3. *If a sequence (u_n) of real numbers is deferred logarithmic summable to a finite limit L and slowly decreasing with respect to summability $(D_{\ell_n},1)$, then (u_n) converges to L.*

It is obvious that if the one-sided sided Tauberian condition with respect to logarithmic summability [24],

$$n\log n\ (u_n-u_{n-1})\ge -H,\quad n=1,2,\dots,$$

is satisfied for some $H>0$, then (u_n) is slowly decreasing with respect to summability $(D_{\ell_n},1)$.

Indeed, in this case we have

$$u_k-u_{q_n}=\sum_{j=q_n+1}^{k}(u_j-u_{j-1})\ge -H\sum_{j=q_n+1}^{k}\frac{1}{j\log j}\ge -H\int_{q_n}^{k}\frac{dx}{x\log x}. \tag{23}$$

It follows from (23) that

$$\min_{q_n<k\le[q_n^\lambda]}(u_k-u_{q_n})\ge -H\int_{q_n}^{[q_n^\lambda]}\frac{dx}{x\log x}$$

and then,

$$\liminf_{n\to\infty}\min_{q_n<k\le[q_n^\lambda]}(u_k-u_{q_n})\ge -H\log\lambda.$$

Since λ can be chosen arbitrarily close to 1, (21) easily follows.

For complex sequences, we prove the following two-sided theorems. First, we give necessary and sufficient Tauberian conditions under which convergence of a certain subsequence of a sequence of complex numbers follows from its deferred logarithmic summability.

Theorem 4.4. *If a sequence (u_n) of complex numbers is deferred logarithmic summable to a finite limit L, then (u_{q_n}) converges to L if and only if one of the following two conditions is satisfied:*

$$\liminf_{\lambda\downarrow 1}\limsup_{n\to\infty}\left|\frac{1}{\ell_n^{q,[q^\lambda]}}\sum_{k=q_n+1}^{[q_n^\lambda]}\left(\frac{u_k-u_{q_n}}{k}\right)\right|=0 \tag{24}$$

and

$$\liminf_{\lambda\uparrow 1}\limsup_{n\to\infty}\left|\frac{1}{\ell_n^{[q^\lambda],q}}\sum_{k=[q_n^\lambda]+1}^{q_n}\left(\frac{u_{q_n}-u_k}{k}\right)\right|=0, \tag{25}$$

in which case we necessarily have (17) for all $\lambda>1$, and (18) for all $0<\lambda<1$.

We can reformulate conditions (24) and (25) as follows: To every $\epsilon>0$ and $\lambda_1>1$, there exist $n_1=n_1(\epsilon)>0$ and $\lambda=\lambda(\epsilon)$ with $1<\lambda<\lambda_1$ such that for every $n\ge n_1$ we have

$$\left|\frac{1}{\ell_n^{q,[q^\lambda]}}\sum_{k=q_n+1}^{[q_n^\lambda]}\left(\frac{u_k-u_{q_n}}{k}\right)\right|\le\epsilon$$

and for another $1 < \lambda < \lambda_1$ we have

$$\left| \frac{1}{\ell_n^{[q^{\lambda^{-1}}],q}} \sum_{k=[q_n^{\lambda^{-1}}]+1}^{q_n} \left(\frac{u_{q_n} - u_k}{k} \right) \right| \leq \epsilon.$$

Motivated by the definition of slow oscillation with respect to logarithmic summability (see, e.g., [24]), we say that a complex sequence (u_n) is slowly oscillating with respect to summability $(D_{\ell_n}, 1)$ if

$$\lim_{\lambda \downarrow 1} \limsup_{n\to\infty} \max_{q_n < k \leq [q_n^\lambda]} |u_k - u_{q_n}| = 0. \tag{26}$$

Expressed in terms of ϵ and λ, this can be formulated as: For given $\epsilon > 0$, there exist $n_1 = n_1(\epsilon) > 0$ and $\lambda = \lambda(\epsilon) > 1$ such that

$$|u_k - u_{q_n}| \leq \epsilon \quad \text{whenever} \quad n \geq n_1 \text{ and } q_n < k \leq [q_n^\lambda].$$

An equivalent reformulation of (26) is the following:

$$\lim_{\lambda \uparrow 1} \limsup_{n\to\infty} \max_{[q_n^\lambda] < k \leq q_n} |u_{q_n} - u_k| = 0. \tag{27}$$

Note that conditions (24) and (25) follow from conditions (26) and (27), respectively.

