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Existence of a weak solution for nonlocal thermistor problem in Sobolev spaces with variable exponent

Ibrahim Dahi[a,*], Moulay Rchid Sidi Ammi[a]

[a] *Department of Mathematics, MAIS Laboratory, AMNEA Group, FST Errachidia, Moulay Ismail University of Meknes, P.O. Box. 509 Boutalamine, 52000 Errachidia, Morocco*

Abstract. The focus of this paper, is to establish the existence of a weak solution for a problem that involves p(x)-Laplacian. We achieve this by utilizing the topological degree theory, which is based on a class of demi-continuous operators of generalized $(S+)$ type, as presented in [6], in conjunction with the theory of variable-exponent Sobolev spaces. Additionally, we provide a numerical example to verify and validate the theoretical results.

1. Introduction

A device constructed from materials whose electrical conductivity is strongly influenced by temperature is called a Thermistor. There are two types of Thermistors, which are PTC and NTC thermistors (Positive and Negative Temperature Coefficient thermistors, respectively) [30]. Both have been employed for temperature sensing, self-resetting overcurrent protection, and inrush speed control since the 1830s, at the end of the industrial revolution. Michael Faraday, a British chemist, is known to be the first scientist who created the first NTC thermistor.

Thermistors offer several advantages as temperature measurement devices, including their affordability, high precision, and ability to be easily customized in terms of size and shape, while the following is a summary of their applications: control and temperature sensing, it is owing to their potential to provide economic and precise temperature sensors for a wide temperature range; thermal relay and switch: due to their voltage regulation, surge protection; indirect measurement of other parameters. In fact, a thermistor's rate of temperature change during heating is influenced by its surroundings [22]. The following system shows how a thermistor, a device for regulating electric current in a circuit, works

$$\begin{cases} \dfrac{\partial v}{\partial s} - \Delta_{p(z)} v = \dfrac{\lambda f(v)}{(\int_\Omega f(v)dx)^2}, & \text{in } Q \\ v(x,s) = 0, & \text{on } \partial\Omega \times]0,T[, \\ v(x,0) = v_0(x), & \text{in } \Omega, \end{cases} \tag{1}$$

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Communicated by Maria Alessandra Ragusa
* Corresponding author: Ibrahim Dahi
Email addresses: masterdabra@gmail.com (Ibrahim Dahi), rachidsiidiammi@gmail.com (Moulay Rchid Sidi Ammi)
ORCID iDs: https://orcid.org/0000-0002-4123-4431 (Ibrahim Dahi), https://orcid.org/0000-0002-4488-9070 (Moulay Rchid Sidi Ammi)

The first equation in (1) contains two key terms. The first one, denoted by $f(v)$, represents the electrical resistance of the conductor. The second term, given by $\frac{f(v)}{(\int_\Omega f(v)dx)^2}$, represents the non-local aspect of the equation. Moving on, the notation Q is defined as the Cartesian product of the open bounded region $\Omega \subset \mathbb{R}^m$ (where $m \geq 1$) and the positive constant T, i.e., $Q := \Omega \times [0, T]$. The "Thermistor Problem" typically consists of two equations. The first one is an elliptic equation that elucidates the quasistatic evolution of the electric potential, while the second one is a non-linear parabolic equation that characterizes the temperature [4]. In reality, the coupling which is in part impacted by the electrical conductivity's substantial temperature dependence is what is responsible for the problem's high nonlinearity. Mathematically, the thermistor problem is formulated as follows.

$$\begin{aligned} \frac{\partial v}{\partial s} &= \nabla \cdot (\kappa(v)\nabla v) + \rho(v)|\nabla\theta|^2, \\ \nabla \cdot (\rho(v)\nabla\theta) &= 0, \end{aligned} \tag{2}$$

where v represent the temperature produced by an electric current passing through a conductor, $k(v)$ and $\rho(v)$ are the electric and thermal conductivities, respectively. While θ is the electric potential (see [3, 18, 21]). The coupling (2) is, partially transformed into a parabolic system, taking into account the boundary and initial conditions. For more details, we refer to [8, 9, 11, 23, 25, 27]. In this work, we shall deal with the non-local model (1) which is the extension of the one appearing in several papers [2, 14–16, 19, 28].

