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Matrix transforms from the set of sequences bounded with speed into the speed-maddox spaces over ultrametric fields

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Abstract.
Let $\mathbb{K}$ be a complete, non-trivially valued, ultrametric (or non-archimedean) field, and $\lambda = \{\lambda_n\}$ - a sequence in $\mathbb{K}$ with the property $0 < |\lambda_n| \nearrow \infty, n \to \infty$, i.e., the speed of convergence. In the present paper, the concepts of boundedness and convergence with speed and speed-Maddox spaces over $\mathbb{K}$, where the speed is defined by λ, have been recalled. Let μ be another speed in $\mathbb{K}$. Necessary and sufficient conditions are found for a matrix A over $\mathbb{K}$ to transform all λ-bounded sequences over $\mathbb{K}$ into speed-Maddox spaces over $\mathbb{K}$, where the speed is defined by μ.

1. Introduction

The present paper is the continuation of [17] and [18]. Therefore we use the notations and concepts from these papers (see also [12] - [14]). To make the paper self-contained, we recall the notations and concepts from mentioned papers, which we need for this paper.

Let, throughout the paper, $\mathbb{K}$ be a complete, non-trivially valued, ultrametric (or non-archimedean) field. Also sequences, infinite series and infinite matrices have entries in $\mathbb{K}$, and indices and summation indices run from 0 to ∞, unless otherwise stated. Given an infinite matrix $A = (a_{nk})$, and a sequence $x = \{x_k\}$, by the A-transform of x, we mean the sequence $A(x) = \{(Ax)_n\}$, where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k,\ \ n = 0, 1, 2, \ldots,$$

it being assumed that the series on the right converge. If $(Ax)_n \to s$, $n \to \infty$, we say that x is A-summable or summable A to s.

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If X, Y are sequence spaces, we write $A = (a_{nk}) \in (X, Y)$ if $\{(Ax)_n\} \in Y$, whenever $x = \{x_k\} \in X$. In the sequel, m, c, c_0 respectively denote the ultrametric Banach spaces of bounded, convergent and null sequences under the ultrametric norm

$$\|x\| = \sup_k |x_k|,\ x = \{x_k\} \in m, c, c_0.$$

Let now, and further in the present paper, $\lambda = \{\lambda_n\}$ is a sequence in $\mathbb{K}$, such that

$$0 < |\lambda_n| \nearrow \infty, n \to \infty.$$

A convergent sequence $\{x_n\}$ in $\mathbb{K}$ with the limit $\lim_{n\to\infty} x_n = s$ is said to be

a) λ-bounded if $\{\lambda_n(x_n - s)\} \in m$,

b) λ-convergent if $\{\lambda_n(x_n - s)\} \in c$,

c) λ-convergent to zero if $\{\lambda_n(x_n - s)\} \in c_0$.

We note that in classical case the concepts of λ-boundedness and λ-convergence have been introduced in [3, 4] by Kangro (see also [1]).

We denote the set of all λ-bounded sequences by m^λ, the set of all λ-convergent sequences by c^λ, and the set of all sequences λ-convergent to zero by c_0^λ. It is not difficult to see that

$$c_0^\lambda \subset c^\lambda \subset m^\lambda \subset c.$$

Moreover, these inclusions are strict.

Let $p = \{p_n\}$ be a sequence of strictly positive real numbers, and let

$$c_0(p) = \{x = \{x_n\} : \lim_n |x_n|^{p_n} = 0\},$$

$$c(p) = \{x = \{x_n\} : \lim_n |x_n - l|^{p_n} = 0 \text{ for some } l \in \mathbb{K}\},$$

$$m(p) = \{x = \{x_n\} : |x_n|^{p_n} = O(1)\}.$$

Earlier the sets $c_0(p)$, $c(p)$ and $m(p)$ were defined over the field of complex numbers, and called as Maddox spaces (see, for example, [5, 6, 19]). In that case, for a bounded sequence p these spaces are also linear spaces. Under some additional conditions (for example, $\inf_n p_n > 0$) these spaces are also paranormed spaces (see [5, 6, 19]). Good overview on these spaces, including the Maddox spaces, has been given, for example, in [2] and [7].

Further, throughout the paper, we assume that p is bounded. Then it is easy to prove (see also Corollary 2.11 of [7]) that

$$c_0(p) \subset c_0,\ c(p) \subset c,\ m \subset m(p).$$

If $p_n \equiv 1$, then

$$c_0(p) = c_0,\ c(p) = c,\ m(p) = m.$$

We note that in the ultrametric set up the Maddox spaces are studied by Natarajan (see, for example, [9]). It appears from [9] that for a bounded p, similarly to the classical case, all Maddox spaces are linear, and under some additional conditions for p these spaces are also paranormed.

