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Global well-posedness in variable exponent Besov-Morrey spaces for the fractional porous medium equation

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Abstract. This paper is concerned with the study of the fractional porous medium equation in variable exponent Besov-Morrey spaces. In the case $1 < \beta \leq 2$, by using the Chemin mono-norm method, we prove global well-posedness for small initial data in the critical variable exponent Besov-Morrey spaces $\dot{\mathcal{N}}^{2-2m-\beta+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),h}(\mathbb{R}^d)$ with $\frac{1-\varepsilon}{2} < m < 1 + \frac{d}{2q(\cdot)}$, $0 < \varepsilon < \beta - 1$, $1 \leq r(\cdot) \leq q(\cdot) < \infty$ and $1 \leq h \leq \infty$. In the limit case $\beta = 1$, we establish the global well-posedness for small initial data in $\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}(\mathbb{R}^d)$ with $\frac{1}{2} < m < 1 + \frac{d}{2q(\cdot)}$ and $1 \leq r(\cdot) \leq q(\cdot) < \infty$.

1. Introduction

We are concerned with the fractional porous medium equation given by the following nonlinear diffusion model that involves fractional Laplacian operators:

$$\begin{cases} v_t + \mu\Lambda^\beta v = \kappa\nabla\cdot(v\nabla\mathcal{K}v) & \text{for} \quad (x,t)\in\mathbb{R}^d\times(0,\infty), \\ \mathcal{K}v = (-\Delta)^{-m}v & \text{for} \quad (x,t)\in\mathbb{R}^d\times(0,\infty), \\ v(x,0) = v_0(x) & \text{for} \quad x\in\mathbb{R}^d, \end{cases} \tag{1.1}$$

where $d \geq 1$, $v = v(x,t)$ denotes the density or concentration, and therefore nonnegative, v_0 is the initial data, $\mu > 0$ is the dissipative coefficient which corresponds the viscous case, while $\mu = 0$ represents the inviscid case, $\kappa = \pm 1$, and here for simplify the notation, we take $\mu = \kappa = 1$. The operator $\Lambda^\beta := (-\Delta)^{\frac{\beta}{2}}$ is the Fourier multiplier with symbol $|\xi|^\beta$, ∇ is the gradient operator, $(-\Delta)^{-m}$ is the inverse fractional Laplacian operator.

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When $\mu = 0$ and $\kappa = m = 1$, the model (1.1) corresponds to the mean field equation

$$\begin{cases} v_t = \nabla\cdot(v\nabla\mathcal{K}v) & \text{for } (x,t)\in\mathbb{R}^d\times(0,\infty),\\ \mathcal{K}v = (-\Delta)^{-1}v & \text{for } (x,t)\in\mathbb{R}^d\times(0,\infty),\\ v(x,0)=v_0(x) & \text{for } x\in\mathbb{R}^d, \end{cases} \tag{1.2}$$

which was introduced for the first time by Lin and Zhang [24]. They demonstrated the existence and uniqueness of positive L^∞ solution in two dimensions. There are many studies on well-posedness results of this equation. For instance, please refer to [29, 35] and related references cited therein.

For the general case of m and β, and $\kappa = 1$, Equation (1.1) presented in this case has been derived starting from the same origin as the classical porous medium equation initially proposed by Caffarelli and Vázquez [11]. Indeed, the model is conceived through the incorporation of the dissipative term $\mu\Lambda^\beta v$ into the continuity equation

$$\partial_t v + \nabla\cdot(vV) = 0, \tag{1.3}$$

where $V = -\nabla p$ is the velocity, and p represents the gas pressure which is related to v through the linear integral operator $\mathcal{K}$, $p = \mathcal{K}v$, with kernel $K(x,y) = c|x-y|^{-(d-2m)}$. For $\mu = 0$, that is, the fractional porous medium equation in the inviscid case, we have many studies on this equation. In [11] the authors stated that the notable feature of this equation is the finite speed of propagation, and they established, the existence of a weak solution with bounded initial data that exponentially decays at infinity, the property of compact support, and also the relevant integral estimates. Please see [10, 15, 29, 35] for more information on this equation. For the viscous case ($\mu > 0$), well-posedness of Equation (1.1) has been proved in the critical Besov spaces $\dot{B}^{-\beta+\frac{d}{q}+2-2m}_{q,h}(\mathbb{R}^d)$ with $1\le\beta\le 2$, $1\le q,h\le\infty$, the Fourier-Besov spaces $F\dot{B}^{-\beta+d(1-\frac{1}{q})+2-2m}_{q,h}(\mathbb{R}^d)$ with $(1-\frac{1}{h})\max\{1,2-2m\} < \beta < 3-2m+d(1-\frac{1}{q})$, $1\le q,h\le\infty$, the critical Fourier-Besov-Morrey spaces $F\dot{N}^{-\beta+\frac{\lambda}{q}+3(1-\frac{1}{q})+2-2m}_{q,\lambda,h}(\mathbb{R}^3)$ with $\max\{0,2-2m\} < \beta < 3-2m+\frac{\lambda}{q}+3(1-\frac{1}{q})$, $1\le q<\infty$, $1\le h\le\infty$, and the variable exponent Fourier-Besov-Morrey spaces $\mathcal{F}\dot{\mathcal{N}}^{1-\beta+3(1-\frac{1}{q(\cdot)})}_{r(\cdot),q(\cdot),h}(\mathbb{R}^3)$ with $1<\beta\le\frac{5}{2}$, $\max\{2,r(\cdot)\}\le q(\cdot)\le\frac{6}{5-2\beta}$, $1\le h<\frac{3}{\beta-1}$, by Xiao & Zhang [31], Xiao & Zhou [32], Toumlilin [30], and Abidin & Chen [2], respectively.

When $\mu = 1$, $\kappa = -1$ and $\beta = m = 1$, Equation (1.1) leads to the following classical Keller-Segel equation:

$$\begin{cases} v_t - \Delta v + \nabla\cdot(v\nabla\mathcal{K}v) = 0 & \text{for } (x,t)\in\mathbb{R}^d\times(0,\infty),\\ -\Delta\mathcal{K}v = v & \text{for } (x,t)\in\mathbb{R}^d\times(0,\infty),\\ v(x,0)=v_0(x) & \text{for } x\in\mathbb{R}^d, \end{cases} \tag{1.4}$$

which describes a model of chemotaxis. The system (1.4) was introduced by Keller and Segel [23]. The well-posedness of classical Keller-Segel models has been studied by several researchers in various spaces. In Morrey spaces by Biler [6]. In the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ and the Besov space $\dot{B}^0_{1,2}(\mathbb{R}^2)$ by Ogawa and Shimizu in [26] and [27], respectively. Iwabuchi [21] demonstrated, by using the smoothing effect of the heat semigroup, the global well-posedness for small initial data in the Besov spaces $\dot{B}^{-2+\frac{d}{q}}_{q,\infty}(\mathbb{R}^d)$ with $d\ge 1$ and $\max\{1, d/2\} < q < \infty$. Later, in 2022, Nogayama and Sawano [25], by the same method, extend these well-posedness results in a closed subspace of Besov-Morrey spaces and in Besov-Morrey spaces $\dot{\mathcal{N}}^{-2+\frac{d}{q}}_{r,q,\infty}(\mathbb{R}^d)$ with $1\le r\le q<\infty$, for local well-posedness and global well-posedness, respectively.

When $\mu = 1$, $\kappa = -1$, $m = 1$ and $1 < \beta < 2$, (1.1) was initially considered by Escudero [14], in which it was utilized to characterize the spatio-temporal distribution of a population density of random walkers subjected to Lévy flights. Furthermore, in that paper, it has been established that (1.1) in this case, has global in time solutions. Biler and Karch [7] have established, in the critical Lebesgue space $L^{\frac{d}{\beta}}(\mathbb{R}^d)$, the existence of both local and global solutions for small initial data. Additionally, they have demonstrated the finite-time blowup of non-negative solutions with specific initial data that satisfy high-concentration or large-mass conditions. Biler and Wu [8] proved, for small initial data, global well-posedness in the critical Besov spaces $\dot{B}^{1-\beta}_{2,q}\left(\mathbb{R}^2\right)$. Zhai [33] has demonstrated the global existence, uniqueness, and stability of solutions with a general potential type nonlinear term in the critical Besov spaces, given that the initial data is sufficiently small. Recently, Zhao [34] obtained well-posedness results of (1.1) in the classical Besov spaces $\dot{B}^{-2\beta+\frac{d}{q}}_{q,h}(\mathbb{R}^d)$ with $1 \le \beta \le 2$ and $1 \le q, h \le \infty$. And here we mention that certain aspects of these results were also extended to the fractional power bipolar type drift-diffusion system. Further information on this topic can be found in [8, 17, 22] and the relevant references cited therein.

