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New estimates for the B-Riesz transform

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Abstract. In this paper, we study the B-Riesz transformation $R_\gamma^{(k)}$ generated by a generalized translation operator $\mathbf{T}^y$ $(y \in \mathbb{R}_+^n)$. We prove that the Cotlar-type inequality holds for these operators. In particular, we prove results related to the Cotlar inequality for the even and odd cases of the kernel of the B-Riesz transform. Thus, new estimates for the B-Riesz transform in weighted Lebesgue spaces $L_{p,\gamma,w}$ are obtained.

1. Introduction

In this study, we establish new estimates for the higher-order Riesz-Bessel transform associated with the generalized translation operator $\mathbf{T}^y$ (where $y \in \mathbb{R}_+^n$). A detailed discussion of these concepts will be provided in Section 2. For the sake of simplicity, we will refer to the higher-order Riesz-Bessel transform as the "B-Riesz transform" throughout this text. Our focus is on a problem derived from the classical Cotlar's inequality:

$$T^*(f)(x) \le C(M(Tf)(x) + M(f)(x)), \tag{1}$$

where T denotes a Calderon-Zygmund singular operator which may not be of convolution type and M stands for the Hardy-Littlewood maximal function defined by

$$Mf(x) = \left(\sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy\right). \tag{2}$$

In (1) the classical Riesz transform (Riesz transform of order one) can replace T. The authors have presented this result in [23]. It is important to note that the classical Riesz transform is a key operator of Calderon-Zygmund type. In their paper, the Riesz transform is denoted by

$$R_j(f)(x) = C \int_{|y|<1} \frac{x_j - y_j}{|x-y|^{n+1}}\, f(y)\, dy.$$

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Hence the definition of classic the Riesz transform for a function f within the Schwartz space is given as

$$R_j(f)(x) = C\ p.v. \int_{\mathbb{R}^n} \frac{y_j}{|y|^{n+1}}\ f(x-y)\,dy. \tag{3}$$

In (3), the kernel as $K_j(y) = C\frac{y_j}{|y|^{n+1}}$. The operator defined in (3) is of convolution type, which means that $f(x-y)$ can be considered as an ordinary translate.

We can also understand that the classical Riesz transform is based on this ordinary translate operator. Instead of using the ordinary translate, which is a specific case of a generalized translation, we can consider the generalized translation $\mathbf{T}^y$. This translation is known as the B.M. Levitan type (see [19]) and corresponds to the solution of the Laplace-Bessel equation Δ_γ. This approach leads to the introduction of a new transform called the B-Riesz transform, which we will denote $R_\gamma^{(k)}$. We remark that the representation of the k-th Riesz transform $R_\gamma^{(k)}$ as a principal value integral operator for every $k \in \mathbb{N}$. Thus, we extend the results in [23] where the result is shown for $k = 1$. Our primary objective is to determine whether this transformation satisfies the Cotlar-type inequality criteria and, if so, to identify the necessary conditions. This leads to the following result

$$R_\gamma^{(k)*}(f)(x) \leq C_\gamma(M_\gamma(R_\gamma^{(k)}f)(x) + M_\gamma(f)(x)), \tag{4}$$

called the Cotlar inequality, where the B-maximum operator M_γ and the B-Riesz transform $R_\gamma^{(k)}$ are generated by the generalized translation operator $\mathbf{T}^y$. Note that if this inequality is satisfied, the kernel of the B-Riesz transform is odd. Also, the inequality

$$R_\gamma^{(k)*}f \leq C_\gamma\, M_\gamma(R_\gamma^{(k)}f) \tag{5}$$

is satisfied if the kernel of the B-Riesz transform is even.

This article is structured as follows: Section 2 gives a brief overview of the general notations, focusing on the context of the B-maximal operator M_γ and the B-Riesz transform $R_\gamma^{(k)}$ derived from the generalized translation operator $\mathbf{T}^y$ in weighted Lebesgue spaces $L_{p,\gamma}$. In section 3, we will study the concept of the B-Riesz transform generated by the generalized translation operator $\mathbf{T}^y$ and the B-maximum operator M_γ. We then study the characterization of the Cotlar's inequality for the B-Riesz transform $R_\gamma^{(k)}$.

2. Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and $\mathbb{R}^n_+ := \{x = (x_1, \dots, x_n) : x_i > 0,\ i = 1, \dots, n\}$. For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r) = \{x_0 \in \mathbb{R}^n : |x - x_0| < r\}$ denotes the open ball centred at x_0 with radius r, $B(x,r)^c$ denotes its complement and $Q = Q(x,r)$ stands for the positive part of the cubes centred at $x \in \mathbb{R}^n_+$, and $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$.

A weight w is a non-negative locally integrable function on $\mathbb{R}^n_+$ that takes values in $(0, +\infty)$ almost everywhere. A weight w is said to belong to Muckenhoupt's class $A_{p,\gamma}$ for $1 < p < \infty$ if there exists a constant $C > 0$ such that

$$\left(\frac{1}{|B|}\int_B w(x)(x_n)^\gamma\,dx\right)^{1/p}\left(\frac{1}{|B|}\int_B w(x)^{-q/p}\,(x_n)^\gamma dx\right)^{1/q} \leq C$$

for every ball $B \subset \mathbb{R}^n_+$, where q is the dual of p such that $1/p + 1/q = 1$. The class A_1 is defined by replacing the above inequality by

$$\frac{1}{|B|}\int_B w(x)\,(x_n)^\gamma dx \leq C\operatorname*{ess\,inf}_{x\in B} w(x)$$

for each ball $B \subset \mathbb{R}^n_+$. We also define $A_\infty = \bigcup_{1\leq p<\infty} A_p$.

