Filomat 39:11 (2025), 3695–3704
https://doi.org/10.2298/FIL2511695B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Impact of gradient solitons on perfect fluid spacetimes admitting a type of semi-symmetric non-metric connection

Ansari Rakesh Baidya[a], **Uday Chand De**[b], **Abul Kalam Mondal**[c,*]

[a]*Department of Mathematics,Dum Dum Motijheel Rabindra Mahavidyalaya, 208/B/2, Dum Dum Road, Kolkata-700074, West Bengal, India*
[b] *Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kolkata- 700019, West Bengal, India*
[c]*Department of mathematics, Acharya Prafulla Chandra College, New Barrackpore, Kolkata-700131, West Bengal, India*

Abstract. This article carries out the investigation of a 4-dimensional Lorentzian manifold M^4 endowed with a type of semi-symmetric non-metric connection (*SSNMC*) whose associated vector field is parallel with respect to the semi-Riemannian connection. We investigate gradient Ricci solitons, gradient η-Ricci solitons and gradient Yamabe soliton with respect to *SSNMC* and we obtain several interesting results on perfect fluid spacetimes..

1. Introduction

The semi-Riemannian geometry plays an important role in differential geometry and theory of relativity. A differentiable manifold M of dimension n with a non-degenerate metric g of signature (p, s) is known as a semi–Riemannian manifold of dimension n, where $p + s = n$ [6]. As a particular case, if we choose $s = n - 1$ and $p = 1$ or $s = 1$ and $p = n - 1$, then the semi-Riemannian manifold converts into a Lorentzian manifold and it has great importance in Physics (specially in the general theory of relativity and cosmology).

In this paper, we study the impact of gradient solitons on perfect fluid spacetimes admitting a type of semi-symmetric non-metric connection (shortly, *SSNMC*) . Many years ago, on a differential manifold, Friedman and Schouten [19] presented the concept of semi-symmetric linear connection. In 1932, Hayden [23] introduced the notion of metric connection with torsion on Riemannian manifold. Later , in 1970 the idea of semi-symmetric metric connection on a Riemannian manifold was further developed by Yano [34] .

Let M^4 be a 4-dimensional Lorentzian manifold with the Lorentzian metric g of signature (-,+,+,+).

Let $\triangledown$ denote the semi-Riemannian connection corresponding to the Lorentzian metric g on M^4. A linear connection $\tilde{\triangledown}$ defined on M^4 is said to be semi-symmetric if the torsion tensor $\tilde{T}$ of $\tilde{\triangledown}$ defined by

$$\tilde{T}(X_1, Y_1) = \tilde{\triangledown}_{X_1} Y_1 - \tilde{\triangledown}_{Y_1} X_1 - [X_1, Y_1] \quad (1)$$

2020 *Mathematics Subject Classification*. Primary 53Z50; Secondary 83C05, 83C10, 83C40.
Keywords. Lorentzian manifolds, perfect fluid spacetimes, gradient Ricci solitons, gradient η- Ricci solitons, gradient Yamabe solitons.
Received: 11 February 2025; Accepted: 05 February 2025
Communicated by Mića Stanković
* Corresponding author: Abul Kalam Mondal
Email addresses: baidyaansarirakesh@gmail.com (Ansari Rakesh Baidya), uc_de@yahoo.com (Uday Chand De), kalam_ju@yahoo.co.in (Abul Kalam Mondal)
ORCID iDs: https://orcid.org/0009-0008-4057-0493 (Ansari Rakesh Baidya), https://orcid.org/0000-0002-8990-4609 (Uday Chand De), https://orcid.org/0009-0003-2345-2624 (Abul Kalam Mondal)

satisfies

$$\tilde{T}(X_1, Y_1) = \omega(Y_1)X_1 - \omega(X_1)Y_1, \tag{2}$$

for all vector fields X_1, Y_1 on M^4, where ω is a 1-form associated with the fixed vector field ρ_1 and satisfies $\omega(X_1)$=$g(X_1, \rho_1)$.

Again, a linear connection $\tilde{\nabla}$ on M^4 is said to be metric if $\tilde{\nabla}g$=0 and if $\tilde{\nabla}g \neq 0$, then it is said to be non-metric [23]. Here, we consider *SSNMC*, that is $\tilde{\nabla}g \neq 0$ and the connection satisfies the equation (1). Agache and Chafle [1] introduced the idea of *SSNMC* on a Riemannian manifold. After that, several researchers studied the properties of *SSNMC* on manifolds with different structures ([7], [10], [17], [18], [24], [25], [28], [29], [32], [33], [35], [36]) and many others.

