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On the b-chromatic number of rooted product graphs

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Abstract. The b-chromatic number of a graph G was defined by Irving and Manlove in 1999 as the largest integer k for which G admits a proper coloring with k colors such that every color class (in this proper coloring) has a vertex that is adjacent to at least one vertex in every other color class. The b-chromatic number has been studied in many contexts, including for various graph products. The rooted product, defined by Godsil and McKay in 1978, is not yet among these. We find bounds for the b-chromatic number of the rooted product of two graphs in terms of the b-chromatic numbers and degrees of the factors, along with some new parameters that we define. Moreover, we give sufficient conditions for equality to hold in these bounds. We refine our results, sometimes to exact values, when one or both of the factors is a path, cycle, complete graph, star, or wheel.

1. Introduction

All graphs considered in this paper are simple, undirected, connected, and finite, with more than one vertex. Throughout this section, let G denote such a graph.

We write $V(G)$ for the set of vertices of G and we call $n(G) = |V(G)|$ the *order* of G. We write $E(G)$ for the set of edges of G. The degree of a vertex v in G is denoted by $d_G(v)$. The notation $\Delta(G)$ stands for the maximum vertex degree. Finally, $\chi(G)$ is the chromatic number of G.

Notations P_n, C_n, and K_n stand for the path, the cycle, and the complete graph, respectively, each of order n. We denote by $K_{s,t}$ the complete bipartite graph with partite sets containing s and t vertices. In particular, we write $S_n (= K_{1,n-1})$ for the star graph of order n. Finally, we denote by W_n the wheel graph of order n.

Consider a proper coloring of the vertices of G. A *color-dominating vertex* (CDV for short) is a vertex that is adjacent to at least one vertex from each other color class. The coloring is a *b-coloring* when each color class includes a CDV. By a *k-b-coloring* we mean a b-coloring with k colors. The *b-chromatic number* of G, denoted by $\varphi(G)$, is the largest integer k such that G admits a k-b-coloring. This concept was introduced

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by Irving and Manlove [15] in 1999 as an alternative to the so-called *achromatic coloring*, which is a proper coloring in which there is an edge between any two distinct color classes. Formally, Irving and Manlove defined a b-coloring with the help of strong partially ordered sets through an iterative process, which can be intuitively described as follows. Suppose we start with a proper coloring of G. If there is a color class in this coloring that does not contain a CDV, then we (properly) recolor every vertex from this class and remove this color from further consideration. We repeat this procedure until we end up with a b-coloring, which can be considered a minimal element in this iterative process. There may be many such minimal elements with respect to the proper coloring of G we started with. These minimal elements may use different numbers of colors, with the smallest being the chromatic number and the largest being the b-chromatic number. Thus $\chi(G) \leq \varphi(G)$. The b-chromatic number can be motivated as placing as many communities as possible in a given area such that every community has a representative that is able to communicate with every other community, thereby preventing conflicts in the area.

We mention the easily-verified values of the b-chromatic number for the graph families we highlighted earlier. If G has order 2, then $\varphi(G) = 2$, so suppose G has at least 3 vertices. If $n \geq 5$, then $\varphi(P_n) = 3$, but $\varphi(P_4) = \varphi(P_3) = 2$. If $n \neq 4$, then $\varphi(C_n) = 3$, but $\varphi(C_4) = 2$. For any n, $\varphi(K_n) = n$ and $\varphi(S_n) = 2$. Now suppose G has at least 4 vertices. If $n \neq 5$, then $\varphi(W_n) = 4$, but $\varphi(W_5) = 3$.

It is clear that $\varphi(G) \leq \Delta(G) + 1$. However, the difference between these values can be arbitrarily large, as can be seen by considering star graphs, since $\varphi(S_n) = 2$ is fixed but $\Delta(S_n) = n - 1$ is not constrained. In this sense, the bound using $\Delta(G)$ does not perform well in general. Therefore, Irving and Manlove introduced another parameter. Suppose that the vertices $v_1, \ldots, v_n$ of G are ordered in such a way that their degrees form a non-increasing sequence: $d_G(v_1) \geq d_G(v_2) \geq \cdots \geq d_G(v_n)$. The *m-degree* of G is defined to be $m(G) = \max\{i : d_G(v_i) \geq i - 1\}$. Intuitively, the idea is that G has $m(G)$ vertices of degree at least $m(G) - 1$, which are the candidates to be CDVs. In [15] it was shown that $\varphi(G) \leq m(G) \leq \Delta(G) + 1$ and that $m(G)$ performs as a much better upper bound to the b-chromatic number.

Returning to our highlighted graph families, we find their m-degrees. If G is a path, cycle with order not 4, complete graph, star, or wheel graph with order not 5, then $m(G) = \varphi(G)$. The exceptions $m(C_4) = 3$ and $m(W_5) = 4$ are one more than the corresponding b-chromatic numbers.

The concept of b-coloring has gained much attention in the scientific arena in the last couple of decades. From the complexity point of view, determining the b-chromatic number of an arbitrary graph is an NP-complete problem, but it turns out that the problem can be solved in polynomial time for trees. Both of these results were proved in the original paper [15]. Approximation algorithms were given in [8, 12]. For many other general results and results on particular graph classes, we refer the reader to a recent survey [19] that covers and cites many papers on this topic.