Given a Lebesgue measurable set E and a weight function w, we denote the characteristic function of E by χ_E, and the weighted measure of E by $w(E)$, where $w(E) = \int_E w(x)\,dx$. It is well known that if $w \in A_p$ with $1 \leq p < \infty$(or $w \in A_\infty$), then w satisfies the doubling condition; that is, for any ball B, there exists a positive constant C such that (see [7])

$$w(2B) \leq C\,w(B). \tag{6}$$

However, if $w \in A_\infty$, then for any ball B and any measurable subset E of the ball B, there exists a number $\delta > 0$ independent of E and B such that (see [7])

$$\frac{w(E)}{w(B)} \leq C\left(\frac{|E|}{|B|}\right)^{\delta}. \tag{7}$$

Let w be a weight function on $\mathbb{R}^n_+$. For $1 \leq p < \infty$, the weighted Lebesgue space $L_{p,\gamma,w}(\mathbb{R}^n_+)$ is defined as the set of all functions f such that

$$\left\|f\right\|_{L_{p,\gamma,w}} := \left(\int_{\mathbb{R}^n_+} \left|f(x)\right|^p w(x)\,(x_n)^\gamma dx\right)^{1/p} < \infty.$$

Furthermore, for $1 \leq p < \infty$, we denote by $WL_{p,\gamma,w}(\mathbb{R}^n)$ the weighted weak Lebesgue space of all measurable functions f such that

$$\left\|f\right\|_{WL_{p,\gamma,w}} := \sup_{\lambda>0} \lambda \cdot \left[w\left(\left\{x \in \mathbb{R}^n : |f(x)| > \lambda\right\}\right)\right]^{1/p} < \infty.$$

It is known that $L_{p,\gamma,w}$ is a Banach space, and it is also known that the following equality holds for the norms of the spaces $L_{p,\gamma,w}$ and $L_{\infty,\gamma,w}$.

$$\|f\|_{\infty,\gamma,w} = \lim_{p\to\infty} \|f\|_{p,\gamma,w}.$$

The Laplace-Bessel operator is defined by

$$\Delta_\gamma = \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} + B_{x_n},$$

where $B_{x_n} = \dfrac{\partial^2}{\partial x_n^2} + \dfrac{\gamma}{x_n}\dfrac{\partial}{\partial x_n}$ and $\gamma > 0$.

The generalized shift operator $\mathbf{T}^y$ is defined by

$$\left(\mathbf{T}^y f\right)(x) = \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\gamma}{2}\right)} \int_0^\pi f\left(x' - y',\, \sqrt{x_n^2 + y_n^2 - 2x_n y_n \cos\varphi}\right) \sin^{\gamma-1}\varphi d\varphi,$$

where, x', $y' \in \mathbb{R}^{n-1}$, $\gamma > 0$ and $C(\gamma) = \pi^{-\frac{1}{2}} \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)}$.

Note that the generalized shift operator is related to the Δ_γ Laplace-Bessel differential operator [5, 14, 15, 20].

$$(f * g)_\gamma(x) = (f * g)_\gamma = \int_{\mathbb{R}^n_+} f(y)\left(\mathbf{T}^y g\right)(x)(y_n)^\gamma dy. \tag{8}$$

is defined as a generalized convolution.

Let J_γ be a Bessel function of the first kind, and let $j_\gamma(x) = \frac{2^\gamma \gamma(\gamma+1)}{x^\gamma} J_\gamma(x)$ is the normalized function of the first kind j_γ. Then the Fourier-Bessel (Hankel) transform of the function $f \in L_{1,\gamma}\left(\mathbb{R}^n_+\right)$ is

$$\mathbf{F}_\gamma[f(x)](\xi) = \mathbf{F}_\gamma[f(x)](\xi) = \widehat{f(\xi)} = \int_{\mathbb{R}^n_+} f(x) j_{\frac{\gamma-1}{2}}(x_n; \xi_n) x_n^\gamma dx.$$

The Fourier-Bessel transform of function $f \in L_{1,\gamma}$ is

$$\mathbf{F}_B[f(x)](\xi) = \int_{\mathbb{R}^n_+} j_{\frac{\gamma-1}{2}}(x_n, \xi_n) \exp^{-ix'.\xi'} f(x) x_n^\gamma dx,$$

where $x'.\xi' = x_1\xi_1 + \cdots + x_{n-1}\xi_{n-1}$ [19, 20].

The main aim of this paper is first to investigate the B-Riesz transform using the B-maximal operator related to the generalized translation operator with weight $L_{p,w,\gamma}$ spaces to obtain new estimates.

