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On a new class of Ψ-Hilfer fractional differential equation involving topological degree method

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Abstract. This paper focuses on investigating the existence and uniqueness of solutions for a novel category of Ψ-Hilfer-type fractional differential equations. We employ the approach of topological degree theory for condensing maps to establish the existence of solutions. Additionally, to deal with the uniqueness, we utilize Banach's contraction principle. To demonstrate the practical implications of our theoretical findings, we present an illustrative example.

1. Introduction

Recently, several mathematicians are interested in fractional differential equations, due to the fact that it can effectively model numerous phenomena and has been shown to do so across a variety of scientific domains. as physics, mechanics, biology, chemistry, and control theory, and other domains for example, see [1, 4, 8, 11, 13, 15, 19, 22–25, 27].

Fractional integrals and derivatives have diverse definitions, the two most well-known ones are the Riemann-Liouville and the Caputo fractional derivatives. Hilfer [14] present the interpolation of these derivatives named the Hilfer fractional derivative of order ϖ and parameter $\sigma \in [0, 1]$, more details for Hilfer and Ψ-Hilfer derivative, we direct readers to the papers [3, 12, 20, 27]. In order to demonstrate the existence and uniqueness of solutions for different classes of fractional differential equations, several versions of fixed point theorems are frequently used, Isaia [15] presented a new fixed theorem for condensing operators that was derived using coincidence degree theory. Researchers used Isaia's fixed point theorem to prove the existence of solutions for a variety of types of nonlinear differential equations [2, 7, 17, 21].

In 2020, Baitiche et al. [8], proved the existence and uniqueness of solutions to some nonlinear fractional differential equations involving the Ψ–Caputo fractional derivative with multi-point boundary conditions based on the technique of topological degree theory for condensing maps and Banach contraction principle. In 2022, Faree and Panchal [11], investigated the existence and uniqueness of solutions to boundary value problems involving the Caputo fractional derivative in Banach space by topological structures with some appropriate conditions.

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Inspired by the works mentioned previously, we merge their concepts in this paper to examine the existence and uniqueness of solution for the following problem

$$\begin{cases} {}^{H}\mathfrak{D}^{\varpi,\sigma;\Psi}w(\tau) = \mathfrak{f}(\tau, w(\tau)), \quad \tau \in \Theta := [\zeta, \iota] \\ w(\zeta) = 0, \qquad w(\iota) = \sum_{i=1}^{n} \omega_i w(\xi_i). \end{cases} \tag{1}$$

Where ${}^{H}\mathfrak{D}^{\varpi,\sigma;\Psi}$ is the Ψ-Hilfer fractional derivative of order ϖ, $1 < \varpi \leq 2$ and parameter σ, $0 \leq \sigma \leq 1$, $\omega_i \in \mathbb{R}$, $i = 1, ..., n$, $\zeta < \xi_1 < ... < \xi_n < \iota$, $\mathfrak{f} \in C\Big(\Theta \times \mathbb{R}, \mathbb{R}\Big)$.

This work's originality lies in its analysis of a current case of fractional derivative named the Ψ-Hilfer fractional derivative [28], this new type of fractional derivative interpolate the well-known fractional derivatives (Riemman-Liouville, Caputo, Ψ-Riemman-Liouville, Hilfer-Hadamard, Katugampola derivetive), for different values of function Ψ and parameter σ such as

- ⋆ If $\Psi(\tau) = \tau$ and $\sigma = 1$, then the problem (1) reduces to Caputo-type.
- ⋆ If $\Psi(\tau) = \tau$ and $\sigma = 0$, then the problem (1) reduces to Riemman-Liouville-type.
- ⋆ If $\sigma = 0$, then the problem (1) reduces to the Ψ-Riemman-Liouville-type.
- ⋆ If $\Psi(\tau) = \tau$, then the problem (1) reduces to Hilfer-type.
- ⋆ If $\Psi(\tau) = log(\tau)$, then the problem (1) reduces to Hilfer-Hadamard-type.
- ⋆ If $\Psi(\tau) = \tau^{\rho}$, then the problem (1) reduces to Katugampola-type.

The remainder of the paperwork is organized as follows : In section 2, we recollect a few notes, definitions, and lemmas from fractional calculus and significant results that our investigation will make advantage of. In section 3, Utilizing topological techniques, we look into the existence result for the problem (1), and by making use of Banach contraction principle we prove the uniqueness of solution. In section 4, an example is given to highlight the main result.

2. Preliminaries

Definition 2.1. *[18] Let $\Psi'(\tau) > 0$ and $\varpi > 0$. The Ψ-Riemann-Liouville fractional integral of order ϖ of a function $\mathfrak{f}$ with respect to another function Ψ on $[\zeta, \iota]$ is defined by*

$$\mathfrak{I}_{\zeta^+}^{\varpi;\Psi}\mathfrak{f}(\tau) = \frac{1}{\Gamma(\varpi)} \int_{\zeta}^{\tau} \Psi'(\tau)(\Psi(\tau) - \Psi(s))^{\varpi-1}\mathfrak{f}(s)ds, \tag{2}$$

where $\Gamma(.)$ represents the Gamma function.