An important and fruitful subtopic is regular graphs. The two upper bounds, $m(G)$ and $\Delta(G) + 1$, coincide if G is a regular graph. It is not hard to see that if G is a regular graph with enough vertices, then $\varphi(G)$ will tend toward $\Delta(G) + 1$. Many articles have tried to determine the minimum order that a regular graph G must have in order to force $\varphi(G) = \Delta(G) + 1$ [1, 3, 10]. The other way to look at this problem is to find the list of regular graphs G for which $\varphi(G) < \Delta(G) + 1$. It is easy to consider 1- and 2-regular graphs, and it turns out that there are precisely four exceptions among 3-regular (cubic) graphs [16]. However, the problem becomes considerably more challenging for higher degrees because the number of exceptions increases rapidly [11]. Another approach is to consider the girth of regular graphs. Larger girth forces vertices of a graph G that are close to each other to have almost disjoint neighborhoods, so they can easily be made into CDVs while the rest of the graph can then be greedily colored with $\Delta(G)+1$ colors (e.g., [2–5]). A very recent contribution is in the article [9], where the authors propose two approaches to showing that the Petersen graph is the only regular graph with girth at least 5 that fails to have b-chromatic number one more than its degree.

The b-chromatic index, for which the edges are the colored objects, was considered in [6, 18, 20], where many results that mimic those for the b-chromatic number were proved.

The b-chromatic number was extensively studied for different graph products. The Cartesian product was considered in [14, 21], where the focus was on the Cartesian product of complete graphs, also known as

Hamming graphs. The generalized Hamming graphs were considered in [7]. As for the Cartesian product of general graphs, only bounds are known (e.g., [22]). Polynomial-time algorithms for determining the b-chromatic number of the Cartesian product of trees with paths/cycles/stars were given in [23]. The direct, the strong, and the lexicographic product of graphs were studied in [17], where many general bounds were obtained and exact results were given for all three products of some families of graphs. The b-chromatic index of direct products of graphs was studied in [20], which provided some general results for many direct products of regular graphs and introduced an integer linear programming model which was used to compute some exact results for the direct products of special graph classes. Another well-known graph product, the rooted product defined by Godsil and McKay [13], is the focus of the present paper.

Let G and H be graphs. We choose a vertex $v \in V(H)$ to be the *root* of H. The *rooted product* of G and H, written $G \circ_v H$, has vertex set $V(G \circ_v H) = V(G) \times V(H)$ and edge set

$$E(G \circ_v H) = \{(x, v)(x', v) : xx' \in E(G)\} \cup \{(x, y)(x, y') : x \in V(G), yy' \in E(H)\}.$$

We observe that $G \circ_v H$ is a subgraph of the Cartesian product $G \,\square\, H$. We refer to the set $V(G) \times \{v\}$ as the *G-layer* of $G \circ_v H$. Similarly, for any vertex $x \in V(G)$, we call the set $\{x\} \times V(H)$ an *H-layer* of $G \circ_v H$. More specifically, we can say it is the ${}^{x}H$-layer. Note that there is only one G-layer, but there are $n(G)$ H-layers. Layers can also be considered to be the subgraphs of $G \circ_v H$ induced by the sets that define them, as appropriate. Obviously, in $G \circ_v H$ the G-layer and each H-layer are isomorphic to G and H, respectively.

Determining the b-chromatic number of the Cartesian product of graphs is hard in general. Since the rooted product is a subgraph of the Cartesian product, it could be enlightening to see what information one can obtain for the b-chromatic number of rooted products. In Section 2 we provide some general results and in Section 3 we give some results where one or both of the factors is fixed. We conclude with Section 4, which poses a few interesting problems that remain open.

2. General results

We begin by presenting some results that apply to broad classes of graphs. All of the bounds in this section are sharp, as demonstrated by examples in Section 3. First we find a lower bound on the b-chromatic number of the rooted product in terms of those of its factors.

Theorem 2.1. *Let G and H be graphs and let v be the root of H. Then*

$$\varphi(G \circ_v H) \geq \max\{\varphi(G), \varphi(H)\}.$$

Proof. We consider two cases: $\varphi(G) \geq \varphi(H)$ and $\varphi(G) < \varphi(H)$.

Suppose $\varphi(G) \geq \varphi(H)$. Color the G-layer of $G \circ_v H$ using a b-coloring of G with $\varphi(G)$ colors. Since $\chi(H) \leq \varphi(H) \leq \varphi(G)$, we can properly color the rest of each H-layer without needing any additional colors. The CDVs from G are CDVs of the product, so this produces a b-coloring of $G \circ_v H$ with $\varphi(G)$ colors. Now $\varphi(G \circ_v H)$ is at least $\varphi(G)$, which is $\max\{\varphi(G), \varphi(H)\}$.

Suppose $\varphi(G) < \varphi(H)$. Color one of the H-layers of $G \circ_v H$ using a b-coloring of H with $\varphi(H)$ colors. Since $\chi(G) \leq \varphi(G) < \varphi(H)$, we can properly color the rest of the G-layer without needing any additional colors. Since $\chi(H) \leq \varphi(H)$, we can also properly color the rest of the remaining H-layers, again without needing additional colors. The CDVs from the first H-layer are CDVs of the product, so this produces a b-coloring of $G \circ_v H$ with $\varphi(H)$ colors. Now $\varphi(G \circ_v H)$ is at least $\varphi(H)$, which is $\max\{\varphi(G), \varphi(H)\}$. □

We next calculate our first b-chromatic number for a rooted product of specific graphs.

Proposition 2.2. *Let $t \geq 4$ and let v be the root of S_t. If $d_{S_t}(v) = 1$, then*

$$\varphi(P_t \circ_v S_t) = t.$$

Proof. Because b-chromatic number is at most one more than maximum degree, we have $\varphi(P_t \circ_v S_t) \leq t$.