In 1982, the notion of Ricci flow was introduced by Hamilton [20], to find the canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian or semi-Riemannian manifold M^4 defined as: $\frac{\partial}{\partial t}g = -2\mathcal{S}$, where $\mathcal{S}$ and g are the Ricci tensor and the metric tensor respectively. A Ricci soliton [21] on a Riemannian or semi-Riemannian manifold (M^4, g) satiesfies the equcation

$$\mathfrak{L}_{W_1}g + 2\lambda g + 2\mathcal{S} = 0, \tag{3}$$

where W_1 is the potential vector field, $\mathfrak{L}$ is the Lie derivative, S is the Ricci tensor of the manifold. If W_1 is a Killing vector field or zero then the Ricci soliton becomes trival or reduces to an Einstein manifold respectively. The Ricci soliton is steady, expanding or shrinking according as $\lambda = 0$, λ>0 or λ<0. If $W_1 = Df$, where D indicates the gradient operator and f is a smooth function on M^4 , then g is called a gradient Ricci soliton and in such case equation (3) becomes

$$\nabla^2 f + \mathcal{S} + \lambda g = 0. \tag{4}$$

As a generalization of Ricci solitons, the notion of η-Ricci solitons was introduced by Cho and Kimura [8]. An η-Ricci solitons is a tuple (g, W_1, λ, μ), where W_1 is the potential vector field, λ and μ are real constants and g is a Riemannian (semi-Riemannian) metric satisfying the equcation

$$\mathfrak{L}_{W_1}g + 2\mathcal{S} + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{5}$$

where η is the g dual 1-form of W_1 and it is closed. Again if, $W_1 = Df$ then it is a gradient η-Ricci solitons and (5) takes the form

$$\nabla^2 f + \mathcal{S} + \lambda g + \mu\eta \otimes \eta = 0. \tag{6}$$

If $\mu = 0$, then η-Ricci soliton(or gradient η-Ricci soliton) turns into Ricci soliton (or gradient Ricci soliton) respectively.

Hamilton [20] proposed the idea of Yamabe flow, defined as follows:

$$\frac{\partial}{\partial t}g(t) + rg(t) = 0, \quad g_0 = g(t), \tag{7}$$

where t indicates the time and r being the scalar curvature of M^4. A Riemannian or semi-Riemannian manifold equipped with a Riemannian or semi-Riemannian metric g is named a Yamabe soliton if it satisfies

$$\mathfrak{L}_{W_1}g - 2(r - \lambda)g = 0, \tag{8}$$

r is the scalar curvature of the manifold and W_1 is the potential vector field. If $W_1 = Df$, then Yamabe soliton is called a gradient Yamabe soliton and the above equation becomes

$$\nabla^2 f = (\lambda - r)g. \tag{9}$$

If f is constant or W_1 is Killing vector field on M^4, then gradient Yamabe soliton(or Yamabe) soliton becomes trivial.

A time-oriented 4-dimensional Lorentzian manifold is said to be a spacetime. Alias et al. ([2], [3]) introduced the concept of generalized Robertson-Walker (shortly, *GRW*) spacetimes. An n-dimensional Lorentzian manifold M is called a *GRW* spacetime if it is constructed as a warped product $M = -I \times_{\varphi^2} M^*$, where I is an open interval of the real line, M^* is an $(n-1)$-dimensional Riemannian manifold and $\varphi > 0$ is a scalar function. It says that φ is a warping function or scale factor. The spacetime turns into Robertson-Walker (shortly, *RW*) spacetime if M^* is of constant curvature and of dimension three. The geometrical and physical properties of *GRW* spacetimes have been studied by many researchers but few are ([5], [14], [26]).

Einstein's field equations without the cosmological constant satisfy

$$S = kT + \frac{r}{2}g, \tag{10}$$

where k and T indicate the gravitational constant and the energy momentum tensor, respectively.

In a perfect fluid spacetime (shortly, *PFS*), T is described by

$$T = (p + \sigma)\omega \otimes \omega + pg, \tag{11}$$

σ and p denote the energy density and isotropic pressure of the *PFS* [22].

In *PFS*, the Ricci tensor S takes the form

$$S = ag + b\omega \otimes \omega, \tag{12}$$

a and b being the smooth functions, ω is a non zero 1-form defined by $\omega(X_1)=g(X_1, \rho_1)$ for all X_1 and $\otimes$ symbolizes the tensor product. Here ρ_1 is a unit time-like vector field, that is, $g(\rho_1, \rho_1) = -1$. Combining the equations (10), (11) and (12) , we infer that

$$a = \frac{k(p - \sigma)}{2 - n}, \quad b = k(p + \sigma). \tag{13}$$

Moreover, σ and p are connected by an equation of state of the type $p = p(\sigma)$ and the *PFS* is said to as isentropic. Further, if $p = \sigma$, the *PFS* is known as stiff matter fluid. If $p + \sigma = 0$, $p = 0$ and $p = \frac{\sigma}{3}$, respectively, the *PFS* is considered to be the dark era, dust matter fluid and the radiation era [9]. The universe is undergoing an accelerating phase when $\frac{p}{\sigma} < -\frac{1}{3}$. It describes the phantom era if $\frac{p}{\sigma} < -1$ and quintessence phase if $-1 < \frac{p}{\sigma} < 0$.