First, let us start by defining the B-maximal operator M_γ

$$M_{\gamma,r} f(x) = M_\gamma(|f|^r)^{\frac{1}{r}}(x) = \left(\sup_{Q \ni x} \frac{1}{|Q|} \int_Q \mathbf{T}^y |f(x)|^r (y_n)^\gamma dy \right)^{\frac{1}{r}},$$

for $r > 0$ as well as its usual sharp B-maximal function $M^\sharp$, ([10, 11, 24, 28, 29]), is defined by

$$M^\sharp_\gamma f(x) = \sup_{Q \ni x} \inf_c \frac{1}{|Q|} \int_Q |\mathbf{T}^y f(x) - c| (y_n)^\gamma dy \ \approx \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |\mathbf{T}^y f(x) - f_Q| (y_n)^\gamma dy \,,$$

where

$$f_Q = \frac{1}{|Q|} \int_Q f(y) dy$$

by f is the average of f over Q. Note also the equality,

$$M^\#_{\gamma,r} f(x) = M^\#_\gamma(|f|^r)(x)^{\frac{1}{r}}.$$

which is useful for the sharp B-maximal operator above.

It is well known that the B-maximum function controls the mean value of a function with respect to any radially decreasing function $L_{1,\gamma}$.

Recall that a function f on $\mathbb{R}^n_+$ is called radial if $f(x) = f(y)$ for $|x| = |y|$. Note that a radial function f on $\mathbb{R}^n_+$ has the form $f(x) = k(|x|)$ for some function k on $\mathbb{R}^n_+$.

We will now introduce some properties of the B-maximal operator. Let's start with a lemma that we plan to use in this paper.

Lemma 2.1. *Let $k \geq 0$ be a continuous function on $[0, \infty)$ except at a finite number of points. Suppose that the function $K(x) = k(|x|)$ is integrable function on $\mathbb{R}^n_+$ and satisfies $K(x) \geq K(y)$ whenever $|x| \leq |y|$ (i.e., k is a decreasing function). Then, we have*

$$\sup(|f| * K)(x) \leq \|K\|_{L_{1,\gamma}} M_\gamma(f)(x) \tag{9}$$

for all locally integrable functions f on $\mathbb{R}^n_+$.

The proof of this lemma can be conducted in a manner similar to that described (see detail [8], Theorem 2.1.10, p.82). However, it is important to note that the inequality is confirmed when K is radial, compactly supported, and continuous. Under these conditions, observe that as $|y| \to \infty$, the kernel K behaves as a radial, compactly supported, and continuous function, increasing. Next, we note that when $y = 0$, it suffices to prove the inequality. Once this is established, replacing f with $\mathbf{T}^y f$ implies that the inequality holds for all values of y.

Theorem 2.2. *Let $0 < p < \infty$, $\gamma > 0$ and let $\omega \in A_\infty$ be a weight functin.*

i) There exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n_+} (M_{\gamma,r} f(x))^p \omega(x) dx \leq C \int_{\mathbb{R}^n_+} (M^\sharp_{\gamma,r} f(x))^p \omega(x) dx, \quad f \in L_{p,\gamma}(\mathbb{R}^n_+).$$

Here, $r > 0$, positive constant C depending only on ω and p.

ii) Let $\varphi : (0,\infty) \to (0,\infty)$ be a function that satisfies the doubling condition. Then, there exists a constant C such that

$$\sup_{t>0} \varphi(t)\omega\Big(\{x \in \mathbb{R}^n_+ : M_{\gamma,r} f(x) > t\}\Big) \leq C \sup_{t>0} \varphi(t)\omega\Big(\{x \in \mathbb{R}^n_+ : M^{\sharp}_{\gamma,r} f(x) > t\}\Big)$$

for $f \in L_{p,\gamma}(\mathbb{R}^n_+)$ Here, $r > 0$, positive constant C depending only on ω and φ.

Let $R^{(k)}_\gamma$ be the B-Riesz transform on $\mathbb{R}^n_+$ with a smooth homogeneous kernel

$$K(y) = \frac{\Omega_k(y)}{|y|^{n+\gamma}}, \quad y \in \mathbb{R}^n_+ \setminus \{0\}, \tag{10}$$

where $\Omega(y) = P_k(y)|y|^{-k}$ is a homogeneous function of order 0. Let $\mathcal{S}^{n-1}$ is a unit sphere and Ω belong to the class $C^\infty(\mathcal{S}^{n-1})$. It satisfies the cancellation property

$$\int_{|y|=1} \Omega_k(y)\, d\tau(y) = 0,$$

where τ denotes the normalized surface measure on $\mathcal{S}^{n-1}$.

Let P_k be a homogeneous harmonic polynomial of degree $k \geq 1$. Recall that $R^{(k)}_\gamma f$ is defined as the principal value convolution operator

$$R^{(k)}_\gamma f(x) = p.v. \int_{\mathbb{R}^n_+} K(y)\, \mathbf{T}^y f(x)\, (y_n)^\gamma dy \equiv \lim_{\epsilon\to 0} R^{(k)}_{\gamma,\epsilon} f(x), \tag{11}$$

where $1 \leq k \leq n$ and $R^{(k)}_{\gamma,\epsilon}$ is that defined by

$$R^{(k)}_{\gamma,\epsilon} f(x) = \int_{|y-x|>\epsilon} K(y)\, \mathbf{T}^y f(x)\, (y_n)^\gamma dy.$$

It is essential to note that the limit in (11) applies to all $x \in \mathbb{R}^n_+$ for functions $f \in L_{p,\gamma}(\mathbb{R}^n_+)$ with $1 \leq p < \infty$. The B-Riesz transform $R^{(k)}_\gamma$ is classified as odd or even depending on the nature of the kernel. In particular, for all $y \in \mathbb{R}^n_+ \setminus \{0\}$, the transform is considered odd if the kernel is odd and even if it is even.