The presence of several applications stimulates our motivation. Furthermore, each layer's non-Ohmic behaviour is described by the exponent $p(x)$, which abruptly changes from one material to another. On the other hand, from a mathematical point of view, this operator poses fascinating problems since it lacks homogeneity, when the exponent $p(x)$ is not a constant. The aim of this study is to utilize topological degree theory to prove the existence of solutions to system (1) for a specific class of demi-continuous operators of generalized $(S+)$ type introduced in [6], along with the theory of variable-exponent Sobolev spaces. Additionally, we provide a numerical example to demonstrate the effectiveness of our proposed approach.

The remainder of this work is structured as follows. We introduce some preliminaries in Section 2, that are used in the sequel. In Section 3, we develop the proof of the existence of a weak solution. While in Section 4, we perform some numerical simulations to validate the theoretical results. Finally, in Section 5, we state a conclusion and some perspectives.

2. Preliminaries

Throughout this paper, we assume that
(H1) $v_0 \in L^2(\Omega)$.
(H2) f is a function and there exists a positive constant σ, C_1 and a function β in $L^1(\Omega)$ such that

$$\sigma \leq f(v) \leq C_1(|v|^{q(z)-1} + \beta(z)), \tag{3}$$

where

$$1 < q^- \leq q(x) \leq q^+ < p^-.$$

We specific the rest parameters of problem (1). Ω will denote an open bounded set of $\mathbb{R}^m$, $m \geq 2$ with smooth boundary and $\Delta_{p(z)}v := \operatorname{div}\left(|\nabla v|^{p(z)-2}\nabla v\right)$ where $\Delta_{p(z)}$ is defined from

$$\mathcal{U} := \left\{ v \in L^{p^-}\left(0, T; W_0^{1,p(\cdot)}(\Omega)\right) : |\nabla v| \in L^{p(\cdot)}(Q) \right\}. \tag{4}$$

where $p : \bar{\Omega} \longrightarrow \mathbb{R}$ is measurable such that:

$$+\infty > p^+ \geq p(z) \geq p^- > 1, \tag{5}$$

where $p^+ = ess\sup_{z\in\Omega} p(z)$, $p^- = ess\inf_{z\in\Omega} p(z)$. We endow the space $\mathcal{U}$ by the following norm $\|v\|_{\mathcal{U}} := \|v\|_{L^{p^-}\left(0,T;W_0^{1,p(\cdot)}(\Omega)\right)} + \|\nabla v\|_{L^{p(\cdot)}(Q)}$. $\mathcal{U}$ is a separable reflexive Banach space [5], and we denote by $\langle .,. \rangle$ the duality between $W_0^{1,p(z)}(\Omega)$ and $W_0^{-1,p\prime(z)}(\Omega)$ where $1/p(z) + 1/p'(z) = 1$. $L^{p(z)}(\Omega) = \{v : \Omega \to \mathbb{R}$ is measurable, such that $\int_\Omega |(v(z))|^{p(z)}\mathrm{d}z < +\infty\Big\}$ endowed with

$$\|v\|_{L^{p(z)}} := \inf\left\{C > 0; \rho_{p(z)}\left(\frac{v}{C}\right) \leq 1\right\},$$

where

$$\rho_{p(z)}(v) = \int_\Omega |v(z)|^{p(z)}\mathrm{d}z, \quad \forall v \in L^{p(z)}(\Omega).$$

We also define the Sobolev space with variable exponent as follows

$$W^{1,p(\cdot)}(\Omega) = \left\{v \in L^{p(\cdot)}(\Omega) : \ |\nabla v| \in L^{p(\cdot)}(\Omega)\right\}.$$

$W^{1,p(\cdot)}(\Omega)$ is endow by

$$\|v\|_{W^{1,p(\cdot)}(\Omega)} = \|v\|_{L^{p(\cdot)}(\Omega)} + \|\nabla v\|_{L^{p(\cdot)}(\Omega)},$$

or

$$\|v\|_{W^{1,p(\cdot)}(\Omega)} = \inf\left\{C > 0 : \ \int_\Omega \left(\left|\frac{\nabla v(x)}{C}\right|^{p(z)} + \left|\frac{v(z)}{C}\right|^{p(z)}\right) dz \leqslant 1\right\}.$$

We define also $\overline{C_c^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)} =: W_0^{1,p(\cdot)}(\Omega)$. We suppose that $p^- > 1$, then the spaces $\left(W^{1,p(\cdot)}(\Omega), \|\cdot\|_{W^{1,p(\cdot)}(\Omega)}\right)$ and $\left(W_0^{1,p(\cdot)}(\Omega), \|\cdot\|_{W^{1,p(\cdot)}(\Omega)}\right)$ are two Banach reflexive separable spaces [20, 26]. The space $\left(W_0^{1,p(\cdot)}(\Omega)\right)^\star$ is the dual of $W_0^{1,p(\cdot)}(\Omega)$. For more details on variable exponent spaces we refer the reader to [17].