Every *RW* spacetime is a *PFS* [27]. For 4-dimension, the *GRW* spacetime becomes a *PFS* if and only if it is a *RW* spacetime. The properties of *PFS* have been investigated by many authors ([4], [13], [30]) and others.

A Lorentzian manifold is said to be a quasi-Einstein spacetime [16] if the Ricci tensor satisfies the relation (12) and a , b are non-zero real constants.

On the otherhand, the Weyl tensor, also known as the conformal curvature tensor, is crucial in both geometry and relativity theory. Several researchers have characterized spacetimes with Weyl tensor. The Weyl tensor C is defined by

$$\begin{aligned} C(X_1, Y_1)Z_1 &= R(X_1, Y_1)Z_1 \\ &\quad - \frac{1}{n-2}[g(QY_1, Z_1)X_1 - g(QX_1, Z_1)Y_1 + g(Y_1, Z_1)QX_1 - g(X_1, Z_1)QY_1] \\ &\quad + \frac{r}{(n-1)(n-2)}[g(Y_1, Z_1)X_1 - g(X_1, Z_1)Y_1] \end{aligned}$$

for all X_1, Y_1, $Z_1 \in \mathfrak{X}(M)$, R being the Riemann curvature tensor and Q being the Ricci operator expressed by $g(QX_1, Y_1) = S(X_1, Y_1)$. Further, we known that

$$\begin{aligned} (div\, C)(X_1, Y_1)Z_1 &= \frac{n-3}{n-2}[\{(\nabla_{X_1} S)(Y_1, Z_1) - (\nabla_{Y_1} S)(X_1, Z_1)\} \\ &\quad - \frac{1}{2(n-1)}\{(X_1 r)g(Y_1, Z_1) - (Y_1 r)g(X_1, Z_1)\}], \end{aligned} \tag{14}$$

'*div*' denotes the divergence.

Here we recall Shepley and Taub's theorem for a 4-dimensional *PFS* to be a *RW* spacetime.

Theorem A ([31]) A 4-dimensional *PFS* with $div\, C = 0$ and subject to an equation of state $p = p(\sigma)$ is conformally flat and the metric is *RW* , the flow is irrotational, shear-free and geodesic.

Some interesting results on perfect fluid spacetime admitting solitons have been investigated in ([11], [12], [15]) and also by others.

Motivated by the above works in this paper we are interested to investigate the impact of gradient solitons in perfect fluid spacetime with a type of semi-symmetric non-metric connection.

The paper is organized as follows: After introduction in Section 1, the fundamental results of *SSNMC* have been mentioned in Section 2. In the next Sections we study gradient Ricci solitons, gradient η-Ricci solitons and gradient Yamabe solitons respectively on *PFS* endowed with a type of *SSNMC*.

2. Semi-symmetric non-metric connection

Agreement. Throughout the paper we assume that the associated vector field of the *SSNMC* is parallel with respect to the semi-Riemannian connection.

A linear connection $\tilde{\nabla}$ on M^4, defined by

$$\tilde{\nabla}_{X_1} Y_1 = \nabla_{X_1} Y_1 + \omega(Y_1)X_1, \tag{15}$$

where ∇ is the semi- Riemannian connection on M^4, is a semi-symmetric non-metric connection. It satisfies [1]

$$(\tilde{\nabla}_{X_1} g)(Y_1, Z_1) = -\omega(Y_1)g(X_1, Z_1) - \omega(Z_1)g(X_1, Y_1). \tag{16}$$

Let $\tilde{R}$ and R denote the curvature tensor with respect to the semi-symmetric non-metric connection $\tilde{\nabla}$ and semi- Riemannian connection ∇ respectively. Then $\tilde{R}$ and R are connected by [1]

$$\tilde{R}(X_1, Y_1)Z_1 = R(X_1, Y_1)Z_1 - \alpha^*(Y_1, Z_1)X_1 + \alpha^*(X_1, Z_1)Y_1, \tag{17}$$

for all X_1, Y_1, Z_1 on M^4, where α^* is a $(0, 2)$- tensor field defined as follows:

$$\alpha^*(X_1, Y_1) = (\nabla_{X_1} \omega)(Y_1) - \omega(X_1)\omega(Y_1). \tag{18}$$

