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Existence and uniqueness of boundary value problems for nonlinear hybrid differential equations with ABC-fractional derivative

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Abstract. In this paper, we establish the existence and uniqueness of solutions to nonlinear hybrid fractional differential equations within the Atangana-Baleanu-Caputo framework. Our analysis incorporates both linear and nonlinear perturbations. The proposed method relies on the nonlinear Leray-Schauder alternative in combination with Banach's fixed-point theorem, ensuring a robust mathematical foundation. To illustrate the practical significance of our findings, we provide a concrete example that demonstrates their applicability.

1. Introduction

Hybrid differential equations form an important and diverse area of study within differential equations.Focusing on quadratic perturbations of nonlinear systems.This area has gained significant attention in recent years due to its extensive applicability in multiple scientific and engineering fields.The authors studied the following hybrid differential equation with Linear and Nonlinear Perturbations[8]:

$$\begin{cases} \frac{d}{d\hat{\varkappa}}[\vartheta(\hat{\varkappa})\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa})) - \eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))] = \theta(\hat{\varkappa},\vartheta(\hat{\varkappa})), \quad \hat{\varkappa} \in I = [0,T], T > 0 \\ \vartheta(0)\varphi(0,\vartheta(0)) + \mu\vartheta(T)\varphi(T,\vartheta(T)) = \eta(0,\vartheta(0)) + \mu\eta(T,\vartheta(T)) + \beta. \end{cases} \tag{1}$$

The extension of hybrid differential equations, incorporating both linear and nonlinear perturbations, to the framework of fractional calculus is presented in this work. [1].

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Motivated by the abovementioned problem, we consider the following boundary value problem for hybrid differential equation:

$$\begin{cases} {}^{ABC}{}_0\mathcal{D}^{\iota}_{\hat{\varkappa}}[\vartheta(\hat{\varkappa})\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa})) - \eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))] = \theta(\hat{\varkappa},\vartheta(\hat{\varkappa})), \quad \hat{\varkappa} \in J = [0,T], T > 0, \\ \vartheta(0)\varphi(0,\vartheta(0)) + \mu\vartheta(T)\varphi(T,\vartheta(T)) = \eta(0,\vartheta(0)) + \mu\eta(T,\vartheta(T)) + \beta, \end{cases} \tag{2}$$

where ${}^{ABC}{}_0\mathcal{D}^{\iota}_{\hat{\varkappa}}$, denote Atangana-Baleanu-Caputo Derivative. of order $\iota, \varphi \in \mathscr{C}(J\times\mathbb{R},\mathbb{R}\backslash\{0\}), \eta \in \mathscr{C}(J\times\mathbb{R},\mathbb{R})$, and $\theta \in \mathscr{C}(J\times\mathbb{R},\mathbb{R})$ are given functions and $\beta, \mu \in \mathbb{R}$ such that $\mu \neq 1$.

The structure of the paper is as follows: Section 2 provides a brief overview of the necessary preliminaries. In Section 3, we investigate the existence and uniqueness of solutions to the initial value problem (2) using the Banach contraction mapping principle (BCMP) and the Leray-Schauder fixed-point theorem. Section 4 presents an example to demonstrate the applicability of our results. Finally, Section 5 offers concluding remarks and suggests directions for future research.

2. Preliminaries

We define some essential definitions related to fractional calculus that is going to be used throughout the paper:

Definition 2.1. *If $\iota \in \mathbb{R}^+$ and $\mathsf{z} \in \mathbb{R}$, the Mittag-Leffler function is defined as:*

$$\mathbb{E}_{\iota}(\mathsf{z}) = \sum_{n=0}^{n=\infty} \frac{z^n}{\Gamma(n\iota+1)},$$

where the Euler gamma function $\Gamma(\cdot)$ is given by

$$\Gamma(\mathsf{z}) = \int_0^{\infty} t^{z-1}e^{-t}\,dt \quad (\mathbb{R}(\mathsf{z}) > 0).$$

Definition 2.2. *[9] Let $\vartheta \in \mathscr{C}(J,\mathbb{R})$ and $\iota \in [0,1]$, the left Atangana-Baleanu-Caputo fractional derivative(ABC) of ϑ of order ι is defined by*

$${}^{ABC}{}_0\mathcal{D}^{\iota}_{\xi}\vartheta(\xi) = \frac{\mathsf{B}(\iota)}{1-\iota}\int_0^{\xi} \mathbb{E}_{\iota}\left[-\frac{\iota}{1-\iota}(\xi-\sigma)^{\iota}\right]\vartheta'(\sigma)d\sigma,$$

where $\mathsf{B}(\iota) = 1-\iota+\frac{\iota}{\Gamma(\iota)} > 0$ is a normalization function satisfying $\mathsf{B}(0) = \mathsf{B}(1) = 1$.

The associated fractional integral is defined by

$${}^{AB}{}_0I^{\iota}_{\xi}\vartheta(\xi) = \frac{1-\iota}{\mathsf{B}(\iota)}\vartheta(\xi) + \frac{\iota}{\mathsf{B}(\iota)}{}_0I^{\iota}_{\xi}\vartheta(\xi).$$

where

$${}_0I^{\iota}_{\xi}\vartheta(\xi) = \frac{1}{\Gamma(\iota)}\int_0^{\xi}(\xi-\sigma)^{\iota-1}\vartheta(\sigma)d\sigma,$$

is the Riemann-Liouville fractional integral [10, 11] of ϑ of order ι.

Lemma 2.3. *[12] If $0 < \iota < 1$, then ${}^{AB}{}_0I^{\iota}_{\xi}\left({}^{ABC}{}_0\mathcal{D}^{\iota}_{\xi}\vartheta(\xi)\right) = \vartheta(\xi) - \vartheta(0)$.*

Lemma 2.4. *[13, 14] The equivalent fractional integral equation to the the ABC-FDEs*

$$\begin{aligned} {}^{ABC}{}_0\mathcal{D}^{\iota}_{\xi}\vartheta(\xi) &= g(\tau,\vartheta(\xi)), \quad \xi \in J = [0,T], T > 0 \\ \vartheta(0) &= \vartheta_0, \end{aligned}$$

is given by

$$\vartheta(\xi) = \vartheta_0 + \frac{1-\alpha}{\mathsf{B}(\iota)}g(\xi,\vartheta(\xi)) + \frac{\iota}{\mathsf{B}(\iota)\Gamma(\iota)}\int_0^{\xi}(\xi-\sigma)^{\iota-1}g(\sigma,\vartheta(\sigma))d\sigma.$$

Lemma 2.5. *(Leray–Schauder alternative see [15]). Let $\Psi : X \longrightarrow X$ be a completely continuous operator and*

$$\mathscr{P}_\Psi = \{x \in X : y = \delta \Psi x \text{ for some } 0 < \delta < 1\}.$$

Then, either the set $\mathscr{P}_\Psi$ is unbounded or Ψ has at least one fixed point.