Proposition 2.1. *(See [20]) Let us $(v_n)_{n\in\mathbb{N}}$ a sequence and $v \in L^{p(\cdot)}(\Omega)$. Then*

$$\|v\|_{p(z)} \leq 1 \text{ equivalent to } \rho_{p(z)}(v) \leq 1, \tag{6}$$

$$\text{if } \|v\|_{p(z)} > 1 \text{ implies } \|v\|_{p(z)}^{p^-} \leq \rho_{p(z)}(v) \leq \|v\|_{p(z)}^{p^+}, \tag{7}$$

$$\|v\|_{p(z)} < 1 \Rightarrow \|v\|_{p(z)}^{p^+} \leq \rho_{p(z)}(v) \leq \|v\|_{p(z)}^{p^-}, \tag{8}$$

$$\lim_{n\to\infty} \|v_n - v\|_{p(z)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{p(z)}\left(v_n - v\right) = 0. \tag{9}$$

Remark 2.2. *If we combine (7) with (8), we get the following inequalities*

$$\|v\|_{p(z)} \leq \rho_{p(z)}(v) + 1, \tag{10}$$

$$\rho_{p(z)}(v) \leq \|v\|_{p(z)}^{p^-} + \|v\|_{p(z)}^{p^+}. \tag{11}$$

Keeping this in mind and using (5), to obtain the following inequalities

$$\|v\|_{L^{p(\cdot)}(Q)}^{p^-} - 1 \leq \int_Q |v|^{p(z)} dxds \leq \|v\|_{L^{p(\cdot)}(Q)}^{p^+} + 1. \tag{12}$$

We make essential use of the following Definition 2.3.

Definition 2.3. *$v \in W_0^{1,p(z)}(\Omega)$ is a weak solution of system (1), if the following identity*

$$-\int_Q v\frac{\partial w}{\partial s}dzds + \int_Q |\nabla v|^{p(z)-2}\nabla v\nabla w dzds = \int_Q \frac{\lambda f(v)}{(\int_\Omega f(v)dz)^2} w dzdt,$$

holds for all $w \in W_0^{1,p(z)}(\Omega)$.

Now, Let us introduce some classes of operators and the notation of topological degree. Let us consider Y and X be two real separable reflexive Banach spaces with dual X^* and Y^* respectively. Let Ω be a non-empty subset of X. The definition of the graph of R, a mapping from X to X^*, is as follows:

$$G(R) := \{(v,w) \in X \times X^* / \ w \in R(v)\}.$$

R is called to be monotone, if for all (v_1,u_1) and (v_2,u_2) in $G(R)$, we get

$$\langle u_1 - u_2, v_1 - v_2\rangle \geq 0,$$

holds. If R is maximal in the sense that it is included in the graph of a monotone multi-valued mapping from X to X^*, then R is considered maximally monotone [12].

Definition 2.4. *(See [7]) Let us Y be real reflexive Banach space. A mapping $R : \Omega \subset X \to Y$ is called to be*

1. *a bounded mapping, if it transforms any bounded set into a bounded set.*
2. *demi continuous mapping, if for all $(v_n)_{n\in\mathbb{N}} \subset \Omega$ such that $v_n \to v$, we have $R(v_n) \to R(v)$.*
3. *a compact mapping, if R is continuous and the image of every bounded set is also compact.*

Definition 2.5. *(See [7]) Let us consider a mapping $R : \Omega \subset X \to X^*$, then*

1. *R is of class (S_+), if for all $(v_n)_{n\in\mathbb{N}} \subset \Omega$ such that $v_n \to v$ and $\limsup_{n\to\infty} \langle R(v_n), v_n - v\rangle \leq 0$, we have $v_n \to v$.*
2. *R is quasi-monotone, if for all sequence $(v_n)_{n\in\mathbb{N}} \subset \Omega$ such that $v_n \to v$, we have $\limsup_{n\to\infty} \langle R(v_n), v_n - v\rangle \geq 0$.*

