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Global existence, extinction, and non-extinction of solutions to a fast diffusion p-Laplace evolution equation with logarithmic nonlinearity

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Abstract. In this paper, we study a class of fast diffusion p-Laplace equation with logarithmic nonlinearity. Under appropriate conditions, by applying energy estimates in combination with the Galerkin method and Sobolev inequality, we establish the global existence of solutions. Moreover, we obtain the extinction and non-extinction properties of these solutions.

1. Introduction

The study of such systems is of significant interest due to their relevance in various physical applications. When a compressible fluid passes through a porous medium, its motion is influenced by the interaction between the fluid and the structure of the medium. These dynamics are captured by a system of partial differential equations that describe the conservation of mass, momentum, and energy. In the case of a compressible fluid, the macroscopic velocity $\vec{V}$, the volumetric moisture content $\theta(x)$ and the density of the fluid ρ are related through the following conservation equation

$$\theta(x)\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{V}) - f(\rho) = 0 \tag{1}$$

where $f(u)$ is a source term that accounts for the generation or depletion of fluid in the medium (see [28, 29]). The velocity of the fluid flow is given by Darcy's law generalized for nonlinear diffusion:

$$\rho\vec{V} = -\lambda|\nabla\rho|^{\alpha-2}\nabla\rho, \tag{2}$$

the quantity α is a characteristic of the medium. The media with $\alpha > 2$ are called dilatant fluids, and those with $1 < \alpha < 2$ are called pseudo-plastics; if $\alpha = 2$, they are called Newtonian fluid.
By substituting the velocity relation in (2) into the conservation equation in (1), we obtain the following diffusion equation for the fluid density ρ

$$\theta(x)\frac{\partial \rho}{\partial t} = \lambda\operatorname{div}(|\nabla\rho|^{\alpha-2}\nabla\rho) + f(\rho). \tag{3}$$

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In this paper, we consider (3) with $\theta(x) = |x|^{-r}$, $r = 0$, $\lambda = 1$ and $f(\rho) = |\rho|^{p-2}\rho\log(|\rho|)$. Then, we focus on the study of the following initial boundary value problem involving logarithmic nonlinearity:

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_p u = |u|^{q-2}u\log(|u|) & \text{in } \Omega\times\mathbb{R}^+, \\ u = 0 & \text{in } \partial\Omega\times\mathbb{R}^+, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{4}$$

where Ω is a bounded open domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and $1 < p, q < 2$.

We first give the definition of weak solutions for problem (4).

Definition 1.1. *Fix $T > 0$ and assume that $u_0 \in W_0$. A function $u \in L^\infty(0,T;W_0)$ is called a weak solution of (4), if $\frac{\partial u}{\partial t} \in L^2(0,T,L^2(\Omega))$ and*

$$\int_\Omega \frac{\partial u}{\partial t}\phi dxdt + \int_\Omega |\nabla u|^{p-2}\nabla u\nabla\phi dx = \int_\Omega |u|^{q-2}u\log(|u|)\phi dx, \tag{5}$$

holds fo a.e $t \in (0,T)$ and for all $\phi \in W_0$.

Problem (4) belongs to the class of quasilinear diffusion problems which have garnered increasing interest in recent years due to their applications in various scientific fields. Moreover, as already mentioned in [10]. In the diffusion theory, the function $u(x,t)$ represents the population density at time t and spatial position x, $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ accounts for the diffusion of the population density and $|u|^{q-2}u\log(|u|)$ is the source.

There are also several studies concerning global solutions for problems similar to (4). In [12], Han et al. studied the existence of global solution and extinction and Non-extinction of solutions for (4) with a nonlocal source and an absorption term by using the energy estimates. We refer also to [1–3, 8, 9, 11, 14, 15, 17, 18, 26, 30] for further studies in this case. When $f(u) = \lambda u^r - \beta u^q$ in (4) where λ, β, r are positive constants, the authors in [22] investigate the extinction properties and decay estimates of solutions, we refer also to [16] and other references therein.