Definition 2.2. *[18] Let $\Psi'(\tau) > 0$ and $\varpi > 0$, $n \in \mathbb{N}$. The Ψ-Riemann-Liouville derivative of a function $\mathfrak{f}$ with respect to another function Ψ of order ϖ is defined by*

$$\mathfrak{D}_{\zeta^+}^{\varpi;\Psi}\mathfrak{f}(\tau) = \left(\frac{1}{\Psi'(\tau)}\frac{d}{dt}\right)^n \mathfrak{I}_{\zeta^+}^{n-\varpi;\Psi}\mathfrak{f}(\tau) \tag{3}$$

$$= \frac{1}{\Gamma(n - \varpi)}\left(\frac{1}{\Psi'(\tau)}\frac{d}{dt}\right)^n \int_{\zeta}^{\tau} \Psi'(\tau)(\Psi(\tau) - \Psi(s))^{n-\varpi-1}\mathfrak{f}(s)ds, \tag{4}$$

where $n - 1 < \varpi < n, n = [\varpi] + 1$, and $[\varpi]$ denotes the integer part of the real number ϖ.

Definition 2.3. *[18] Let $n-1<\varpi<n$ with $n\in\mathbb{N}$ and $\mathfrak{f},\Psi\in C^n(\Theta,\mathbb{R})$ two functions such that $\Psi'(\tau)>0$ for all $\tau\in\Theta$. The Ψ-Hilfer fractional derivative of a function $\mathfrak{f}$ of order ϖ and type $0\leq\sigma\leq 1$, is defined by*

$$\begin{aligned} {}^{H}\mathfrak{D}^{\varpi,\sigma;\Psi}_{\zeta^+}\mathfrak{f}(\tau) &= \mathfrak{I}^{\sigma(n-\varpi);\Psi}_{\zeta^+}\left(\frac{1}{\Psi'(\tau)}\frac{d}{dt}\right)^{n}\mathfrak{I}^{(1-\sigma)(n-\varpi);\Psi}_{\zeta^+}\mathfrak{f}(\tau) \\ &= \mathfrak{I}^{\gamma-\varpi;\Psi}_{\zeta^+}\mathfrak{D}^{\gamma;\Psi}_{\zeta^+}\mathfrak{f}(\tau), \end{aligned} \tag{5}$$

where $n-1<\varpi<n, n=[\varpi]+1$, and $\gamma=\varpi+\sigma(n-\varpi)$.

Lemma 2.4. *[18] Let $\varpi,\sigma>0$. Then we have*

$$\mathfrak{I}^{\varpi;\Psi}_{\zeta^+}\mathfrak{I}^{\sigma;\Psi}_{\zeta^+}\mathfrak{f}(\tau)=\mathfrak{I}^{\varpi+qa;\Psi}_{\zeta^+}\mathfrak{f}(\tau), \tau>\zeta. \tag{6}$$

Proposition 2.5. *[18] Let $\zeta\geq 0$ and $\tau>\zeta$. Then, we have*

(i) $\mathfrak{I}^{\varpi;\Psi}_{\zeta^+}(\Psi(s)-\Psi(\zeta))^{\tau-1}(\tau)=\dfrac{\Gamma(\tau)}{\Gamma(\tau+\varpi)}(\Psi(s)-\Psi(\zeta))^{\tau+\varpi-1}(\tau)$.

(ii) ${}^{H}\mathfrak{D}^{\varpi;\Psi}_{\zeta^+}(\Psi(s)-\Psi(\zeta))^{\tau-1}(\tau)=\dfrac{\Gamma(\tau)}{\Gamma(\tau+\varpi)}(\Psi(s)-\Psi(\zeta))^{\tau-\varpi-1}(\tau)$.

Lemma 2.6. *[18] If $\mathfrak{f}\in C^n([\zeta,\iota],\mathbb{R})$, $n-1<\varpi<n$, $\quad 0\leq\sigma\leq 1$ and $\gamma=\varpi+\sigma(n-\varpi)$, then*

$$\mathfrak{I}^{\varpi;\Psi}_{\zeta^+}({}^{H}\mathfrak{D}^{\varpi,\sigma;\Psi}_{\zeta^+}\mathfrak{f})(\tau)=\mathfrak{f}(\tau)-\sum_{k=1}^{n}\frac{(\Psi(\tau)-\Psi(\zeta))^{\gamma-k}}{\Gamma(\gamma-k+1)}\mathfrak{f}^{[n-k]}_{\Psi}\mathfrak{I}^{(1-\sigma)(n-\varpi);\Psi}_{\zeta^+}\mathfrak{f}(\zeta), \tag{7}$$

for all $\tau\in\Theta$, where $\mathfrak{f}^{[n-k]}_{\Psi}\mathfrak{f}(\tau):=\left(\dfrac{1}{\Psi'(\tau)}\dfrac{d}{dt}\right)^{n-k}\mathfrak{f}(\tau)$.