Throughout this article, we consider the vector field ρ_1 is a unit timelike vector field on M^4 and it is parallel with respect to the semi-Riemannian connection ∇, that is $\nabla \rho_1 = 0$ and $\omega(\rho_1) = g(\rho_1, \rho_1) = -1$. Then $\nabla_{X_1} \rho_1 = 0$ infers that

$$R(X_1, Y_1)\rho_1 = 0 \tag{19}$$

and

$$\mathcal{S}(X_1, \rho_1) = 0. \tag{20}$$

Also, using $\nabla_{X_1} \rho_1 = 0$, we get

$$(\nabla_{X_1} \omega)Y_1 = 0. \tag{21}$$

Using (21) in (18) infers $\alpha^*(X_1, Y_1) = -\omega(X_1)\omega(Y_1)$. Then using the above equation in (17) we obtain,

$$\tilde{R}(X_1, Y_1)Z_1 = R(X_1, Y_1)Z_1 + \omega(Z_1)[\omega(Y_1)X_1 - \omega(X_1)Y_1]. \tag{22}$$

From the foregoing equation, we have

$$\tilde{\mathcal{S}}(X_1, Y_1) = \mathcal{S}(X_1, Y_1) + 3\omega(X_1)\omega(Y_1), \tag{23}$$

where $\tilde{S}$ and S denote the Ricci tensor of $\tilde{\nabla}$ and ∇ respectively.

Contracting the above equation, we reveal

$$\tilde{r} = r - 3, \tag{24}$$

$\tilde{r}$ and r are the scalar curvature of $\tilde{\nabla}$ and ∇ respectively.

Using (19), we have from (22)

$$\tilde{R}(X_1, Y_1)\rho_1 = \omega(X_1)Y_1 - \omega(Y_1)X_1. \tag{25}$$

So, we get relations

$$\omega(\tilde{R}(X_1, Y_1)Z_1) = 0, \tag{26}$$

and

$$\tilde{\mathcal{S}}(X_1, \rho_1) = -3\omega(X_1), \quad \tilde{Q}\rho_1 = 3\rho_1. \tag{27}$$

where $\tilde{Q}$ is the Ricci operator with respect to the semi-symmetric non-metric connection $\tilde{\nabla}$ defined by $g(\tilde{Q}X_1, Y_1) = \tilde{S}(X_1, Y_1)$.

Existence of a unit timelike parallel vector field on a 4-dimensional Lorentzian manifold endowed with a *SSNMC* has been proved [7].

3. Gradient Ricci solitons

The soliton equation (4) can be written as

$$\tilde{\nabla}_{X_1} Df = -\tilde{Q}X_1 - \lambda X_1, \tag{28}$$

for all $X_1 \in \mathfrak{X}(M^4)$.Then using the above equation ,the subsequent relation

$$\tilde{R}(X_1, Y_1)Df = \tilde{\nabla}_{X_1}\tilde{\nabla}_{Y_1}Df - \tilde{\nabla}_{Y_1}\tilde{\nabla}_{X_1}Df - \tilde{\nabla}_{[X_1,Y_1]}Df \tag{29}$$

yields

$$\tilde{R}(X_1, Y_1)Df = -(\tilde{\nabla}_{X_1}\tilde{Q})(Y_1) + (\tilde{\nabla}_{Y_1}\tilde{Q})(X_1). \tag{30}$$

Contracting the above equation, we have

$$\tilde{\mathcal{S}}(Y_1, Df) = \frac{1}{2}(Y_1\tilde{r}). \tag{31}$$

With the help of equations (23) and (24), equation (31) takes the shape

$$\mathcal{S}(Y_1, Df) = \frac{1}{2}(Y_1 r) - 3\omega(Y_1)(\rho_1 f). \tag{32}$$

Again, from the equation (12) we have

$$\mathcal{S}(Y_1, Df) = a(Y_1 f) + b\omega(Y_1)(\rho_1 f). \tag{33}$$

Combining the equations (32) and (33) we get

$$\frac{1}{2}(Y_1 r) - 3\omega(Y_1)(\rho_1 f) = a(Y_1 f) + b\omega(Y_1)(\rho_1 f). \tag{34}$$

Setting $Y_1 = \rho_1$ in (34) and using $\omega(\rho_1) = -1$, we find

$$\rho_1 r = 2(a - b - 3)(\rho_1 f). \tag{35}$$

From (30) it follows

$$g(\tilde{R}(X_1, Y_1)\rho_1, Df) = 0. \tag{36}$$

Again , equation (25) infers

$$g(\tilde{R}(X_1, Y_1)\rho_1, Df) = \omega(X_1)(Y_1 f) - \omega(Y_1)(X_1 f). \tag{37}$$

Comparing the equations (36) and (37) we get

$$(Y_1 f)\omega(X_1) - (X_1 f)\omega(Y_1) = 0. \tag{38}$$

Setting $Y_1 = \rho_1$ and using $\omega(\rho_1) = -1$ in (38), we obtain

$$X_1 f + (\rho_1 f)\omega(X_1) = 0. \tag{39}$$

Now we suppose that the scalar curvature remains invariant under the velocity vector field, that is, $(\rho_1 r) = 0$. Then (35) entails that either $(\rho_1 f) = 0$ or $(\rho_1 f) \neq 0$.