3. Existence Result

In this section, we will prove the existence of a integral solution for problem (2). To obtain the existence of a integral solution, we will need the following assumptions:

(P_1): The map $\vartheta \longmapsto \vartheta\varphi(\hat{\varkappa}, \vartheta) - \eta(\hat{\varkappa}, \vartheta)$ is increasing in $\mathbb{R}$ for each $\hat{\varkappa} \in J$.

(P_2) : There exist positive constants γ_ϕ and γ_η, such that

$$|\varphi(\hat{\varkappa}, \vartheta)| \geq \gamma_\varphi,$$
$$|\eta(\hat{\varkappa}, \vartheta)| \leq \gamma_\eta.$$

(P_3) : There exists positive constants $\lambda_\varphi, \lambda_\eta$, and λ_η such that

$$|\varphi(\hat{\varkappa}, \vartheta) - \varphi(\hat{\varkappa}, \omega)| \leq \lambda_\varphi |\vartheta - \omega|,$$
$$|\eta(\hat{\varkappa}, \vartheta) - \eta(\hat{\varkappa}, \omega)| \leq \lambda_\eta |\vartheta - \omega|,$$
$$|\theta(\hat{\varkappa}, \vartheta) - \theta(\hat{\varkappa}, \omega)| \leq \lambda_\theta |\vartheta - \omega|.$$

for each $\hat{\varkappa} \in J$ and $\vartheta, \omega \in \mathbb{R}$.

Denote $\mathscr{C} := \mathscr{C}(J, \mathbb{R})$, the space of all continuous mapping defined on J into $\mathbb{R}$ endowed with the norm $\|\vartheta\| = \sup_{\hat{\varkappa} \in J} \|\vartheta(\hat{\varkappa})\|$.

Lemma 3.1. *Let $h \in \mathscr{C}(J, \mathbb{R})$, then ϑ is an integral solution of (2) if and only if it satisfies the following integral equation:*

$$\begin{aligned}\vartheta(\hat{\varkappa}) =& \frac{\beta}{(1+\mu)\phi(\hat{\varkappa}, \vartheta(\hat{\varkappa}))} - \frac{\mu(1-\iota)\theta(T, \vartheta(T))}{\mathrm{B}(\iota)(1+\mu)\phi(\hat{\varkappa}, \vartheta(\hat{\varkappa}))} - \frac{\mu\iota\iota}{(1+\mu)\mathrm{B}(\iota)\Gamma(\iota)\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}))} \int_0^T (T-s)^{\iota-1}\theta(s, \vartheta(s))\, \mathrm{d}s \\ &+ \frac{\eta(\hat{\varkappa}, \vartheta(\hat{\varkappa}))}{\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}))} + \frac{1-\iota}{\mathrm{B}(\iota)} \frac{\theta(\hat{\varkappa}, \vartheta(\hat{\varkappa}))}{\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}))} + \frac{\iota}{\mathrm{B}(\iota\iota)\Gamma(\iota)} \frac{1}{\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}))} \int_0^{\hat{\varkappa}} (\hat{\varkappa} - s)^{\iota-1}\theta(s, \vartheta(s))\, \mathrm{d}s, \quad \hat{\varkappa} \in J.\end{aligned}$$

Proof. Suppose that ϑ is a solution for (2), then we obtain

$$\begin{aligned}\vartheta(\hat{\varkappa})\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa})) =& \vartheta(0)\varphi(0, \vartheta(0)) - \eta(0, \vartheta(0)) + \eta(\hat{\varkappa}, \vartheta(\hat{\varkappa})) + \frac{1-\iota}{\mathrm{B}(\iota)}\theta(\hat{\varkappa}, \vartheta(\hat{\varkappa})) \\ &+ \frac{\alpha}{\mathrm{B}(\iota)\Gamma(\iota)} \int_0^{\hat{\varkappa}} (\hat{\varkappa} - s)^{\iota-1}\theta(s, \vartheta(s))\, \mathrm{d}s, \quad \text{for } \hat{\varkappa} \in J.\end{aligned} \tag{3}$$

Then we get

$$\begin{aligned}\mu\vartheta(T)\varphi(T, \vartheta(T)) =& \mu\vartheta(0)\varphi(0, \vartheta(0)) - \mu\eta(0, \vartheta(0)) + \mu\eta(T, \vartheta(T)) \\ &+ \mu\frac{1-\iota}{\mathrm{B}(\iota)}\theta(T, \vartheta(T)) + \frac{\mu\iota}{\mathrm{B}(\iota\iota)\Gamma(\iota)} \int_0^T (T-s)^{\iota-1}\theta(s, \vartheta(s))\, \mathrm{d}s.\end{aligned} \tag{4}$$

Hence, we get

$$\vartheta(0)\varphi(0, \vartheta(0)) + \mu\vartheta(T)\varphi(T, \vartheta(a)) = (\mu+1)\vartheta(0)\varphi(0, \vartheta(0)) - \mu\eta(0, \vartheta(0)) + \mu\eta(T, \vartheta(T)) + \mu\frac{1-\iota}{\mathrm{B}(\iota)}\theta(T, \vartheta(T))$$

$$+\frac{\mu\iota}{\mathrm{B}(\iota)\Gamma(\iota)}\int_0^T (T-s)^{\iota-1}\theta(s,\vartheta(s))\,\mathrm{d}s. \tag{5}$$

By using the second equation in (2), we obtain

$$\vartheta(0)\varphi(0,\vartheta(0))-\eta(0,\vartheta(0))=\frac{\beta}{1+\mu}-\mu\frac{1-\iota}{\mathrm{B}(\iota)(1+\mu)}\theta(T,\vartheta(T))-\frac{\mu\iota}{\mathrm{B}(\iota)\Gamma(\iota)(\mu+1)}\int_0^T (T-s)^{\iota-1}\theta(s,\vartheta(s))\,\mathrm{d}s. \tag{6}$$