Definition 2.7. *[9] The Kuratowski measure of non-compactness is the mapping $\delta:\Gamma_X\longrightarrow\mathbb{R}_+$ defined by:*

$$\delta(B)=inf\{\varepsilon>0: B\ can\ be\ covered\ by\ finitely\ many\ sets\ with\ diameter\leq\varepsilon\}. \tag{8}$$

Where Γ_X is the class of non-empty and bounded subsets of X.

Proposition 2.8. *[9] The Kuratowski measure of noncompactness δ satisfies the following properties*

1- $\delta(\mathfrak{A})=0$ *iff* $\mathfrak{A}$ *is relatively compact,*

2- $\mathfrak{A}\subset\mathfrak{B}\Rightarrow\delta(\mathfrak{A})\leq\delta(\mathfrak{B})$,

3- $\delta(\mathfrak{A})=\delta(\overline{\mathfrak{A}})=\delta(conv(\mathfrak{A}))$, *where* $\overline{\mathfrak{A}}$ *and* $conv(\mathfrak{A})$ *is the closure and the convex hull of* $\mathfrak{A}$ *respectively,*

4- $\delta(\mathfrak{A}+\mathfrak{B})\leq\delta(\mathfrak{A})+\delta(\mathfrak{B})$,

5- $\delta(k\mathfrak{A})=|k|\delta(\mathfrak{A})$, $k\in\mathbb{R}$,

Definition 2.9. *Let $\mathcal{H}:\mathfrak{A}\longrightarrow X$ be a continuous bounded map. The operator $\mathcal{H}$ is said to be δ-Lipschitz if there exists $l\geq 0$ such that*

$$\delta(\mathcal{H}(\mathfrak{B}))<l\delta(\mathfrak{B}),\ for\ every\mathfrak{B}\subset\mathfrak{A}. \tag{9}$$

Furthermore, if $l<1$, then $\mathcal{H}$ is a strict δ-contraction.

Definition 2.10. *$\mathcal{H}:\mathfrak{A}\longrightarrow X$ is called δ-condensing if*

$$\delta(\mathcal{H}(\mathfrak{B}))<\delta(\mathfrak{B}), \tag{10}$$

for every bounded and nonprecompact subset $\mathfrak{B}$ of A, with $\delta(\mathfrak{B})>0$.

Definition 2.11. *We say that the function $\mathcal{H} : \mathfrak{A} \longrightarrow X$ is Lipschitz if there exists $l > 0$ such that*

$$\|\mathcal{H}(w) - \mathcal{H}(v)\| \leq l\|w - v\|, \text{ for all } w, v \in A, \tag{11}$$

Furthermore, if $l < 1$, then $\mathcal{H}$ is a strict contraction.

Proposition 2.12. *[9, 15] If $\mathcal{H}, \mathcal{K} : \mathfrak{A} \longrightarrow X$, are δ-Lipschitz mapping with constants l_1 and l_2 respectively, then $\mathcal{H} + \mathcal{K} : \mathfrak{A} \longrightarrow X$ is δ-Lipschitz mapping with constant $l_1 + l_2$.*

Proposition 2.13. *[9, 15] If $\mathcal{H} : \mathfrak{A} \longrightarrow X$, is compact, then $\mathcal{H}$ is δ-Lipschitz mapping with constant $l = 0$.*

Proposition 2.14. *[9, 15] If $\mathcal{H} : \mathfrak{A} \longrightarrow X$, is Lipschitz mapping with constant l, then $\mathcal{H}$ is δ-Lipschitz mapping with the same constant l.*

Theorem 2.15. *[15] Let $\mathcal{S} : \mathfrak{A} \longrightarrow X$ be δ-condensing and*

$$\Xi_\beta = \{w \in X : w = \beta\mathcal{S}w, \text{ for some } 0 \leq \beta \leq 1\}. \tag{12}$$

If Ξ_β is a bounded set in X, so there exists $r > 0$, such that $\Xi_\beta \in \mathcal{B}_r(0)$, then the degree

$$deg\Big(I - \beta\mathcal{S}, \mathcal{B}_r(0), 0\Big) = 1, \text{ for all } \beta \in [0, 1]. \tag{13}$$

Consequently, $\mathcal{S}$ has at least one fixed point and the set of the fixed points of $\mathcal{S}$ lies in $\mathcal{B}_r(0)$.