Inspired by the recent works of the above results, in this paper we aim to investigate, by using the Chemin mono-norm method, the global well-posedness of the fractional porous medium equation (1.1) with initial data in the critical variable exponent Besov-Morrey spaces $\dot{\mathcal{N}}^{2-2m-\beta+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),h}(\mathbb{R}^d)$ with $1 < \beta \le 2$, $\frac{1-\varepsilon}{2} < m < 1 + \frac{d}{2q(\cdot)}$, $0 < \varepsilon < \beta - 1$, $1 \le r(\cdot) \le q(\cdot) < \infty$ and $1 \le h \le \infty$, and for the limit case $\beta = 1$, in $\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}(\mathbb{R}^d)$ with $\frac{1}{2} < m < 1 + \frac{d}{2q(\cdot)}$, $1 \le r(\cdot) \le q(\cdot) < \infty$.

The variable exponent Besov-Morrey space $\dot{\mathcal{N}}^{2-2m-\beta+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),h}$ is critical for Equation (1.1). In fact, if $v(x,t)$ is the solution of Equation (1.1), then

$$v_\lambda(x,t) := \lambda^{\beta-1} v(\lambda x, \lambda^\beta t)$$

is also a solution of the same equation, and

$$\|v(\cdot,0)\|_{\dot{\mathcal{N}}^{2-2m-\beta+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),h}} \sim \|v_\lambda(\cdot,0)\|_{\dot{\mathcal{N}}^{2-2m-\beta+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),h}}.$$

In general, variable exponent function spaces have garnered significant attention from researchers in recent times. This interest extends beyond theoretical aspects, encompassing their pivotal role in various applications, such as image processing [13], fluid dynamics [28] and resolving specific equations [12, 18, 20]. Notably, variable exponent Besov-Morrey space, based on variable exponent Morrey spaces, is a new large framework compared to variable exponent Besov space, i.e. variable exponent Besov-Morrey space is strictly larger than the latter. However, there are many challenges in addressing the well-posedness of equations in these spaces. Simply substituting the classical L^p-norm with the $\mathcal{M}^{q(\cdot)}_{r(\cdot)}$-norm is insufficient for transitioning from classical Besov spaces or classical Besov-Morrey spaces to variable exponent Besov-Morrey spaces. One significant hindrance arises from the failure of certain crucial embedding properties and the inapplicability of certain classical theories, like the multiplier theorem and Young's inequality, within variable exponent Besov-Morrey spaces, unlike classical Besov spaces or classical Besov-Morrey spaces. To overcome these challenges, the present paper primarily relies on the properties described in Section 2 to look at the global well-posedness result. Moreover, variable exponent function spaces have different structures from each other. In particular, an analysis of the structure of a variable exponent Besov-Morrey space reveals its notable distinctions from a variable exponent Besov space. In contrast to the latter, this specific space is better suited for studying the boundedness of semigroup operators and for estimating nonlinear terms. For an in-depth exploration of these variable exponent function spaces, we direct the reader to [1–4, 16, 19, 28] and the associated references therein.

To address Equation (1.1), we think about the following equivalent integral equations:

$$v(t) = e^{-t\Lambda^\beta} v_0 + \int_0^t e^{-(t-\tau)\Lambda^\beta} \nabla \cdot (v \nabla (-\Delta)^{-m} v)\, d\tau, \tag{1.5}$$

where $e^{-t\Lambda^\beta} := \mathcal{F}^{-1}(e^{-t|\xi|^\beta}\mathcal{F})$ is the fractional heat semigroup operator.

Throughout this paper, C will represent constants that may differ at different places, $E \lesssim F$ denotes the existence of a constant $C > 0$ such that $E \le CF$ and $E \sim F$ denotes the existence of constants $C_1, C_2 > 0$ such that $C_1 F \le E \le C_2 F$. We define, for two Banach spaces X and Y, and $v \in X \cap Y$, the norm $\|\cdot\|_{X\cap Y}$ as

$$\|v\|_{X\cap Y} := \|v\|_X + \|v\|_Y,$$

for $v \in \mathcal{S}(\mathbb{R}^d)$, the Fourier transform as

$$\mathcal{F}v(\xi) = \widehat{v}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} v(x)dx,$$

and its inverse Fourier transform as

$$\mathcal{F}^{-1}v(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} v(\xi)d\xi.$$

This article is structured as follows: In Section 2, we present some basic background information on the Littlewood-Paley theory and some different laws on products in variable exponent Besov-Morrey spaces, and then we state our main theorem. In order to demonstrate this theorem, in Section 3 we present and prove some of its crucial estimates. Finally, in Section 4, we establish Theorem 2.10.

2. Preliminaries and Main Theorem

This section presents some definitions of different variable exponent function spaces, basic knowledge of Littlewood-Paley theory and some propositions that are pertinent to our purposes, and lastly states the main theorem of the present paper.

Definition 2.1. *[4] For the measurable function $r(\cdot)$, let*

$$\mathcal{P}_0(\mathbb{R}^d) := \left\{ r(\cdot) : \mathbb{R}^d \to (0,\infty];\ 0 < r_- = \operatorname*{essinf}_{x\in\mathbb{R}^d} r(x), \operatorname*{esssup}_{x\in\mathbb{R}^d} r(x) = r_+ < \infty \right\}.$$

The Lebesgue space with variable exponent is defined by

$$L^{r(\cdot)}\left(\mathbb{R}^d\right) = \left\{ u : \mathbb{R}^d \to \mathbb{R} \text{ is measurable}, \int_{\mathbb{R}^d} |u(x)|^{r(x)}dx < \infty \right\},$$

with norm

$$\begin{aligned} \|u\|_{L^{r(\cdot)}} &:= \inf\left\{ \lambda > 0 : \varrho_{r(\cdot)}(u/\lambda) \le 1 \right\} \\ &= \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|u(x)|}{\lambda} \right)^{r(x)} dx \le 1 \right\}. \end{aligned}$$

We use the following notation to separate variable exponents from constant exponents: $r(\cdot)$ for variable exponents, r for constant exponents. Also $(L^{r(\cdot)}\left(\mathbb{R}^d\right), \|u\|_{L^{r(\cdot)}})$ is a Banach space.

$L^{r(\cdot)}$ doesn't have the same features as L^r. Therefore, to assure the boundedness of the maximal Hardy-Littlewood operator M on $L^{r(\cdot)}(\mathbb{R}^d)$, the following standard conditions are assumed:

1. (Locally log-Hölder's continuous)[4] There exists a constant $C_{\log}(r)$ such that

$$|r(x) - r(y)| \le \frac{C_{\log}(r)}{\log\left(e + |x-y|^{-1}\right)}, \text{ for all } x, y \in \mathbb{R}^d,\ x \ne y.$$

2. (Globally log-Hölder's continuous)[4] There exist two constants $C_{\log}(r)$ and r_∞ such that

$$|r(x) - r_\infty| \le \frac{C_{\log}(r)}{\log(e + |x|)}, \text{ for all } x \in \mathbb{R}^n.$$

$C_{\log}(\mathbb{R}^d)$ denotes the set of all functions $r(\cdot) : \mathbb{R}^d \to \mathbb{R}$ that satisfy 1 and 2.

Definition 2.2. *[3] Let $r(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^d)$ with $0 < r_- \leq r(\cdot) \leq q(\cdot) \leq \infty$, the variable exponent Morrey space $\mathcal{M}^{q(\cdot)}_{r(\cdot)} := \mathcal{M}^{q(\cdot)}_{r(\cdot)}(\mathbb{R}^d)$ is defined as the set of all measurable functions on $\mathbb{R}^d$ such that*

$$\|u\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} := \sup_{x_0\in\mathbb{R}^d, R>0} \left\| R^{\frac{d}{q(x)} - \frac{d}{r(x)}} u \right\|_{L^{r(\cdot)}(B(x_0,R))} < \infty.$$

From the definition of $L^{r(\cdot)}$-norm, the quasinorm $\|\cdot\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}}$ is also expressed as follows:

$$\|u\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} := \sup_{x_0\in\mathbb{R}^d, R>0} \inf\left\{ \lambda > 0 : \varrho_{r(\cdot)}\left(R^{\frac{d}{q(x)} - \frac{d}{r(x)}} \frac{u}{\lambda} \chi_{B(x_0,R)} \right) \leq 1 \right\}.$$

Here we give an important lemma.

Lemma 2.3. *[3] Let $r(\cdot) \in \mathcal{P}_0(\mathbb{R}^d)$, then*

1. *If $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^d)$ satisfies $r(\cdot) \leq q(\cdot)$. Then for any measurable function u, we have*

$$\|u\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} := \inf\left\{ \lambda > 0 : \sup_{x_0\in\mathbb{R}^d, R>0} \varrho_{r(\cdot)}\left(R^{\frac{d}{q(x)} - \frac{d}{r(x)}} \frac{u}{\lambda} \chi_{B(x_0,R)} \right) \leq 1 \right\}.$$

2. *For any measurable function u*

$$\sup_{x_0\in\mathbb{R}^d, R>0} \varrho_{r(\cdot)}\left(u\chi_{B(x_0,R)} \right) = \varrho_{r(\cdot)}(u),$$

and $\|u\|_{\mathcal{M}^{r(\cdot)}_{r(\cdot)}} = \|u\|_{L^{r(\cdot)}}$.