The research with logarithmic nonlinearity can be found in many physical applications, including the theory of superfluidity, diffusion and transport phenomena, and nuclear physics. We refer the readers to [34] for more information. Within the framework of partial differential equations (see [21, 23, 27]), especially in the classical case, the authors [6] studied the existence of global solution to the following semilinear pseudo-parabolic problem

$$u_t - \Delta u_t - \Delta u = u\log|u|, \tag{6}$$

with Dirichlet boundary condition by applying the logarithmic Sobolev inequality. The authors in [19] studied the global existence or nonexistence of the problem (4) by applying the potential well method and a logarithmic Sobolev inequality. We refer the reader to see [7, 13, 20, 31] where the theory of logarithmic nonlinearity find its applications for the same evolution equations. In [24, 25, 32, 33], the authors have studied the existence results to p-Kirchhoff problems with logarithmic nonlinearity.

In the present work, we aim to study the combined effects of the p-Laplacian and logarithmic nonlinearity to discuss the global existence and extinction properties of solutions. Our approach relies on energy estimates and Sobolev inequalities to establish the global existence of weak solutions for the problem (4). To the best of our knowledge, this is the first study to explore both the global existence and extinction properties of solutions for evolution equations involving the p-Laplacian and logarithmic nonlinearity.

Next, we will introduce the energy functional $E : W_0^{1,p}(\Omega) \to \mathbb{R}$ associated with problem (4) defined by

$$E(u) = \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)} - \frac{1}{q}\int_\Omega |u|^q\log(|u|)dx + \frac{1}{q^2}\int_\Omega |u|^q dx \tag{7}$$

For simplicity reasons, throughout this paper, we adopt the following abbreviations:

$$\|u\|_p = \|u\|_{L^p(\Omega)}, \quad \|u\|_2 = \|u\|_{L^2(\Omega)}.$$

2. Preliminaries

In this section, we give some lemmas and definitions which will be needed in our proofs of the main results.
In the first instance, for $1 \leq p < \infty$, we introduce the Hilbert space

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{\partial u}{\partial x_i} \in L^p(\Omega), \quad i = 1, 2, \ldots, n \right\}$$

endowed with the norm

$$\|u\|^p_{W^{1,p}(\Omega)} = \|u\|^p_p + \|\nabla u\|^p_p.$$

Denote

$$W_0 := W^{1,p}_0(\Omega) = \left\{ u \in W^{1,p}(\Omega) : \, u|_{\partial\Omega} = 0 \right\}.$$

Due to Poincaré's inequality, one can know that $\|\nabla u\|_p$ is an equivalent norm to $\|u\|_{W^{1,p}(\Omega)}$ in W_0.
Secondly, we denote the maximal existence time of a solution $u = u(t)$ to problem (4) by T_m, which is defined as follows:

Definition 2.1. *(1) If there exists a $\hat{t} \in (0, +\infty)$ such that u exists for $0 \leq t < \hat{t}$, but u blows up in W_0 as $t \to \hat{t}^-$ i.e*

$$\lim_{t \to \hat{t}^-} \|u(t)\|_{W_0} = +\infty, \tag{8}$$

then $T_m = \hat{t}$.

(2) If (8) *does not happen at any finite time, then $T_m = +\infty$ and we say u exists globally.*

Definition 2.2. *Let u be a solution to the problem* (4)*. We say $u = u(t)$ vanishes in finite time if there exists a $T^* > 0$ such that*

$$u(x,t) \equiv 0 \quad on\ \Omega, \quad t \geq T^*. \tag{9}$$

Lemma 2.3. *[5] Let $1 \leq p \leq \infty$, we have*

$$\|u\|_{L^{\frac{np}{n-p}}} \leq C_p \|u\|_{W_0} \quad \forall u \in W_0. \tag{10}$$

Lemma 2.4. *[16] Suppose that l, m and s are the positive constants, and $\varphi(t)$ is absolutely continuous and nonnegative function such that $\varphi'(t) + m\varphi^l(t) \geq s$, $t > 0$. Then*

$$\varphi(t) \geq \min \left\{ \varphi(0), \left(\frac{s}{m} \right)^{\frac{1}{l}} \right\}.$$

Lemma 2.5. *[22] If $0 < r < s \leq 1$ and $h(t)$ solve*

$$\begin{cases} \frac{dh}{dt} + \gamma_1 h^r \leq \gamma_2 h^s, & t > 0 \\ h(0) = h_0 > 0 \end{cases} \tag{11}$$

with $\gamma_1 > 0, 0 < \gamma_2 < \frac{1}{2}\gamma_1 h_0^{r-s}$. Then, there exists $q_1, q_2 > 0$ such that

$$0 \leq h(t) \leq q_2 e^{-q_1 t} \text{ for all } t \geq 0.$$

Lemma 2.6. *[4] Let σ be a positive constant. Then for all $r \geqslant 1$, we have:*

$$|\log(r)| \leq \frac{1}{\sigma} r^\sigma,$$

for all $r \in [1, +\infty)$.