By replacing in (3) , we obtain:

$$\begin{aligned}\vartheta(\hat{\varkappa})=&\frac{\beta}{(1+\mu)\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}))}-\frac{\mu(1-\iota)\theta(T,\vartheta(T))}{\mathrm{B}(\iota)(1+\mu)\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}))}-\frac{\mu\iota}{(1+\mu)\mathrm{B}(\iota)\Gamma(\iota)\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}))}\int_0^T (T-s)^{\iota\iota-1}\theta(s,\vartheta(s))\,\mathrm{d}s\\&+\frac{\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))}{\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}))}+\frac{1-\iota}{\mathrm{B}(\iota)}\frac{\theta(\hat{\varkappa},\vartheta(\hat{\varkappa}))}{\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}))}+\frac{\iota}{\mathrm{B}(\iota)\Gamma(\iota)\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}))}\int_0^{\hat{\varkappa}} (\hat{\varkappa}-s)^{\iota-1}\theta(s,\vartheta(s))\,\mathrm{d}s,\quad \hat{\varkappa}\in J.\end{aligned} \tag{7}$$

□

To reduce the form of mathematical expressions, consider the following notations:

$$\begin{aligned}\pi_1=&\frac{|\beta|}{\gamma_\varphi^2|\mu+1|}\lambda_\varphi+\frac{\mu(1-\iota)}{B(\iota)(1+\mu)}\left(\frac{\lambda_\theta r}{\gamma_\varphi^2}\lambda_\varphi+\frac{\lambda_k}{\gamma_\varphi^2}\lambda_\varphi+\frac{1}{\gamma_\varphi}\lambda_\theta\right)\\&+\frac{\mu\iota}{(1+\mu)\mathrm{B}(\iota)\Gamma(\iota)}\frac{T^\iota}{\iota}\left(\frac{\lambda_\theta}{\gamma_\varphi}+\frac{\lambda_\varphi}{\gamma_\varphi^2}\lambda_\theta r+\frac{\lambda_\varphi}{\gamma_\varphi^2}\lambda_k\right)+\frac{\lambda_\eta}{\gamma_\varphi}+\frac{\gamma_\eta}{\gamma_\varphi^2}\lambda_\varphi\\&+\frac{(1-\iota)}{\mathrm{B}(\iota)}\left(\frac{\lambda_\theta r}{\gamma_\varphi^2}\lambda_\varphi+\frac{\lambda_k}{\gamma_\varphi^2}\lambda_\varphi+\frac{\lambda_\theta}{\gamma_\varphi}\right)+\frac{\varphi}{\mathrm{B}(\varphi)\Gamma(\varphi)\gamma_\varphi}\lambda_\theta\frac{T^\iota}{\alpha}+\frac{\iota}{\mathrm{B}(\iota)\Gamma(\iota)\gamma_\varphi^2}\lambda_\varphi\frac{T^\iota}{\iota}(\lambda_\theta r+\lambda_k),\\\pi_2=&\frac{|\beta|}{|1+\mu|\gamma_\iota}+\frac{|\mu(1-\iota)|\lambda_k}{|\mathrm{B}(\iota)(1+\mu)|\gamma_\varphi}+\frac{(\mu\iota\lambda_k+|\mu+1|\iota\lambda_k)}{|(1+\mu)\mathrm{B}(\iota)|\Gamma(\iota)\gamma_\iota}\frac{T^\iota}{\iota}\\&+\frac{\gamma_\eta}{\gamma_\varphi}+\frac{|1-\iota|}{\mathrm{B}(\iota)}\frac{\lambda_k}{\gamma_\iota},\\\pi_3=&\frac{|1-\iota|}{\mathrm{B}(\iota)}\frac{\lambda_\theta}{\gamma_\iota}+\frac{|\mu(1-\iota)|\lambda_\theta}{|\mathrm{B}(\iota)(1+\mu)|\gamma_\varphi}+\frac{(\mu\iota\lambda_\theta+|\mu+1|\iota\lambda_\theta)}{|(1+\mu)\mathrm{B}(\iota)|\Gamma(\iota)\gamma_\varphi}\frac{T^\iota}{\iota}.\end{aligned}$$

Theorem 3.2. *Suppose that $(P_1)-(P_3)$ are satisfied. In addition, assume that the following condition is verified:*

$$\pi_1<1.$$

Then, the problem (2) has a unique solution.

Proof. Let us set $\sup_{\hat{\varkappa}\in J}\theta(\hat{\varkappa},0)=\lambda_k<\infty$, and define a closed ball B_r as follows:

$$B_r=\{\vartheta\in\mathscr{C}:\|\vartheta\|\le r\},$$

where

$$r\ge\frac{\pi_2}{1-\pi_3}. \tag{8}$$

Also, we define the following operator ψ on $\mathscr{C}$ by

$$\psi(\vartheta)(\hat{\varkappa}\hat{\varkappa})=\frac{\beta}{(1+\mu)\varphi(\hat{\varkappa},\varphi(\hat{\varkappa}))}-\frac{\mu(1-\iota)\theta(T,\vartheta(T))}{\mathrm{B}(\iota)(1+\mu)\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}))}-\frac{\mu\iota}{(1+\mu)\mathrm{B}(\iota)\Gamma(\iota)\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}))}\int_0^T (T-s)^{\iota-1}\theta(s,\vartheta(s))\,\mathrm{d}s$$

$$+\frac{\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))}{\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}))}+\frac{1-\iota}{\mathrm{B}(\iota)}\frac{\theta(\hat{\varkappa},\vartheta(\hat{\varkappa}))}{\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}))}+\frac{\iota}{\mathrm{B}(\iota)\Gamma(\iota)\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}))}\int_0^{\hat{\varkappa}}(\hat{\varkappa}-s)^{\iota-1}\theta(s,\vartheta(s))\,\mathrm{d}s,\quad \hat{\varkappa}\in J. \quad (9)$$

The proof will be made in two steps:

(i) : $\Psi\mathrm{B}_r \subseteq \mathrm{B}_r$. Indeed, for $\vartheta \in \mathrm{B}_r$ and $\hat{\varkappa} \in J$, we have