3. Main Result

Lemma 3.1. *Let $\zeta \geq 0$, $1 < \varpi \leq 2$, $0 \leq \sigma \leq 1$, $\gamma = \varpi + 2\sigma - \varpi\sigma$ and $\mathfrak{h} \in C(\Theta, \mathbb{R})$. Then the function w is a solution of the following boundary value problem:*

$$\begin{cases} {}^H\mathfrak{D}^{\varpi,\sigma;\Psi} w(\tau) = \mathfrak{h}(\tau), \quad \tau \in \Theta := [\zeta, \iota] \\ w(\zeta) = 0 \quad , \quad w(\iota) = \displaystyle\sum_{i=1}^{n} \omega_i w(\xi_i), \ \zeta < \xi_i < \iota, \end{cases} \tag{14}$$

if and only if

$$w(\tau) = \mathfrak{I}^{\varpi;\Psi}\mathfrak{h}(\tau) + \frac{(\Psi(\tau) - \Psi(\zeta))^{\gamma-1}}{\Lambda\Gamma(\gamma)}\Big(\mathfrak{I}^{\varpi;\Psi}\mathfrak{h}(\iota) - \sum_{i=1}^{n} \omega_i \mathfrak{I}^{\varpi;\Psi}\mathfrak{h}(\xi_i)\Big), \tag{15}$$

where

$$\Lambda = \sum_{i=1}^{n} \omega_i \frac{(\Psi(\xi_i) - \Psi(\zeta))^{\gamma-1}}{\Gamma(\gamma)} - \frac{(\Psi(\iota) - \Psi(\zeta))^{\gamma-1}}{\Gamma(\gamma)} \neq 0 \tag{16}$$

Proof. The problem (14) can be written as

$$\mathfrak{I}^{\gamma-\varpi;\Psi}\mathfrak{D}^{\gamma;\Psi} w(\tau) = \mathfrak{h}(\tau). \tag{17}$$

Where $\gamma = \varpi + 2\sigma - \varpi\sigma$, applying the Ψ-Riemann-Liouville fractional integral of order ϖ to both sides of (17) and by using Lemma 2.6 we obtain

$$w(\tau) = \mathfrak{I}^{\varpi;\Psi}\mathfrak{h}(\tau) + \frac{c_0}{\Gamma(\gamma)}((\Psi(\tau) - \Psi(\zeta))^{\gamma-1} + \frac{c_1}{\Gamma(\gamma - 1)}((\Psi(\tau) - \Psi(\zeta))^{\gamma-2}, \tag{18}$$

where c_0, c_1. Next, from using the boundary condition $w(\zeta) = 0$ in (18) we obtain that $c_1 = 0$. Then, we get

$$w(\tau) = \mathfrak{I}^{\varpi;\Psi}\mathfrak{h}(\tau) + \frac{c_0}{\Gamma(\gamma)}((\Psi(\tau) - \Psi(\zeta))^{\gamma-1}. \tag{19}$$

From using the boundary condition $w(\iota) = \sum_{i=1}^{n} \omega_i w(\xi_i)$, in (19) we find

$$c_0 = \frac{1}{\Lambda}\left[\mathfrak{I}^{\varpi;\Psi}\mathfrak{h}(\iota) - \sum_{i=1}^{n} \omega_i \mathfrak{I}^{\varpi;\Psi}\mathfrak{h}(\xi_i)\right]. \tag{20}$$

Substituting the value of c_0 in (19) we obtain the integral equation (15). The converse follows by direct computation. □

The following notations are used to make the calculations clearer.

$$\Phi = \frac{(\Psi(\iota) - \Psi(\zeta))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left[\frac{(\Psi(\iota) - \Psi(\zeta))^{\varpi}}{\Gamma(\varpi+1)} + \sum_{i=1}^{n} |\omega_i| \frac{(\Psi(\xi_i) - \Psi(\zeta))^{\varpi}}{\Gamma(\varpi+1)}\right]. \tag{21}$$

We adopt the following hypotheses

(H_1): There exists a constant $L_{\mathfrak{f}} > 0$ such that

$$|\mathfrak{f}(\tau, w) - \mathfrak{f}(\tau, v)| \leq L_{\mathfrak{f}}|w - v|, \textit{for each } \tau \in \Theta \textit{ and } w, v \in C(\Theta, \mathbb{R}). \tag{22}$$

(H_2): There exist two constants $K_{\mathfrak{f}}, N_{\mathfrak{f}} > 0$ and $\alpha \in (0, 1)$ such that

$$|\mathfrak{f}(\tau, w)| \leq K_{\mathfrak{f}}|w|^{\alpha} + N_{\mathfrak{f}} \textit{ for each } \tau \in \Theta \textit{ and } w \in C(\Theta, \mathbb{R}). \tag{23}$$