We now recall the Littlewood-Paley decomposition (see [5] for more details). Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ be a smooth radial function such that

$$0 \leq \varphi \leq 1,$$
$$\operatorname{supp} \varphi \subset \left\{ \xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \right\},$$
$$\sum_{j\in Z} \varphi\left(2^{-j}\xi\right) = 1, \quad \text{for all } \xi \neq 0,$$

and we denote $\varphi_j(\xi) = \varphi(2^{-j}\xi)$. Then for every $u \in \mathcal{S}'(\mathbb{R}^d)$, we define the frequency localization operators for all $j \in \mathbb{Z}$, as follows

$$\Delta_j u = \mathcal{F}^{-1}\varphi_j * u \qquad \text{and} \qquad S_j u = \sum_{k\leq j-1} \Delta_k u. \tag{2.1}$$

Here, we observe that Δ_j is a frequency to $\{|\xi| \sim 2^j\}$ and S_j is a frequency to $\{|\xi| \lesssim 2^j\}$, and we denote also that the almost orthogonality property of the Littlewood-Paley decomposition is satisfied, i.e. for any $u, v \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}$,

$$\Delta_i\Delta_j u = 0 \quad \text{if} \quad |i-j| \geq 2 \quad \text{and} \quad \Delta_i\left(S_{j-1}u\Delta_j v\right) = 0 \quad \text{if} \quad |i-j| \geq 5, \tag{2.2}$$

where $\mathcal{P}$ is the set of all polynomials on $\mathbb{R}^d$.

Throughout this paper, the following Bony paraproduct decomposition will be used:

$$uv = T_u v + T_v u + R(u,v), \tag{2.3}$$

with

$$T_u v = \sum_j S_{j-1} u \Delta_j v, \quad R(u,v) = \sum_j \sum_{|j-l|\leq 1} \Delta_j u \Delta_l v.$$

Definition 2.4. *[3] Let $r(\cdot), q(\cdot), h(\cdot) \in \mathcal{P}_0(\mathbb{R}^d)$ with $r(\cdot) \leq q(\cdot)$, the mixed Morrey-sequence space $\ell^{h(\cdot)}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})$ is the set of all sequences $\left\{a_j\right\}_{j\in\mathbb{Z}}$ of measurable functions on $\mathbb{R}^d$ such that*

$$\|\left\{a_j\right\}_{j\in\mathbb{Z}}\|_{\ell^{h(\cdot)}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} := \inf\left\{\lambda > 0 : \varrho_{\ell^{h(\cdot)}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}(\left\{a_j/\lambda\right\}_{j\in\mathbb{Z}}) \leq 1\right\},$$

where

$$\varrho_{\ell^{h(\cdot)}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}(\left\{a_j\right\}_{j\in\mathbb{Z}}) := \sum_{j\in\mathbb{Z}} \inf\left\{\nu > 0 : \int_{\mathbb{R}^n}\left(\frac{|R^{\frac{d}{q(x)}-\frac{d}{r(x)}} a_j \chi_{B(x_0,R)}|}{\nu^{\frac{1}{h(x)}}}\right)^{r(x)} dx \leq 1\right\}.$$

Notice that if $h_+ < \infty$ and $r(\cdot) \leq h(\cdot)$, then

$$\varrho_{\ell^{h(\cdot)}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}(\left\{a_j\right\}_{j\in\mathbb{Z}}) := \sum_{j\in\mathbb{Z}} \sup_{x_0\in\mathbb{R}^d, R>0} \left\|\left(R^{\frac{d}{q(x)}-\frac{d}{r(x)}} u\right)^{h(x)}\right\|_{L^{\frac{r(\cdot)}{h(\cdot)}}(B(x_0,R))}.$$

Definition 2.5. *[3] Let $s(\cdot) \in C_{\log}(\mathbb{R}^d)$ and $r(\cdot), q(\cdot), h(\cdot) \in \mathcal{P}_0(\mathbb{R}^d) \cap C_{\log}(\mathbb{R}^d)$ with $0 < r_- \leq r(\cdot) \leq q(\cdot) \leq \infty$. The variable exponent homogeneous Besov-Morrey space $\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)}$ is defined by*

$$\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)} := \left\{u \in \mathcal{D}'\left(\mathbb{R}^d\right) : \|u\|_{\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)}} < \infty\right\},$$

with norm

$$\|u\|_{\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)}} := \left\|\left\{2^{js(\cdot)}\Delta_j u\right\}_{j\in\mathbb{Z}}\right\|_{\ell^{h(\cdot)}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})},$$

and $\mathcal{D}'\left(\mathbb{R}^d\right)$ represents the dual space of

$$\mathcal{D}\left(\mathbb{R}^d\right) = \left\{u \in \mathcal{S}\left(\mathbb{R}^d\right) : (D^\alpha u)(0) = 0, \text{ for all multi-index } \alpha\right\}.$$

For $T > 0$ and $1 \leq h, \rho \leq \infty$. The mixed space-time space $\mathcal{L}^\rho\left(0, T; \dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right)$ is the set of all tempered distribution u satisfying

$$\|u\|_{\mathcal{L}^\rho\left(0,T;\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right)} := \left(\sum_{j\in\mathbb{Z}} \left\|2^{js(\cdot)}\Delta_j u\right\|^h_{L^\rho\left(0,T;\mathcal{M}^{q(\cdot)}_{r(\cdot)}\right)}\right)^{\frac{1}{h}} < \infty,$$

where

$$\|u\|_{L^\rho\left(0,T;\mathcal{M}^{q(\cdot)}_{r(\cdot)}\right)} := \left(\int_0^T \|u(\cdot,t)\|^\rho_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \mathrm{d}t\right)^{\frac{1}{\rho}}.$$

With the standard modification if $h = \infty$ or $\rho = \infty$. Due to Minkowski's inequality, we have

$$\mathcal{L}^\rho\left(0, T; \dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right) \hookrightarrow L^\rho\left(0, T; \dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right), \quad \text{if } \rho \geq h,$$
$$L^\rho\left(0, T; \dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right) \hookrightarrow \mathcal{L}^\rho\left(0, T; \dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right), \quad \text{if } \rho \leq h.$$

Proposition 2.6. *The following inclusions hold for variable exponent Morrey spaces.*

1. *(Hölder's inequality)[1] Let $r(\cdot), r_1(\cdot), r_2(\cdot), q(\cdot), q_1(\cdot), q_2(\cdot) \in \mathcal{P}_0(\mathbb{R}^d)$ satisfying $r(\cdot) \leq q(\cdot)$, $r_i(\cdot) \leq q_i(\cdot)(i = 1, 2)$, $\frac{1}{r(\cdot)} = \frac{1}{r_1(\cdot)} + \frac{1}{r_2(\cdot)}$ and $\frac{1}{q(\cdot)} = \frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)}$. Then for all $u \in \mathcal{M}^{q_1(\cdot)}_{r_1(\cdot)}$ and $v \in \mathcal{M}^{q_2(\cdot)}_{r_2(\cdot)}$, there is a constant C depending only on r_- and r_+ such that*

$$\|uv\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \leq C\,\|u\|_{\mathcal{M}^{q_1(\cdot)}_{r_1(\cdot)}} \|v\|_{\mathcal{M}^{q_2(\cdot)}_{r_2(\cdot)}}. \tag{2.4}$$

   And for all $u \in L^\infty(\mathbb{R}^d)$ and $v \in \mathcal{M}^{q(\cdot)}_{r(\cdot)}$, there is a constant C such that

$$\|uv\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \leq C\,\|u\|_{L^\infty} \|v\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}}. \tag{2.5}$$

2. *(Sobolev-type embedding) [1?] Let* $r_1(\cdot), r_2(\cdot), q_1(\cdot), q_2(\cdot) \in \mathcal{P}_0(\mathbb{R}^d)$, $0 < h < \infty$ *and* $s_1(\cdot), s_2(\cdot) \in L^\infty \cap C_{\log}(\mathbb{R}^d)$ *with* $s_1(\cdot) > s_2(\cdot)$. *If* $\frac{1}{h}$ *and*
$$s_1(\cdot) - \frac{d}{r_1(\cdot)} = s_2(\cdot) - \frac{d}{r_2(\cdot)}$$
are locally log-Hölder continuous, then
$$\dot{\mathcal{N}}^{s_1(\cdot)}_{r_1(\cdot),q_1(\cdot),h} \hookrightarrow \dot{\mathcal{N}}^{s_2(\cdot)}_{r_2(\cdot),q_2(\cdot),h}. \tag{2.6}$$
3. *(Mollification inequality) [3] Let* $r(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^d) \cap C_{\log}(\mathbb{R}^d)$ *and* $\phi \in L^1(\mathbb{R}^d)$, *suppose* $\Phi(y) = \sup_{x \notin B(0,|y|)} |\phi(x)|$ *is integrable. Then for all* $u \in \mathcal{M}^{q(\cdot)}_{r(\cdot)}$, *there is a constant C depending only on d such that*
$$\left\| u * \phi_\varepsilon \right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \le C \left\| u \right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \left\| \Phi \right\|_{L^1}, \tag{2.7}$$
where $\phi_\varepsilon = \frac{1}{\varepsilon^d}\phi(\varepsilon)$.