Let $R^{(k)*}_\gamma$ be the B-maximal Riesz transform

$$R^{(k)*}_\gamma f(x) = \sup_{\epsilon>0} |R^{(k)}_{\gamma,\epsilon} f(x)|$$

for $x \in \mathbb{R}^n_+$. Consider the inequality $R^{(k)}_\gamma f$ associated with $R^{(k)*}_\gamma f$. The first thing that comes to mind is the basic estimate

$$\|R^{(k)*}_\gamma f\|_{2,\gamma} \leq C \|R^{(k)}_\gamma f\|_{2,\gamma}, \quad f \in L_{2,\gamma}(\mathbb{R}^n_+). \tag{12}$$

in $L_{2,\gamma}$.

The transform $R^{(k)*}_\gamma f$ dominated by $R^{(k)}_\gamma f$ satisfies the inequality (5). Although weaker than (5), the condition (12) still holds the inequality

$$R^{(k)*}_\gamma f(x) \leq C\, M^2_\gamma (R^{(k)}_\gamma f)(x), \quad x \in \mathbb{R}^n_+, \tag{13}$$

in the space $L_{2,\gamma}$, where $M^2_\gamma = M_\gamma \circ M_\gamma \circ M_\gamma$ represents the iterated B-maximal operator. Therefore, the problem with this operator when working with a single kernel is inherent in the problem of multiple kernels. As a result, condition (13) implies the inequality

$$\|R^{(k)*}_\gamma f\|_{p,\gamma} \leq C\, \|R^{(k)}_\gamma f\|_{p,\gamma}, \quad f \in L_{p,\gamma}(\mathbb{R}^n_+), \quad 1 < p \leq \infty \tag{14}$$

in the $L_{p,\gamma}$.

Finally, we showed that if the kernel of the B-Riesz transformation $R_\gamma^{(k)}$ is even, then (4) holds, and the stronger (13) inequality holds when the kernel of the B-Riesz transformation is odd. Recall that $R_\gamma^{(k)}$ is a B-Riesz transform with a kernel represented by the function Ω_k, which has the form given in (10), where P_k is a homogeneous harmonic polynomial of degree $k \geq 1$. If $P_k(x) = x_k$, then the B-Riesz transformation $R_\gamma^{(k)}$ corresponds to a classical Riesz transformation R_γ. If the homogeneous polynomial P_k need not be harmonic but still has a zero integral on the unit sphere, then we call $R_\gamma^{(k)}$ a polynomial operator. Recall that P_k must be harmonic such that it satisfies the Laplace-Bessel equation.

Let us present our results, starting with the case of the odd kernel for the operator.

Theorem 2.3. *Let $R_\gamma^{(k)}$ be B-Riesz transforms with odd smooth homogeneous kernel (10). Then the following statements are equivalent.*

i)

$$R_\gamma^{(k)*} f(x) \leq C\, M^2(R_\gamma^{(k)} f)(x), \quad x \in \mathbb{R}_+^n.$$

ii)

$$\|R_\gamma^{(k)*} f\|_{2,\gamma} \leq C\, \|R_\gamma^{(k)} f\|_{2,\gamma}, \quad f \in L_2(\mathbb{R}_+^n).$$

Clearly, in Theorem 2.3, the condition (i) implies (ii) is a consequence of the bounded-ness of the Hardy-Littlewood maximal operator on weighted $L_{p,w,\gamma}$ spaces when $p = 2$. The proof of *(ii)* implies *(i)* in Theorem 2.3 is proved in [21, 23].

We extend the results in [23],by considering higher order higher order Riesz Bessel transform.

Theorem 2.4. *Let $R_{\gamma_1}^{(k)}$ and $R_{\gamma_2}^{(k)}$ be two the B-Riesz transforms. If $f \in L_{p,\gamma}$, $0 < p < \infty$ and $\omega \in A_\infty$,*

i) Then there exists a constant C such that

$$\int_{\mathbb{R}_+^n} (R_{\gamma_1}^{(k)*} \circ R_{\gamma_2}^{(k)})(f)(x)^p \omega(x)dx \leq C \int_{\mathbb{R}_+^n} (M_\gamma^2(f)(x))^p \omega(x)dx, \tag{15}$$

where the constant C depends on ω, and

$$\sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} \omega(\{x \in \mathbb{R}_+^n : (R_{\gamma_1}^{(k)*} \circ R_{\gamma_2}^{(k)})(f)(x) > t\}) \leq C \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} \omega(\{x \in \mathbb{R}_+^n : M_\gamma^2(f)(x) > t\}), \tag{16}$$

where $\Phi : [0, 1) \to [0, 1)$ a Young function and $\Phi(t) = t(1 + \log +t) \approx t \log(e + t)$.

ii) The estimates (15) and (16) still hold if $R_{\gamma_1}^{(k)*} \circ R_{\gamma_2}^{(k)}$ is replaced by $R_{\gamma_1}^{(k)} \circ R_{\gamma_2}^{(k))*}$ on the left-hand side.