Case I: $(\rho_1 f) \neq 0$ implies $(a - b - 3) = 0$ follows from (35). Therefore from equation (13), we obtain $3p + \sigma = constant$ which represents the equation of the state $p = p(\sigma)$ of the *PFS* and the *PFS* is isentropic.

Case II: $(\rho_1 f) = 0$. Then equation (39) gives $f = constant$ and hence equation (28) gives

$$\tilde{S}(X_1, Y_1) = (-\lambda)g(X_1, Y_1). \tag{40}$$

Now using (23) in the above equation we infer

$$S(X_1, Y_1) = (-\lambda)g(X_1, Y_1) - 3\omega(X_1)\omega(Y_1), \tag{41}$$

which implies that the *PFS* becomes a quasi-Einstein spacetime.

Hence, we can state the result as:

Theorem 3.1. *If a PFS endowed with a type of SSNMC admitting a gradient Ricci soliton whose scalar curvature remains invariant under the velocity vector field ,then either*
(i) The perfect fluid spacetime is isentropic , or
(ii)The perfect fluid spacetime reduces to a quasi-Einstein spacetime..

If we assume $div\, C = 0$ in a 4-dimensional *PFS* , then in view of the theorem **A**, we can state the following:

Corollary 3.2. *A PFS endowed with a type of SSNMC admitting a gradient Ricci soliton whose scalar curvature remains invariant under the velocity vector field and satisfying divergence free conformal curvature tensor is conformally flat and the metric is RW, the flow is irrotational, shear-free and geodesic, provided $f \neq constant$.*

4. Gradient η−Ricci solitons

The soliton equation (6) can be written as

$$\tilde{\nabla}_{X_1} Df = -\tilde{Q}X_1 - \lambda X_1 - \mu\eta(X_1)\rho_1, \tag{42}$$

for all $X_1 \in \mathfrak{X}(M^4)$. Using (42) and the definition

$$\tilde{R}(X_1, Y_1)Df = \tilde{\nabla}_{X_1}\tilde{\nabla}_{Y_1} Df - \tilde{\nabla}_{Y_1}\tilde{\nabla}_{X_1} Df - \tilde{\nabla}_{[X_1,Y_1]} Df \tag{43}$$

we reveal

$$\tilde{R}(X_1, Y_1)Df = -(\tilde{\nabla}_{X_1}\tilde{Q})(Y_1) + (\tilde{\nabla}_{Y_1}\tilde{Q})(X_1) - \mu[\eta(Y_1)X_1 - \eta(X_1)Y_1]. \tag{44}$$

Contracting the above equation, we have

$$\tilde{S}(Y_1, Df) = \frac{1}{2}(Y_1\tilde{r}) - 3\mu\eta(Y_1). \tag{45}$$

In the light of equations (23) and (24), equation (45) becomes

$$S(Y_1, Df) = \frac{1}{2}(Y_1 r) - 3\mu\eta(Y_1) - 3\omega(Y_1)(\rho_1 f). \tag{46}$$

Again, from the equation (12) we have

$$S(Y_1, Df) = a(Y_1 f) + b\omega(Y_1)(\rho_1 f). \tag{47}$$

Combining the equations (46) and (47) we get

$$\frac{1}{2}(Y_1 r) - 3\mu\eta(Y_1) - 3\omega(Y_1)(\rho_1 f) = a(Y_1 f) + b\omega(Y_1)(\rho_1 f). \tag{48}$$

Setting $Y_1 = \rho_1$ in (48) and using $\omega(\rho_1) = -1$, we find

$$\rho_1 r = 2(a - b - 3 + 3\mu)(\rho_1 f). \tag{49}$$

From (44)

$$g(\tilde{R}(X_1, Y_1)\rho_1, Df) = \mu[(Y_1 f)\omega(X_1) - (X_1 f)\omega(Y_1)]. \tag{50}$$

Again , from equation (25), we obtain

$$g(\tilde{R}(X_1, Y_1)\rho_1, Df) = \omega(X_1)(Y_1 f) - \omega(Y_1)(X_1 f). \tag{51}$$

Comparing the equations (50) and (51) we get

$$\mu[(Y_1 f)\omega(X_1) - (X_1 f)\omega(Y_1)] = \omega(X_1)(Y_1 f) - \omega(Y_1)(X_1 f). \tag{52}$$

Setting $Y_1 = \rho_1$ and using $\omega(\rho_1) = -1$, in (52) we find

$$(\mu - 1)(X_1 f + (\rho_1 f)\omega(X_1)) = 0. \tag{53}$$

For $\mu \neq 1$, equation (53 gives $Df = -(\rho_1 f)\rho_1$.
Thus we can state the following:

Theorem 4.1. *If the Lorentzian metric of a perfect fluid spacetime equipped with a SSNMC be a gradient η-Ricci soliton satisfying $\mu \neq 1$, then the potential function of the gradient η-Ricci soliton is pointwise collinear with the velocity vector field of the perfect fluid spacetime.*

Suppose potential function remains invariant under the velocity vector ρ_1, then from equation (53) we get $f = constant$. Then (49) infers $\rho_1 r = 0$ and hence equation (48) gives $\mu = 0$.
Considering $f = constant$ and $\mu = 0$ we obtain from equation (42)

$$\tilde{S}(X_1, Y_1) = (-\lambda)g(X_1, Y_1). \tag{54}$$

Now using (23) in the above equation we get

$$S(X_1, Y_1) = (-\lambda)g(X_1, Y_1) - 3\omega(X_1)\omega(Y_1), \tag{55}$$

which infers that the *PFS* becomes a quasi-Einstein spacetime.

Corollary 4.2. *A PFS endowed with a type of SSNMC admitting a gradient η-Ricci solitons whose potential function remains invariant under the velocity vector field represents a quasi-Einstein spacetime with $\mu \neq 1$.*

5. Gradient Yamabe solitons

Let M^4 admit a gradient Yamabe soliton, then (9) implies

$$\tilde{\nabla}_{Y_1} Df = (\tilde{r} - \lambda)Y_1. \tag{56}$$

Differentiating (56) covariantly along the vector field X_1, we obtain

$$\tilde{\nabla}_{X_1}\tilde{\nabla}_{Y_1} Df = (X_1\tilde{r})Y_1 + (\tilde{r} - \lambda)\tilde{\nabla}_{X_1} Y_1. \tag{57}$$

Interchanging X_1 and Y_1 in the above equation and then utilizing the preceding equation in $\tilde{R}(X_1, Y_1)Df = \tilde{\nabla}_{X_1}\tilde{\nabla}_{Y_1} Df - \tilde{\nabla}_{Y_1}\tilde{\nabla}_{X_1} Df - \tilde{\nabla}_{[X_1,Y_1]} Df$, we lead

$$\tilde{R}(X_1, Y_1)Df = (X_1\tilde{r})Y_1 - (Y_1\tilde{r})X_1. \tag{58}$$

Contracting the previous equation over X_1, we get

$$\tilde{S}(Y_1, Df) = -3(Y_1\tilde{r}). \tag{59}$$

In the light of equations (23) and (24), equation (59) takes the form

$$S(Y_1, Df) = -3(Y_1 r) - 3\omega(Y_1)(\rho_1 f). \tag{60}$$

Again, from the equation (12) we have

$$S(Y_1, Df) = a(Y_1 f) + b\omega(Y_1)(\rho_1 f). \tag{61}$$

Comparing the equations (60) and (61) we get

$$-3(Y_1 r) - 3\omega(Y_1)(\rho_1 f) = a(Y_1 f) + b\omega(Y_1)(\rho_1 f). \tag{62}$$

Now putting $Y_1 = \rho_1$ in (62) and using $\omega(\rho_1) = -1$, we find

$$\rho_1 r = \frac{(3 - a + b)}{3}(\rho_1 f). \tag{63}$$

From (58)

$$g(\tilde{R}(X_1, Y_1)\rho_1, Df) = \omega(X_1)(Y_1\tilde{r}) - \omega(Y_1)(X_1\tilde{r}). \tag{64}$$

Again, from equation (25), we find

$$g(\tilde{R}(X_1, Y_1)\rho_1, Df) = \omega(X_1)(Y_1 f) - \omega(Y_1)(X_1 f). \tag{65}$$

Combining the equations (64) and (65) we obtain

$$\omega(X_1)(Y_1 f) - \omega(Y_1)(X_1 f) = \omega(X_1)(Y_1\tilde{r}) - \omega(Y_1)(X_1\tilde{r}). \tag{66}$$

Using equation (24) in (66) we get

$$\omega(X_1)(Y_1 f) - \omega(Y_1)(X_1 f) = \omega(X_1)(Y_1 r) - \omega(Y_1)(X_1 r). \tag{67}$$

Setting $Y = \rho_1$ and using $\omega(\rho_1) = -1$, we infer

$$\omega(X_1)(\rho_1 r) + (X_1 r) = \omega(X_1)(\rho_1 f) + (X_1 f). \tag{68}$$

Using equation (63) in the above equation we get

$$(X_1 r) = \frac{(a - b)}{3}\omega(X_1)(\rho_1 f) + (X_1 f). \tag{69}$$

Using equation (69) in equation (62) we find

$$(a+3)(X_1 f + (\rho_1 f)\omega(X_1)) = 0, \tag{70}$$

which shows that either $X_1 f + (\rho_1 f)\omega(X_1) = 0$ or, $X_1 f + (\rho_1 f)\omega(X_1) \neq 0$.
Case I:
$X_1 f + (\rho_1 f)\omega(X_1) = 0$ implies $Df = -(\rho_1 f)\rho_1$.
Case II:
$X_1 f + (\rho_1 f)\omega(X_1) \neq 0$ implies $(a+3) = 0$ follows from equation (70).Therefore from equation (13), we can conclude that $p = \sigma + constant$. This represents the equation of state of the *PFS*.