$$\begin{aligned}
|\psi(\vartheta)(\hat{\varkappa})| \leq & \frac{|\beta|}{|1+\mu||\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}))|}+\frac{|\mu(1-\iota)||\theta(T,\vartheta(T))-\theta(T,0)+\theta(T,0)|}{|\mathrm{B}(\iota)(1+\mu)||\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}))|}\\
&+\frac{\mu\iota}{|(1+\mu)\mathrm{B}(\iota)|\Gamma(\iota)|\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}))|}\int_0^T(T-s)^{\iota-1}\left(|\theta(s,\vartheta(s))-\theta(s,0)|+|\theta(s,0)|\right)\mathrm{d}s\\
&+\frac{|\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))|}{|\vartheta(\hat{\varkappa},\vartheta(\hat{\varkappa}))|}+\frac{1-\iota}{\mathrm{B}(\iota)}\frac{|\theta(\hat{\varkappa},\vartheta(\hat{\varkappa}))-\theta(\hat{\varkappa},0)+\theta(\hat{\varkappa},0)|}{|\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}))|}\\
&+\frac{\alpha}{\mathrm{B}(\iota)\Gamma(\iota)|\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}))|}\int_0^{\vartheta(\hat{\varkappa})}(\vartheta(\hat{\varkappa})-s)^{\iota-1}\left(|\theta(s,\vartheta(s))-\theta(s,0)|+|\theta(s,0)|\right)\mathrm{d}s\\
\leq & \frac{|\beta|}{|1+\mu|\gamma_\varphi}+\frac{|\mu(1-\iota)|\lambda_\theta|\vartheta(T)|}{|\mathrm{B}(\iota)(1+\mu)|\gamma_\varphi}+\frac{|\mu(1-\iota)|\lambda_k}{|B(\iota)(1+\mu)|\gamma_\varphi}\\
&+\frac{\mu\iota}{|(1+\mu)\mathrm{B}(\iota)|\Gamma(\iota)\gamma_\varphi}\int_0^T(T-s)^{\iota-1}\left(\lambda_\theta|\vartheta(s)|+\lambda_k\right)\mathrm{d}s\\
&+\frac{\gamma_\eta}{\gamma_\varphi}+\frac{|1-\iota|}{\mathrm{B}(\iota)}\frac{\lambda_\theta|\vartheta(\vartheta(\hat{\varkappa}))|}{\gamma_\varphi}+\frac{|1-\iota|}{\mathrm{B}(\iota)}\frac{\lambda_k}{\gamma_\varphi}\\
&+\frac{\iota}{\mathrm{B}(\iota)\Gamma(\iota)\gamma_\varphi}\int_0^{\vartheta(\hat{\varkappa})}(\vartheta(\hat{\varkappa})-s)^{\iota-1}\left(\lambda_\theta|\vartheta(s)|+\lambda_k\right)\mathrm{d}s\\
\leq & \frac{|\beta|}{|1+\mu|\gamma_\varphi}+\frac{|\mu(1-\iota)|\lambda_\theta r}{|\mathrm{B}(\iota)(1+\mu)|\gamma_\varphi}+\frac{|\mu(1-\iota)|\lambda_k}{|\mathrm{B}(\iota)(1+\mu)|\gamma_\varphi}\\
&+\frac{(\mu\iota\lambda_\theta+|\mu+1|\iota\lambda_\theta)}{|(1+\iota)\mathrm{B}(\iota)|\Gamma(\iota)\gamma_\varphi}\frac{T^\iota}{\iota}r+\frac{(\mu\iota\lambda_k+|\mu+1|\iota\lambda_k)}{|(1+\mu)\mathrm{B}(\iota)|\Gamma(\iota)\gamma_\phi}\frac{T^\iota}{\iota}\\
&+\frac{\gamma_\eta}{\gamma_\varphi}+\frac{|1-\iota|}{\mathrm{B}(\iota)}\frac{\lambda_\theta r}{\gamma_\varphi}+\frac{|1-\iota|}{\mathrm{B}(\iota)}\frac{\lambda_k}{\gamma_\varphi}\\
\leq & \frac{|\beta|}{|1+\mu|\gamma_\varphi}+\frac{|\mu(1-\iota)|\lambda_k}{|\mathrm{B}(\iota)(1+\mu)|\gamma_\varphi}\\
&+\frac{(\mu\iota\lambda_k+|\mu+1|\iota\lambda_k)}{|(1+\mu)\mathrm{B}(\iota)|\Gamma(\iota)\gamma_\varphi}\frac{T^\iota}{\iota}+\frac{\gamma_\eta}{\gamma_\varphi}+\frac{|1-\iota|}{\mathrm{B}(\iota)}\frac{\lambda_k}{\gamma_\varphi}\\
&+\left(\frac{|1-\iota|}{\mathrm{B}(\iota)}\frac{\lambda_\theta}{\gamma_\varphi}+\frac{|\mu(1-\iota)|\lambda_\theta}{|\mathrm{B}(\iota)(1+\mu)|\gamma_\varphi}+\frac{(\mu\iota\lambda_\theta+|\mu+1|\iota\lambda_\theta)}{|(1+\mu)\mathrm{B}(\iota)|\Gamma(\iota)\gamma_\varphi}\frac{T^\iota}{\iota}\right)r.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
|\psi(\vartheta)(\hat{\varkappa})| \leq & \frac{|\beta|}{|1+\mu|\gamma_\varphi}+\frac{|\mu(1-\iota)|\lambda_k}{|\mathrm{B}(\iota)(1+\mu)|\gamma_\varphi}\\
&+\frac{(\mu\iota\lambda_k+|\mu+1|\iota\lambda_k)}{|(1+\mu)\mathrm{B}(\iota)|\Gamma(\iota)\gamma_\varphi}\frac{T^\iota}{\iota}+\frac{\gamma_\eta}{\gamma_\varphi}+\frac{|1-\iota|}{\mathrm{B}(\iota)}\frac{\lambda_k}{\gamma_\varphi}\\
&+\left(\frac{|1-\iota|}{\mathrm{B}(\iota)}\frac{\lambda_\theta}{\gamma_\varphi}+\frac{|\mu(1-\iota)|\lambda_\theta}{|\mathrm{B}(\iota)(1+\mu)|\gamma_\varphi}+\frac{(\mu\iota\lambda_\theta+|\mu+1|\iota\lambda_\theta)}{|(1+\mu)\mathrm{B}(\iota)|\Gamma(\iota)\gamma_\varphi}\frac{T^\iota}{\iota}\right)r.
\end{aligned}$$