Corollary 2.5. *Let $R_{\gamma_1}^{(k)}$ and $R_{\gamma_2}^{(k)}$ be two the B-Riesz transforms and let $\omega \in A_1$. Then there is a constant C such that*

$$\omega(\{x \in \mathbb{R} : (R_{\gamma_1}^{(k)*} \circ R_{\gamma_2}^{(k)})(f)(x) > t\}) \leq C \int_{\mathbb{R}_+^n} \Phi\Big(\frac{\mathbf{T}^y|f(x)|}{t}\Big)\omega(y)dy, \quad t > 0, \tag{17}$$

where $\Phi(t) = t \log(e + t)$ and the constant C depends on ω, The estimate (17) holds with $R_{\gamma_1}^{(k)*} \circ R_{\gamma_2}^{(k)}$ replaced by $R_{\gamma_1}^{(k)} \circ R_{\gamma_2}^{(k)*}$ in the left hand side.

As previously mentioned, the composition of the B-Riesz transforms $R_{\gamma_1}^{(k)}$ and $R_{\gamma_2}^{(k)}$, referred to as $R_{\gamma_1}^{(k)} \circ R_{\gamma_2}^{(k)} = R_{\gamma_1\gamma_2}^{(k)}$, is not of the weak type $(1, 1)$. This is in contrast to the case of Fourier-Bessel multipliers of $R_\gamma^{(k)}$, where the multiplier γ satisfies the classical Mihlin condition. In fact, according to well-established classical results, if $R_{\gamma_1}^{(k)}$ and $R_{\gamma_2}^{(k)}$ are two valid multipliers, then their composition operator $R_{\gamma_1} \circ R_{\gamma_2}$ is also classified as being of weak type $(1,1)$. In fact, R_{γ_1} and R_{γ_2} are two multiplier operators. Each multiplier γ_j, $j=1,2$ is bounded, belongs to $C^{|\gamma|+[\frac{n}{2}]+k}$ in the complement of the origin, and satisfies the classical Mihlin condition,

$$(\Delta_\gamma^\alpha \gamma_j)(\xi) \leq C|\xi|^{\gamma+\frac{n}{2}-\alpha}, \quad \xi \neq 0$$

for every α such that $|\alpha| \leq \gamma + [\frac{n}{2}] + k$.

The Riesz transformation is a smooth Calderón-Zygmund operator that possesses an additional cancellation property, which is essential for developing Theorem 2.4. Since $R_\gamma^{(k)*}$ is precisely the adjoint of $R_\gamma^{(k)}$, the pointwise inequality (5) directly implies that $R_\gamma^{(k)*} \circ R_\gamma^{(k)}$ is of weak type $(1,1)$.

Riesz transforms are not bounded on $L_1(R^n)$. Instead, we consider the weak type $(1,1)$ property for Riesz maps. Riesz transformations have the weak type $(1,1)$ property. We think that the operator $R_\gamma^{(k)*} \circ R_\gamma^{(k)}$ is also of weak type $(1,1)$. This is not easy to prove, but from the following pointwise inequality

$$R_\gamma^{(k)*}(R_\gamma^{(k)}(f))(x) \leq C_\gamma((R_\gamma^{(k)})^2)^*(f)(x) + M_\gamma(f)(x), \quad x \in \mathbb{R}_+^n \tag{18}$$

can be shown. Here $(R_\gamma^{(k)})^2 = R_\gamma^{(k)} \circ R_\gamma^{(k)} = -I$ is a smooth singular integral operator and the B-maximal operator $(1,1)$ is of weak type.

Now we specify the case of even operators.

Theorem 2.6. *Let $R_\gamma^{(k)}$ be an even B-Riesz transform. Then there exists a smooth homogeneous B-Riesz transform such that*

$$R_\gamma^{(k)^*}(R_\gamma^{(k)}(f))(x) \leq C(R_\gamma^{(k)^*})^2(f)(x) + M_\gamma(f)(x)), \quad x \in \mathbb{R}_+^n,$$

where C_γ is a constant that depends on γ. The operator $R_\gamma^{(k)}$ is defined by the identity $R_\gamma^{(k)^*} \circ R_\gamma^{(k)} = C_\gamma I$, and the operator $R_\gamma^{(k)^*} \circ R_\gamma^{(k)}$ is of the weak type $(1,1)$.

3. Cotlar's pointwise inequality for B-Riesz transforms

The B-Riesz transform $R_\gamma^{(k)}$ we consider here is a continuous linear operator on $L_{p,\gamma}(\mathbb{R}_+^n)$ and its kernel K satisfies the following inequalities:

$$|K(x)| \leq \frac{|P_k(x)|}{|x|^{n+k+\gamma}} \leq \frac{C}{|x|^{n+k+\gamma}}$$

and the regularity condition

$$|K(x) - K(y)| \lesssim \frac{|x|^\epsilon}{|y|^{n+k+\gamma+\epsilon}}, \tag{19}$$

for some $\epsilon > 0$ and whenever $|y| < \frac{1}{2}|x|$. Here P_k is a homogeneous polynomial of order k and $\sup|P_k(x)| = M < \infty$ [5].

For the results of the Calderon-Zygmund operators, see [9]. Considering the higher order Riesz Bessel transform $R_\gamma^{(k)}$ as an operator of Calderon-Zygmund type, it is natural to show that the Cotlar's inequality (4) is satisfied. We can find a useful improvement of the Cotlar's inequality in [9] Then, it led to the following main result, which is a source of inspiration for us.