Lemma 2.7. *[3] Let C be a ring, and* $\mathcal{B}$ *a ball in* $\mathbb{R}^d$, *and let* $k \in \mathbb{N}$, $j \in \mathbb{Z}$, $\lambda > 0$, *and* $r(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^d) \cap C_{\log}(\mathbb{R}^d)$ *with* $r(\cdot) \le q(\cdot) < \infty$.

1. *Assume that* $u \in \mathcal{M}^{q(\cdot)}_{r(\cdot)}$ *satisfying supp*$\mathcal{F}(u) \subset \lambda\mathcal{B}$, *then*
$$\sup_{|\alpha|=k} \left\| \partial^\alpha u \right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \le C^{k+1} \lambda^k \left\| u \right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}}.$$
2. *Assume that* $u \in \mathcal{M}^{q(\cdot)}_{r(\cdot)}$ *satisfying supp*$\mathcal{F}(u) \subset \lambda^j\mathcal{B}$, *then*
$$\left\| u \right\|_{L^\infty} \le C\lambda^{j\frac{d}{q(\cdot)}} \left\| u \right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}}.$$

where C is a constant independent of λ.

Lemma 2.8. *Let* $m \in \mathbb{R}$, $s(\cdot) \in C_{\log}(\mathbb{R}^d)$, $r(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^d) \cap C_{\log}(\mathbb{R}^d)$ *with* $r(\cdot) \le q(\cdot)$, *and let* $0 < h < \infty$. *Then*
$$\partial^m_\xi : \dot{\mathcal{N}}^{s(\cdot)+m}_{r(\cdot),q(\cdot),h} \to \dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h}$$
is bounded.

Proof. By using the fact that $\{|\xi| \sim 2^j\}$ for all $j \in \mathbb{Z}$, we have
$$\begin{aligned}
\left\| \partial^m_\xi u \right\|_{\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h}} &= \left(\sum_{j \in Z} 2^{jsh} \left\| \dot{\Delta}_j \partial^m_\xi u \right\|^h_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \right)^{1/h} \\
&= \left(\sum_{j \in Z} 2^{jsh} \left\| \mathcal{F}^{-1}\left(\varphi_j \mathcal{F}\left(\partial^m_\xi u \right) \right) \right\|^h_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \right)^{1/h} \\
&= \left(\sum_{j \in Z} 2^{jsh} \left\| |\xi|^m \dot{\Delta}_j u \right\|^h_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \right)^{1/h} \\
&\lesssim \left(\sum_{j \in Z} 2^{jsh} 2^{jmh} \left\| \dot{\Delta}_j u \right\|^h_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \right)^{1/h} \\
&\lesssim \left\| u \right\|_{\dot{\mathcal{N}}^{s(\cdot)+m}_{r(\cdot),q(\cdot),h}}.
\end{aligned}$$

□

Lemma 2.9. *([5, Lemma 5.5]) Let X be a Banach space with norm $\|\cdot\|_X$ and B be a bounded bilinear operator from $X \times X$ to X satisfying*

$$\|B(x_1, x_2)\|_X \leq C_0 \|x_1\|_X \|x_2\|_X ,$$

for all $x_1, x_2 \in X$ and a constant $C_0 > 0$. Then for any $a \in X$ such that $\|a\|_X < \frac{1}{4C_0}$, the equation $x = a + B(x, x)$ has a solution x in X. Moreover, the solution is such that $\|x\|_X \leq 2\|a\|_X$, and it is the only one such that $\|x\|_X < \frac{1}{2C_0}$.

2.1. Main Theorem

Our main result is as follows.

Theorem 2.10. *Let $d \geq 1$, $1 \leq r(\cdot) \leq q(\cdot) < \infty$ and $1 \leq h \leq \infty$.*

1. *Let $1 < \beta \leq 2$, $\frac{1-\varepsilon}{2} < m < 1 + \frac{d}{2q(\cdot)}$ and $0 < \varepsilon < \beta - 1$. Then there exists a constant $\delta > 0$ such that for any $v_0 \in \dot{\mathcal{N}}^{2-2m-\beta+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),h}$ satisfying $\|v_0\|_{\dot{\mathcal{N}}^{2-2m-\beta+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),h}} \leq \delta$, Equation (1.1) has a unique global solution $v \in \mathcal{X}$, where*

$$\mathcal{X} := \mathcal{L}^{\infty}\left(\mathbb{R}_+; \dot{\mathcal{N}}^{2-2m-\beta+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),h}\right) \cap \mathcal{L}^{\gamma_1}\left(\mathbb{R}_+; \dot{\mathcal{N}}^{s_1(\cdot)}_{r(\cdot),q(\cdot),h}\right) \cap \mathcal{L}^{\gamma_2}\left(\mathbb{R}_+; \dot{\mathcal{N}}^{s_2(\cdot)}_{r(\cdot),q(\cdot),h}\right),$$

   with

$$s_1(\cdot) = 1 - 2m + \frac{d}{q(\cdot)} + \varepsilon, \;\; s_2(\cdot) = 1 - 2m + \frac{d}{q(\cdot)} - \varepsilon, \;\; \gamma_1 = \frac{\beta}{\beta - 1 + \varepsilon}, \;\; \gamma_2 = \frac{\beta}{\beta - 1 - \varepsilon}.$$

2. *Let $\beta = 1$ and $\frac{1}{2} < m < 1 + \frac{d}{2q(\cdot)}$. Assume that $v_0 \in \dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}$ is small enough. Then Equation (1.1) has a unique global solution v satisfying*

$$v \in \mathcal{L}^{\infty}\left(\mathbb{R}_+; \dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right).$$

Remark 2.11. *The results of this work remain valid if we take variable exponent Besov space $\mathcal{B}^{s(\cdot)}_{q(\cdot),h(\cdot)}$ instead of variable exponent Besov-Morrey space $\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)}$. Indeed, if we have $q(\cdot) = r(\cdot)$, then $\dot{\mathcal{N}}^{s(\cdot)}_{q(\cdot),r(\cdot),h(\cdot)} = \mathcal{B}^{s(\cdot)}_{r(\cdot),h(\cdot)}$.*

Remark 2.12. *It has been proved global well-posedness of Equation (1.1) in:*

- $\dot{B}^{-\beta+\frac{d}{q}+2-2m}_{q,h}(\mathbb{R}^d)$ *with* $1 \leq \beta \leq 2$, $1 \leq q, h \leq \infty$ *by [31];*
- $F\dot{B}^{-\beta+d(1-\frac{1}{q})+2-2m}_{q,h}(\mathbb{R}^d)$ *with* $(1 - \frac{1}{h})\max\{1, 2 - 2m\} < \beta < 3 - 2m + d(1 - \frac{1}{q})$, $1 \leq q, h \leq \infty$, *by [32];*
- $F\dot{N}^{-\beta+\frac{\lambda}{q}+3(1-\frac{1}{q})+2-2m}_{q,\lambda,h}(\mathbb{R}^3)$ *with* $\max\{0, 2 - 2m\} < \beta < 3 - 2m + \frac{\lambda}{q} + 3(1 - \frac{1}{q})$, $1 \leq q < \infty$, $1 \leq h \leq \infty$ *by [30];*
- $\mathcal{F}\dot{\mathcal{N}}^{1-\beta+3(1-\frac{1}{q(\cdot)})}_{r(\cdot),q(\cdot),h}(\mathbb{R}^3)$ *with* $1 < \beta \leq \frac{5}{2}$, $\max\{2, r(\cdot)\} \leq q(\cdot) \leq \frac{6}{5-2\beta}$, $1 \leq h < \frac{3}{\beta-1}$ *by [2].*

While in this work we have the result in $\dot{\mathcal{N}}^{2-2m-\beta+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),h}(\mathbb{R}^d)$ with $1 \leq \beta \leq 2$, $\frac{1-\varepsilon}{2} < m < 1 + \frac{d}{2q(\cdot)}$, $0 < \varepsilon < \beta - 1$, $1 \leq r(\cdot) \leq q(\cdot) < \infty$ and $1 \leq h \leq \infty$. So, compared to the above works, Theorem 2.10 gives a new class of initial data for which Equation (1.1) is globally solvable, and may be viewed as a new global well-posedness result for this equation.