Let P_t have vertices $\{1, \ldots, t\}$ and edges $\{ij : 1 \leq i \leq t-1, j = i+1\}$. Let S_t have vertices $\{0, \ldots, t-1\}$ and edges $\{0i : 1 \leq i \leq t-1\}$. Without loss of generality, let v be vertex 1 in S_t. In $P_t \circ_v S_t$, give vertex (i, j) color $i + j \pmod t$. This is a b-coloring with t colors, so $\varphi(P_t \circ_v S_t) \geq t$. □

Recall that $\varphi(P_n) \leq 3$ and $\varphi(S_n) = 2$. Proposition 2.2 shows that, without restricting G and H, no function of $\varphi(G)$ and $\varphi(H)$ can bound $\varphi(G \circ_v H)$ from above. In fact, even the amount by which $\varphi(G \circ_v H)$ exceeds the lower bound in Theorem 2.1 can be arbitrarily large.

To improve the lower bound from Theorem 2.1, we make the following definition.

Definition 2.3. *Let G be a graph. Denote the maximum number of CDVs in any $\varphi(G)$-b-coloring of G by $n_\varphi(G)$.*

Again addressing our highlighted graph families, we note their values of our new parameter. A graph G of order 2 has $n_\varphi(G) = 2$, so suppose $n \geq 3$. If n is at most 4, then $n_\varphi(P_n) = n$; otherwise, $n_\varphi(P_n) = n-2$. If n is at most 4 or is a multiple of 3, then $n_\varphi(C_n) = n$; otherwise, $n_\varphi(C_n) = n-2$. For any n, $n_\varphi(K_n) = n$. Similarly, $n_\varphi(S_n) = n$. Now suppose $n \geq 4$. If $n-1$ is at most 4 or is a multiple of 3, then $n_\varphi(W_n) = n$; otherwise, $n_\varphi(W_n) = n-2$.

Clearly, $n_\varphi(G) \geq \varphi(G)$ for any graph G. The difference between the two can be arbitrarily large, as seen by considering cycles, for example.

When this new parameter and degrees from the second factor are helpfully related, we find a better lower bound on the b-chromatic number of a rooted product.

Theorem 2.4. *Let H be a graph with root v and let G be a graph with*

$$n_\varphi(G) \geq \varphi(G) + d_H(v) \geq \Delta(H) + 1.$$

Then

$$\varphi(G \circ_v H) \geq \varphi(G) + d_H(v).$$

Proof. We start by choosing a $\varphi(G)$-b-coloring of G with $n_\varphi(G)$ CDVs. Let these CDVs be denoted $x_1, \ldots, x_{n_{\varphi(G)}}$. Without loss of generality, we assume that x_i is colored with color i for each $i \in \{1, \ldots, \varphi(G)\}$.

Keep in mind that $n_{\varphi(G)} \geq \varphi(G) + d_H(v)$ and $d_H(v) \geq 1$. For each $i \in \{1, \ldots, n_{\varphi(G)}\}$, since x_i is a CDV, we know that $d_G(x_i) \geq \varphi(G) - 1$.

For every $i \in \{\varphi(G)+1, \ldots, \varphi(G)+d_H(v)\}$, we recolor the vertex x_i with (new) color i. From this, we obtain a proper coloring of G which is in general not a b-coloring anymore.

We color the G layer in $G \circ_v H$ with the obtained proper coloring of G. Every vertex (x_i, v), $i \in \{1, \ldots, \varphi(G)+d_H(v)\}$, has exactly $d_H(v)$ uncolored neighbors, which are precisely the neighbors of (x_i, v) in the ${}^{x_i}H$-layer. For all $i \in \{1, \ldots, \varphi(G)+d_H(v)\}$, at least $\varphi(G)$ different colors were already used on vertex (x_i, v) and its neighbors in the G-layer, since x_i was a CDV in the original b-coloring. For every color from the set $\{1, \ldots, \varphi(G)+d_H(v)\}$, which is not used on the vertex (x_i, v) and its neighbors in the G-layer, we color exactly one neighbor of (x_i, v) in the ${}^{x_i}H$-layer with this color. Note that we need at most $d_H(v)$ such neighbors, which is precisely what we have. We properly color the rest of the vertices in $G \circ_v H$ with colors from the set $\{1, \ldots, \varphi(G) + d_H(v)\}$ by using the greedy algorithm. This is possible because $d_{G \circ_v H}(x, y) = d_H(y) \leq \Delta(H) \leq \varphi(G) + d_H(v) - 1$ for every $x \in V(G)$ and every $y \in V(H) \backslash \{v\}$.

Observe that we have constructed a b-coloring of $G \circ_v H$ with exactly $\varphi(G) + d_H(v)$ colors such that each (x_i, v), $i \in \{1, \ldots, \varphi(G) + d_H(v)\}$, is a CDV. Thus, $\varphi(G \circ_v H) \geq \varphi(G) + d_H(v)$. □

It turns out that many graphs have b-chromatic number equal to m-degree, which makes the following corollary particularly useful.

Corollary 2.5. *Let H be a graph with root v and let G be a graph with*

$$n_\varphi(G) \geq \varphi(G) + d_H(v) \geq \Delta(H) + 1.$$

If $\varphi(G) = m(G)$, then

$$\varphi(G \circ_v H) = \varphi(G) + d_H(v).$$

Proof. By Theorem 2.4, $\varphi(G \circ_v H) \geq \varphi(G) + d_H(v)$. Suppose that $\varphi(G \circ_v H) > \varphi(G) + d_H(v)$. Since $m(G \circ_v H) \geq \varphi(G \circ_v H)$, there are at least $\varphi(G) + d_H(v) + 1$ vertices of degree at least $\varphi(G) + d_H(v)$ in $G \circ_v H$. These vertices must be in the G-layer because $\varphi(G) + d_H(v) > \Delta(H)$. Thus there are at least $\varphi(G) + 1$ vertices of degree at least $\varphi(G)$ in G. But this means that $m(G) > \varphi(G)$, which is a contradiction. □

One of the hypotheses of Theorem 2.4 is that $\varphi(G) + d_H(v)$ is at least $\Delta(H) + 1$. What if the direction of the inequality is reversed? This will be partially answered by Theorem 2.7, which requires the following definition.