3. Main results

We are now in a position to present the main results given by the following two theorems.

Theorem 3.1. *Let $u_0 \in W_0$. The problem* (4) *admits a global weak solution $u = u(t)$ if there exists a constant $\sigma > 0$ such that* $1 < q + \sigma \leq 2$.

Proof. Assume that u is a weak solution of (4). As in Theorem 3.2 of [19], the local existence of weak solutions to problem (4) is established using the Galerkin method. Moreover, we establish the global existence of the weak solution. For this reason, let assume that the weak solution of (4) blows up in finite time as in definition 2.1.
We let $\phi = u$ in (5), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 + \|u(t)\|_{W_0}^p = \int_\Omega |u(t)|^q \log|u(t)|dx. \tag{12}$$

According to Lemma 2.6, we have

$$\int_\Omega |u(t)|^q \log|u(t)|dx \leq \frac{1}{\sigma}\int_\Omega |u(t)|^{q+\sigma}dx = \frac{1}{\sigma}\|u(t)\|_{q+\sigma}^{q+\sigma}. \tag{13}$$

Combining (12), (13) and the Hölder inequality, we obtain

$$\|u(t)\|_2 \frac{d}{dt}\|u(t)\|_2 + \|u(t)\|_{W_0}^p \leq \frac{1}{\sigma}\|u(t)\|_{q+\sigma}^{q+\sigma} \leq \frac{|\Omega|^{\frac{2-(q+\sigma)}{2}}}{\sigma}\|u(t)\|_2^{q+\sigma}.$$

Therefore

$$\frac{d}{dt}\|u(t)\|_2 \leq \frac{|\Omega|^{\frac{2-(q+\sigma)}{2}}}{\sigma}\|u(t)\|_2^{q+\sigma-1}. \tag{14}$$

Next, by choosing $\phi = \frac{du}{dt}$ in (5) and by using the Hölder inequality and the Young inequality, we get

$$\begin{aligned}\left\|\frac{du(t)}{dt}\right\|_2^2 + \frac{1}{p}\frac{d}{dt}\|u(t)\|_{W_0}^p &\leq \int_\Omega |u(t)|^{q-2}u(t)\log|u(t)|\frac{du(t)}{dt}dx \\ &\leq \int_\Omega \frac{1}{\sigma}|u(t)|^{q+\sigma-1}\left|\frac{du(t)}{dt}\right|dx \\ &\leq \frac{1}{2\sigma}\int_\Omega |u(t)|^{2(q+\sigma-1)}dx + \frac{1}{2}\int_\Omega \left|\frac{du(t)}{dt}\right|^2 dx \\ &\leq \frac{|\Omega|^{1-\frac{2(q+\sigma-1)}{p}}}{2\sigma}\|u(t)\|_p^{2(q+\sigma-1)} + \frac{1}{2}\left\|\frac{du(t)}{dt}\right\|_2^2.\end{aligned} \tag{15}$$

Case 1: If $1 < q + \sigma < 2$, after a simple calculation on (14), we obtain

$$\|u(t)\|_2 \leq \left(\frac{(2-q-\sigma)\,|\Omega|^{1-\frac{q+\sigma}{2}}}{\sigma}t + \|u_0\|_2^{2-q-\sigma}\right)^{\frac{1}{2-q-\sigma}} \quad \forall t \in (0, T) \tag{16}$$

for sufficiently large T.
Since $p < 2$, combining (16) with Hölder inequality, we conclude that

$$\|u(t)\|_p \leq |\Omega|^{\frac{2-p}{2p}}\|u(t)\|_2 \leq \left(\frac{(2-q-\sigma)\,|\Omega|^{\frac{2-(q+\sigma)}{2}}}{\sigma}t + \|u_0\|_2^{2-q-\sigma}\right)^{\frac{1}{2-q-\sigma}}|\Omega|^{\frac{2-p}{2p}} \tag{17}$$