3. A Priori Estimates

In this section, we get our key estimates. The first one is the estimate for the localisations of the fractional heat semigroup $\{e^{-t\Lambda^\beta}\}_{t\geq 0}$ in our framework.

Lemma 3.1. *Let $t > 0$, $j \in \mathbb{Z}$ and $1 \leq r(\cdot) \leq q(\cdot) < \infty$. Then for all $v \in \mathcal{S}'(\mathbb{R}^d)$ satisfying $\Delta_j v \in \mathcal{M}^{q(\cdot)}_{r(\cdot)}$, we have*

$$\left\|\Delta_j(e^{-t\Lambda^\beta}v)\right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \leq \mathcal{K}e^{-\kappa t 2^{\beta j}} \left\|\Delta_j v\right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}}.$$

where $\mathcal{K}$ and κ are tow constants independent of j and t.

Proof. Recalling that $supp(\mathcal{F}(\Delta_j v)) \subset 2^j\mathcal{C}$ (Δ_j is a frequency to $\{|\xi| \sim 2^j\}$), and considering a function $\phi \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$ with $\phi \equiv 1$ in a neighborhood of the ring $\mathcal{C}$, then one has

$$\begin{aligned}
\Delta_j(e^{-t\Lambda^\beta}v) &= e^{-t\Lambda^\beta}\Delta_j v \\
&= \phi(2^j\cdot)e^{-t\Lambda^\beta}\Delta_j v \\
&= \mathcal{F}^{-1}\left(\phi(2^j\xi)e^{-t|\xi|^\beta}\mathcal{F}(\Delta_j v)\right) \\
&= \mathcal{F}^{-1}\left(\phi(2^j\xi)e^{-t|\xi|^\beta}\right) * \Delta_j v.
\end{aligned}$$

Hence, Mollification inequality in $\mathcal{M}^{q(\cdot)}_{r(\cdot)}$-space (2.7) and estimate
$\|\mathcal{F}^{-1}(\phi(2^j\xi)e^{-t|\xi|^\beta})\|_{L^1} \leq \mathcal{K}e^{-\kappa t 2^{\beta j}}$, give

$$\begin{aligned}
\left\|\Delta_j(e^{-t\Lambda^\beta}v)\right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} &\leq \left\|\mathcal{F}^{-1}\left(\phi(2^j\xi)e^{-t|\xi|^\beta}\right)\right\|_{L^1}\left\|\Delta_j v\right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \\
&\leq \mathcal{K}e^{-\kappa t 2^{\beta j}}\left\|\Delta_j v\right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}},
\end{aligned}$$

as desired. □

Second, we have the following linear estimates.

Lemma 3.2. *Let $1 \leq \beta \leq 2$, $s(\cdot) \in C_{\log}(\mathbb{R}^d)$, $1 \leq h, \rho \leq \infty$, and let $r(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^d) \cap C_{\log}(\mathbb{R}^d)$ with $1 \leq r(\cdot) \leq q(\cdot) < \infty$. Assume that $v_0 \in \dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h}$. Then there holds*

$$\|e^{-t\Lambda^\beta}v_0\|_{\mathcal{L}^\rho\left(0,\infty;\dot{\mathcal{N}}^{s(\cdot)+\frac{\beta}{\rho}}_{r(\cdot),q(\cdot),h}\right)} \leq C\|v_0\|_{\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h}}, \tag{3.1}$$

where $C > 0$ is a constant depending only on β and d.

Proof. According to Lemma 3.1, we have

$$\begin{aligned}
\left\|\Delta_j(e^{-t\Lambda^\beta}v_0)\right\|_{L^\rho\left(0,\infty;\mathcal{M}^{q(\cdot)}_{r(\cdot)}\right)} &\lesssim \left\|e^{-\kappa t 2^{j\beta}}\right\|_{L^\rho(\mathbb{R}_+)}\left\|\Delta_j v_0\right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \\
&\lesssim \left(\frac{1}{\kappa 2^{\beta j}\rho}\right)^{\frac{1}{\rho}}\left\|\Delta_j v_0\right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \\
&\lesssim 2^{-\frac{\beta}{\rho}j}\left\|\Delta_j v_0\right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}}.
\end{aligned}$$

Thus, multiplying by $2^{(s(\cdot)+\frac{\beta}{\rho})j}$, and taking ℓ^h-norm of both sides in the above inequality, we obtain

$$\left\|e^{-t\Lambda^\beta}v_0\right\|_{\mathcal{L}^\rho\left(0,\infty;\dot{\mathcal{N}}^{s(\cdot)+\frac{\beta}{\rho}}_{r(\cdot),q(\cdot),h}\right)} \lesssim \left\|2^{s(\cdot)j}\left\|\Delta_j v_0\right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}}\right\|_{\ell^h} = \|v_0\|_{\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h}}.$$

□

The last one is as follows:

Lemma 3.3. *Let* $1 \le \beta \le 2$, $s(\cdot) \in C_{\log}(\mathbb{R}^d)$, $1 \le h, \gamma \le \infty$, *and let* $r(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^d) \cap C_{\log}(\mathbb{R}^d)$ *with* $1 \le r(\cdot) \le q(\cdot) < \infty$. *Assume that* $f \in \mathcal{L}^\gamma(\mathbb{R}_+; \dot{\mathcal{N}}^{s(\cdot)+\frac{\beta}{\gamma}-\beta}_{r(\cdot),q(\cdot),h})$. *Then there holds, for any* $\rho \in [\gamma, \infty]$,

$$\left\| \int_0^t e^{-(t-\tau)\Lambda^\beta} f(\tau) \mathrm{d}\tau \right\|_{\mathcal{L}^\rho\left(0,\infty; \dot{\mathcal{N}}^{s(\cdot)+\frac{\beta}{\rho}}_{r(\cdot),q(\cdot),h}\right)} \le C \|f\|_{\mathcal{L}^\gamma\left(\mathbb{R}_+; \dot{\mathcal{N}}^{s(\cdot)+\frac{\beta}{\gamma}-\beta}_{r(\cdot),q(\cdot),h}\right)}, \tag{3.2}$$

where $C > 0$ *is a constant depending only on* β *and* d.

Proof. Set $\frac{1}{\theta} = 1 + \frac{1}{\rho} - \frac{1}{\gamma}$. Lemma 3.1 and Young's inequality in L^ρ give us,

$$\begin{aligned}
\left\| \Delta_j \int_0^t e^{-(t-\tau)\Lambda^\beta} f(\tau) \mathrm{d}\tau \right\|_{L^\rho\left(0,\infty; \mathcal{M}^{q(\cdot)}_{r(\cdot)}\right)} &\lesssim \left\| \int_0^t \left\| e^{-(t-\tau)\Lambda^\beta} \Delta_j f(\tau) \right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \mathrm{d}\tau \right\|_{L^\rho(\mathbb{R}_+)} \\
&\lesssim \left\| \int_0^t e^{-\kappa(t-\tau)2^{j\beta}} \left\| \Delta_j f(\tau) \right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \mathrm{d}\tau \right\|_{L^\rho(\mathbb{R}_+)} \\
&\lesssim \left\| e^{-\kappa(\cdot)2^{j\beta}} * \left\| \Delta_j f(\cdot) \right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \right\|_{L^\rho(\mathbb{R}_+)} \\
&\lesssim \left(\frac{1}{\kappa 2^{\beta j}\theta} \right)^{\frac{1}{\theta}} \left\| \Delta_j f \right\|_{L^\gamma\left(0,\infty; \mathcal{M}^{q(\cdot)}_{r(\cdot)}\right)} \\
&\lesssim 2^{-\beta(1+\frac{1}{\rho}-\frac{1}{\gamma})j} \left\| \Delta_j f \right\|_{L^\gamma\left(0,\infty; \mathcal{M}^{q(\cdot)}_{r(\cdot)}\right)}.
\end{aligned}$$

Finally, multiplying by $2^{(s(\cdot)+\frac{\beta}{\rho})j}$, and taking ℓ^h-norm of both sides in the above inequality, we obtain the desired estimate. □

4. Proof of Theorem 2.10

This part proves our main result, for the case $1 < \beta \le 2$ and the case $\beta = 1$ in Subsection 4.1 and Subsection 4.2, respectively.

4.1. The case $1 < \beta \le 2$

In this subsection, we demonstrate the global well-posedness result for Equation (1.1) with small initial data of in the critical variable exponent Besov-Morrey spaces $\dot{\mathcal{N}}^{2-2m-\beta+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),h}$ where $1 < \beta \le 2$, $\frac{1-\varepsilon}{2} < m < 1 + \frac{d}{2q(\cdot)}$, $0 < \varepsilon < \beta - 1$, $1 \le r(\cdot) \le q(\cdot) < \infty$ and $1 \le h \le \infty$. Firstly, we get the following key bilinear estimate.