Hence, we state the result as:

Theorem 5.1. *If the Lorentzian metric of a perfect fluid spacetime equipped with a SSNMC be a gradient Yamabe soliton, then either*

(i)The gradient of Yamabe soliton potential function is pointwise collinear with the velocity vector field of the perfect fluid spacetime, or

(ii) The state equation is $p = \sigma + constant$.

Next, we consider $(a+3) \neq 0$. The covariant derivative of $Df = -(\rho_1 f)\rho_1$ yields

$$\tilde{\nabla}_{X_1} Df = -(\rho_1 f)\tilde{\nabla}_{X_1}\rho_1 - (X_1(\rho_1 f))\rho_1.$$

Using equation (56) we get

$$(\tilde{r} - \lambda)X_1 = -(\rho_1 f)\tilde{\nabla}_{X_1}\rho_1 - (X_1(\rho_1 f))\rho_1.$$

If f is invariant under velocity vector field ρ_1 that is $\rho_1 f = 0$, then we can find from the foregoing equation $\tilde{r} = \lambda$, a constant. Hence from (24), we infer that $r = constant$ i.e M^4 is of constant scalar curvature.

Again $\rho_1 f = 0$ implies $Df = 0$, gives that f is constant and hence the gradient Yamabe soliton becomes trivial.

Corollary 5.2. *Let a PFS endowed with a SSNMC admits a gradient Yamabe soliton with $a \neq -3$. If f is invariant under the velocity vector field , then*

(i) The scalar curvature is constant in M^4.

(ii)The soliton becomes trivial.

6. Conclusion

The semi-symmetric connections have applications in sub-manifold theory, *HSU*-unified structure manifolds, warped product manifolds, statistical manifolds and theoretical physies. In [25] the authors have provided the application of *SSNMC* in a nice way.

Solitons are nothing but the waves which is physically propagate with some loss of energy and hold their speed and shape after colliding with one more such wave. In non-linear partial differential equations describing wave propagation, solitons play an important role in the treatment of initial-value problems.

In the present paper we characterize perfect fluid spacetime endowed with a *SSNMC* admitting gradient Ricci solitons, gradient η-Ricci solitons and gradient Yamabe solitons. Among others we establish that a *PFS* endowed with a type of *SSNMC* admitting a gradient Ricci soliton whose scalar curvature remains invariant under the velocity vector field and satisfying divergence free conformal curvature tensor is conformally flat and the metric is *RW*, the flow is irrotational, shear-free and geodesic, provided $f \neq constant$. Moreover, we prove that a PFS endowed with a type of *SSNMC* admitting a gradient η-Ricci solitons whose potential function remains invariant under the velocity vector field represents a quasi-Einstein spacetime with $\mu \neq 1$. Finally, it is shown that a PFS endowed with a *SSNMC* admits a gradient Yamabe soliton with $a \neq -3$ and if f is invariant under the velocity vector field, then the scalar curvature is constant in M^4 and the soliton becomes trivial.

In near future we may consider the same problem for the generalized Robertson-Walker spacetimes.