Definition 2.6. *Let H be a graph. Let D denote the set of all maximum-degree vertices of H. That is, $D = \{y \in V(H) : d_H(y) = \Delta(H)\}$. We will call a subset $F \subseteq D$ a* far set *of H if for every two distinct vertices y and y' in F, the distance between y and y' is at least 4. Finally, we define*

$$n_f(H) = \max\{|F| : F \text{ is a far set of } H\}.$$

Theorem 2.7. *Let H be a graph with root v and let G be a graph with*

$$\Delta(G) + d_H(v) \leq \Delta(H).$$

If

$$n(G) \geq \frac{\Delta(H) + 1}{n_f(H)},$$

then

$$\varphi(G \circ_v H) = \Delta(H) + 1.$$

Proof. Since $\Delta(G) + d_H(v) \leq \Delta(H)$, we know $\Delta(G \circ_v H) = \Delta(H)$ and therefore $\varphi(G \circ_v H) \leq \Delta(H) + 1$. It thus suffices to construct a b-coloring of $G \circ_v H$ using $\Delta(H) + 1$ colors.

Let F be a far set of H such that $|F| = n_f(H)$. We will use the colors $0, 1, \ldots, \Delta(H)$. Any colors given via formula should be evaluated modulo $\Delta(H) + 1$. Let the vertices of G be denoted $x_1, \ldots, x_{n(G)}$ and let those of F be $y_1, \ldots, y_{n_f(H)}$.

First give each vertex (x_i, y_j) color $i \cdot n_f(H) + j$. Next, if v is a neighbor of any vertex in F, then give each vertex (x_i, v) color i. This guarantees no color conflicts in the G-layer. Note that v itself is not in F because $\Delta(G) + d_H(v) \leq \Delta(H)$ and $\Delta(G) > 0$, so this second round of assignments does not interfere with the first. Now color the non-G-layer neighbors of each (x_i, y_j) appropriately to make that (x_i, y_j) a CDV. This is possible because any two members of F are no closer than distance 4 and thus cannot have neighbors that are adjacent to each other. We assumed $n(G) \cdot n_f(H) \geq \Delta(H) + 1$, so we have created a full complement of CDVs. Finally, color any remaining vertices greedily. This works because the maximum degree is less than the number of colors available. □

Essentially, the difference between Theorems 2.4 and 2.7 is whether we locate our CDVs in the G-layer or the H-layers. We note that Proposition 2.2 follows from Theorem 2.7.

The remaining results in this section are about m-degree, which we recall functions as an upper bound on the b-chromatic number.

Theorem 2.8. *Let G and H be graphs and let v be the root of H. Define $M = \{x : d_G(x) \geq m(G) - 1\}$. If*

$$m(G) > \Delta(H) - d_H(v) + 1,$$

then

$$m(G) \leq \min\{|M|, m(G) + d_H(v)\} \leq m(G \circ_v H) \leq m(G) + d_H(v).$$

Proof. Define $M' = \{x : d_G(x) \geq m(G)\}$. From the definition of m-degree, we know that $|M| \geq m(G)$ and $|M'| < m(G) + 1$.

Observe that $d_{G\circ_v H}(x, v) < m(G) + d_H(v) - 1$ if $x \notin M$. When $y \neq v$, we have $d_{G\circ_v H}(x, y) = d_H(y)$ being at most $\Delta(H)$, which we assume to be strictly less than $m(G) + d_H(v) - 1$. To have $d_{G\circ_v H}(x, y) \geq m(G) + d_H(v)$ thus requires $x \in M'$ and $y = v$. Since there are fewer than $m(G) + 1$ such (x, y), we have $m(G \circ_v H) \leq m(G) + d_H(v)$.

Now observe that $d_{G\circ_v H}(x, y)$ is at least $m(G) + d_H(v) - 1$ if $x \in M$ and $y = v$. If $|M| \geq m(G) + d_H(v)$, then $G \circ_v H$ has at least $m(G) + d_H(v)$ vertices with degree at least $m(G) + d_H(v) - 1$, and $m(G \circ_v H) \geq m(G) + d_H(v)$. On the other hand, if $|M| \leq m(G) + d_H(v)$, then $G \circ_v H$ has at least $|M|$ vertices with degree at least $|M| - 1$, and $m(G \circ_v H) \geq |M|$. □

One might think that the value of $m(G \circ_v H)$ can only be one of the two extremes from Theorem 2.8. However, it can in fact be strictly between them. For an example, suppose G has ten vertices of degree 9 and ten of degree 8, while H is 5-regular. Then $|M| = m(G) = 10$ and $m(G) + d_H(v) = 15$, but $m(G \circ_v H) = 14$.

If G is regular, then its m-degree is easily seen to be one more than its degree, making the bound $\varphi(G) \leq m(G)$ redundant. The rooted product of regular graphs is not itself regular, though, so this bound is still relevant there. However, calculating the m-degree of a rooted product is not trivial even when both of the factors are regular. We give a complete list of the values the m-degree can take in this situation.