Theorem 3.1. *Let $R_\gamma^{(k)}$ and $R_\gamma^{(k)*}$ as before and let $0 < \gamma < 1$. Then there is a positive constant C_γ such that*

$$R_\gamma^{(k)*}(f)(x) \leq C_\gamma M_\gamma(R_\gamma^{(k)} f)(x) + C_\gamma M_\gamma f(x), \quad x \in \mathbb{R}^n. \tag{20}$$

Proof. Let $f \in L_{p,\gamma}(\mathbb{R}^n_+)$ be a function such that $1 \le p < \infty$ and $\gamma > 0$. To prove (20), we also fix $\varepsilon > 0$ and we set $f_1 = f\chi_{B_\varepsilon(x)}$ and $f_2 = f\chi_{B^c_\varepsilon(x)}$. Since $x \notin \text{supp } f_2$ whenever $|y| \ge \varepsilon$, we have

$$(R^{(k)}_\gamma f_2)(x) = \int_{\mathbb{R}^n_+} K(y)\, \mathbf{T}^y f_2(x)(y_n)^\gamma dy = \int_{|y|\ge\varepsilon} K(y)\, \mathbf{T}^y f(x)(y_n)^\gamma dy = R^{(k),\varepsilon}_\gamma f(x).$$

Considering (19), for $z \in B_{\frac{\varepsilon}{2}}(x)$ we get $2|z| \le |y|$, whenever $|y| \ge \varepsilon$ and thus

$$\begin{aligned}|R^{(k),\varepsilon}_\gamma f_2(x) - R^{(k)}_\gamma f_2(z)| &= \left| \int_{|y|\ge\varepsilon} [K(z) - K(y)]\mathbf{T}^y f(x)\,(y_n)^\gamma\, dy \right| \le |z|^\gamma \int_{|y|\ge\varepsilon} \frac{\mathbf{T}^y f(x)}{|y|^{n+k+\gamma+\varepsilon}}\,(y_n)^\gamma\, dy \\ &\le \left(\frac{\varepsilon}{2}\right)^\gamma \int_{|y|\ge\varepsilon} \frac{\mathbf{T}^y f(x)}{|y|^{n+k+\gamma+\varepsilon}}\,(y_n)^\gamma\, dy \le C_{\gamma,\varepsilon}\, M_\gamma f(x),\end{aligned}$$

where the last estimate is a consequence of Lemma 2.1. We conclude that for all $z \in B_{\frac{\varepsilon}{2}}(x)$, we have

$$\begin{aligned}|R^{(k),\varepsilon}_\gamma f(x)| &= \left|R^{(k)}_\gamma f_2(x)\right| \le \left|R^{(k)}_\gamma f_2(x) - R^{(k)}_\gamma f_2(z)\right| + \left|R^{(k)}_\gamma f_2(z)\right| \\ &\le C_{\gamma,\varepsilon}\, M_\gamma f(x) + \left|R^{(k)}_\gamma f_1(z)\right|.\end{aligned} \tag{21}$$

Integration over $z \in B_{\frac{\varepsilon}{2}}(x)$, dividing by it follows from (21) that for $z \in B_{\frac{\varepsilon}{2}}(x)$ we get $|B_{\frac{\varepsilon}{2}}(x)|$, we have

$$|R^{(k),\varepsilon}_\gamma f(x)| \le C_{\gamma,\varepsilon}\, M_\gamma f(x) + \left(\frac{1}{|B_{\frac{\varepsilon}{2}}(x)|} \int_{B_{\frac{\varepsilon}{2}}(x)} \left|R^{(k)}_\gamma f_1(z)\right| dz\right) + M_\gamma\left(\left|R^{(k)}_\gamma f(x)\right|\right).$$

From the last inequality we obtain the desired result. □

Focusing on the problem of the Cotlar inequality in (20), it is essential to derive the inequality for the B-Riesz transform $R^{(k)}_\gamma$ and the B-maximal operator M_γ studied by Ekincioglu in [5]. Therefore, the following inequality for the B-Riesz transform $R^{(k)}_\gamma$ and the B-maximal operator M_γ can be derived and proved.

Theorem 3.2. *Let $R^{(k)}_\gamma$ be B-Riesz transform. Then*

i) If $0 < p < \infty$ and $\omega \in A_\infty$, then there exists a positive constant C_γ that depends on ω such that

$$\int_{\mathbb{R}^n} |R^{(k)*}_\gamma f(x)|^p \omega(x) dx \le C_\gamma \int_{\mathbb{R}^n} M_\gamma f(x)^p \omega(x) dx. \tag{22}$$

ii) Let $\varphi : (0,\infty) \to (0,\infty)$ satisfy the doubling condition, then there exists a positive constant C_γ depending on ω and the doubling condition of φ such that

$$\sup_{t>0} \varphi(t)\omega\Big(\{y \in \mathbb{R}^n : |R^{(k)*}_\gamma f(x)| > t\}\Big) \le C_\gamma \sup_{t>0} \varphi(t)\omega\Big(\{y \in \mathbb{R}^n : M_\gamma f(x) > t\}\Big).$$

We will use a local version of (22) in the proof of the following Lemma 3.3. If $0 < p < \infty$, $\omega \in A_\infty$ and Q is an arbitrary cube, then there exists a constant $\omega \in A_\infty$ such that

$$\int_{2Q} |R^{(k)*}_\gamma f(x)|^p \omega(x) dx \le C_\gamma \int_{2Q} M_\gamma f(x)^p \omega(x) dx, \tag{23}$$

for any function f that is supported in Q.