Let

$$(c_0(p))^\lambda = \{x = \{x_n\} : \lim_{n\to\infty} x_n = s \text{ (say) and } \{\lambda_n(x_n - s)\} \in c_0(p)\},$$

$$(c(p))^\lambda = \{x = \{x_n\} : \lim_{n\to\infty} x_n = s \text{ (say) and } \{\lambda_n(x_n - s)\} \in c(p)\},$$

$$(m(p))^\lambda = \{x = \{x_n\} : \lim_{n\to\infty} x_n = s \text{ (say) and } \{\lambda_n(x_n - s)\} \in m(p)\}.$$

We call the sets $(c_0(p))^\lambda, (c(p))^\lambda$ and $(m(p))^\lambda$ as speed-Maddox spaces. These spaces over the field of complex numbers are studied in [15, 16]. The notions of paranormed zero-convergence, paranormed convergence and paranormed boundedness with speed λ over the field of complex numbers are also defined in [15, 16]. Using these definitions we can say that for a bounded sequence p, in classical case $(c_0(p))^\lambda$ consists of all paranormally zero-convergent sequences with speed λ, $(c(p))^\lambda$ consists of all paranormally λ-convergent sequences, and $(m(p))^\lambda$ consists of all paranormally λ-bounded sequences.

Let μ be another speed. Necessary and sufficient conditions in ultrametric set up for a matrix A would transform c_0^λ into $(c_0(p))^\mu, (c(p))^\mu$ or $(m(p))^\mu$ are proved in [17], and from c^λ into $(c_0(p))^\mu$, $(c(p))^\mu$ or $(m(p))^\mu$ in [18]. In the present paper we continue the studies started in [17] and [18]. We give the characterization of matrix classes $(m^\lambda, (c_0(p))^\mu)$, $(m^\lambda, (c(p))^\mu)$ and $(m^\lambda, (m(p))^\mu)$ over the ultrametric field $\mathbb{K}$.

2. Auxiliary results

First we present two lemmas, which are important in ultrametric analysis and used for the proof of next results of this section. Using Theorem 1.1 of [10] or Theorem 2.1 of [11] we can formulate the following result.

Lemma 2.1. *If $x_n \in \mathbb{K}$, then*

$$\left|\sum_{k=0}^{\infty} x_k\right| \leq \sup_k |x_k|.$$

Lemma 2.2 ([10], Theorem 1.3, see also [11], Theorem 2.5). *A sequence $x = (x_k)$ in $\mathbb{K}$* is a Cauchy sequence if and only if

$$|x_{k+1} - x_k| \to \infty; \;\; k \to \infty.$$

Now we note some results that will be used for the proof of main results of the paper in next section. Throughout this section we assume that $A = (a_{nk})$ has entries in $\mathbb{K}$.

Proposition 2.3 *A matrix $A \in (m, c)$ if and only if*

$$\lim_{k\to\infty} a_{nk} = 0, \;\; n = 0, 1, 2, \ldots \tag{2.1}$$

and

$$\lim_{n\to\infty} \sup_k |a_{n+1,k} - a_{nk}| = 0. \tag{2.2}$$

In such a case,

$$\lim_{n\to\infty} (Ax)_n = \sum_{k=0}^{\infty} a_k x_k \tag{2.3}$$

for every $x = (x_k) \in m$, where

$$\lim_{n\to\infty} a_{nk} = a_k \; uniformly \; with \; respect \; to \; k. \tag{2.4}$$

Proof. Here we prove (2.3) only. The proof of the rest of the theorem can be found in [8] (see also [10], Theorem 4.2). First we show that condition (2.4) follows from (2.2). Indeed, using (2.2), we note that for every $\varepsilon > 0$ there exists a positive integer N, such that

$$\sup_k |a_{n+1,k} - a_{nk}| < \varepsilon$$

for all $n > N$, i.e.,

$$|a_{n+1,k} - a_{nk}| < \varepsilon, \;\; n > N$$

for every $k = 0, 1, \dots$. Hence $\{(a_{nk}\}_{n=0}^{\infty}$ is uniformly Cauchy with respect to $k = 0, 1, \dots$ by Lemma 2.2. Since $\mathbb{K}$ is complete, then condition (2.4) holds.