Theorem 2.9. *Suppose G and H are regular graphs and let v be the root of H. Then*

$$m(G \circ_v H) = \begin{cases} \Delta(G) + \Delta(H) + 1 & \Delta(G) + \Delta(H) + 1 \leq n(G) \\ n(G) & \Delta(H) + 1 < n(G) < \Delta(G) + \Delta(H) + 1 \\ \Delta(H) + 1 & n(G) \leq \Delta(H) + 1 \end{cases}.$$

Proof. In $G \circ_v H$ there are $n(G)$ vertices with degree $\Delta(G) + \Delta(H)$. The remaining (more than $\Delta(H) + 1$) vertices have (lower) degree $\Delta(H)$.

If $n(G)$ is at least $\Delta(G) + \Delta(H) + 1$, then $m(G \circ_v H) = \Delta(G) + \Delta(H) + 1$. On the other hand, if $n(G)$ is at most $\Delta(H) + 1$, then $m(G \circ_v H) = \Delta(H) + 1$. In either case, there are enough vertices, so degree is the constraint. If $n(G)$ is strictly between $\Delta(G) + \Delta(H) + 1$ and $\Delta(H) + 1$, though, it is not degree but number of vertices with at least that degree that becomes relevant, and $m(G \circ_v H) = n(G)$. □

3. Fixing one or both factors

In this section, we present some results for specific choices of the first or second factor. If we fix one of G and H and impose some constraints on the other, we can often determine $\varphi(G \circ_v H)$ fairly well. In particular, Theorem 2.4 and Corollary 2.5 give us information when either G or H is a path, cycle, star, or wheel, sometimes with one of the hypothesis inequalities holding automatically. When H is S_t or W_t and the root is not the vertex of degree $t - 1$, Theorem 2.7 applies. Before developing new results concerning fixing a factor, we explore the above applications.

If we fix G, Corollary 2.5 specializes in the following manner. For simplicity, suppose $s \geq 6$. In the case that G is P_s or C_s, we get that $\varphi(G \circ_v H) = 3 + d_H(v)$ whenever $\Delta(H) - 2 \leq d_H(v) \leq s - 5$. (If G is C_s with s a multiple of 3, the upper bound can be relaxed to $s - 3$.) In the case that G is S_s, we get that $\varphi(G \circ_v H) = 2 + d_H(v)$ whenever $\Delta(H) - 1 \leq d_H(v) \leq s - 2$. In the case that G is W_s, we get that $\varphi(G \circ_v H) = 4 + d_H(v)$ whenever $\Delta(H) - 3 \leq d_H(v) \leq s - 6$. (If $s - 1$ is a multiple of 3, the upper bound can be relaxed to $s - 4$.)

Now, let G be an arbitrary graph and let us focus on fixing the second factor, H.

Suppose that H is P_t or C_t. Theorem 2.4 shows that $\varphi(G \circ_v H) \geq \varphi(G) + d_H(v)$ whenever $n_\varphi(G) \geq \varphi(G) + d_H(v)$. In this result, we get equality via Corollary 2.5 whenever $\varphi(G) = m(G)$. Theorem 2.7 is not useful because $\Delta(H)$ is so small.

If we now take H to be S_t, $\varphi(G \circ_v H)$ will depend on whether we take the root to be the central vertex of S_t or a leaf. Suppose that v is a leaf, so that $d_H(v) = 1$. If $t \leq \varphi(G) + 1 \leq n_\varphi(G)$, then $\varphi(G \circ_v H) \geq \varphi(G) + 1$ by Theorem 2.4, with equality via Corollary 2.5 whenever $\varphi(G) = m(G)$. If $\Delta(G) + 2 \leq t \leq n(G)$, then

$\varphi(G \circ_v H) = t$ by Theorem 2.7. (Note that this covers a generalized version of Proposition 2.2.) Now suppose that v is the central vertex of S_t, so that $d_H(v) = t-1$. In this case, we see that if $n_\varphi(G) \geq \varphi(G) + t - 1$, then $\varphi(G \circ_v H) \geq \varphi(G) + t - 1$ by Theorem 2.4, with equality via Corollary 2.5 whenever $\varphi(G) = m(G)$. Theorem 2.7 is not useful in this context because $d_H(v) = \Delta(H)$.

We now consider the case where H is W_t. Like when H is a star, $\varphi(G \circ_v H)$ will depend on whether we take v to be the central vertex or a vertex on the outer cycle. Suppose that v is the central vertex, so $d_H(v) = t - 1$. If $n_\varphi(G) \geq \varphi(G) + t - 1$, then $\varphi(G \circ_v H) \geq \varphi(G) + t - 1$ by Theorem 2.4, with equality via Corollary 2.5 whenever $\varphi(G) = m(G)$. This is another situation where Theorem 2.7 does not help. Now suppose that v is on the outer cycle, so $d_H(v) = 3$. If $t \leq \varphi(G) + 3 \leq n_\varphi(G)$, then $\varphi(G \circ_v H) \geq \varphi(G) + 3$ by Theorem 2.4, with equality via Corollary 2.5 whenever $\varphi(G) = m(G)$. Applying Theorem 2.7 instead, we get $\varphi(G \circ_v H) = t$ if $\Delta(G) + 4 \leq t \leq n(G)$.

For more details, including how small cases are dealt with separately, we refer the reader to Propositions 3.1–3.4, which give the b-chromatic number and m-degree of rooted products whose factors are paths or cycles. In the literature, these are the most commonly treated factors. Their Cartesian products are sometimes called "toroidal" graphs. Other examples mentioned in the above discussion can be tackled in a similar manner.