Definition 2.6. *(See [7]) For each operator $R : \Omega \subset X \to X$ and all bounded operator $F : \Omega_1 \subset X \to X^*$ such that $\Omega \subset \Omega_0$, get that R*

1. *is of class $(S_+)_F$, if for each $(v_n)_{n\in\mathbb{N}} \subset \Omega$ such that $v_n \to v$, $y_n := F(v_n) \to y$ and $\limsup_{n\to\infty} \langle R(v_n), y_n - y\rangle \leq 0$, we get $v_n \to v$.*
2. *admits property $(QM)_F$, if for all $(v_n)_{n\in\mathbb{N}} \subset \Omega$ with $v_n \to v$, $y_n := F(v_n) \to y$, we obtain $\limsup_{n\to\infty} \langle R(v_n), y - y_n\rangle \geq 0$.*

The next result contains some fundamental properties of the operators:

$$\mathcal{F}_1(\Omega) := \{S + K : D(K) \cap \Omega \to X^* : K \text{ is demi-continuous, bounded and of class } (S_+) \text{ with respect to } D(K)\},$$

$$\mathcal{H}_G := \{S + K(s) : D(S) \cap \bar{G} \to X^*/K(s) \text{ is a bounded homotopy of class } (S_+) \text{ with respect to } D(S) \text{ from } \bar{G} \text{ to } X^*\},$$

where $\mathcal{H}_G$ (class) includes all affine homotopies $S + (1-s)K_1 + sK_2$ with $\left(K_j + L\right) \in \mathcal{F}_G$, $j = 1, 2$.

Lemma 2.7. *(See [24, Lemma 2.3]) Let X a real reflexive Banach space, and let us $S : D(S) \subset X^* \to X$ to be demi-continuous operator and F a continuous mapping in $\mathcal{F}_1(\bar{E})$, where $F(\bar{E}) \subset D(S)$ and $E \subset X$ is an open bounded domain. Then*

1. *If S is quas-imonotone, we get* $\mathcal{I} + SoF \in \mathcal{F}_F(\overline{G})$, *where* $\mathcal{I}$ *is the identity.*
2. *If S is of class* (S_+), *we obtain* $SoF \in \mathcal{F}_F(\overline{G})$.

The following result owing to Mustonen and Berkovits [6] provides us with the existence of a topological degree function fulfil (a)-(d).

Theorem 2.8. *(See [6]) Let us consider* $L : D(L) \subset X \longrightarrow X^*$ *a densely maximal monotone linear mapping. There exists a topological degree function* $d : \{(R, G, h) : R \in \mathcal{F}_G,\ G \subset X \text{ an open bounded, } h \notin R(\partial G \cap D(L))\} \rightarrow \mathbb{Z}$ *fulfilled the following:*

(a) If $d(R, G, h) \neq 0$, *we get* $R(v) = h$ *admit a solution in* $G \cap D(L)$. *(Existence).*

(b) If $G_1, G_2 \in G$ *and* $G_1 \neq G_2$ *such that* $h \notin R\left[(\bar{G}\setminus (G_1 \cup G_2)) \cap D(L)\right]$, *then we have*

$$d(R, G, h) = d\left(R, G_1, h\right) + d\left(R, G_2, h\right). \text{ (Additivity).}$$

(c) If $R(s) \in \mathcal{H}_G$ *and* $h(s) \notin R(s)(\partial G \cap D(L))$ *for all* $s \in [0, 1]$, *where* $h(s)$ *is a continuous curve in* X^*, *then*

$$d(R(s), G, h(s)) = \text{ constant, for all } s \in [0, 1]. \text{ (Invariance under homotopies)}$$

(d) $J + L$ *is a normalizing map, where* J *is the duality mapping of* X *into* X^*, *that is,*

$$d(J + L, G, h) = 1, \text{ whenever } h \in (J + L)(G \cap D(L)). \text{ (Normalization).}$$

Proposition 2.9. *Under assumptions* (H1)-(H2), *the operator*

$$S : W_0^{1,p(z)}(\Omega) \longrightarrow W_0^{1,p'(z)}(\Omega)$$

defined by

$$\langle Sv, w\rangle = -\int_\Omega \frac{\lambda f(v)}{(\int_\Omega f(v)dz)^2} w dz, \text{ for all } v, w \in W_0^{1,p(z)}(\Omega) \text{ is compact.}$$