The proof of this theorem is similar to the proof in [2]. However, it is worth noting that there is a different approach to the above theorem, which can be found in [1]. This approach is based on the combination of the well-known inequality, which is much simpler. Fefferman-Stein theorem 2.2 and is a pointwise approximation of the next lemma (25) used in the paper.

Lemma 3.3. *If $R_\gamma^{(k)}$ is a B-Riesz transform, then we have*

$$M_\gamma^\sharp(R_\gamma^{(k)} f)(x) \le C_\gamma\, M_\gamma^2(f)(x), \tag{24}$$

and, for $0 < r < 1$,

$$M_{\gamma,r}^\sharp(R_\gamma^{(k)} f)(x) \le C_{\gamma,r}\, M_\gamma f(x). \tag{25}$$

The inequality given in equation (24) is known to hold if we replace the right-hand side with the larger operator$M_\gamma(f)$ for $\gamma > 1$. We will begin by discussing the proof of (24) as presented in Lemma 3.3, which utilizes standard arguments. Additionally, it is worth noting that an alternative argument can be found in [16].

It suffices to show that $C > 0$ for some constants $c = c_Q$ such that

$$\frac{1}{|Q|}\int_Q |R_\gamma^{(k)} f(y) - c|\, dy \le C_\gamma\, M_\gamma f(x). \tag{26}$$

Let $f = f_1 + f_2$ such that $f_1 = f\,\chi_{2Q}$. If we choose $c = (R_\gamma^{(k)}(f_2))_Q$, we can estimate the left-hand side of the (26) by a multiple of

$$\frac{1}{|Q|}\int_Q |R_\gamma^{(k)}(f_1)(y)|\, dy + \frac{1}{|Q|}\int_Q |R_\gamma^{(k)}(f_2) - (R_\gamma^{(k)}(f_2))_Q|\, dy = I + II.$$

To deal with (II), we use the regularity of the kernel in [9, p. 153]. So we have

$$II \le C_\gamma\, M_\gamma f(x).$$

We will utilize (23) to verify I. Since f_1 is supported on $2Q$, we have

$$\begin{aligned} I &\le \frac{C}{|Q|}\int_{4Q} |R_\gamma^{(k)}(f_1)(y)|\, dy \le \frac{C}{|Q|}\int_{4Q} M_\gamma(f_1)(y)\, dy \le C\,\frac{C}{|4Q|}\int_{4Q} M_\gamma(f)(y)\, dy \\ &\le C_\gamma\, M_\gamma^2(f)(x). \end{aligned}$$

Proof. [Proof of Theorem 2.4] According to [[2] Theorem 3.2 and [6] Theorem 2.2] we have

$$\begin{aligned} \int_{\mathbb{R}_+^n} \left(R_{\gamma_1}^{(k)*} \circ R_{\gamma_2}^{(k)}(f)(x)\right)^p w &\le \int_{\mathbb{R}_+^n} \left(M_\gamma \circ R_{\gamma_2}^{(k)} f(x)\right)^p w(x)dx \le C_\gamma \int_{\mathbb{R}_+^n} \left(M_\gamma^\# \circ R_{\gamma_2}^{(k)} f(x)\right)^p w(x)dx \\ &\le C_\gamma \int_{\mathbb{R}_+^n} (M_\gamma^2 f)^p\, w, \end{aligned}$$

In our previous estimate, we used (24) from Lemma 3.3, which leads to the equation (15) and concludes the proof of the first part of the theorem. To establish (16), we will use similar reasoning, but this time we will refer to part ii) of both Theorems 3.2 and 2.2. Consequently, we obtain

$$\begin{aligned} &\sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w(\{y \in \mathbb{R}_+^n \,:\, |R_{\gamma_1}^{(k)*} \circ R_{\gamma_2}^{(k)} f| > t\}) \le \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w(\{y \in \mathbb{R}_+^n : M_\gamma(R_{\gamma_2}^{(k)} f)(y) > t\}) \\ &\le \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w(\{y \in \mathbb{R}_+^n \,:\, M_\gamma^\#(R_{\gamma_2}^{(k)} f)(y) > t\}) \le \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w(\{y \in \mathbb{R}_+^n : M_\gamma^2(f)(y) > t\}), \end{aligned}$$

where $\Phi : [0, 1) \to [0, 1)$ a Young function and $\Phi(t) = t(1 + \log + t) \approx t\log(e + t)$.

We use a similar argument to prove ii) in Theorem 2.4. The main difference is that we first use Cotlar's inequality estimate from Theorem 3.1. Therefore, we have

$$\int_{\mathbb{R}^n_+}\left(R^{(k)*}_{\gamma_1}\circ R^{(k)*}_{\gamma_2}(f)(x)\right)^p w\le\int_{\mathbb{R}^n_+}\left(M_\gamma\circ R^{(k)*}_{\gamma_2}f(x)\right)^p w(x)dx$$
$$\le\int_{\mathbb{R}^n_+}\left(M_\gamma\circ M_\delta\circ R^{(k)}_{\gamma_2}f(x)\right)^p w(x)dx+\int_{\mathbb{R}^n_+}M^2_\gamma f(x)^p\,w(x)dx=I+II.$$

We only need to check I. So it is sufficient to show that

$$M_\gamma\circ M_{\gamma,r}f\le C_{\gamma,r}Mf(x). \tag{27}$$

is satisfied. Thus, by Theorem 2.2, we obtain

$$I\le C_{\gamma,r}\int_{\mathbb{R}^n_+}\left(M_\gamma\circ R^{(k)}_{\gamma_2}f(x)\right)^p w(x)dx\le C_{\gamma,r}\int_{\mathbb{R}^n_+}\left(M^{\#}_\gamma\circ R^{(k)}_{\gamma_2}f(x)\right)^p w(x)dx\le C_\gamma\int_{\mathbb{R}^n_+}\left(M^2_\gamma f(x)\right)^p w(x)dx.$$

Here we have used (24) in the lemma 3.3 in the last estimate.