Proposition 3.1. *Let $s, t \geq 2$ and let v be the root of P_t.*
If $d_{P_t}(v) = 1$, then

$$\varphi(P_s \circ_v P_t) = m(P_s \circ_v P_t) = \begin{cases} 2 & s = t = 2 \\ 3 & (s = 2 \text{ and } t \geq 3) \text{ or } 3 \leq s \leq 5 \\ 4 & s \geq 6 \end{cases}.$$

If $d_{P_t}(v) = 2$, then

$$\varphi(P_s \circ_v P_t) = m(P_s \circ_v P_t) = \begin{cases} 2 & s = 2 \text{ and } t = 3 \\ 3 & (s = 2 \text{ and } t \geq 4) \text{ or } s = 3 \\ 4 & 4 \leq s \leq 6 \\ 5 & s \geq 7 \end{cases}.$$

Proof. We note that $\varphi(P_s) = m(P_s)$. We also note that $\varphi(P_s) + d_{P_t}(v) \geq 2 + 1 \geq \Delta(P_t) + 1$. Together with Corollary 2.5, these show that $n_\varphi(P_s) \geq \varphi(P_s) + d_{P_t}(v)$ suffices to get $\varphi(P_s \circ_v P_t) = \varphi(P_s) + d_{P_t}(v)$. We will use this repeatedly without further comment.

Suppose that $d_{P_t}(v) = 1$. For $s \geq 6$, we have $n_\varphi(P_s) = s - 2 \geq 3 + 1 = \varphi(P_s) + d_{P_t}(v)$, so $\varphi(P_s \circ_v P_t) = 4$. Because $P_5 \circ_v P_t$ has only three vertices of degree at least 3, but at least three vertices of degree at least 2, its m-degree is 3. It is easy to find a 3-b-coloring of $P_5 \circ_v P_t$, so its b-chromatic number is also 3. For $s = 4$ and $s = 3$, we have $n_\varphi(P_s) \geq 2 + 1 = \varphi(P_s) + d_{P_t}(v)$, so $\varphi(P_s \circ_v P_t) = 3$. The graph $P_2 \circ_v P_t$ is isomorphic to P_{2t}, which we know has m-degree and b-chromatic number 2 for $t = 2$, and 3 for $t \geq 3$.

Suppose that $d_{P_t}(v) = 2$. Note that this forces $t \geq 3$. For $s \geq 7$ we have $n_\varphi(P_s) = s-2 \geq 3+2 = \varphi(P_s)+d_{P_t}(v)$, so $\varphi(P_s \circ_v P_t) = 5$. For $s \in \{6, 5, 4\}$, $P_s \circ_v P_t$ has fewer than five vertices of degree at least 4, but at least four vertices of degree at least 3, so its m-degree is 4. In these cases, it is easy to find a 4-b-coloring of $P_s \circ_v P_t$, so its b-chromatic number is also 4. Because $P_3 \circ_v P_t$ has only three vertices of degree at least 3, but at least three vertices of degree at least 2, its m-degree is 3. It is easy to find a 3-b-coloring of $P_3 \circ_v P_t$, so its b-chromatic number is also 3.

The graph $P_2 \circ_v P_t$ has only two vertices of degree at least 3. It has at least three vertices of degree at least 2 when $t \geq 4$ but only two when $t = 3$. Of course, it has more than two vertices of degree at least 1. Thus, its m-degree is 3 when $t \geq 4$ and 2 when $t = 3$. In either case, one can easily find a b-coloring of $P_2 \circ_v P_t$ to show that the b-chromatic number is equal to the m-degree. □

Proposition 3.2. *Let $s \geq 2$ and $t \geq 3$ and let v be the root of C_t. Then*

$$\varphi(P_s \circ_v C_t) = m(P_s \circ_v C_t) = \begin{cases} 3 & s \leq 3 \\ 4 & 4 \leq s \leq 6 \\ 5 & s \geq 7 \end{cases}.$$

Proof. Essentially like the proof of Proposition 3.1, with the root having degree 2. □

Proposition 3.3. *Let $s \geq 3$ and $t \geq 2$ and let v be the root of P_t.*
If $d_{P_t}(v) = 1$, then

$$\varphi(C_s \circ_v P_t) = \begin{cases} 3 & s = 3 \text{ or } s = 5 \\ 4 & s = 4 \text{ or } s \geq 6 \end{cases}$$

and

$$m(C_s \circ_v P_t) = \begin{cases} 3 & s = 3 \\ 4 & s \geq 4 \end{cases}.$$

If $d_{P_t}(v) = 2$, then

$$\varphi(C_s \circ_v P_t) = m(C_s \circ_v P_t) = \min\{5, s\}.$$

Proof. Most of the cases can be handled as in the proof of Proposition 3.1. We draw attention, though, to the exceptional situation where $s = 5$ and $d_{P_t}(v) = 1$. The m-degree of $C_5 \circ_v P_t$ is easily checked to be 4. However, it is impossible to make a 4-b-coloring of this graph. There would have to be four CDVs in the C_5-layer, each with degree 3 and thus no repeated colors on neighbors. There is no choice for the color of the fifth vertex in the C_5-layer that makes this work. It is easy to find a 3-b-coloring. □

Proposition 3.4. *Let $s, t \geq 3$ and let v be the root of C_t. Then*

$$\varphi(C_s \circ_v C_t) = m(C_s \circ_v C_t) = \min\{5, s\}.$$

Proof. Again, this is essentially like the proof of Proposition 3.1. □

The next calculation is particularly interesting because the b-chromatic number of the Cartesian product of complete graphs is still open [21].