Proof. First, we demonstrate

$$\varphi(v) := \frac{\lambda f(v)}{(\int_\Omega f(v)dz)^2}$$

is bounded and continuous. For all $v \in W_0^{1,p(z)}(\Omega)$, from (3) we get

$$\begin{aligned}
\left\|\varphi(v)\right\|_{p'(z)} &\leq \rho_{p'(z)}(\varphi(v)) + 1 \\
&\leq \frac{\lambda^{p^+}}{(\sigma meas(\Omega))^{2p^-}} \int_\Omega |f(v(z,s))|^{p'(z)} dz + 1 \\
&\leq \frac{\lambda^{p^+}}{(\sigma meas(\Omega))^{2p^-}} \int_\Omega \left(C_1(|v|^{q(z)-1} + \beta(z))\right)^{p'(z)} + 1 \\
&\leq \frac{\lambda^{p^+}}{(\sigma meas(\Omega))^{2p^-}} \left(\rho_{p'(z)}(\beta) + \rho_{r(z)}(v)\right) + 1.
\end{aligned} \tag{13}$$

Since

$$\rho_{p(z)}(v) \leq \|v\|_{p(z)}^{p^-} + \|v\|_{p(z)}^{p^+},$$

there exits a positive constant C_2 such that

$$\left\|\varphi(v)\right\|_{p'(z)} \leq \frac{\lambda^{p^+} C_2}{(\sigma meas(\Omega))^{2p^-}} \left(\left\|\beta\right\|_{p'(z)}^{p'^+} + \|v\|_{r(z)}^{r^+}\right) + 1.$$

Since $r(z) := p'(z)(q(z)-1)$, we have $L^{p(z)}(\Omega) \hookrightarrow L^{r(z)}(\Omega)$. Keeping this in mind and using Poincarée's Inequality [13], to obtain that

$$\|v\|_{r(z)}^{r^+} \leq \|v\|_{p(z)}^{r^+} \leq \|v\|_{p(z)}^{p^+} \leq C_3 \|v\|_{1,p(z)}^{p^+}. \tag{14}$$

From (13) and (14), it yields that

$$\left\|\varphi(v)\right\|_{p'(z)} \leq \frac{\lambda^{p^+} C_2}{(\sigma \cdot meas(\Omega))^{2p^-}} \left(\left\|\beta\right\|_{p'(z)}^{p'^+} + C_3 \|v\|_{1,p(z)}^{p^+}\right) + 1. \tag{15}$$

All the term in the right side of the above inequality are bounded. One can conclude that φ is bounded in $W_0^{1,p(z)}(\Omega)$. It can be shown that φ is a continuous function. It is worth mentioning that

$$\mathcal{I} : W_0^{1,p(z)}(\Omega) \to L^{p(z)}(\Omega)$$

is compact and knowing that the adjoint operator

$$\mathcal{I}^* : L^{p'(z)}(\Omega) \to W^{-1,p'(z)}(\Omega)$$

is also compact, we get $I^* o\phi : W_0^{1,p(z)}(\Omega) \to W^{-1,p'(z)}(\Omega)$ is compact. This concludes the proof. □

2.1. Properties of p(z)-Laplacian operator

We devote this Section to presents some properties of $p(z)$-Laplacian, which useful in the sequel. We define $p(z)$-Laplacian operator as follows:

$$-\Delta_{p(z)} v := -\operatorname{div}\left(|\nabla v|^{p(z)-2} \nabla v\right).$$

It is worth mentioning that

$$K(v) := \int_\Omega \frac{1}{p(z)} |\nabla v|^{p(z)} dz, \quad v \in W_0^{1,p(z)}(\Omega).$$

From [10], we obtain $K \in C^1\left(W_0^{1,p(z)}(\Omega), \mathbb{R}\right)$, and the $p(z)$-Laplacian operator is the derivative operator of K in the weak sense. We denote

$$J := K' : W_0^{1,p(z)}(\Omega) \to W^{-1,p'(z)}(\Omega),$$

then we have

$$\langle Jv, w\rangle = \int_\Omega |\nabla v|^{p(z)-2} \nabla v \nabla w dz, \text{ for all } v, w \in W_0^{1,p(z)}(\Omega).$$

Theorem 2.10. *(See [10, Theorem 3.1])*

- $J : W_0^{1,p(z)}(\Omega) \to W^{-1,p'(z)}(\Omega)$ *is a strictly monotone, continuous and bounded operator;*
- J *is a mapping of class* (S_+);
- J *is a homeomorphism.*

3. Existence of a weak solution

This section is devoted to prove existence of a weak solution for problem (1).