Let $\mathcal{H}, \mathcal{K} : C(\Theta, \mathbb{R}) \longrightarrow C(\Theta, \mathbb{R})$ defined by

$$\mathcal{H}w(\tau) = \mathfrak{I}^{\alpha;\Psi}\mathfrak{f}(\tau, w(\tau)),\ \tau \in \Theta, \tag{24}$$

and

$$\mathcal{K}w(\tau) = \frac{(\Psi(\tau) - \Psi(\zeta))^{\gamma-1}}{\Lambda\Gamma(\gamma)}\left(\mathfrak{I}^{\varpi;\Psi}\mathfrak{f}(\iota, w(\iota)) - \sum_{i=1}^{n} \omega_i \mathfrak{I}^{\varpi;\Psi}\mathfrak{f}(\xi_i, w(\xi_i))\right),\ \tau \in \Theta. \tag{25}$$

Then, (15) can be expressed by

$$\mathcal{S}w(\tau) = \mathcal{H}w(\tau) + \mathcal{K}w(\tau),\ \tau \in \Theta. \tag{26}$$

Theorem 3.2. *Suppose that* (H_1)–(H_2) *holds. If* $L_{\mathfrak{f}}\Phi < 1$*, then the problem (1) has at least one solution* $w \in C(\Theta, \mathbb{R})$*.*

Lemma 3.3. $\mathcal{K}$ *is* δ*-Lipschitz with the constant* $L_{\mathfrak{f}}\Phi$*. In addition* $\mathcal{K}$ *satisfies the inequality given below*

$$\|\mathcal{K}\| \leq \Phi\left(K_{\mathfrak{f}}\|w\|^{\alpha} + N_{\mathfrak{f}}\right), \tag{27}$$

where Φ *is given by (21).*

Proof. Let $w, v \in C(\Theta, \mathbb{R})$, then we get

$$\begin{aligned}
|\mathcal{K}w(\tau) - \mathcal{K}v(\tau)| &\leq \frac{(\Psi(\tau) - \Psi(\zeta))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\Big[\Im^{\varpi;\Psi}|\mathfrak{f}(\iota, w(\iota)) - \mathfrak{f}(\iota, v(\iota))| + \sum_{i=1}^{n}|\omega_i|\Im^{\varpi;\Psi}|\mathfrak{f}(\xi_i, w(\xi_i)) - \mathfrak{f}(\xi_i, v(\xi_i))|, \\
&\leq \frac{L_{\mathfrak{f}}(\Psi(\iota) - \Psi(\zeta))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left[\frac{(\Psi(\iota) - \Psi(\zeta))^{\varpi}}{\Gamma(\varpi+1)} + \sum_{i=1}^{n}|\omega_i|\frac{(\Psi(\xi_i) - \Psi(\zeta))^{\varpi}}{\Gamma(\varpi+1)}\right]\|w - v\|, \\
&\leq L_{\mathfrak{f}}\Phi\|w - v\|.
\end{aligned}$$

Next, we obtain

$$\|\mathcal{K}w - \mathcal{K}v\| \leq L_{\mathfrak{f}}\Phi\|w - v\|.$$

Then $\mathcal{K}$ is Lipschitz with the constant $L_{\mathfrak{f}}\Phi$ and by the Proposition 2.14, $\mathcal{K}$ is δ-Lipschitz with the same constant $L_{\mathfrak{f}}\Phi$. Besides for $w \in C(\Theta, \mathbb{R})$ and by using (H_2) we get

$$\begin{aligned}
|\mathcal{K}w(\tau)| &\leq \frac{(\Psi(\iota) - \Psi(\zeta))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\Big[\Im^{\varpi;\Psi}|\mathfrak{f}(\iota, w(\iota))| + \sum_{i=1}^{n}|\omega_i|\Im^{\varpi;\Psi}|\mathfrak{f}(\xi_i, w(\xi_i))|\Big], \\
&\leq \frac{(K_{\mathfrak{f}}\|w\|^{\alpha} + N_{\mathfrak{f}})(\Psi(\iota) - \Psi(\zeta))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left[\frac{(\Psi(\iota) - \Psi(\zeta))^{\varpi}}{\Gamma(\varpi+1)} + \sum_{i=1}^{n}|\omega_i|\frac{(\Psi(\xi_i) - \Psi(\zeta))^{\varpi}}{\Gamma(\varpi+1)}\right], \\
&\leq \Phi(K_{\mathfrak{f}}\|w\|^{\alpha} + N_{\mathfrak{f}}).
\end{aligned}$$

Implies that $\|\mathcal{K}w\| \leq \Phi(K_{\mathfrak{f}}\|w\|^{\alpha} + N_{\mathfrak{f}})$. □

Lemma 3.4. *$\mathcal{H}$ is continuous and satisfies the inequality given below*

$$\|\mathcal{H}w\| \leq \frac{(\Psi(\iota) - \Psi(\zeta))^{\varpi}}{\Gamma(\varpi+1)}\Big(K_{\mathfrak{f}}\|w\|^{\alpha} + N_{\mathfrak{f}}\Big), \tag{28}$$