Let $x = (x_k)$ be arbitrary sequence in m. Then there exists a positive number H, such that

$$\sup_k |x_k| < H. \tag{2.5}$$

Moreover, from (2.4) we conclude that for every $\varepsilon > 0$ there exists a positive integer L, such that

$$\sup_k |a_{nk} - a_k| < \frac{\varepsilon}{H} \tag{2.6}$$

for all $n > L$. Hence, by Lemma 2.1 and relations (2.5), (2.6) we obtain

$$\left|\sum_{k=0}^{\infty} (a_{nk} - a_k) x_k\right| \leq \sup_k |a_{nk} - a_k||x_k| < \frac{\varepsilon}{H} H = \varepsilon$$

for all $n > L$. It means

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} (a_{nk} - a_k) x_k = 0. \tag{2.7}$$

As $\{(Ax)_n\}$ converges, from (2.7) we obtain that the series $\sum_{k=0}^{\infty} a_k x_k$ also converges. Therefore

$$(Ax)_n = \sum_{k=0}^{\infty} (a_{nk} - a_k) x_k + \sum_{k=0}^{\infty} a_k x_k,$$

which implies by (2.7) that relation (2.3) holds for every $x \in m$. □

For formulating following Propositions 2.4 - 2.6 we assume that $p = \{p_n\}$ is a sequence of strictly positive real numbers. Using Lemmas 2.1 and 2.2, these propositions can be proved like Statements 17, 21 and 25 of Theorem 4.13 for $p_k \equiv 1$ in [7] have been proved in classical case (see also [2], p. 232-233).

Proposition 2.4 *A matrix $A \in (m, c(p))$ if and only if*

$$\sup_{n,k} |a_{nk}| < \infty, \tag{2.8}$$

and there exists a sequence (c_k) such that

$$\lim_{n\to\infty} \left(M \sup_k |a_{nk} - c_k|\right)^{p_n} = 0 \ for \ all \ M > 0. \tag{2.9}$$

Proposition 2.5 *A matrix $A \in (m, c_0(p))$ if and only if*

$$\lim_{n\to\infty} \left(M \sup_k |a_{nk}|\right)^{p_n} = 0 \ for \ all \ M > 0. \tag{2.10}$$

Proposition 2.6 *A matrix $A \in (m, m(p))$ if and only if*

$$\sup_n \left(M \sup_k |a_{nk}|\right)^{p_n} < \infty \ for \ all \ M > 0. \tag{2.11}$$

3. Main results

Now we are able to prove the main results of the paper. Let further $\lambda = \{\lambda_n\}$, $\mu = \{\mu_n\}$ be speeds of convergence over $\mathbb{K}$, $p = \{p_n\}$ - a bounded sequence of strictly positive real numbers, and $B = (b_{nk})$ - the matrix, defined by

$$b_{nk} := \frac{\mu_n(a_{nk} - a_k)}{\lambda_k}, n, k = 0, 1, 2, \dots,$$

provided that

$$\textit{there exists the limit } \lim_{n\to\infty} a_{nk} = a_k;\ k = 0, 1, 2, \dots. \tag{3.1}$$

Also, for the proof of the main results we need the special sequences

$$e_k = \{0, ..., 0, 1, 0, ...\},$$

where 1 is in the k-th position only ($k = 0, 1, 2, \dots$), and

$$e := (1, 1, \dots, 1, \dots).$$

We note that $e_k, e \in m^\lambda$.

Theorem 3.1. *A matrix $A = (a_{nk}) \in (m^\lambda, (c(p))^\mu)$ if and only if*

$$A(e), A(e_k) \in (c(p))^\mu,\ k = 0, 1, 2, \dots, \tag{3.2}$$

$$\lim_{k\to\infty} \frac{a_{nk}}{\lambda_k} = 0,\ n = 0, 1, 2, \dots, \tag{3.3}$$

$$\lim_{n\to\infty} \sup_k \left| \frac{a_{n+1,k} - a_{nk}}{\lambda_k} \right| = 0, \tag{3.4}$$

$$\sup_{n,k} |b_{nk}| < \infty, \tag{3.5}$$

and there exists a sequence (c_k) such that

$$\lim_{n\to\infty} \sup_k |(b_{nk} - c_k)M|^{p_n} = 0 \textit{ for all } M > 0. \tag{3.6}$$

Proof. Necessity. Assume that $A = (a_{nk}) \in (m^\lambda, (c(p))^\mu)$. Then condition (3.2) holds, since $e, e_k \in m^\lambda$. As in this case also $A(e_k), A(e) \in c$, then condition (3.1) holds and

$$\text{there exists } \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = a \text{ (say)}. \tag{3.7}$$

Let, now, $x = \{x_k\}$ be an arbitrary sequence in m^λ. Then there exists the limit $\lim_{k\to\infty} x_k = s$ (say), since $x \in c$, and $\{v_k\} \in m$, where

$$v_k = \lambda_k (x_k - s), , k = 0, 1, 2, \dots. \tag{3.8}$$

As it follows from (3.8) that

$$x_k = \frac{v_k}{\lambda_k} + s,\ k = 0, 1, 2, \dots,$$

then we obtain

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} \frac{a_{nk}}{\lambda_k} v_k + s \sum_{k=0}^{\infty} a_{nk}. \tag{3.9}$$