□

From (8), it follows that

$$\|\psi(v)\| \leq r.$$

(ii) Ψ is a contraction:
For $v, \omega \in B_r$ and $t \in I$, we have

$$\begin{aligned}
|\psi(\vartheta)(\hat{\varkappa}) - \psi(\omega)(\hat{\varkappa})| &\leq |\frac{\beta}{(1+\mu)\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}))} - \frac{\beta}{(1+\mu)\varphi(\hat{\varkappa}, \omega(\hat{\varkappa}))}| \\
&+ \frac{\mu(1-\iota)}{\mathrm{B}(\iota)(1+\mu)}|\frac{\theta(T, \vartheta(T))}{\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}))} - \frac{\theta(T, \vartheta(T))}{\varphi(\hat{\varkappa}, \omega(\hat{\varkappa}))} + \frac{\theta(T, \vartheta(T))}{\vartheta(\hat{\varkappa}, \omega(\hat{\varkappa}))} - \frac{\theta(T, \omega(T))}{\varphi(\hat{\varkappa}, \omega(\hat{\varkappa}))}| \\
&+ \frac{\mu\iota}{(1+\mu)\mathrm{B}(\iota)\Gamma(\iota)}\left(\frac{1}{\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}))}\int_0^T (T-s)^{\iota-1}\theta(s, \vartheta(s))\,\mathrm{d}s - \frac{1}{\varphi(t, \omega(\hat{\varkappa}))}\int_0^T (T-s)^{\iota-1}\theta(s, \omega(s))\,\mathrm{d}s\right) \\
&+ |\frac{\eta(\hat{\varkappa}, \vartheta(\hat{\varkappa}))}{\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}))} - \frac{\eta(\hat{\varkappa}, \omega(\hat{\varkappa}))}{\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}))} + \frac{\eta(\hat{\varkappa}, \omega(\hat{\varkappa}))}{\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}))} - \frac{\eta(\hat{\varkappa}, \omega(\hat{\varkappa}))}{\varphi(\hat{\varkappa}, \omega(\hat{\varkappa}))}| \\
&+ |\frac{1-\iota}{\mathrm{B}(\iota)}||\frac{\theta(\hat{\varkappa}, \vartheta(\hat{\varkappa}))}{\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}))} - \frac{\theta(\hat{\varkappa}, \vartheta(\hat{\varkappa}))}{\varphi(\hat{\varkappa}, \omega(\hat{\varkappa}))} + \frac{\theta(\hat{\varkappa}, \vartheta(\hat{\varkappa}))}{\varphi(\hat{\varkappa}, \omega(\hat{\varkappa}))} - \frac{\theta(\hat{\varkappa}, \omega(\hat{\varkappa}))}{\varphi(\hat{\varkappa}, \omega(\hat{\varkappa}))}| \\
&+ \frac{\iota}{\mathrm{B}(\iota)\Gamma(\iota)}\left(\frac{1}{\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}))}\int_0^{\hat{\varkappa}} (\hat{\varkappa}-s)^{\iota-1}\theta(s, \vartheta(s))\,\mathrm{d}s - \frac{1}{\varphi(\hat{\varkappa}, \omega(\hat{\varkappa}))}\int_0^{\hat{\varkappa}} (\hat{\varkappa}-s)^{\iota-1}\theta(s, \omega(s))\,\mathrm{d}s\right) \\
&\leq \frac{|\beta|}{\gamma_\varphi^2|\mu+1|}|\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa})) - \varphi(\hat{\varkappa}, \omega(\hat{\varkappa}))| \\
&+ \frac{\mu(1-\iota)}{\mathrm{B}(\hat{\varkappa})(1+\mu)}\left(\frac{\theta(T, \vartheta(T))}{\gamma_\varphi^2}|\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa})) - \varphi(\hat{\varkappa}, \omega(\hat{\varkappa}))| + \frac{1}{\gamma_\varphi}|\theta(T, \vartheta(T)) - \theta(T, \omega(T))|\right) \\
&+ \frac{\mu\iota}{(1+\mu)\mathrm{B}(\iota)\Gamma(\iota)}\left(\frac{1}{\gamma_\varphi}\int_0^T (T-s)^{\iota-1}|\theta(s, \vartheta(s)) - \theta(s, \omega(s))|\,\mathrm{d}s\right) \\
&+ \frac{\mu\iota}{(1+\mu)\mathrm{B}(\iota)\Gamma(\iota)\gamma_\varphi^2}|\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa})) - \varphi(\hat{\varkappa}, \omega(\hat{\varkappa}))|\int_0^T (T-s)^{\iota-1}\theta(s, \omega(s))\,\mathrm{d}s \\
&+ \frac{1}{\gamma_\varphi}|\eta(\hat{\varkappa}, \vartheta(\hat{\varkappa})) - \eta(\hat{\varkappa}, \omega(\hat{\varkappa}))| + \frac{\lambda_\eta}{\gamma_\varphi^2}|\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa})) - \vartheta(\hat{\varkappa}, \omega(\hat{\varkappa}))| \\
&+ \frac{(1-\iota)}{\mathrm{B}(\iota)}\left(\frac{|\theta(\hat{\varkappa}, \vartheta(\hat{\varkappa}))|}{\gamma_\varphi^2}|\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa})) - \varphi(\hat{\varkappa}, \omega(\hat{\varkappa}))| + \frac{1}{\gamma_\varphi}|\theta(\hat{\varkappa}, \vartheta(\hat{\varkappa})) - \theta(\hat{\varkappa}, \omega(\hat{\varkappa}))|\right) \\
&+ \frac{\iota}{\mathrm{B}(\iota)\Gamma(\alpha)}\left(\frac{1}{\gamma_\varphi}\int_0^{\hat{\varkappa}} (\hat{\varkappa}-s)^{\iota-1}|\theta(s, \vartheta(s)) - \theta(s, \omega(s))|\,\mathrm{d}s\right) \\
&+ \frac{\iota}{\mathrm{B}(\iota)\Gamma(\iota)\gamma_\varphi^2}|\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa})) - \varphi(\hat{\varkappa}, \omega(\hat{\varkappa}))|\int_0^{\hat{\varkappa}} (\hat{\varkappa}-s)^{\iota-1}\theta(s, \omega(s))\,\mathrm{d}s \\
&\leq \frac{|\beta|}{\gamma_\varphi^2|\mu+1|}\lambda_\varphi\|\vartheta - \omega\| \\
&+ \frac{\mu(1-\iota)}{\mathrm{B}(\iota)(1+\mu)}\left(\frac{\lambda_\theta r}{\gamma_\varphi^2}\lambda_\varphi\|\vartheta - \omega\| + \frac{\lambda_k}{\gamma_\varphi^2}\lambda_\varphi\|\vartheta - \omega\| + \frac{1}{\gamma_\varphi}\lambda_\theta\|\vartheta - \omega\|\right) \\
&+ \frac{\mu\iota}{(1+\mu)\mathrm{B}(\iota)\Gamma(\iota)}\frac{T^\iota}{\iota}\left(\frac{\lambda_\theta}{\gamma_\varphi}\|\vartheta - \omega\| + \frac{\lambda_\varphi}{\gamma_\varphi^2}\lambda_\theta r\|\vartheta - \omega\| + \frac{\lambda_\varphi}{\gamma_\varphi^2}\lambda_k\|\vartheta - \omega\|\right) \\
&+ \frac{\lambda_\eta}{\gamma_\varphi}\|\vartheta - \omega\| + \frac{\gamma_\eta}{\gamma_\varphi^2}\lambda_\varphi\|\vartheta - \omega\|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(1-\alpha)}{\mathrm{B}(\iota)}\left(\frac{\lambda_\theta r}{\gamma_\varphi^2}\lambda_\varphi\|\vartheta-\omega\|+\frac{\lambda_k}{\gamma_\varphi^2}\lambda_\varphi\|\vartheta-\omega\|+\frac{\lambda_\theta}{\gamma_\varphi}\|\vartheta-\omega\|\right)\\
&+\frac{\iota}{\mathrm{B}(\iota)\Gamma(\iota)\gamma_\varphi}\lambda_\theta\frac{T^\iota}{\iota}\|\vartheta-\omega\|+\frac{\iota}{B(\iota)\Gamma(\iota)\gamma_\varphi^2}\lambda_\varphi\|\vartheta-\omega\|\frac{T^\iota}{\iota}\left(\lambda_\theta r+\lambda_k\right)\\
\le&\left(\frac{|\beta|}{\gamma_\varphi^2|\mu+1|}\lambda_\varphi+\frac{\mu(1-\iota)}{\mathrm{B}(\iota)(1+\mu)}\left(\frac{\lambda_\theta r}{\gamma_\iota^2}\lambda_\varphi+\frac{\lambda_k}{\gamma_\varphi^2}\lambda_\varphi+\frac{\lambda_\theta}{\gamma_\varphi}\right)\right.\\
&+\frac{\mu\iota}{(1+\mu)\mathrm{B}(\iota)\Gamma(\iota)}\frac{T^\iota}{\iota}\left(\frac{\lambda_\theta}{\gamma_\varphi}+\frac{\lambda_\varphi}{\gamma_\varphi^2}\lambda_\theta r+\frac{\lambda_\varphi}{\gamma_\varphi^2}\lambda_k\right)+\frac{\lambda_\eta}{\gamma_\varphi}+\frac{\gamma_\eta}{\gamma_\varphi^2}\lambda_\varphi\\
&+\frac{(1-\iota)}{\mathrm{B}(\iota)}\left(\frac{\lambda_\theta r}{\gamma_\varphi^2}\lambda_\varphi+\frac{\lambda_k}{\gamma_\varphi^2}\lambda_\varphi+\frac{\lambda_\theta}{\gamma_\varphi}\right)+\frac{\alpha}{\mathscr{B}(\iota)\Gamma(\iota)\gamma_\varphi}\lambda_\theta\frac{T^\iota}{\iota}\\
&\left.+\frac{\iota}{\mathrm{B}(\iota)\Gamma(\iota)\gamma_\varphi^2}\lambda_\varphi\frac{T^\iota}{\iota}\left(\lambda_\theta r+\lambda_k\right)\|\vartheta-\omega\|\right)\le\pi_1\|\vartheta-\omega\|,
\end{aligned}
$$