References

[1] S. Agashe and M. R. Chafle , *A semi-symmetric non-metric connection on a Riemannian manifold*, Indian J. Pure Appl. Math. **23** (1992), 399–409.
[2] L. Alias, A. Romero and M. Sanchez, *Compact spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetimes* , in Geometry and Topology of Submanifolds VII, River Edge NJ, USA, World Scientific, **67**, 1995.
[3] L. Alias, A. Romero and M. Sanchez, *Uniqueness of complete spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetimes*, Gen. Relat. Gravit. **27** (1995), 71.
[4] A. M. Blaga, *Solitons and geometrical structures in a perfect fluid spacetime*, Rocky Mountain J. Math. **50** (2020), 41–53.
[5] B. Y. Chen, *A simple characterization of generalized Robertson-Walker spacetimes*, Gen. Relativ. Gravit. **46**(2014), 1833.
[6] B. Y. Chen, *Pseudo-Riemannian Geometry, δ-invariants and Applications*, World Scientific, Hackensack , **2011**.
[7] S. K. Chaubey , U. C. De and M. D. Siddiqi, *Characterization Lorentzian manifolds with a semi-symmetric linear connection.*, J. Geom. Phys. **166** (2021) 104269.
[8] J. T. Cho and M. Kimura, *Ricci Solutions and real hypersurfaces in a complex space form*, Tohoku Math. J. **61**, no. 2(2009), 205-212.
[9] P. H. Chavanis, *Cosmology with a stiff matter era*, Phys. Rev. D, **92** (2015), 103004.
[10] A. De, *On almost pseudo symmetric manifolds admitting a type of semi-symmetric non-metric connection*, Acta Math. Hungar, **125** (1-2) (2009), 183–190.
[11] K. De and U. C. De , *Investigation on gradient solitons in perfect fluid spacetimes* , Reports Math. Phys. **91** (2023), 277-289.
[12] K. De, U. C. De, A. A. Syied, N. B. Turki and S. Alsaeed , *Perfect Fluid Spacetimes and Gradient Solitons*, Journal of Nonlinear Mathematical Physics, https://doi.org/10.1007/s44198-022-00066-5.
[13] U. C. De and A. Sardar, *Static perfect fluid spacetimes on GRW spacetimes*, Anal. Math. Phys. **13**, 44(2023).
[14] U. C. De and S. Shenawy, *Generalized quasi-Einstein GRW space-times*, Int. J. Geom. Methods Mod. Phys. **16**(2019), 1950124.
[15] U. C. De , S. Shenawy and S. K. Chaubey, *Perfect fluid space-times and Yamabe solitons* , J. Math. Phys. **62**(2021), 032501.
[16] R. Deszcz and L. Verstraelen, *Hypersurfaces of semi-Riemannian conformally flat manifolds*, Geometry and Topology of Submanifolds, III, World Sci. Publishing, River Edge, NJ. **1991**, 131-147.
[17] Y. Dogru, C. Özgür and C. Murathan, *Riemannian manifolds with a semi-symmetric nonmetric connection satisfying some semisymmetry conditions*, Bull. Math. Anal. Appl. **3** (2)(2011), 206–212.
[18] Y. Dogru, *η-Bourguignon solitons with a semi-symmetric metric and semi-symmetric metric non-metric connection* , AIMS Math. **8**(2023), no. 5, 11943-11952.
[19] A. Friedmann and J. A. Schouten, *Uber die geometric der holbsymmetrischen Ubertragurgen*, Math. Zeitschr. **21** (1924), 211–233.
[20] R. S. Hamilton, *The Ricci flow on surfaces*, Contemp. Math. **71** (1988), 237–261.
[21] R. S. Hamilton, *Three-manifolds with positive Ricci curvature* , J. Differ. Geom. **17** (1982), 255-306.
[22] S. W. Hawking and G. F. R. Ellis, *The large scale structure of spacetime*, Cambridge University Press, Cambridge, 1973.
[23] H. A. Hayden, *Subspace of space with torsion*, Proc. Lon. Math. Soc., **34** (1932), 27–50.
[24] A. Mihai, *A note on derived connections from semi-symmetric metric connections* , Math. Slovaca, **67** (2017), 221-226.
[25] A. Mihai and I. Mihai, *A note on well defined sectional curvature of a semi-symmetric non-metric connection* , Int. Electronic J. Geometry, **17** (2024), 15-23.
[26] C. A. Mantica and L. G. Molinari, *Generalized Robertson-Walker spacetimes-A survey*, Int. J. Geom. Methods Mod. Phys. **14** (2017), 1730001.
[27] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.
[28] C. Özgür and S. Sular, *Warped products with a semi-symmetric non-metric connection*, Arab J. Math. Sci. **36** (2011), 461–473.
[29] J. Sengupta, U. C. De and T. Q. Binh, *On a type of semi-symmetric non-metric connection on a Riemannian manifold*, Indian J. Pure Appl. Math. **31** (12) (2000), 1659–1670.
[30] R. Sharma and A.Ghosh, *Perfect fluid space-times whose energy-momentum tensor is conformal Killing*, J. Math. Phys. **51** (2010), 022504.
[31] L.C. Shepley and A.H. Taub, *Space-times containing perfect fluids and having a vanishing conformal divergence* , Commun. Math. Phys. **5** (1967), 237-256.
[32] S. Sundriyal and J.Upreti, *On a type of semi-symmetric non-metric connection in HSU-unified structure manifold* , Int. Electronic J. Geometry, **14** (2021), 383-390.
[33] Y. Wang, *Affine connections of non-integrable distributions* , Int. J. Geom. Methods Mod. Phys. **17**(2020), 2050127.
[34] K.Yano, *On semi-symmetric metric connection*, Revue Roumaine de Math. Pures et Appliques. **15** (1970), 1579–1586.
[35] M.Yildirim, *semi-symmetric non-metric connection on Statistical manifolds*, J. Geo. and Phys. **176** (2022), 104505.
[36] A.Yucesan and E. Yasar, *Non-degenerate hyperspaces of a semi-Riemannian manifold with a semi-symmetric non-metric connection*, Math. Reports, **14** (2012), 209-219.