Proof. Let $w_n, w \in C(\Theta, \mathbb{R})$ such that w_n converging to w in $C(\Theta, \mathbb{R})$, implies that there exists $\mu > 0$ such that $\|w_n\| \leq \mu$ for all $n \geq 1$, in addition by taking limits, we get $\|w\| \leq \mu$. By using the fact that $\mathfrak{f}$ is continuous and (H_2), for every $\tau \in \Theta$ we get

$$\begin{aligned}
|\mathfrak{f}(\tau, w_n(\tau)) - \mathfrak{f}(\tau, w(\tau))| &\leq |\mathfrak{f}(\tau, w_n(\tau))| + |\mathfrak{f}(\tau, w(\tau)| \\
&\leq 2(K_{\mathfrak{f}}\mu^{\alpha} + N_{\mathfrak{f}}).
\end{aligned}$$

The function $s \longrightarrow 2(K_{\mathfrak{f}}\mu^{\alpha} + N_{\mathfrak{f}})$ is integrable for $s \in [0, \tau], \tau \in \Theta$ by making use of Lebesgue Dominated Convergence theorem we get

$$\|\mathcal{H}w_n(\tau) - \mathcal{H}w(\tau)\| \leq \Im^{\varpi;\Psi}|\mathfrak{f}(\tau, w_n(\tau)) - \mathfrak{f}(\tau, w(\tau))| \longrightarrow 0 \; as \; n \longrightarrow \infty,$$

then $\|\mathcal{H}w_n(\tau) - \mathcal{H}w(\tau)\| \longrightarrow 0$ *as* $n \longrightarrow \infty$, implies that $\mathcal{H}$ is continuous.
Moreover,

$$\begin{aligned}
|\mathcal{H}w(\tau)| &\leq \Im^{\varpi;\Psi}|\mathfrak{f}(\tau, w(\tau))| \\
&\leq \frac{(\Psi(\iota) - \Psi(\zeta))^{\varpi}}{\Gamma(\varpi+1)}\Big(K_{\mathfrak{f}}\|w\|^{\alpha} + N_{\mathfrak{f}}\Big).
\end{aligned}$$

Implies that $\|\mathcal{H}w\| \leq \dfrac{(\Psi(\iota) - \Psi(\zeta))^{\varpi}}{\Gamma(\varpi+1)}\Big(K_{\mathfrak{f}}\|w\|^{\alpha} + N_{\mathfrak{f}}\Big)$. □

Lemma 3.5. *$\mathcal{H}$ is compact, as a consequence $\mathcal{H}$ is δ-Lipschitz with zero constant.*

Proof. To prove that $\mathcal{H}$ is compact, we take a bounded set Δ such that $\Delta \subset \mathcal{B}_\rho$, it remain to prove that $\mathcal{H}(\Delta)$ is relatively compact in $C(\Theta,\mathbb{R})$. For this reason let $w \in \Delta \subset \mathcal{B}_\rho$ and by making use of (28), we obtain

$$\|\mathcal{H}w\| \leq \frac{(\Psi(\iota)-\Psi(\zeta))^{\varpi}}{\Gamma(\varpi+1)}\Big(K_{\mathfrak{f}}\|\rho\|^{\alpha}+N_{\mathfrak{f}}\Big) := \kappa, \tag{29}$$

then $\mathcal{H}(\Delta) \subset \mathcal{B}_\kappa$, as consequence $\mathcal{H}(\Delta)$ is bounded.

For the equicontinuity of $\mathcal{H}$, let $\tau_1, \tau_2 \in \Theta$ with $\tau_1 < \tau_2$ and for $w \in \Delta$ we have

$$\begin{aligned}
|\mathcal{H}w(\tau_2)-\mathcal{H}w(\tau_1)| &\leq \mathfrak{I}^{\varpi;\Psi}|\mathfrak{f}(\tau_2,w(\tau_2))-\mathfrak{f}(\tau_1,w(\tau_1))| \\
&\leq \Big(K_{\mathfrak{f}}\|w\|^{\alpha}+N_{\mathfrak{f}}\Big)\frac{1}{\Gamma(\varpi)}\int_{\tau_1}^{\tau_2}\Psi'(s)(\Psi(\tau_2)-\Psi(s))^{\varpi-1}ds \\
&\leq \Big(K_{\mathfrak{f}}\|w\|^{\alpha}+N_{\mathfrak{f}}\Big)\frac{(\Psi(\tau_2)-\Psi(\tau_1))^{\varpi}}{\Gamma(\varpi+1)}.
\end{aligned}$$

By using the continuity of the function Ψ, the right hand side of the above inequality tends to 0 as τ_2 tends to τ_1 this implies that $\mathcal{H}(\Delta)$ is equicontinuous. It follows by using Ascoli–Arzelà theorem that the operator $\mathcal{H}$ is compact as a consequence of Proposition 2.13 $\mathcal{H}$ is δ-Lipschitz with zero constant. □

Since all the conditions are satisfied we demonstrate the validity of our main result as Theorem 3.2.