Lemma 4.1. *Let* $s(\cdot) \in C_{\log}(\mathbb{R}^d)$, $r(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^d) \cap C_{\log}(\mathbb{R}^d)$ *with* $s(\cdot) > \max\{-1, -m\}$, $1 \le r(\cdot) \le q(\cdot) < \infty$, *and let* $1 \le h, \gamma, \gamma_1, \gamma_2 \le \infty$ *satisfying* $\frac{1}{\gamma} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}$. *Then for any* $\varepsilon > 0$ *and* $m > \max\{\frac{1-\varepsilon}{2}, 0\}$, *one has*

$$\begin{aligned}
\left\| f\nabla(-\Delta)^{-m} g + g\nabla(-\Delta)^{-m} f \right\|_{\mathcal{L}^\gamma\left(\mathbb{R}_+; \dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right)} \lesssim{}& \|f\|_{\mathcal{L}^{\gamma_1}\left(\mathbb{R}_+; \dot{\mathcal{N}}^{s(\cdot)+\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \|g\|_{\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+; \dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \\
&+ \|g\|_{\mathcal{L}^{\gamma_1}\left(\mathbb{R}_+; \dot{\mathcal{N}}^{s(\cdot)+\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \|f\|_{\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+; \dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)}.
\end{aligned} \tag{4.1}$$

Proof. Using the following paraproduct decomposition due to J. M. Bony [9],

$$f\nabla(-\Delta)^{-m} g + g\nabla(-\Delta)^{-m} f := J_1 + J_2 + J_3, \tag{4.2}$$

where,

$$J_1 := \sum_{l\in\mathbb{Z}} \Delta_l f \nabla(-\Delta)^{-m} S_{l-1} g + \Delta_l g \nabla(-\Delta)^{-m} S_{l-1} f,$$
$$J_2 := \sum_{l\in\mathbb{Z}} S_{l-1} f \nabla(-\Delta)^{-m} \Delta_l g + S_{l-1} g \nabla(-\Delta)^{-m} \Delta_l f,$$
$$J_3 := \sum_{l\in\mathbb{Z}} \sum_{|l-l'|\le 1} \Delta_l f \nabla(-\Delta)^{-m} \Delta_{l'} g + \Delta_l g \nabla(-\Delta)^{-m} \Delta_{l'} f.$$

In what follows, we estimate J_1, J_2 and J_3 separately. For J_1, we consider the estimate of its first term only, while the second one can be treated similarly. So, by the facts (2.1) and (2.2), Proposition 2.6, Hölder's inequality in L^p-space, and Lemmas 2.7 and 2.8, when $\varepsilon > 0$, one has

$$\begin{aligned}
\left\|\Delta_j \sum_{l\in\mathbb{Z}} \Delta_l f \nabla(-\Delta)^{-m} S_{l-1} g\right\|_{L^{\gamma}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} &\lesssim \sum_{|l-j|\le 4} \left\|\Delta_l f\right\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \left\|\nabla(-\Delta)^{-m} S_{l-1} g\right\|_{L^{\gamma_2}_T(L^\infty)} \\
&\lesssim \sum_{|l-j|\le 4} \left\|\Delta_l f\right\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \sum_{k\le l-2} 2^{k\left(1-2m+\frac{d}{q(\cdot)}\right)} \left\|\Delta_k g\right\|_{L^{\gamma_2}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \\
&\lesssim \sum_{|l-j|\le 4} \left\|\Delta_l f\right\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \left(\sum_{k\le l-2} 2^{\varepsilon k h'}\right)^{1/h'} \left\|g\right\|_{\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \\
&\lesssim 2^{-s(\cdot)j} \sum_{|l-j|\le 4} 2^{-s(\cdot)(l-j)} 2^{(s(\cdot)+\varepsilon)l} \left\|\Delta_l f\right\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \left\|g\right\|_{\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)}.
\end{aligned}$$

Multiplying by $2^{s(\cdot)j}$, and taking ℓ^h-norm of both sides in the above estimate, we obtain

$$\left\|\sum_{l\in\mathbb{Z}} \Delta_l f \nabla(-\Delta)^{-m} S_{l-1} g\right\|_{\mathcal{L}^{\gamma}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right)} \lesssim \left\|f\right\|_{\mathcal{L}^{\gamma_1}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{s(\cdot)+\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \left\|g\right\|_{\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)},$$

which implies that

$$\|J_1\|_{\mathcal{L}^{\gamma}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right)} \lesssim \left\|f\right\|_{\mathcal{L}^{\gamma_1}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{s(\cdot)+\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \left\|g\right\|_{\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)} + \left\|g\right\|_{\mathcal{L}^{\gamma_1}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{s(\cdot)+\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \left\|f\right\|_{\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)}. \tag{4.3}$$

Analogously for J_2, using Proposition 2.6, Hölder's inequality, and Lemmas 2.7 and 2.8 again, since $m > \frac{1-\varepsilon}{2}$, we obtain

$$\begin{aligned}
\left\|\Delta_j \sum_{l\in\mathbb{Z}} S_{l-1} f \nabla(-\Delta)^{-m} \Delta_l g\right\|_{L^{\gamma}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} &\lesssim \sum_{|l-j|\le 4} \left\|S_{l-1} f \nabla(-\Delta)^{-m} \Delta_l g\right\|_{L^{\gamma}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \\
&\lesssim \sum_{|l-j|\le 4} \sum_{k\le l-2} 2^{k\frac{d}{q(\cdot)}} \left\|\Delta_k f\right\|_{L^{\gamma_2}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \left\|\nabla(-\Delta)^{-m} \Delta_l g\right\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \\
&\lesssim \sum_{|l-j|\le 4} \sum_{k\le l-2} 2^{(1-2m+\frac{d}{q(\cdot)}-\varepsilon)k} 2^{(-1+2m+\varepsilon)k} \left\|\Delta_k f\right\|_{L^{\gamma_2}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} 2^{(1-2m)l} \left\|\Delta_l g\right\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \\
&\lesssim \sum_{|l-j|\le 4} 2^{(1-2m)l} \left\|\Delta_l g\right\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \left(\sum_{k\le l-2} 2^{(-1+2m+\varepsilon)kh'}\right)^{1/h'} \left\|f\right\|_{\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \\
&\lesssim \sum_{|l-j|\le 4} 2^{-s(\cdot)l} 2^{(s(\cdot)+\varepsilon)l} \left\|\Delta_l g\right\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \left\|f\right\|_{\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)},
\end{aligned}$$

which gives us that

$$\left\|\sum_{l\in\mathbb{Z}} S_{l-1}f\nabla(-\Delta)^{-m}\Delta_l g\right\|_{\mathcal{L}^{\gamma}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right)} \lesssim \left\|g\right\|_{\mathcal{L}^{\gamma_1}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{s(\cdot)+\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \left\|f\right\|_{\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)}.$$

Thus, we get

$$\|J_2\|_{\mathcal{L}^{\gamma}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right)} \lesssim \left\|f\right\|_{\mathcal{L}^{\gamma_1}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{s(\cdot)+\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \left\|g\right\|_{\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)} + \left\|g\right\|_{\mathcal{L}^{\gamma_1}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{s(\cdot)+\varepsilon}_{r(\cdot),q(\cdot),h}\right)} \left\|f\right\|_{\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)}. \tag{4.4}$$

We now move on to the final term J_3. We use the following formula, based on an analysis of the algebraic structure of Equation (1.1) [34]:

$$(J_3)_i = \sum_{l\in\mathbb{Z}}\sum_{|l-l'|\leq 1} \Delta_l f\partial_i(-\Delta)^{-m}\Delta_{l'} g + \Delta_l g\partial_i(-\Delta)^{-m}\Delta_{l'} f = K_i^1 + K_i^2 + K_i^3,$$

for $i = 1, 2, ..., n$. Where $(J_3)_i$ is the i-th exponent of (J_3) and

$$\begin{aligned}
K_i^1 &:= \sum_{l\in\mathbb{Z}}\sum_{|l-l'|\leq 1} (-\Delta)^m \left[\left((-\Delta)^{-m}\Delta_l f\right)\left(\partial_i(-\Delta)^{-m}\Delta_{l'} g\right)\right],\\
K_i^2 &:= \sum_{l\in\mathbb{Z}}\sum_{|l-l'|\leq 1} 2\nabla^m \cdot \left[\left((-\Delta)^{-m}\Delta_l f\right)\left(\partial_i \nabla^m(-\Delta)^{-m}\Delta_{l'} g\right)\right],\\
K_i^3 &:= \sum_{l\in\mathbb{Z}}\sum_{|l-l'|\leq 1} \partial_i \left[\left((-\Delta)^{-m}\Delta_l f\right)\Delta_{l'} g\right].
\end{aligned}$$