Proposition 3.5. *Let $s, t \geq 2$ and let v be the root of K_t. Then*

$$\varphi(K_s \circ_v K_t) = m(K_s \circ_v K_t) = \max\{s, t\}.$$

Proof. Recall that $\varphi(K_n) = m(K_n) = n$ for any n. By Theorem 2.1,

$$\varphi(K_s \circ_v K_t) \geq \max\{\varphi(K_s), \varphi(K_t)\} = \max\{s, t\}.$$

By Theorem 2.9, $m(K_s \circ_v K_t)$ is the larger of s and t. The result follows from these comments along with the fact that the m-degree bounds the b-chromatic number from above. □

We can generalize this a bit. When the first factor is a complete graph, it suffices to restrict the maximum degree of the second.

Proposition 3.6. *Let $s \geq 2$, let H be a graph, and let v be the root of H. If*

$$s > \Delta(H),$$

then

$$\varphi(K_s \circ_v H) = m(K_s \circ_v H) = s.$$

Proof. By Theorem 2.1, we know $\varphi(K_s \circ_v H) \geq \varphi(K_s) = s$. Since $\Delta(H) < s$ and K_s only has s vertices, we also know $m(K_s \circ_v H) = s$. The result now follows from the fact that $\varphi(K_s \circ_v H) \leq m(K_s \circ_v H)$. □

We include the next calculation to set up a problem in Section 4.

Proposition 3.7. *Let $t \geq 3$, let $G = K_{t+1,t+1}$, let $H = K_{t,t}$, and let v be the root of H. Then*

$$\varphi(G \circ_v H) = m(G \circ_v H) = 2t + 2.$$

Proof. The degree of each vertex in the G-layer is $2t + 1$, and there are exactly $2t + 2$ such vertices. Non-root vertices in the H-layers only have degree t. Hence, $m(G \circ_v H) = 2t + 2$.

We now describe how to obtain a b-coloring of $G \circ_v H$ with $2t + 2$ colors, which will complete the result. First, give each vertex in the G-layer a different color. Next, each of the vertices in the G-layer can be made a CDV by appropriately coloring the remaining vertices in their neighborhoods. This works because these sets of neighbors are disjoint and independent. Finally, the rest of the vertices can be colored greedily because each has degree $t < 2t + 2$. □

As with the previous section, we finish this one by switching our focus to the m-degree.

Proposition 3.8. *Let G be a graph, let $t \geq 2$, and let v be the root of P_t. Then*

$$m(G \circ_v P_t) = m(G) + r \text{ for some } r \in \{0, 1, 2\}.$$

Proof. Unless $m(G) = 2$ and $d_H(v) = 1$, this follows from Theorem 2.8. In this remaining case, G has fewer than three vertices of degree at least 2, $G \circ_v H$ has fewer than three vertices of degree at least 3, and $m(G \circ_v H) \leq 3$, so the result still holds. □

4. Directions for future research

We conclude with some avenues for further exploration. Throughout, let G and H be graphs and let v be a vertex of H.

While the primary focus of this paper is the b-chromatic number of rooted products, it is worthwhile to consider the possible implications of our results for Cartesian products. The rooted product is a subgraph of the Cartesian product, so insights from analysis of the one may guide study of the other. We certainly invite such progress. However, it does not appear that our theorems extend immediately, as having more than one G-layer complicates matters considerably. We briefly describe obstacles in the context of two of our main results.

Theorem 2.7 includes an inequality involving $d_H(v)$ and $\Delta(H)$. In the Cartesian product, v is not distinguished as a root. The natural extension would replace $d_H(v)$ with $\Delta(H)$, making the inequality impossible to satisfy. This inequality is used for an upper bound on the b-chromatic number of the product. The lower bound is produced by construction, but even the first step of the process (coloring the (x_i, y_j) vertices) can fail to maintain propriety.

The natural extension of the analogous inequality in Theorem 2.4 would replace $d_H(v)$ with the minimum degree of H, which does not cause trouble. If we simply choose a G-layer to use for the first step of the construction in the proof, though, we can again run into impropriety when coloring the H-layer neighbors of these vertices, as they may be adjacent to each other.

Nevertheless, we find rooted products to be interesting for their own sake and offer a few specific problems to pursue in this area.

Theorem 2.1 gives a lower bound on the b-chromatic number of a rooted product. Propositions 3.5 and 3.6 provide some examples of rooted products for which this bound is tight, but we do not yet have a characterization.

Problem 4.1. *When is $\varphi(G \circ_v H)$ simply the larger of $\varphi(G)$ and $\varphi(H)$?*

Corollary 2.5 provides sufficient conditions to have $\varphi(G \circ_v H) = \varphi(G) + d_H(v)$. We wonder if there is a more general result.

Problem 4.2. *When is $\varphi(G \circ_v H)$ the sum of $\varphi(G)$ and $d_H(v)$?*

Recall that the m-degree is an upper bound on the b-chromatic number. It is interesting to consider how equality between the two in the factors relates to equality in the product.

We observe that $\varphi(G)$ and $\varphi(H)$ can be quite far from $m(G)$ and $m(H)$, respectively, while having $\varphi(G \circ_v H)$ equal to $m(G \circ_v H)$. For example, consider $G = K_{t+1,t+t}$ and $H = K_{t,t}$ with $t \geq 3$. It is easy to see that $\varphi(G) = \varphi(H) = 2$ with $m(G) = t + 1$ and $m(H) = t$, but Proposition 3.7 shows that $\varphi(G \circ_v H) = m(G \circ_v H)$.