Proof. Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{S}$, the operators given by (24), (25), (26) respectively. These operators are continuous and bounded. Furthermore, By making use of Lemma 3.3 , $\mathcal{K}$ is is δ-Lipschitz with constant $L_{\mathfrak{f}}\Phi$, and by using Lemma 3.5, $\mathcal{H}$ is δ-Lipschitz with constant zero, hence $\mathcal{S}$ is a strict δ-contraction with constant $L_{\mathfrak{f}}\Phi$, finally $\mathcal{S}$ is δ-condensing because $L_{\mathfrak{f}}\Phi < 1$.

Next, considering the following set

$$\Xi_\beta = \{w \in w : w = \beta\mathcal{S}w, for\ some\ 0 \leq \beta \leq 1\}. \tag{30}$$

It remain to show that the set Ξ_β is bounded in $C(\Theta,\mathbb{R})$, for that let $w \in \Xi_\beta$ then we have $w = \beta\mathcal{S}w = \beta(\mathcal{H}w+\mathcal{K}w)$. It follows by using Lemmas 3.4 and 3.3

$$\begin{aligned}
\|w\| &= \beta\|\mathcal{H}w+\mathcal{K}w\| \\
&\leq \|\mathcal{H}w\|+\|\mathcal{K}w\| \\
&\leq \frac{(\Psi(\iota)-\Psi(\zeta))^{\varpi}}{\Gamma(\varpi+1)}\Big(K_{\mathfrak{f}}\|w\|^{\alpha}+N_{\mathfrak{f}}\Big)+\Phi\Big(K_{\mathfrak{f}}\|w\|^{\alpha}+N_{\mathfrak{f}}\Big) \\
&\leq \Big(\frac{(\Psi(\iota)-\Psi(\zeta))^{\varpi}}{\Gamma(\varpi+1)}+\Phi\Big)\Big(K_{\mathfrak{f}}\|w\|^{\alpha}+N_{\mathfrak{f}}\Big),
\end{aligned}$$

where Φ is given by (21), then the set Ξ_β is bounded in $C(\Theta,\mathbb{R})$. If the set Ξ_β is not bounded, then we suppose that $\chi := \|w\| \longrightarrow \infty$ and by using the above inequality we get

$$1 \leq \lim_{\chi\to+\infty}\frac{\Big(\frac{(\Psi(\iota)-\Psi(\zeta))^{\varpi}}{\Gamma(\varpi+1)}+\Phi\Big)\Big(K_{\mathfrak{f}}\chi^{\alpha}+N_{\mathfrak{f}}\Big)}{\chi} = 0 \tag{31}$$

which is a contradiction. Thus by using Theorem 2.15, $\mathcal{S}$ has at least one fixed point which is the solution of the problem (1). Moreover, the set of the solution of the problem (1) is bounded in $C(\Theta,\mathbb{R})$. □

To deal with the uniqueness of solution for the problem (1), we use Banach's contraction principle.

Theorem 3.6. *Assume that* (H_1) *hold. If* $\Big[\frac{(\Psi(\iota)-\Psi(\zeta))^{\varpi}}{\Gamma(\varpi+1)}+\Phi\Big]L_{\mathfrak{f}} < 1$ *then the problem (1) has a unique solution in* $C(\Theta,\mathbb{R})$.

Proof. For every $w, v \in C(\Theta, \mathbb{R})$ and $\tau \in \Theta$ we have

$$\begin{aligned}|\mathcal{S}w(\tau) - \mathcal{S}v(\tau)| &\leq \mathfrak{I}^{\varpi;\Psi}|\mathfrak{f}(\tau, w(\tau)) - \mathfrak{f}(\tau, v(\tau))| \\ &+ \frac{(\Psi(\tau) - \Psi(\zeta))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\Big[\mathfrak{I}^{\varpi;\Psi}|\mathfrak{f}(\iota, w(\iota)) - \mathfrak{f}(\iota, v(\iota))| \\ &+ \sum_{i=1}^{n}|\omega_i|\mathfrak{I}^{\varpi;\Psi}|\mathfrak{f}(\xi_i, w(\xi_i)) - \mathfrak{f}(\xi_i, v(\xi_i))|\Big], \\ &\leq L_{\mathfrak{f}}\frac{(\Psi(\iota) - \Psi(\zeta))^{\varpi}}{\Gamma(\varpi+1)}|w(\tau) - v(\tau)| + \frac{L_{\mathfrak{f}}(\Psi(\iota) - \Psi(\zeta))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \\ &\times\left[\frac{(\Psi(\iota) - \Psi(\zeta))^{\varpi}}{\Gamma(\varpi+1)} + \sum_{i=1}^{n}|\omega_i|\frac{(\Psi(\xi_i) - \Psi(\zeta))^{\varpi}}{\Gamma(\varpi+1)}\right]|w(\tau) - v(\tau)|, \\ &\leq \left[\frac{(\Psi(\iota) - \Psi(\zeta))^{\varpi}}{\Gamma(\varpi+1)} + \Phi\right]L_{\mathfrak{f}}|w(\tau) - v(\tau)|,\end{aligned}$$

taking the supremum over τ we get

$$\|\mathcal{S}w - \mathcal{S}v\| \leq \left[\frac{(\Psi(\iota) - \Psi(\zeta))^{\varpi}}{\Gamma(\varpi+1)} + \Phi\right]L_{\mathfrak{f}}\|w - v\|.$$