which shows ψ is a contraction.
Thus, ψ is a contraction. Then, the existence and uniqueness of the solution is guaranteed by Banach's fixed-point theorem.
Our second result focuses on establishing the existence of solutions for the problem (2) using the Leray-Schauder alternative. For brevity, let us set:

$$\Lambda_1 = 1-\gamma_2\Lambda_0, \tag{10}$$

$$\Lambda_0 = \frac{T^\alpha}{\mathrm{B}(\iota)\Gamma(\iota)\gamma_\varphi}. \tag{11}$$

Theorem 3.3. *Suppose that (P_1) and (P_3) are satisfied. In addition, assume that there exist $\gamma_1, \gamma_2 > 0$, such that*

$$|\theta(\hat{\varkappa},\vartheta)| \le \gamma_1+\gamma_2\|\vartheta\|, \textit{ for each } (\hat{\varkappa},\vartheta)\in J\times\mathbb{R}.$$

Also, $\gamma_2\Lambda_0 < 1$. Then, problem (2) has at least one solution.

Proof. Let $\mathscr{B}\subseteq\mathscr{C}$ be a bounded subset. Then, there exists $\nu_\theta > 0$ such that

$$|\theta(\hat{\varkappa},\vartheta(\hat{\varkappa}))| \le \nu_\theta.$$

for each $\vartheta\in\mathscr{B}$ and $\hat{\varkappa}\in J$. The proof will be given in several steps:

(i) Ψ is uniformly bounded:
For $\vartheta\in\mathscr{B}$ and $\hat{\varkappa}\in J$, we have

$$
\begin{aligned}
|\psi(\vartheta)(\hat{\varkappa})| \le&\frac{|\beta|}{|1+\mu||\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}))|}+\frac{|\mu(1-\iota)||\theta(T,\vartheta(T))|}{|\mathscr{B}(\iota)(1+\mu)||\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}))|}\\
&+\frac{\mu\iota}{|(1+\mu)\mathscr{B}(\iota)|\Gamma(\iota)|\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}))|}\int_0^T(T-s)^{\iota-1}(|\theta(s,\vartheta(s))|)\mathrm{d}s\\
&+\frac{|\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))|}{|\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}))|}+\frac{1-\iota}{\mathscr{B}(\iota)}\frac{|\theta(\hat{\varkappa},\vartheta(\hat{\varkappa}))|}{|\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}))|}\\
&+\frac{\iota}{\mathscr{B}(\iota)\Gamma(\iota)|\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}))|}\int_0^{\hat{\varkappa}}(\hat{\varkappa}-s)^{\iota-1}(|\theta(s,\vartheta(s))|)\mathrm{d}s\\
\le&\frac{|\beta|}{|1+\mu|\gamma_\vartheta}+\frac{|\mu(1-\iota)|\nu_\theta}{|\mathscr{B}(\iota)(1+\mu)|\gamma_\vartheta}
\end{aligned}
$$

$$+\frac{\mu\iota\nu_\theta}{|(1+\mu)\mathscr{B}(\iota)|\Gamma(\iota)\gamma_\varphi}\frac{T^\iota}{\iota}+\frac{\gamma_\eta}{\gamma_\varphi}+\frac{|1-\iota|}{\mathscr{B}(\iota)}\frac{\nu_\theta}{\gamma_\varphi}+\frac{\iota\nu_\theta}{\mathscr{B}(\iota)\Gamma(\iota)\gamma_\varphi}\frac{T^\iota}{\iota}.$$

Then, ψ is uniformly bounded.