Theorem 3.1. *Let* $S + J \in \mathcal{F}_X$ *and* $\varphi \in X^*$. *Suppose that there exists* $r > 0$ *such that*

$$\langle S(v) + J(v) - \varphi(v), v\rangle > 0, \tag{16}$$

for all $v \in \partial B_r(0) \cap D(S)$. *Hence,* $(S + J)(D(S)) = X^*$.

Proof. Let ε be a positive constant with τ in $[0,1]$ and

$$\mathcal{H}_\varepsilon(\tau, v) = S(v) + (1-\tau)J(v) + \tau(J(v) + \varepsilon L(v) - \varphi(v)).$$

Using the boundary condition of (1), we get

$$\begin{aligned}\langle \mathcal{H}_\varepsilon(\tau, v), v\rangle &= \langle \tau(S(v) + J(v) - \varphi(v)), v\rangle + \langle (1-\tau)S(v) + (1+\varepsilon\tau-\tau)L(v), v\rangle \\ &\geq \langle (1-\tau)S(v) + (1+\varepsilon\tau-\tau)S(v), v\rangle.\end{aligned}$$

In view of the fact that $0 \in S(0)$, we obtain the following inequalities

$$\begin{aligned}\langle \mathcal{H}_\varepsilon(\tau, v), v\rangle &\geq (1-\tau)\langle S(v), v\rangle + (1+\varepsilon\tau-\tau)\langle L(v), v\rangle \\ &\geq (1+\varepsilon\tau-\tau)\|v\|^2 = (1+\varepsilon\tau-\tau)r^2 > 0.\end{aligned}$$

That is $0 \notin \mathcal{H}_\varepsilon(\tau, v)$. By reason of $S + \varepsilon J$ and J are bounded, continuous and of type (S_+), $\{\mathcal{H}_\varepsilon(\tau,\cdot)\}_{\tau\in[0,1]}$ is an admissible homotopy. Consequently, by invariance, homotopy and normalisation, we get

$$d\left(\mathcal{H}_\varepsilon(\tau,\cdot), B_r(0), 0\right) = d\left(S + L, B_r(0), 0\right) = 1.$$

Therefore, there exists $v_\varepsilon \in D(S)$ such that $0 \in \mathcal{H}_\varepsilon(\tau,\cdot)$. Letting $\varepsilon \to 0^+$and $\tau = 1$, we get $\varphi(v) \in Sv + Jv$ for some $v \in D(S)$. In view of the fact that $\varphi \in X^*$, we obtain $(S + J)(D(S)) = X^*$. □

The following theorem gives the existence result.

Theorem 3.2. *Let $\varphi \in \mathcal{U}^*$ and $v_0 \in L^2(\Omega)$. There exists a weak solution $v \in D(S)$ of problem (1) such that*

$$-\int_Q v\frac{\partial w}{\partial s}dzds + \int_Q |\nabla v|^{p(z)-2}\nabla v\nabla w dzds = \int_Q \varphi(v)w dzds,$$

for all $w \in \mathcal{U}$.

Proof. Let us $S : D(S) \subset \mathcal{U} \longrightarrow \mathcal{U}^*$, where

$$D(S) = \{v \in \mathcal{U} : v' \in \mathcal{U}^*, v(0) = 0\}$$

and

$$\langle S(v), w\rangle = -\int_Q v\frac{\partial w}{\partial s}dzds, \text{ for all } v \in D(S), w \in \mathcal{U}.$$

We define S by the following identity

$$\langle S(v), w\rangle = \int_0^T \langle v'(s), w(s)\rangle\, ds, \text{ for all } v \in D(S), w \in \mathcal{U}.$$

The existence of S as a densely defined maximal monotone operator can be established, as in [29]. Using the fact that $\langle S(v), v\rangle \geq 0$ for all $v \in D(S)$ and by (12), we get

$$\langle S(v) + J(v), v\rangle \geq \langle J(v), v\rangle = \int_Q |\nabla v|^{p(z)}dzds \geq \|\nabla v\|^{p^-}_{L^{p(\cdot)}(Q)} - 1 = \|v\|^{p^-}_{\mathcal{U}} - 1, \tag{17}$$

for all $v \in \mathcal{U}$. In view of the fact that the right side of inequality (17) approaches ∞ as $\|v\|_{\mathcal{U}} \to \infty$. So, for any function $\varphi(v) \in \mathcal{U}^*$, there exists a positive constant r such that $\langle S(v) + J(v) - \varphi(v), v\rangle > 0$ for all $v \in B_r(0) \cap D(S)$. Thanks to Theorem 2.8. Hence, the identity $S(v) + J(v) = \varphi(v)$ is solvable in $D(S)$, that is, the problem (1) has a weak solution. □