In order to estimate the above three terms, we use Proposition 2.6, Hölder's inequality in L^p-space, and Lemmas 2.7 and 2.8 as follows: From (2.2) we have existence of a positive integer d_0 such that

$$\begin{aligned}
\left\|\Delta_j K_i^1\right\|_{L^{\gamma}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} &\lesssim 2^{2mj} \sum_{l\geq j-d_0}\sum_{|l-l'|\leq 1} \left\|\left((-\Delta)^{-m}\Delta_l f\right)\left(\partial_i(-\Delta)^{-m}\Delta_{l'} g\right)\right\|_{L^{\gamma}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\
&\lesssim 2^{2mj} \sum_{l\geq j-d_0}\sum_{|l-l'|\leq 1} 2^{-2ml}\left\|\Delta_l f\right\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} 2^{(1-2m)l'}\left\|\Delta_{l'} g\right\|_{L^{\gamma_2}_T(L^{\infty})}\\
&\lesssim 2^{2mj} \sum_{l\geq j-d_0}\sum_{|l-l'|\leq 1} 2^{-2ml}\left\|\Delta_l f\right\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} 2^{(1-2m+\frac{d}{q(\cdot)})l'}\left\|\Delta_{l'} g\right\|_{L^{\gamma_2}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\
&\lesssim \sum_{l\geq j-d_0} 2^{-s(\cdot)j}2^{-(2m+s(\cdot))(l-j)}2^{(s(\cdot)+\varepsilon)l}\left\|\Delta_l f\right\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\left\|g\right\|_{\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)},
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
\left\|\Delta_j K_i^2\right\|_{L^{\gamma}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} &\lesssim 2^{mj} \sum_{l\geq j-d_0}\sum_{|l-l'|\leq 1} \left\|\left((-\Delta)^{-m}\Delta_l f\right)\left(\partial_i\nabla^m(-\Delta)^{-m}\Delta_{l'} g\right)\right\|_{L^{\gamma}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\
&\lesssim 2^{mj} \sum_{l\geq j-d_0}\sum_{|l-l'|\leq 1} 2^{-2ml}\left\|\Delta_l f\right\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} 2^{(1-m+\frac{d}{q(\cdot)})l'}\left\|\Delta_{l'} g\right\|_{L^{\gamma_2}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\
&\lesssim \sum_{l\geq j-d_0} 2^{-s(\cdot)j}2^{-(m+s(\cdot))(l-j)}2^{(s(\cdot)+\varepsilon)l}\left\|\Delta_l f\right\|_{L^{\gamma_1}_T(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\left\|g\right\|_{\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)},
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\left\|\Delta_j K_i^3\right\|_{L_T^{\gamma}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} &\lesssim 2^j \sum_{l\geq j-d_0}\sum_{|l-l'|\leq 1}\left\|((-\Delta)^{-m}\Delta_l f)\,\Delta_{l'}g\right\|_{L_T^{\gamma}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\ &\lesssim 2^j \sum_{l\geq j-d_0}\sum_{|l-l'|\leq 1} 2^{-2ml}\left\|\Delta_l f\right\|_{L_T^{\gamma_1}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} 2^{\frac{d}{q(\cdot)}l'}\left\|\Delta_{l'} g\right\|_{L_T^{\gamma_2}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\\ &\lesssim \sum_{l\geq j-d_0} 2^{-s(\cdot)j}2^{-(1+s(\cdot))(l-j)}2^{(s(\cdot)+\varepsilon)l}\left\|\Delta_l f\right\|_{L_T^{\gamma_1}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}\left\|g\right\|_{\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)}.\end{aligned}\tag{4.7}$$

Thus, since $m>0$ and $s(\cdot)>\max\{-1,-m\}$, (4.5), (4.6) and (4.7) give us,

$$\begin{aligned}\|J_3\|_{\mathcal{L}^{\gamma}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right)} &\leq \sum_{i=1}^{d}\sum_{k=1}^{3}\left\|K_i^k\right\|_{\mathcal{L}^{\gamma}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h}\right)}\\ &\lesssim \left\|f\right\|_{\mathcal{L}^{\gamma_1}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{s(\cdot)+\varepsilon}_{r(\cdot),q(\cdot),h}\right)}\left\|g\right\|_{\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}-\varepsilon}_{r(\cdot),q(\cdot),h}\right)}.\end{aligned}\tag{4.8}$$

Finally, by combining (4.3), (4.4) and (4.8), with (4.2), we get (4.1). This completes the proof of Lemma 4.1. □

Now, by using Lemma 2.9, we can start to prove the existence of global solutions of Equation (1.1) in the case $1<\beta\leq 2$. We define

$$\mathcal{X}^1 := \mathcal{L}^{\gamma_1}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{s_1(\cdot)}_{r(\cdot),q(\cdot),h}\right)\cap\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{s_2(\cdot)}_{r(\cdot),q(\cdot),h}\right),$$

with

$$s_1(\cdot)=1-2m+\frac{d}{q(\cdot)}+\varepsilon,\ s_2(\cdot)=1-2m+\frac{d}{q(\cdot)}-\varepsilon,\ \gamma_1=\frac{\beta}{\beta-1+\varepsilon},\ \gamma_2=\frac{\beta}{\beta-1-\varepsilon},$$
$$0<\varepsilon<\beta-1,\ \ \frac{1-\varepsilon}{2}<m<1+\frac{d}{2q(\cdot)}.$$

Due to Duhamel's principle, the solution of Equation (1.1) can be written as

$$v(t)=e^{-t\Lambda^{\beta}}v_0+\int_0^t e^{-(t-\tau)\Lambda^{\beta}}\nabla\cdot(v\nabla(-\Delta)^{-m}v)\,\mathrm{d}\tau.\tag{4.9}$$

Set

$$\mathcal{B}_{\beta}(v,w):=\int_0^t e^{-(t-\tau)\Lambda^{\beta}}\nabla\cdot(v\nabla(-\Delta)^{-m}w)\,\mathrm{d}\tau.$$

By applying Lemma 3.3 and Lemma 4.1, with $\gamma=\frac{\beta}{2\beta-2}$, and since $\gamma_1\geq\gamma$ and $s_1(\cdot)=2-2m+\frac{d}{q(\cdot)}-\beta+\frac{\beta}{\gamma_1}$, we see that

$$\begin{aligned}\left\|\mathcal{B}_{\beta}(v,w)\right\|_{\mathcal{L}^{\gamma_1}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{s_1(\cdot)}_{r(\cdot),q(\cdot),h}\right)} &\lesssim \left\|\nabla\cdot(v\nabla(-\Delta)^{-m}w)\right\|_{\mathcal{L}^{\frac{\beta}{2\beta-2}}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),h}\right)}\\ &\lesssim \left\|v\nabla(-\Delta)^{-m}w\right\|_{\mathcal{L}^{\frac{\beta}{2\beta-2}}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),h}\right)}\\ &\lesssim \|v\|_{\mathcal{L}^{\gamma_1}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{s_1(\cdot)}_{r(\cdot),q(\cdot),h}\right)}\|w\|_{\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{s_2(\cdot)}_{r(\cdot),q(\cdot),h}\right)}\\ &\quad+\|w\|_{\mathcal{L}^{\gamma_1}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{s_1(\cdot)}_{r(\cdot),q(\cdot),h}\right)}\|v\|_{\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{s_2(\cdot)}_{r(\cdot),q(\cdot),h}\right)}\\ &\lesssim \|v\|_{\mathcal{X}^1}\|w\|_{\mathcal{X}^1},\end{aligned}$$

and similarly,

$$\left\|\mathcal{B}_\beta(v,w)\right\|_{\mathcal{L}^{\gamma_2}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{s_2(\cdot)}_{r(\cdot),q(\cdot),h}\right)} \lesssim \|v\|_{\mathcal{X}^1}\,\|w\|_{\mathcal{X}^1}\,.$$

Thus,

$$\left\|\mathcal{B}_\beta(v,w)\right\|_{\mathcal{X}^1} \le C\,\|v\|_{\mathcal{X}^1}\,\|w\|_{\mathcal{X}^1}\,. \tag{4.10}$$

On the other hand, we can directly deduce from Lemma 3.2 that,

$$\left\|e^{-t\Lambda^\beta}v_0\right\|_{\mathcal{X}^1} \le C\,\|v_0\|_{\dot{\mathcal{N}}^{2-2m-\beta+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),h}}\,.$$

So, if $\|v_0\|_{\dot{\mathcal{N}}^{2-2m-\beta+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),h}} \le \delta$ with $\delta = \frac{1}{4C^2}$, then by Lemma 2.9, Equation (1.1) admits a unique solution v in $\mathcal{X}^1$. Furthermore, the integral equation (4.9) and Lemmas 3.2, 3.3 and 4.1, again give us

$$\|v\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{2-2m-\beta+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),h}\right)} \lesssim \|v_0\|_{\dot{\mathcal{N}}^{2-2m-\beta+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),h}} + \|v\|^2_{\mathcal{X}^1} < \infty.$$

Finally,

$$v \in \mathcal{X} := \mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{2-2m-\beta+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),h}\right) \cap \mathcal{L}^{\gamma_1}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{s_1(\cdot)}_{r(\cdot),q(\cdot),h}\right) \cap \mathcal{L}^{\gamma_2}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{s_2(\cdot)}_{r(\cdot),q(\cdot),h}\right).$$

This completes the proof of the first assertion of Theorem 2.10.