Substituting the inequality (17) into (15), we get

$$\frac{d}{dt}\|u\|_{W_0}^p \leq \left(\frac{(2-q-\sigma)\,|\Omega|^{\frac{2-(q+\sigma)}{2}}}{\sigma} t + \|u_0\|_2^{2-q-\sigma} \right)^{\frac{2(q+\sigma-1)}{2-q-\sigma}} \frac{p|\Omega|^{2-q-\sigma}}{2\sigma}. \tag{18}$$

Since the right-hand sides of (17) and (18) is defined on $t \in [0, +\infty)$ and

$$\left(\frac{(2-q-\sigma)\,|\Omega|^{\frac{2-(q+\sigma)}{2}}}{\sigma} t + \|u_0\|_2^{2-q-\sigma} \right)^{\frac{1}{2-q-\sigma}} |\Omega|^{\frac{2-p}{2p}} \in [0, +\infty) \quad \forall t \geq 0,$$

$$\left(\frac{(2-q-\sigma)\,|\Omega|^{\frac{2-(q+\sigma)}{2}}}{\sigma} t + \|u_0\|_2^{2-q-\sigma} \right)^{\frac{2(q+\sigma-1)}{2-q-\sigma}} \frac{p|\Omega|^{2-q-\sigma}}{2\sigma} \in [0, +\infty) \quad \forall t \geq 0.$$

Moreover, for $\hat{t} \in (0, +\infty)$,

$$\lim_{t \to \hat{t}} \left(\frac{(2-q-\sigma)\,|\Omega|^{\frac{2-(q+\sigma)}{2}}}{\sigma} t + \|u_0\|_2^{2-q-\sigma} \right)^{\frac{1}{2-q-\sigma}} |\Omega|^{\frac{2-p}{2p}} < +\infty,$$

$$\lim_{t \to \hat{t}} \left(\frac{(2-q-\sigma)\,|\Omega|^{\frac{2-(q+\sigma)}{2}}}{\sigma} t + \|u_0\|_2^{2-q-\sigma} \right)^{\frac{2(q+\sigma-1)}{2-q-\sigma}} \frac{p|\Omega|^{2-q-\sigma}}{2\sigma} < +\infty,$$

this contradicts (8). As a result, the problem (4) has a weak solution u that exists globally.
Case 2: If $q + \sigma = 2$, according to inequality (14) that

$$\frac{d}{dt}\|u(t)\|_2 \leq \frac{1}{\sigma}\|u(t)\|_2,$$

we obtain

$$\|u(t)\|_2 \leq \|u_0\|_2\, e^{\frac{t}{\sigma}}. \tag{19}$$

According to the Hölder inequality combined with (19) with $p < 2$, we have

$$\|u(t)\|_p \leq |\Omega|^{\frac{2-p}{2p}} \|u(t)\|_2 \leq |\Omega|^{\frac{2-p}{2p}} \|u_0\|_2\, e^{\frac{t}{\sigma}}. \tag{20}$$

Substituting (20) into (15) with $q + \sigma = 2$, we obtain that

$$\frac{d}{dt}\|u\|_{W_0}^p \leq \frac{p|\Omega|^{1-\frac{2}{p}}}{2\sigma}\|u(t)\|_p^2 \leq \frac{p\,\|u_0\|_2^2}{2\sigma^2} e^{\frac{2t}{\sigma}}. \tag{21}$$

Since the right hand sides in (20) and (21) is defined on $t \in [0, +\infty)$ and

$$|\Omega|^{\frac{2-p}{2p}} \|u_0\|_2\, e^{\frac{t}{\sigma}} \in [0, +\infty) \quad t \geq 0,$$

$$\frac{p\,\|u_0\|_2^2}{2\sigma^2} e^{\frac{2t}{\sigma}} \in [0, +\infty) \quad t \geq 0.$$

Moreover, for $\hat{t} \in (0, +\infty)$,

$$\lim_{t \to \hat{t}} |\Omega|^{\frac{2-p}{2p}} \|u_0\|_2\, e^{\frac{t}{\sigma}} < +\infty,$$

$$\lim_{t \to \hat{t}} \frac{p\,\|u_0\|_2^2}{2\sigma^2} e^{\frac{2t}{\sigma}} < +\infty$$

this contradicts (8). Consequently, the global solution is obtained. □

Theorem 3.2. *Let $u_0 \in W_0 \cap L^2(\Omega)$. Assume that problem (4) admits a global weak solution. Then*

i) *if $\frac{2n}{n+2} < p < \min\{n, q+\sigma\}$ and*

$$\|u_0\|_2^{q+\sigma-p} < \frac{\sigma}{2C_p}|\Omega|^{\frac{n(q+\sigma-p)-2p}{2n}} \tag{22}$$

with some $0 < \sigma \leq 2-q$, then u vanishes in finite time.

ii) *Suppose that:*

$$\|u_0\|_2 \neq 0 \text{ and } E(u_0) \leq 0, \text{ when } q = p$$

or

$$\|u_0\|_2 \neq 0 \text{ and } E(u_0) < 0, \text{ when } q < p.$$

Then u cannot vanish in finite time.