$$\begin{aligned}
|\psi(\vartheta)(\varepsilon_1)-\varphi(\vartheta)(\varepsilon_2)| \le& \left|\frac{\beta}{(1+\mu)\varphi(\varepsilon_1,\vartheta(\varepsilon_1))}-\frac{\beta}{(1+\mu)\varphi(\varepsilon_2,\vartheta(\varepsilon_2))}\right|+\frac{\mu(1-\iota)}{\mathscr{B}(\iota)(1+\mu)}\left|\frac{\theta(T,\vartheta(T))}{\varphi(\varepsilon_1,\vartheta(\varepsilon_1))}-\frac{\theta(T,\vartheta(T))}{\varphi(\varepsilon_2,\vartheta(\varepsilon_2))}\right|\\
&+\frac{\mu\iota}{(1+\mu)\mathscr{B}(\iota)\Gamma(\iota)}\left(\frac{1}{\varphi(\varepsilon_1,\vartheta(\varepsilon_1))}-\frac{1}{\varphi(\varepsilon_2,\vartheta(\varepsilon_2))}\right)\int_0^T(T-s)^{\iota-1}|\theta(s,\vartheta(s))|\,\mathrm{d}s\\
&+\|\frac{\eta(\varepsilon_1,\vartheta(\varepsilon_1))}{\varphi(\varepsilon_1,\vartheta(\varepsilon_1))}-\frac{\eta(\varepsilon_2,\vartheta(\tau_2))}{\varphi(\varepsilon_1,\vartheta(\varepsilon_1))}\|+\left|\frac{\eta(\varepsilon_2,\vartheta(\varepsilon_2))}{\varphi(\varepsilon_1,\vartheta(\varepsilon_1))}-\frac{\eta(\varepsilon_2,\vartheta(\varepsilon_2))}{\varphi(\varepsilon_2,\vartheta(\varepsilon_2))}\right|\\
&+\left|\frac{1-\iota}{\mathscr{B}(\iota)}\right|\left|\frac{\theta(\varepsilon_1,\vartheta(\varepsilon_1))}{\varphi(\varepsilon_1,\vartheta(\varepsilon_1))}-\frac{\theta(\varepsilon_2,\vartheta(\varepsilon_2))}{\varphi(\varepsilon_2,\vartheta(\varepsilon_2))}\right|\\
&+\left|\frac{\iota}{\mathscr{B}(\iota)\Gamma(\iota)}\right|\,|\frac{1}{\varphi(\varepsilon_1,\vartheta(\varepsilon_1))}\int_0^{\varepsilon_1}(\varepsilon_1-s)^{\iota-1}\theta(s,\vartheta(s))\,\mathrm{d}s\\
&-\frac{1}{\varphi(\varepsilon_1,\vartheta(\varepsilon_1))}\int_0^{\varepsilon_2}(\varepsilon_2-s)^{\iota-1}\theta(s,\vartheta(s))\,\mathrm{d}s|\\
&+\left|\frac{\iota}{\mathscr{B}(\iota)\Gamma(\iota)}\right|\,|\frac{1}{\varphi(\varepsilon_1,\vartheta(\varepsilon_1))}\int_0^{\varepsilon_2}(\varepsilon_2-s)^{\iota-1}\theta(s,\vartheta(s))\,\mathrm{d}s\\
&-\frac{1}{\varphi(\varepsilon_2,\vartheta(\varepsilon_2))}\int_0^{\varepsilon_2}(\varepsilon_2-s)^{\iota-1}\theta(s,\vartheta(s))\,\mathrm{d}s|\\
\le&\frac{\beta}{(1+\mu)\gamma_\varphi^2}\left|\varphi(\varepsilon_1,\vartheta(\tau_1))-\varphi(\varepsilon_2,\vartheta(\varepsilon_2))\right|+\frac{\mu(1-\iota)\nu_\theta}{\gamma_\varphi^2}\left|\varphi(\varepsilon_1,\vartheta(\varepsilon_1))-\varphi(\varepsilon_2,\vartheta(\varepsilon_2))\right|\\
&+\frac{\mu\iota\nu_\theta}{(1+\mu)\mathscr{B}(\iota)\Gamma(\iota)\gamma_\varphi^2}\frac{T^\iota}{\iota}\left|\varphi(\varepsilon_1,\vartheta(\varepsilon_1))-\varphi(\varepsilon_2,\vartheta(\varepsilon_2))\right|+\frac{1}{\gamma_\varphi}\left|\eta(\varepsilon_1,\vartheta(\varepsilon_1))-\eta(\varepsilon_2,\vartheta(\varepsilon_2))\right|\\
&+\frac{\gamma_\eta}{\gamma_\varphi^2}\left|\varphi(\varepsilon_1,\vartheta(\tau_1))-\varphi(\varepsilon_2,\vartheta(\varepsilon_2))\right|+\frac{|1-\iota|\nu_\theta}{\mathscr{B}(\iota)\gamma_\varphi^2}\left|\varphi(\varepsilon_1,\vartheta(\varepsilon_1))-\varphi(\varepsilon_2,\vartheta(\varepsilon_2))\right|\\
&+\frac{\iota\nu_\theta}{\mathscr{B}(\iota)\Gamma(\iota)\gamma_\varphi}\left(\frac{\varepsilon_1^\iota}{\iota}-\frac{\tau_2^\iota}{\iota}\right)+\frac{\iota\nu_\theta}{\mathscr{B}(\iota)\Gamma(\iota)\gamma_\varphi^2}\frac{\varepsilon_2^\iota}{\iota}\left|\varphi(\varepsilon_1,\vartheta(\varepsilon_1))-\varphi(\varepsilon_2,\vartheta(\varepsilon_2))\right|\longrightarrow 0,\\
&\text{as}\quad \varepsilon_1\longrightarrow\varepsilon_2. \qquad (12)
\end{aligned}$$

Hence, ψ is equicontinuous.

(iii) $\mathscr{P}_\Psi$ is bounded:
We denote by

$$\mathscr{P}_\Psi=\{\vartheta\in\mathbb{R}:\vartheta=\delta\Psi(\vartheta),0\le\delta\le1\}.$$

Let $v\in\mathscr{P}_\Psi$ and $t\in J$, we have

$$|\vartheta(\hat{\varkappa})|\le\frac{|\beta|}{|1+\mu|\gamma_\varphi}+\frac{|\mu(1-\iota)|\nu_\theta}{|\mathscr{B}(\iota)(1+\mu)|\gamma_\varphi}+\frac{\mu\iota\nu_\theta}{|(1+\mu)\mathscr{B}(\iota)|\Gamma(\iota)\gamma_\varphi}\frac{T^\iota}{\iota}+\frac{\gamma_\eta}{\gamma_\varphi}+\frac{|1-\iota|}{\mathscr{B}(\iota)}\frac{\nu_\theta}{\gamma_\varphi}+\frac{\iota(\gamma_1+\gamma_2\|\vartheta\|)}{\mathscr{B}(\iota)\Gamma(\iota)\gamma_\varphi}\frac{T^\iota}{\iota}.$$