4.2. The case $\beta = 1$

In this subsection, we establish the global well-posedness result for Equation (1.1) in the limit case $\beta = 1$, with initial data in the critical variable exponent Besov-Morrey spaces $\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}$ with $\frac{1}{2} < m < 1 + \frac{d}{2q(\cdot)}$ and $1 \le r(\cdot) \le q(\cdot) < \infty$. Firstly, by making a slight modification to the proof of Lemma 4.1, we obtain the following estimate:

Lemma 4.2. *For any $f, g \in \mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)$ and $\frac{1}{2} < m < 1 + \frac{d}{2q(\cdot)}$, one has*

$$\left\|f\nabla(-\Delta)^{-m}g + g\nabla(-\Delta)^{-m}f\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \lesssim \left\|f\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \left\|g\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}. \tag{4.11}$$

Proof. We estimate the first term of J_1 as follows:

$$\begin{aligned}
\left\|\Delta_j \sum_{l\in\mathbb{Z}} \Delta_l f \nabla(-\Delta)^{-m} S_{l-1} g\right\|_{L^\infty(\mathbb{R}_+;\mathcal{M}^{q(\cdot)}_{r(\cdot)})} &\lesssim \sum_{|l-j|\le 4} \left\|\Delta_l f\right\|_{L^\infty(\mathbb{R}_+;\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \left\|\nabla(-\Delta)^{-m} S_{l-1} g\right\|_{L^\infty(\mathbb{R}_+;L^\infty)} \\
&\lesssim \sum_{|l-j|\le 4} \left\|\Delta_l f\right\|_{L^\infty(\mathbb{R}_+;\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \sum_{k\le l-2} 2^{k\left(1-2m+\frac{d}{q(\cdot)}\right)} \left\|\Delta_k g\right\|_{L^\infty(\mathbb{R}_+;\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \\
&\lesssim \sum_{|l-j|\le 4} \left\|\Delta_l f\right\|_{L^\infty(\mathbb{R}_+;\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \left\|g\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}.
\end{aligned}$$

Multiplying by $2^{(1-2m+\frac{d}{q(\cdot)})j}$, and taking ℓ^1−norm of both sides in the above estimate, we obtain

$$\left\|\sum_{l\in\mathbb{Z}} \Delta_l f \nabla(-\Delta)^{-m} S_{l-1} g\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \lesssim \left\|f\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \left\|g\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}.$$

And then,

$$\|J_1\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \lesssim \|f\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \|g\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}. \tag{4.12}$$

Similarly for J_2, when $m > \frac{1}{2}$,

$$\begin{aligned}
&\left\|\Delta_j \sum_{l\in\mathbb{Z}} S_{l-1} f \nabla(-\Delta)^{-m}\Delta_l g\right\|_{L^\infty(\mathbb{R}_+;\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \\
&\lesssim \sum_{|l-j|\le 4}\sum_{k\le l-2} 2^{(1-2m+\frac{d}{q(\cdot)})k} 2^{-(1-2m)k} \|\Delta_k f\|_{L^\infty(\mathbb{R}_+;\mathcal{M}^{q(\cdot)}_{r(\cdot)})} 2^{(1-2m)l} \|\Delta_l g\|_{L^\infty(\mathbb{R}_+;\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \\
&\lesssim \sum_{|l-j|\le 4} \|\Delta_l g\|_{L^\infty(\mathbb{R}_+;\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \|f\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)},
\end{aligned}$$

which gives us that

$$\left\|\sum_{l\in\mathbb{Z}} S_{l-1} f \nabla(-\Delta)^{-m}\Delta_l g\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \lesssim \|g\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \|f\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}.$$

Thus, we get

$$\|J_2\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \lesssim \|f\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \|g\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}. \tag{4.13}$$

Moreover for the final term $J_3 = K^1 + K^2 + K^3$, we estimate K^1, K^2 and K^3 as follows:

$$\begin{aligned}
\|\Delta_j K^1_i\|_{L^\infty(\mathbb{R}_+;\mathcal{M}^{q(\cdot)}_{r(\cdot)})} &\lesssim 2^{2mj} \sum_{l\ge j-d_0}\sum_{|l-l'|\le 1} 2^{-2ml} \|\Delta_l f\|_{L^\infty(\mathbb{R}_+;\mathcal{M}^{q(\cdot)}_{r(\cdot)})} 2^{(1-2m+\frac{d}{q(\cdot)})l'} \|\Delta_{l'} g\|_{L^\infty(\mathbb{R}_+;\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \\
&\lesssim 2^{-(1-2m+\frac{d}{q(\cdot)})j} \sum_{l\ge j-d_0} 2^{-(1+\frac{d}{q(\cdot)})(l-j)} 2^{(1-2m+\frac{d}{q(\cdot)})l} \|\Delta_l f\|_{L^\infty(\mathbb{R}_+;\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \|g\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)},
\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}
\|\Delta_j K^2_i\|_{L^\infty(\mathbb{R}_+;\mathcal{M}^{q(\cdot)}_{r(\cdot)})} &\lesssim 2^{mj} \sum_{l\ge j-d_0}\sum_{|l-l'|\le 1} 2^{-2ml} \|\Delta_l f\|_{L^\infty(\mathbb{R}_+;\mathcal{M}^{q(\cdot)}_{r(\cdot)})} 2^{(1-m+\frac{d}{q(\cdot)})l'} \|\Delta_{l'} g\|_{L^\infty(\mathbb{R}_+;\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \\
&\lesssim 2^{-(1-2m+\frac{d}{q(\cdot)})j} \sum_{l\ge j-d_0} 2^{-(1-m+\frac{d}{q(\cdot)})(l-j)} 2^{(1-2m+\frac{d}{q(\cdot)})l} \|\Delta_l f\|_{L^\infty(\mathbb{R}_+;\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \|g\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)},
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
\|\Delta_j K^3_i\|_{L^\infty(\mathbb{R}_+;\mathcal{M}^{q(\cdot)}_{r(\cdot)})} &\lesssim 2^{j} \sum_{l\ge j-d_0}\sum_{|l-l'|\le 1} 2^{-2ml} \|\Delta_l f\|_{L^\infty(\mathbb{R}_+;\mathcal{M}^{q(\cdot)}_{r(\cdot)})} 2^{\frac{d}{q(\cdot)}l'} \|\Delta_{l'} g\|_{L^\infty(\mathbb{R}_+;\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \\
&\lesssim 2^{-(1-2m+\frac{d}{q(\cdot)})j} \sum_{l\ge j-d_0} 2^{-(2-2m+\frac{d}{q(\cdot)})(l-j)} 2^{(1-2m+\frac{d}{q(\cdot)})l} \|\Delta_l f\|_{L^\infty(\mathbb{R}_+;\mathcal{M}^{q(\cdot)}_{r(\cdot)})} \|g\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}.
\end{aligned} \tag{4.16}$$

Hence, from (4.14), (4.15) and (4.16), and since $m < 1 + \frac{d}{2q(\cdot)}$, we arrive at

$$\begin{aligned}
\|J_3\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} &\le \sum_{i=1}^{d}\sum_{k=1}^{3} \|K^k_i\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \\
&\lesssim \|f\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \|g\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}.
\end{aligned} \tag{4.17}$$

Finally, putting the estimates (4.12), (4.13) and (4.17) together, we get (4.11). The proof of Lemma 4.2 is complete. □

We are now in a position to demonstrate the second assertion of Theorem 2.10. By considering the resolution space $\mathcal{L}^\infty(\mathbb{R}_+; \dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1})$ and returning to the integral equation (4.9) with $\beta = 1$, Lemmas 3.3 and 4.2, give us

$$\begin{aligned}\|\mathcal{B}_1(v,w)\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} &\leq C\left\|\nabla\cdot(v\nabla(-\Delta)^{-m}w)\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}\\ &\leq C\|v\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}\|w\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)}.\end{aligned}$$

On the other hand, from Lemma 3.2, we have

$$\left\|e^{-t\Lambda}v_0\right\|_{\mathcal{L}^\infty\left(\mathbb{R}_+;\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)} \leq C\|v_0\|_{\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}}.$$

If $\|v_0\|_{\dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}}$ is sufficiently small, by using the fixed point argument as in Subsection 4.1, we get the global solution of Equation (1.1) in $\mathcal{L}^\infty\left(\mathbb{R}_+; \dot{\mathcal{N}}^{1-2m+\frac{d}{q(\cdot)}}_{r(\cdot),q(\cdot),1}\right)$. The proof of Theorem 2.10 is complete, as desired.

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