Proof. Theorem 3.1 leads to the conclusion that the problem (4) has a global weak solution u. First, we prove conclusion i). From (5), we deduced that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 + \|u(t)\|_{W_0}^p \leq \int_\Omega |u(t)|^q \log|u(t)|\mathrm{d}x \tag{23}$$

Plugging (13) into (23) and using Lemma 2.3, we get

$$\|u(t)\|_2\frac{d}{dt}\|u(t)\|_2 + \frac{1}{C_p}\|u(t)\|^p_{L^{\frac{np}{n-p}}(\Omega)} \leq \frac{1}{\sigma}\|u(t)\|^{q+\sigma}_{L^{q+\sigma}(\Omega)}. \tag{24}$$

Using again the Hölder inequality to (24), it follows that

$$\frac{d}{dt}\|u(t)\|_2 + \frac{|\Omega|^{1-\frac{p(n+2)}{2n}}}{C_p}\|u(t)\|_2^{p-1} \leq \frac{|\Omega|^{\frac{2-(q+\sigma)}{2}}}{\sigma}\|u(t)\|_2^{q+\sigma-1}.$$

Let $z(t) = \|u(t)\|_2$, the inequality mentioned above can be reduced to

$$\frac{dz}{dt} + \frac{|\Omega|^{1-\frac{p(n+2)}{2n}}}{C_p}z^{p-1}(t) \leq \frac{|\Omega|^{\frac{2-(q+\sigma)}{2}}}{\sigma}z^{q+\sigma-1}(t). \tag{25}$$

Furthermore, from inequality (22) and $1 \leq \frac{2n}{n+2} < p < \min\{n, q+\sigma\}$, we ensures that $p-1 < q+\sigma-1$ and

$$0 < \frac{|\Omega|^{\frac{2-(q+\sigma)}{2}}}{\sigma} < \frac{|\Omega|^{\frac{2n-p(n+2)}{2n}}}{2C_p}\|u_0\|_2^{p-q-\sigma}.$$

Thus, it follows from Lemma 2.5 that constants $q_1 > 0$ and $q_2 > 0$ exist such that

$$0 \leq z(t) \leq q_2 e^{-q_1 t}, \quad t \geq 0. \tag{26}$$

We take the constant $T^* > 0$ such that

$$\left(q_2 e^{-q_1 t}\right)^{q+\sigma-p} \leq \frac{\sigma}{2C_p}|\Omega|^{\frac{n(q+\sigma-p)-2p}{2n}}, \quad t \geq T^*. \tag{27}$$

Combining inequalities (26) and (27), it is concluded that

$$z^{q+\sigma-1}(t) = \left(z^{q+\sigma-p}z^{p-1}\right)(t) \leq \frac{\sigma}{2C_p}|\Omega|^{\frac{n(q+\sigma-p)-2p}{2n}}z^{p-1}(t), \quad t \geq T^*. \tag{28}$$

Plugging (28) into (25), we get

$$\frac{dz}{dt}+\frac{|\Omega|^{1-\frac{p(n+2)}{2n}}}{2C_p}z^{p-1}(t)\leq 0,\quad t\geq T^*.$$

After a simple calculation, we obtain

$$\begin{cases} \|u(t)\|_2\leq\left(z^{2-p}\left(T^*\right)-\frac{(2-p)|\Omega|^{1-\frac{p(n+2)}{2n}}}{2C_p}\left(t-T^*\right)\right)^{\frac{1}{2-p}}, & T^*\leq t<T_v^* \\ \|u(t)\|_2\equiv 0 & t\geq T_v^* \end{cases}$$

where $T_v^*=\frac{2C_p}{(2-p)|\Omega|^{1-\frac{p(n+2)}{2n}}}z^{2-p}\left(T^*\right)+T^*$. Then, u vanishes in finite time.