Next, we show that condition of being slowly oscillating with respect to summability $(D_{\ell_n}, 1)$ is sufficient for a deferred logarithmic summable sequence to be convergent.

Theorem 4.5. *If a sequence (u_n) of complex numbers is deferred logarithmic summable to a finite limit L and slowly oscillating with respect to summability $(D_{\ell_n}, 1)$, then (u_n) converges to Lt*

If the classical two-sided Tauberian condition with respect to logarithmic summability [24]

$$n \log n |u_n - u_{n-1}| \leq H, \quad n = 1, 2, \ldots,$$

is satisfied for some $H > 0$, then (u_n) is slowly oscillating with respect to summability $(D_{\ell_n}, 1)$.

Indeed, in this case we have

$$|u_k - u_{q_n}| \leq \sum_{j=q_n+1}^{k} |u_j - u_{j-1}| \leq H \sum_{j=q_n+1}^{k} \frac{1}{j \log j} \leq H \int_{q_n}^{k} \frac{dx}{x \log x}. \tag{28}$$

It follows from (28) that

$$\max_{q_n < k \leq [q_n^\lambda]} |u_k - u_{q_n}| \leq H \int_{q_n}^{[q_n^\lambda]} \frac{dx}{x \log x}$$

and then

$$\limsup_{n\to\infty} \max_{q_n < k \leq [q_n^\lambda]} |u_k - u_{q_n}| \leq H \log \lambda.$$

Since λ can be chosen arbitrarily close to 1, it is easy to verify that (26) holds.

5. Proofs of theorems

Proof of Theorem 4.1. Necessity. It follows from (4) and (14) that

$$\lim_{n\to\infty} (u_{q_n} - D_{\ell_n}^{p,q}(u)) = 0. \tag{29}$$

Case $\lambda > 1$.By (4) and (12),we have

$$\lim_{n\to\infty} \frac{\ell_n^{p,[q^\lambda]}}{\ell_n^{q,[q^\lambda]}} \left(D_{\ell_n}^{p,[q^\lambda]}(u) - D_{\ell_n}^{p,q}(u) \right) = 0. \tag{30}$$

Case $0 < \lambda < 1$. By (4) and (13),

$$\lim_{n\to\infty} \frac{\ell_n^{[p^\lambda],q}}{\ell_n^{[q^\lambda],q}} \left(D_{\ell_n}^{[p^\lambda],q}(u) - D_{\ell_n}^{[p^\lambda],[q^\lambda]}(u) \right) = 0. \tag{31}$$

Then, (17) (respectively (18)) is obtained from (8) (respectively (9)), (29) and (30) (respectively (31)).
Sufficiency. ssume that (4), (15) and (16) are satisfied. In the case $\lambda > 1$, by (8), we have

$$u_{q_n} - D_{\ell_n}^{p,q}(u) = \frac{\ell_n^{p,[q^\lambda]}}{\ell_n^{q,[q^\lambda]}} \left(D_{\ell_n}^{p,[q^\lambda]}(u) - D_{\ell_n}^{p,q}(u) \right) - \frac{1}{\ell_n^{q,[q^\lambda]}} \sum_{k=q_n+1}^{[q_n^\lambda]} \left(\frac{u_k - u_{q_n}}{k} \right). \tag{32}$$

From (15) it follows that there exists a sequence $\lambda_J \downarrow 1$ such that

$$\lim_{J\to\infty} \liminf_{n\to\infty} \frac{1}{\ell_n^{q,[q^{\lambda_J}]}} \sum_{k=q_n+1}^{[q_n^{\lambda_J}]} \left(\frac{u_k - u_{q_n}}{k} \right) \geq 0, \tag{33}$$

where $\lambda_J := [\lambda J]$.

By (32), we have

$$\limsup_{n\to\infty} \left(u_{q_n} - D_{\ell_n}^{p,q}(u) \right) \leq \lim_{J\to\infty} \limsup_{n\to\infty} \frac{\ell_n^{p,[q^{\lambda_J}]}}{\ell_n^{q,[q^{\lambda_J}]}} \left(D_{\ell_n}^{p,[q^{\lambda_J}]}(u) - D_{\ell_n}^{p,q}(u) \right) + \lim_{J\to\infty} \limsup_{n\to\infty} \left(-\frac{1}{\ell_n^{q,[q^{\lambda_J}]}} \sum_{k=q_n+1}^{[q_n^{\lambda_J}]} \left(\frac{u_k - u_{q_n}}{k} \right) \right).$$