We still need to establish the validity of (27). Let $x\in\mathbb{R}^n_+$ and let $Q=Q(x,r)$ denote an arbitrary cube centered at$x$ with side length r. We need to show that

$$\frac{1}{|Q|}\int_Q M_{\gamma,r}f(y)\,dy\le C_\gamma\,M_\gamma f(x).$$

Then let $f=f_1+f_2$ so that $f_1=f\,\chi_{2Q}$. We can estimate the left side by a multiple of

$$\frac{1}{|Q|}\int_Q M_{\gamma,r}f_1(y)\,dy+\frac{1}{|Q|}\int_Q M_{\gamma,r}f_2(y)\,dy=I+II.$$

To deal with (II), we use that it is roughly constant over Q in [9, p. 299]. Hence, we get

$$II\le C_\gamma\,M_{\gamma,r}f(x)\le C_\gamma\,M_\gamma f(x).$$

To prove (I), let us take $r<1$ and observe that the maximal operator is bounded on $L_{1/r}(\mathbb{R}^n_+)$. Therefore, we conclude that

$$I\le\frac{C_{\gamma,r}}{|Q|}\int_{2Q}\mathbf{T}^y|f(x)|\,dy\le C_\gamma\,M_\gamma(f)(x).$$

This completes the proof of part (i) of Theorem 2.4. The proof of the second part is similar to that of part (i). □

Proof. [Proof of Corollary 2.5] According to homogeneity, it is sufficient to assume $t=1$, and therefore we only have to prove

$$w(\{y\in\mathbb{R}^n:|R^{(k)*}_{\gamma_1}\circ R^{(k)}_{\gamma_2}f(y)|>1\})\le C_\gamma\int_{\mathbb{R}^n_+}\Phi(|f(y)|)w(y)dy.$$

Now, $\Phi=t(\log(e+t))\approx t(1+\log^+t)$ is submultiplicative, i.e. $\Phi(ab)\le\Phi(a)\,\Phi(b)$, $a,b\ge0$. Especially, Φ is doubling. According to Theorem 2.4 and Corallary 2.5, we get

$$\begin{aligned} w(\{y\in\mathbb{R}^n : |R^{(k)*}_{\gamma_1}\circ R^{(k)}_{\gamma_2}f(y)|>1\}) &\le C\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}w(\{y\in\mathbb{R}^n : |R^{(k)*}_{\gamma_1}\circ R^{(k)}_{\gamma_2}f(y)|>t\})\\ &\le C\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}w(\{y\in\mathbb{R}^n : M^2_\gamma f(y)>t\})\\ &\le C\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}\int_{\mathbb{R}^n}\Phi(\frac{\mathbf{T}^y|f(x)|}{t})w(y)dy\\ &\le C\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}\int_{\mathbb{R}^n}\Phi(\mathbf{T}^y|f(x)|)\Phi(\frac{1}{t})w(y)dy\\ &= C\int_{\mathbb{R}^n}\Phi(\mathbf{T}^y|f(x)|)w(y)dy\,. \end{aligned}$$

The proof is now complete. □

Proof. [Proof of Theorem 2.6] For classic Riesz transforms, for the B-Riesz transforms it is sufficient to prove that

$$|R^{k,*}_\gamma(R^k_\gamma(f))(0)|\le C(|(R^{k,*}_\gamma)^2f(0)|+Mf(0)),$$

where $R^{k,*}_\gamma$ represent the truncations of R^k_γ, and note that$(R^{k,*}_\gamma)^2=-I$. Define K_γ and K^*_γ as the kernels of R^k_γ and $R^{k,*}_\gamma$ respectively. Let B denote the unit ball in $\mathbb{R}^n_+$. Since R^k_γ is an even B-Riesz transform, its kernel outside the unit ball lies within the region ofR^k_γ (see [21, 23]). More specifically, there exists a polynomial b such that

$$K_\gamma(y)\chi_{B^c}(y)=R^k_\gamma(b\chi_B)(y)\,,\quad y\in\mathbb{R}^n_+\,.$$

Therefore, if $R^k_\gamma\circ R^k_\gamma=-I$, we have:

$$\begin{aligned} R^k_\gamma(R^k_\gamma f(0)) &= \int_{|y|\ge1}K_\gamma(y)\mathbf{T}^y(R^k_\gamma f(x))(y_n)^\gamma dy=\int R^k_\gamma(b\chi_B)(y)R^k_\gamma f(y)dy\\ &= C\int b(y)\chi_B(y)\mathbf{T}^yf(x)(y_n)^\gamma dy. \end{aligned}$$

Obviously this integral is bounded by $C\|b\|_{L^\infty(B)}\,(M_\gamma f)(0)$. So the proof is complete. □

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