Secondly, we prove conclusion ii). Choosing $\phi=\frac{du}{dt}$ in (5), we have

$$\int_\Omega\left(\frac{du(t)}{dt}\right)^2dx+\frac{1}{p}\frac{d}{dt}\|u(t)\|_{W_0}^p=\int_\Omega|u(t)|^{q-2}u(t)\log|u(t)|\frac{du(t)}{dt}dx. \tag{29}$$

We can see that (29) simplifies as follows:

$$\int_\Omega\left(\frac{du(t)}{dt}\right)^2dx+\frac{1}{p}\frac{d}{dt}\|u(t)\|_{W_0}^p=\frac{1}{q}\frac{d}{dt}\int_\Omega|u(t)|^q\log|u(t)|dx-\frac{1}{q^2}\frac{d}{dt}\int_\Omega|u(t)|^qdx. \tag{30}$$

Differentiating (7) with respect to t, we get

$$\frac{d}{dt}E(u(t))=\frac{1}{p}\frac{d}{dt}\|u(t)\|_{W_0}^p-\frac{1}{q}\frac{d}{dt}\int_\Omega|u|^q\log|u|\mathrm{d}x+\frac{1}{q^2}\frac{d}{dt}\int_\Omega|u|^qdx \tag{31}$$

According to (30) and (31), we can conclude that

$$\frac{d}{dt}E(u(t))=-\int_\Omega\left(\frac{du(t)}{dt}\right)^2dx\leq 0. \tag{32}$$

Thus, $E(u)$ is non-increasing with respect to t. Consequently, by integrating (32) from 0 to t, we have

$$E(u(t))=E\left(u_0\right)-\int_0^t\int_\Omega\left(\frac{du(s)}{ds}\right)^2dxds \tag{33}$$

Let $g(t)=\frac{1}{2}\|u(t)\|_2^2$. Then, by (7), (12) and (33), we can get

$$\begin{aligned} g'(t)&=-\|u(t)\|_{W_0}^p+\int_\Omega|u(t)|^q\log|u(t)|\mathrm{d}x\\ &=-pE(u(t))+\frac{q-p}{q}\int_\Omega|u(x,t)|^q\log|u(t)|dx+\frac{p}{q^2}\int_\Omega|u(t)|^qdx\\ &=-pE\left(u_0\right)+p\int_0^t\int_\Omega\left(\frac{d}{ds}u(t)\right)^2dxds+\frac{q-p}{q}\int_\Omega|u(t)|^q\log|u(t)|\mathrm{d}x\\ &+\frac{p}{q^2}\int_\Omega|u(t)|^qdx\\ &\geq-pE\left(u_0\right)+\frac{q-p}{q}\int_\Omega|u(t)|^q\log|u(t)|dx. \end{aligned} \tag{34}$$

Case 1: If $q=p$, hence, (34) can be reduced to $g'(t)\geq-pE\left(u_0\right)$, which gives

$$g(t)\geq g(0)-pE\left(u_0\right)>0,\quad\forall t>0$$

therefore, u cannot vanish in finite time.

Case 2: If $q < p$. Thanks to $1 < q < 2$, there exists a constant $\sigma > 0$ such that $q + \sigma < 2$. Consequently, inequality (34) can be simplified as follows

$$\begin{aligned} g'(t) &\geq -pE(u_0) + \frac{q-p}{q\sigma}\int_\Omega |u(t)|^{q+\sigma}dx \\ &\geq -pE(u_0) + \frac{|\Omega|^{\frac{2-q-\sigma}{2}}(q-p)}{q\sigma}\left(\int_\Omega |u(t)|^2\,\mathrm{d}x\right)^{\frac{q+\sigma}{2}} \\ &= -pE(u_0) - \frac{(p-q)(\sqrt{2})^{q+\sigma}|\Omega|^{\frac{2-q-\sigma}{2}}}{q\sigma} g^{\frac{q+\sigma}{2}}(t). \end{aligned}$$

Consequently, using Lemma 2.4 allow us to get

$$g(t) \geq \min\left\{ g(0), \frac{-pq\sigma E(u_0)}{(p-q)(\sqrt{2})^{q+\sigma}|\Omega|^{\frac{2-q-\sigma}{2}}} \right\} > 0, \quad \forall t > 0$$

As a results, u cannot vanish in finite time. This completes the proof of theorem 3.2. □

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