Considering (4), (30) and (33), we conclude that

$$\limsup_{n\to\infty} \left(u_{q_n} - D_{\ell_n}^{p,q}(u) \right) \leq -\lim_{J\to\infty} \liminf_{n\to\infty} \left(\frac{1}{\ell_n^{q,[q^{\lambda_J}]}} \sum_{k=q_n+1}^{[q_n^{\lambda_J}]} \left(\frac{u_k - u_{q_n}}{k} \right) \right) \leq 0. \tag{34}$$

In the case $0 < \lambda < 1$, by (9), we obtain

$$u_{q_n} - D_{\ell_n}^{p,q}(u) = \left(D_{\ell_n}^{[p^\lambda],[q^\lambda]}(u) - D_{\ell_n}^{p,q}(u) \right) + \frac{\ell_n^{[p^\lambda],q}}{\ell_n^{[q^\lambda],q}} \left(D_{\ell_n}^{[p^\lambda],q}(u) - D_{\ell_n}^{[p^\lambda],[q^\lambda]}(u) \right) + \frac{1}{\ell_n^{[q^\lambda],q}} \sum_{k=[q_n^\lambda]+1}^{q_n} \left(\frac{u_{q_n} - u_k}{k} \right).$$

From (16) it follows that for some sequence $\lambda_J \uparrow 1$, we have

$$\lim_{J\to\infty} \liminf_{n\to\infty} \frac{1}{\ell_n^{[q^{\lambda_J}],q}} \sum_{k=[q_n^{\lambda_J}]+1}^{q_n} \left(\frac{u_{q_n} - u_k}{k} \right) \geq 0.$$

Consequently, in a similar way as above,

$$\begin{aligned} \liminf_{n\to\infty} \left(u_{q_n} - D_n^{p,q}(u) \right) \geq \lim_{J\to\infty} \liminf_{n\to\infty} \left(D_{\ell_n}^{[p^{\lambda_J}],[q^{\lambda_J}]}(u) - D_{\ell_n}^{p,q}(u) \right) + \lim_{J\to\infty} \liminf_{n\to\infty} \frac{\ell_n^{[p^{\lambda_J}],q}}{\ell_n^{[q^{\lambda_J}],q}} \left(D_{\ell_n}^{[p^{\lambda_J}],q}(u) - D_{\ell_n}^{[p^{\lambda_J}],[q^{\lambda_J}]}(u) \right) \\ + \lim_{J\to\infty} \liminf_{n\to\infty} \frac{1}{\ell_n^{[q^{\lambda_J}],q}} \sum_{k=[q_n^{\lambda_J}]+1}^{q_n} \left(\frac{u_{q_n} - u_k}{k} \right) \geq 0. \end{aligned} \tag{35}$$

Combining (34) and (35) yields (29), which implies (14) since (4). □

Proof of Theorem 4.3. Assume (21) is satisfied, then so is (22). It is clear that conditions (21) and (22) imply (15) and (16), respectively. Then, from Theorem 4.1, we have convergence of (u_{q_n}) to L: For given $\epsilon > 0$, there exists $N = N(\epsilon) > 0$ such that

$$-\frac{\epsilon}{2} \leq u_{q_n} - L \leq \frac{\epsilon}{2} \tag{36}$$

whenever $n \geq N$

It follows from the equivalent form of (21) that for given $\epsilon > 0$, there exist $n_1 = n_1(\epsilon) > 0$ and $\lambda = \lambda(\epsilon) > 1$ such that

$$u_k - u_{q_n} \geq -\frac{\epsilon}{2} \tag{37}$$

whenever $n \geq n_1$ and $q_n < k \leq [q_n^\lambda]$.

It follows from the equivalent form of (22) that for given $\epsilon > 0$, there exist $n_2 = n_2(\epsilon) > 0$ and $0 < \lambda = \lambda(\epsilon) < 1$ such that

$$u_{q_n} - u_k \geq -\frac{\epsilon}{2} \tag{38}$$

whenever $n \geq n_2$ and $[q_n^\lambda] < k \leq q_n$.

Taking (36) and (37) into account, we have

$$u_k - L = u_k - u_{q_n} + u_{q_n} - L \geq -\frac{\epsilon}{2} - \frac{\epsilon}{2} = -\epsilon \tag{39}$$

whenever $k \geq n \geq N_1 = \max\{n_1, N\}$.

Taking (36) and (38) into account, we have

$$u_k - L = u_k - u_{q_n} + u_{q_n} - L \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{40}$$

whenever $k \geq n \geq N_2 = \max\{n_2, N\}$.