Using the fact that $\left[\frac{(\Psi(\iota) - \Psi(\zeta))^{\varpi}}{\Gamma(\varpi+1)} + \Phi\right]L_{\mathfrak{f}} < 1$, then $\mathcal{S}$ is a contraction, finally by the Banach fixed point theorem, $\mathcal{S}$ has a unique fixed point which is a unique solution of problem (1). □

4. Example

Consider the following problem

$$\begin{cases} {}^{H}\mathfrak{D}^{\frac{7}{4}, \frac{2}{5}; \frac{e^{\tau}}{6}} w(\tau) = \frac{1}{6 + e^{\tau}}\Big(|w(\tau)| + 1\Big), \ 0 \leq \tau \leq 1, \\ w(0) = 0\ , \ w(1) = \frac{2}{5}w(\frac{1}{3}) + \frac{3}{5}w(\frac{1}{2}). \end{cases} \tag{32}$$

Where $\varpi = \frac{7}{4}$, $\sigma = \frac{2}{5}$, $\zeta = 0$, $\iota = 1$, $\Theta = [0, 1]$, $n = 2$, $\omega_1 = \frac{2}{5}$, $\omega_2 = \frac{3}{5}$, $\xi_1 = \frac{1}{3}$, $\xi_2 = \frac{1}{2}$ and $\psi(\tau) = \frac{e^{\tau}}{6}$.

Set, $\mathfrak{f} : \Theta \times \mathbb{R} \longrightarrow \mathbb{R}$, a function defined by

$$(\tau, w) \longrightarrow \mathfrak{f}(\tau, w) = \frac{1}{6 + e^{\tau}}\Big(|w| + 1\Big).$$

$\mathfrak{f}$ is a continuous function, furthermore for every $\tau \in \Theta$ and $w, v \in \mathbb{R}$ we have

$$\begin{aligned}|\mathfrak{f}(\tau, w) - \mathfrak{f}(\tau, v)| &\leq \frac{1}{|6 + e^{\tau}|}|w - v| \\ &\leq \frac{1}{7}|w - v|.\end{aligned}$$

Thus the hypotheses (H_1) holds with $L_{\mathfrak{f}} = \frac{1}{7} > 0$. In addition, for every $\tau \in \Theta$ and $w, v \in \mathbb{R}$ we have

$$\begin{aligned}|\mathfrak{f}(\tau, w)| &\leq \frac{1}{|6 + e^{\tau}|}\Big(|w| + 1\Big) \\ &\leq \frac{1}{7}\Big(|w| + 1\Big).\end{aligned}$$

Then the hypotheses (H_2) holds with $K_{\mathfrak{f}} = N_{\mathfrak{f}} = \dfrac{1}{7} > 0$ and $\alpha = 1$. Finally, all the conditions of Theorem 3.2 are satisfied, thus the problem (32) has at least one solution defined on $[0, 1]$.

For the uniqueness we use the data given above, we get $\gamma = \varpi + 2\sigma - \varpi\sigma = \frac{37}{20}$, $\dfrac{(\Psi(1) - \Psi(0))^{\frac{7}{4}}}{\Gamma(\frac{11}{4})} \simeq 0.111167$, $\Phi \simeq 0.398559$ and $L_{\mathfrak{f}} = \dfrac{1}{7} = 0.142857$.

Then $\left[\dfrac{(\Psi(1) - \Psi(0))^{\varpi}}{\Gamma(\varpi + 1)} + \Phi\right] L_{\mathfrak{f}} = (0.111167 + 0.398559) * 0.142857 \simeq 0.072817 < 1.$

Hence by Theorem 3.6 the problem (32) has a unique solution on $[0, 1]$.

5. Conclusion

In the current article, we have studied and investigated the existence and uniqueness of solution for a new class of Ψ-Hilfer-type fractional differential equation. The novelty of this work is that it is more general than the works based on the well-known fractional derivatives such as (Caputo, Riemann-Liouville, Ψ-Riemman-Liouville, Hilfer, Hilfer-Hadamard, Katugampola) for different values of function Ψ and parameter σ. In this paper we established the existence and uniqueness results for the problem (1), by using the topological degree method and Banach's fixed point theorem. Finally a numerical example is presented to clarify the obtained result.

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Data Availability

The data used to support the finding of this study are available from the corresponding author upon request.

Conflicts Of Interest

The authors declare that they have no conflict of interest.

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