Problem 4.3. *Suppose $\varphi(G \circ_v H) = m(G \circ_v H)$. What extra conditions are necessary to have $\varphi(G) = m(G)$ and $\varphi(H) = m(H)$?*

On the other hand, we also can have $\varphi(G) = m(G)$ and $\varphi(H) = m(H)$ but have $\varphi(G \circ_v H)$ different from $m(G \circ_v H)$. Take the example $G = C_5$ and $H = P_t$ with $t \geq 5$ and let $d_H(v) = 1$. Here we have $\varphi(G) = \varphi(H) = m(G) = m(H) = 3$, but according to Proposition 3.3 we have $\varphi(G \circ_v H) = 3$ and $m(G \circ_v H) = 4$. Note that this example is not as broad as the one in the previous paragraph, in two senses. First, the b-chromatic number is still quite close to the m-degree in the product. Second, there may only be a small number of possibilities for G. Except when G is C_5 or W_6, every example we considered with $\varphi(G) = m(G)$ and $\varphi(H) = m(H)$ also has $\varphi(G \circ_v H) = m(G \circ_v H)$.

Problem 4.4. *Suppose $\varphi(G) = m(G)$ and $\varphi(H) = m(H)$. How far apart can $\varphi(G \circ_v H)$ and $m(G \circ_v H)$ be? What extra conditions are necessary to conclude that $\varphi(G \circ_v H) = m(G \circ_v H)$?*

Our final problem is again about the m-degree rather than directly about the b-chromatic number. It is inspired by Theorem 2.8.

Problem 4.5. *What can be said about $m(G \circ_v H)$ when $m(G) \leq \Delta(H) + 1 - d_H(v)$, or perhaps when $m(G) < m(H)$?*

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References

[1] El Sahili Amine, Kouider Mekkia, and Mortada Maidoun. On the b-chromatic number of regular bounded graphs. *Discrete Appl. Math.*, 193:174–179, 2015.
[2] Mostafa Blidia, Frédéric Maffray, and Zoham Zemir. On b-colorings in regular graphs. *Discrete Appl. Math.*, 157(8):1787–1793, 2009.
[3] Sergio Cabello and Marko Jakovac. On the b-chromatic number of regular graphs. *Discrete Appl. Math.*, 159(13):1303–1310, 2011.
[4] Victor A. Campos, Carlos V. Lima, and Ana Silva. Graphs of girth at least 7 have high b-chromatic number. *European J. Combin.*, 48:154–164, 2015.
[5] Victor Campos, Victor A. E. de Farias, and Ana Silva. b-coloring graphs with large girth. *J. Braz. Comput. Soc.*, 18(4):375–378, 2012.
[6] Victor A. Campos, Carlos V. Lima, Nicolas A. Martins, Leonardo Sampaio, Marcio C. Santos, and Ana Silva. The b-chromatic index of graphs. *Discrete Math.*, 338(11):2072–2079, 2015.
[7] Affif F. Chaouche and Abdelhafid Berrachedi. Some bounds for the b-chromatic number of a generalized Hamming graphs. *Far East J. Appl. Math.*, 26(3):375–391, 2007.
[8] Sylvie Corteel, Mario Valencia-Pabon, and Juan-Carlos Vera. On approximating the b-chromatic number. *Discrete Appl. Math.*, 146(1):106–110, 2005.
[9] Magda Dettlaff, Hanna Furmańczyk, Iztok Peterin, Adriana Roux, and Radosław Ziemann. A new approach to b-coloring of regular graphs. *Appl. Math. Comput.*, 481:Paper No. 128914, 9, 2024.
[10] Amine El Sahili and Mekkia Kouider. About b-colouring of regular graphs. *Util. Math.*, 80:211–215, 2009.
[11] Iztok Fister, Iztok Peterin, Marjan Mernik, and Matej Črepinšek. Hybrid evolutionary algorithm for the b-chromatic number. *Journal of Heuristics*, 21:501–521, 08 2015.
[12] František Galčík and Ján Katrenič. A note on approximating the b-chromatic number. *Discrete Appl. Math.*, 161(7-8):1137–1140, 2013.
[13] Christopher D. Godsil and Brendan D. McKay. A new graph product and its spectrum. *Bull. Austral. Math. Soc.*, 18(1):21–28, 1978.
[14] Chuan Guo and Mike Newman. On the b-chromatic number of cartesian products. *Discrete Appl. Math.*, 239:82–93, 2018.
[15] Robert W. Irving and David F. Manlove. The b-chromatic number of a graph. *Discrete Appl. Math.*, 91(1-3):127–141, 1999.

[16] Marko Jakovac and Sandi Klavžar. The *b*-chromatic number of cubic graphs. *Graphs Combin.*, 26(1):107–118, 2010.
[17] Marko Jakovac and Iztok Peterin. On the b-chromatic number of some graph products. *Studia Sci. Math. Hungar.*, 49(2):156–169, 2012.
[18] Marko Jakovac and Iztok Peterin. The b-chromatic index of a graph. *Bull. Malays. Math. Sci. Soc.*, 38(4):1375–1392, 2015.
[19] Marko Jakovac and Iztok Peterin. The b-chromatic number and related topics—a survey. *Discrete Appl. Math.*, 235:184–201, 2018.
[20] Ivo Koch and Iztok Peterin. The b-chromatic index of direct product of graphs. *Discrete Appl. Math.*, 190/191:109–117, 2015.
[21] Mekkia Kouider and Maryvonne Mahéo. Some bounds for the *b*-chromatic number of a graph. *Discrete Math.*, 256(1-2):267–277, 2002.
[22] Mekkia Kouider and Maryvonne Mahéo. The *b*-chromatic number of the Cartesian product of two graphs. *Studia Sci. Math. Hungar.*, 44(1):49–55, 2007.
[23] Frédéric Maffray and Ana Silva. b-colouring the Cartesian product of trees and some other graphs. *Discrete Appl. Math.*, 161(4-5):650–669, 2013.