4. Numerical results

In this section, we present numerical example to illustrate the evolution of solution. We use explicit scheme of finite difference

$$\begin{cases} v_s(z,s) = \left(|v_z(z,s)|^{p(z)-2}\, v_z(z,s)\right)_z + \dfrac{\lambda f(v(z,s))}{(\int_\Omega f(v(z,s))dz)^2}, & z \in [0,1], s > 0, \\ v(z,0) = v_0(z), & z \in [0,1], \\ v(z,s) = 0, & z \in \partial\Omega, s > 0. \end{cases} \tag{18}$$

For the space discretization, we take

$$U_h = \{z_i : \quad 0 = z_0 < z_1 < \ldots < z_{M+1} = 1\}$$

with $z_i = ih$ on Ω and substitute $\left(|v_z|^{p(z)-2}\, v_z\right)_z$ by the central difference approximation for $i \in \{1, 2, ..., M\}$. System (18) becomes as in (19), for $V_i^n \approx v\,(z_i, s_n)$ $(1 \le i \le M)$ and $n = 0, \ldots, m-1$, we have

$$\begin{cases} \dfrac{V_i^{n+1} - V_i^n}{\Delta s} = h^{-p(z_i)} B\left(V_i^n\right) + \dfrac{\lambda f\left(V_i^n\right)}{\left(\frac{1}{2}\left[f(V_M^n) + f(V_1^n)\right]\right)^2}, & 1 \le i \le M, \\ \quad V_0^{n+1} = V_{M+1}^{n+1}, \\ \quad V(0,s) = V(1,s) = 0, \end{cases} \tag{19}$$

where $q_i + 2 := p\,(z_i)$ and

$$\begin{aligned} B\left(V_i^n\right) = & \left|V_{i+1}^n - V_i^n\right|^{q_i}\left(V_{i+1}^n - V_i^n\right) \\ & - \left|V_i^n - V_{i-1}^n\right|^{q_i}\left(V_i^n - V_{i-1}^n\right). \end{aligned}$$

Using MATLAB, we illustrate our previous results with the numerical experiments. We take the initial data $v_0(z) = (z-1)sin(z)$, $z \in [0,1]$. The other parameters are specified in Figure 1.

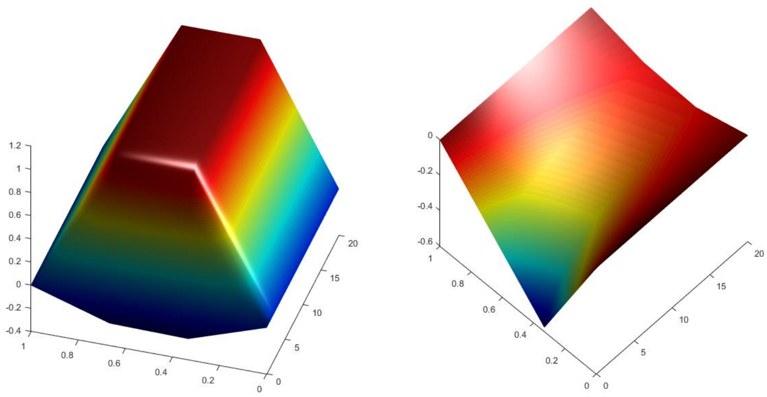

Figure 1: Evolution of Solution for $p(z) = \pi z + 2$ at the left and $p(z) = 1$ at the right for Example 1

The region starts as blue, indicating a temperature of zero degrees. The temperature then steadily rises until it reaches its maximum in the middle. After that, the material returns to its initial temperature.

5. Conclusion

In this work, we showed existence of a weak solution in Sobolev spaces with variable exponent for a nonlocal thermistor problem in the presence of doubly nonlinear terms. We also present a numerical example to illustrate the evolution of solution. As future work, we plan to study the regularity of solution in Sobolev spaces generalized for the same problem (1).

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