Since $\{(Ax)_n\} \in c$ and (3.7) holds, then the matrix

$$A_\lambda := \left(\frac{a_{nk}}{\lambda_k}\right)$$

transforms this sequence $\{v_k\} \in m$ into c. Moreover, it is not difficult to show that for every sequence $\{v_k\} \in m$ there exists a sequence $\{x_k\} \in m^\lambda$, such that (3.8) holds. Hence, $A_\lambda \in (m, c)$. Therefore, using Proposition 2.3 and conditions (3.1), (3.7), we can conclude that conditions (3.3) and (3.4) hold, and

$$\eta := \lim_{n\to\infty} (Ax)_n = \sum_{k=0}^{\infty} \frac{a_k}{\lambda_k} v_k + sa \tag{3.10}$$

for every $x = \{x_k\} \in m^\lambda$, where $v := \lim_{k\to\infty} v_k$. From (3.9) and (3.10) we have

$$(Ax)_n - \eta = \sum_{k=0}^{\infty} \frac{(a_{nk} - a_k)}{\lambda_k} v_k + s\left(\sum_{k=0}^{\infty} a_{nk} - a\right),$$

and so

$$\mu_n\left((Ax)_n - \eta\right) = \sum_{k=0}^{\infty} b_{nk} v_k + s\mu_n\left(\sum_{k=0}^{\infty} a_{nk} - a\right) \tag{3.11}$$

for every $x \in m^\lambda$. By assumption, $\{(Ax)_n\} \in (c(p))^\mu$ for every $x \in m^\lambda$, hence

$$\{\mu_n\left((Ax)_n - \eta\right)\} \in c(p) \tag{3.12}$$

for every $x \in m^\lambda$. In addition, since $A(e) \in (c(p))^\mu$, then

$$\left\{\mu_n\left(\sum_{k=0}^{\infty} a_{nk} - a\right)\right\} \in c(p). \tag{3.13}$$

Thus, from (3.11) we can conclude that $B = (b_{nk}) \in (m, c(p))$. Therefore conditions (3.5) and (3.6) are satisfied by Proposition 2.4, completing the proof of the necessity part.

Sufficiency. Assume that conditions (3.2) - (3.6) hold. Then conditions (3.1) and (3.7) are satisfied by (3.2), and relation (3.9) holds for every $x \in m^\lambda$, where $\lim_{k\to\infty} x_k = s$ and v_k is defined by (3.8). We note that

$$\lim_{n\to\infty} \frac{a_{nk}}{\lambda_k} = \frac{a_k}{\lambda_k}, \ k = 0, 1, 2, \ldots$$

by (3.1). Hence, with the help of Proposition 2.3 we have by (3.3) and (3.4) that $A_\lambda \in (m, c)$ and (3.10) holds for every $x \in m^\lambda$. Then also (3.11) is satisfied for every $x \in m^\lambda$ by (3.9) and (3.10). Now, relation (3.13) holds by 3.2, and, using Proposition 2.4, we can conclude that $B \in (m, c(p))$ by (3.5) - (3.6). Therefore, from (3.11) we obtain that (3.12) is also satisfied for every $x \in m^\lambda$, completing the proof of the theorem. □

Theorem 3.2. *A matrix* $A = (a_{nk}) \in \left(m^\lambda, (c_0(p))^\mu\right)$ *if and only if conditions* (3.3), (3.4) *hold, and*

$$A(e), A(e_k) \in (c_0(p))^\mu, \ k = 0, 1, 2, \ldots, \tag{3.14}$$

$$\lim_{n\to\infty} \left(M \sup_k |b_{nk}|\right)^{p_n} = 0 \ for\ all\ M > 0. \tag{3.15}$$

Theorem 3.3. *A matrix* $A = (a_{nk}) \in \left(m^\lambda, (m(p))^\mu\right)$ *if and only if conditions* (3.3), (3.4) *hold, and*

$$A(e), A(e_k) \in (m(p))^\mu, \ k = 0, 1, 2, \ldots, \tag{3.16}$$

$$\sup_n \left(M \sup_k |a_{nk}| \right)^{p_n} < \infty \; for\ all\ M > 0. \tag{3.17}$$

The proofs of Theorems 3.2 and 3.3 are similar to the proof of Theorem 3.1. Therefore we omit them. We note only that for the proof of Theorem 3.2, instead of Proposition 2.4 we need to use Proposition 2.5, and (3.2) it is necessary to replace by (3.14), and in the proof of Theorem 3.3, instead of Proposition 2.4 we need to use Proposition 2.6, and (3.2) it is necessary to replace by (3.16).

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