which implies that

$$\|\vartheta(\hat{\varkappa})\|\le\frac{|\beta|}{|1+\mu|\gamma_\varphi}+\frac{|\mu(1-\iota)|\nu_\theta}{|\mathscr{B}(\iota)(1+\mu)|\gamma_\varphi}+\frac{\mu\iota\nu_\theta}{|(1+\mu)\mathscr{B}(\iota)|\Gamma(\iota)\gamma_\varphi}\frac{T^\iota}{\iota}+\frac{\gamma_\eta}{\gamma_\varphi}+\frac{|1-\iota|}{\mathscr{B}(\iota)}\frac{\nu_\theta}{\gamma_\varphi}+\frac{\iota(\gamma_1+\gamma_2\|\vartheta\|)}{\mathscr{B}(\iota)\Gamma(\iota)\gamma_\varphi}\frac{T^\iota}{\iota}.$$

which, in view of (10), can be expressed as:

$$\|\vartheta(\hat{\varkappa})\|\le\frac{1}{\Lambda_1}\left(\frac{|\beta|}{|1+\mu|\gamma_\varphi}+\frac{|\mu(1-\iota)|\nu_\theta}{|\mathscr{B}(\iota)(1+\mu)|\gamma_\varphi}+\frac{\mu\iota\nu_\theta}{|(1+\mu)\mathscr{B}(\iota)|\Gamma(\iota)\gamma_\varphi}\frac{T^\iota}{\iota}+\frac{\gamma_\eta}{\gamma_\varphi}+\frac{|1-\iota|}{\mathscr{B}(\iota)}\frac{\nu_\theta}{\gamma_\varphi}+\frac{\iota\gamma_1}{\mathscr{B}(\iota)\Gamma(\iota)\gamma_\varphi}\frac{T^\iota}{\iota}\right).$$

This demonstrates that the set $\mathscr{P}$ is bounded. As a result, all the conditions of Lemma 2.5 are satisfied. Therefore, the operator Ψ has at least one fixed point, which corresponds to a solution of problem (2). This completes the proof. □

Now, we give an example to illustrate the obtained results.

4. Example

$$\begin{cases} {}^{ABC}{}_0\mathcal{D}^{\frac{1}{2}}_{\hat{\varkappa}}[\vartheta(\hat{\varkappa})\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}))-\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))]=\theta(\hat{\varkappa},\vartheta(\hat{\varkappa})), \quad \hat{\varkappa}\in J=[0,1] \\ \vartheta(0)\varphi(0,\vartheta(0))+17\vartheta(T)\vartheta(T,\vartheta(T))=\eta(0,\vartheta(0))+17\eta(T,\vartheta(T))+1, \end{cases} \tag{13}$$

Here, we have

$$\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}))=\frac{arctan(\hat{\varkappa})}{4}|\vartheta(\hat{\varkappa})|+\frac{1}{4},$$

$$\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))=\frac{1}{5}+\frac{1}{7}\vartheta(\hat{\varkappa}),$$

$$\theta(\hat{\varkappa},\vartheta(\hat{\varkappa}))=\hat{\varkappa}^2(2+\frac{\vartheta(\hat{\varkappa})}{5}).$$

We can easily verify that

$$|\eta(\hat{\varkappa},\vartheta_2)-\eta(\hat{\varkappa},\vartheta_1|\leq\frac{1}{7}|\vartheta_2-\vartheta_1|,$$

and

$$|\theta(\hat{\varkappa},\vartheta_2)-\theta(\hat{\varkappa},\vartheta_1|\leq\frac{1}{5}|\vartheta_2-\vartheta_1|.$$

We can easily verify that

$$\iota=\frac{1}{2},\beta=1,\mu=17,\lambda_k=0,\lambda_\varphi=\frac{1}{4},\lambda_\theta=\frac{1}{5},\lambda_\eta=\frac{1}{7},\gamma_\varphi=\frac{1}{4},\gamma_\varphi=\frac{1}{5}.$$

and

$$\begin{aligned}\pi_1 =&\frac{|\beta|}{\gamma_\varphi^2|\mu+1|}\lambda_\varphi+\frac{\mu(1-\iota)}{\mathscr{B}(\iota)(1+\mu)}\left(\frac{\lambda_\theta r}{\gamma_\varphi^2}\lambda_\varphi+\frac{\lambda_k}{\gamma_\varphi^2}\lambda_\varphi+\frac{1}{\gamma_\varphi}\lambda_\theta\right)\\ &+\frac{\mu\iota}{(1+\mu)\mathscr{B}(\iota)\Gamma(\iota)}\frac{T^\iota}{\iota}\left(\frac{\lambda_\theta}{\gamma_\varphi}+\frac{\lambda_\varphi}{\gamma_\varphi^2}\lambda_\theta r+\frac{\lambda_\varphi}{\gamma_\varphi^2}\lambda_k\right)+\frac{\lambda_\eta}{\gamma_\varphi}+\frac{\gamma_\eta}{\gamma_\varphi^2}\lambda_\varphi\\ &+\frac{(1-\iota)}{\mathscr{B}(\iota)}\left(\frac{\lambda_\theta r}{\gamma_\varphi^2}\lambda_\varphi+\frac{\lambda_k}{\gamma_\varphi^2}\lambda_\varphi+\frac{\lambda_\theta}{\gamma_\varphi}\right)+\frac{\iota}{\mathscr{B}(\iota)\Gamma(\iota)\gamma_\varphi}\lambda_\theta\frac{T^\iota}{\iota}\\ &+\frac{\iota}{\mathscr{B}(\iota)\Gamma(\iota)\gamma_\varphi^2}\lambda_\varpi\frac{T^\iota}{\iota}(\lambda_\theta r+\lambda_k)=0.2987564231.\end{aligned}$$

As all of the assumptions in Theorem 3.2 are satisfied, our results can be directly applied to the problem (13).

5. Conclusion

The study on the existence and uniqueness of solutions for boundary value problems involving nonlinear hybrid differential equations with the Atangana-Baleanu-Caputo ABC-fractional derivative establishes that unique solutions can be ensured under specific assumptions, utilizing fixed-point theorems such as Banach's and nonlinear Leray-Schauder alternative. The findings highlight that the ABC-fractional derivative, with its non-singular kernel and non-local properties, offers a powerful framework for modeling complex systems characterized by memory effects and hereditary behavior. The role of boundary conditions is emphasized as crucial for ensuring the mathematical coherence and robustness of the solutions. Additionally, the study underscores the significance of hybrid equations in representing real-world systems that integrate both discrete and continuous dynamics. This research also opens pathways for future exploration, including the development of numerical methods for approximating solutions and the study of alternative fractional derivatives, with potential applications spanning control theory, viscoelasticity, and biological modeling.

Declarations:

Data availability:

No data set were used in this study.

Conflict of interest:

We declare that we do not have any commercial or associative interest that represents a confict of interest in connection with the work submitted.

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