By (39) and (40), we have for given $\epsilon > 0$ there exists $N_3 = \max\{N_1, N_2\}$ such that

$$-\epsilon \leq u_n - L \leq \epsilon$$

whenever $n \geq N_3$. This completes the proof. □

Proof of Theorem 4.4. Necessity. The proof runs along similar lines to the proof of the necessity part in Theorem 4.1.

Sufficiency. Assume that (4) and (24) are satisfied. It follows from (24) that there exists a sequence $\lambda_j \downarrow 1$ satisfying

$$\lim_{j\to\infty} \limsup_{n\to\infty} \left| \frac{1}{\ell_n^{q,[q^{\lambda_j}]}} \sum_{k=q_n+1}^{[q_n^{\lambda_j}]} \left(\frac{u_k - u_{q_n}}{k} \right) \right| = 0. \tag{41}$$

By (8), we have

$$\limsup_{n\to\infty} \left| u_{q_n} - D_{\ell_n}^{p,q}(u) \right| \leq \lim_{j\to\infty} \limsup_{n\to\infty} \frac{\ell_n^{p,[q^{\lambda_j}]}}{\ell_n^{q,[q^{\lambda_j}]}} \left| D_{\ell_n}^{p,[q^{\lambda_j}]}(u) - D_{\ell_n}^{p,q}(u) \right| + \lim_{j\to\infty} \limsup_{n\to\infty} \left| \frac{1}{\ell_n^{q,[q^{\lambda_j}]}} \sum_{k=q_n+1}^{[q_n^{\lambda_j}]} \left(\frac{u_k - u_{q_n}}{k} \right) \right|.$$

Taking (4), (30) and (41) into account, we obtain

$$\limsup_{n\to\infty}\left|u_{q_n}-D^{p,q}_{\ell_n}(u)\right|\le\lim_{j\to\infty}\limsup_{n\to\infty}\left|\frac{1}{\ell_n^{q,[q^{\lambda_j}]}}\sum_{k=q_n+1}^{[q_n^{\lambda_j}]}\left(\frac{u_k-u_{q_n}}{k}\right)\right|=0,$$

which concludes the proof of convergence of (u_{q_n}) to L.

A similar proof can be given if (25) is satisfied. □

Proof of Theorem 4.5. Assume that (u_n) is deferred logarithmic summable to L and condition (26) is satisfied. By Theorem 4.4, we have convergence of (u_{q_n}) to L : For given $\epsilon > 0$ there exists $N = N(\epsilon) > 0$ such that

$$|u_{q_n}-L|\le\frac{\epsilon}{2} \tag{42}$$

whenever $n \ge N$.

It follows from the equivalent form of (26) that for given $\epsilon > 0$, there exist $n_1 = n_1(\epsilon) > 0$ and $\lambda = \lambda(\epsilon) > 1$ such that

$$|u_k-u_{q_n}|\le\frac{\epsilon}{2} \tag{43}$$

whenever $n \ge n_1$ and $q_n < k \le [q_n^\lambda]$.

Taking (42) and (43) into account, we have

$$|u_k-L|\le|u_k-u_{q_n}|+|u_{q_n}-L|\le\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$$

whenever $k \ge n \ge N_1 = \max\{n_1, N\}$.

A similar proof can be given if (27) is satisfied. □

6. Conclusion

The concept of summability has been extensively studied over the years, with various transformations being proposed to address different challenges. Although established summability methods serve as important milestones within the field, there remains a persistent need for further exploration and refinement. Recently, numerous studies have focused on the generalization of classical summability methods, resulting in new approaches such as the deferred Cesàro mean. In this study, we introduced the deferred logarithmic transformation by modifying the logarithmic method, aiming to achieve a more general and effective transformation compared to logarithmic and other well-known methods in terms of its properties. In line with this aim, upon examining the implication relations between the summability methods discussed, we established that if $(D_{\ell_n}, 1)$ is proper, then the $(D_{\ell_n}, 1)$ summability method is more effective than both the $(\ell, 1)$ and $(C, 1)$ summability methods. Additionally, for the deferred logarithmic method, for which we demonstrated regularity under conditions (2) and (3), we presented the converse results, namely Tauberian theorems. We provided necessary and sufficient Tauberian conditions under which convergence of a sequence and convergence of its certain subsequences follows from deferred logarithmic summability. The results obtained in this paper anticipated to be beneficial for deriving more interesting and useful results concerning other advanced summability methods and for extending these findings to additional areas of research.

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