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Generalizations of the numerical radius, Crawford number and numerical index functions in the weighted case

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Abstract. In this article, firstly, some simple and smoothness properties of the weighted numerical radius and the weighted Crawford number functions are investigated. Then, some generalization formulas for lower and upper bounds of the weighted numerical radius function are obtained. Later on, some evaluations for lower and upper bounds of the weighted numerical index are given. The obtained results are generalized some well-known famous results about the special weighted numerical radius and the special weighted Crawford number functions in the recently literature. Also, important contribution is made to existing literature by different and useful results.

1. Introduction

Throughout this article, H denotes a complex Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and associated norm $\|\cdot\|$. Let $\mathbb{B}(H)$ stand for the C^*-algebra of all bounded linear operators acting on H. The numerical radius of an operator A is given by

$$\omega(A) = \sup\{|\langle Ax, x\rangle| : \ x \in H, \ \|x\| = 1\}.$$

The usual operator norm and the Crawford number of an operator A are, respectively, defined by

$$\|A\| = \sup\{\|Ax\| : \ x \in H, \ \|x\| = 1\}$$

and

$$c(A) = \inf\{|\langle Ax, x\rangle| : x \in H, \ \|x\| = 1\}$$

[15].

Recall that for any $A \in \mathbb{B}(H)$ the classical numerical radius $\omega(A)$ is a norm on $\mathbb{B}(H)$ and equivalent to the operator norm that satisfies the relation

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\| \tag{1}$$

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(see, e.g., [15]).

For any $A \in \mathbb{B}(H)$ one of improved formula for the numerical radius in form

$$\frac{1}{4}\|A^*A + AA^*\| \leq \omega^2(A) \leq \frac{1}{2}\|A^*A + AA^*\| \tag{2}$$

has been obtained in [22].

For more results related to lower and upper bounds of the classical numerical radius we refer to studies in recent years (see, [2], [3], [5], [11], [12], [22], [29]).

Assume that $\varphi : [0,1] \to \mathbb{R}$ and $\psi : [0,1] \to \mathbb{R}$ are continuous functions. For $A \in \mathbb{B}(H)$ and $t \in [0,1]$ the weighted numerical radius and the weighted Crawford number functions will be defined by

$$\omega_t(\varphi,\psi;A) = \sup_{x \in S_1(H)} |\langle(\varphi(t)A + \psi(t)A^*)\,x, x\rangle|$$

and

$$c_t(\varphi,\psi;A) = \inf_{x \in S_1(H)} |\langle(\varphi(t)A + \psi(t)A^*)\,x, x\rangle|,$$

respectively, where $S_1(H)$ is a unit sphere in H.

Similarly, the weighted operator norm of any operator $A \in \mathbb{B}(H)$ will be defined as

$$\|A\|_t = \|\varphi(t)A + \psi(t)A^*\|,\ t \in [0,1].$$

It is clear that if $\varphi(t) = 1,\ \psi(t) = 0,\ t \in [0,1]$, then $\omega_t(1,0;A)$ and $c_t(1,0;A)$ coincide with the classical numerical radius $\omega(A)$ and the classical Crawford number $c(A)$, respectively.

In case $\varphi(t) = 1,\ \psi(t) = 1 - 2t,\ 0 \leq t \leq 1$ and $\varphi(\nu) = \nu,\ \psi(\nu) = 1 - \nu,\ 0 \leq \nu \leq 1$, these special weighted numerical radii have been investigated in [26] and [28], respectively. There in some upper bounds for the weighted numerical radius have also been researched.

If we take $\varphi,\ \psi \in C[0,1]$ real-valued functions and

$$A = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix},\ A : \mathbb{C}^2 \to \mathbb{C}^2,$$

then we obtain that

$$\varphi A + \psi A^* = \begin{pmatrix} 0 & \varphi + 3\psi \\ 3\varphi + \psi & 0 \end{pmatrix}.$$

For $x = (x_1, x_2) \in \mathbb{C}^2$, we also have

$$\langle(\varphi A + \psi A^*)x, x\rangle = (\varphi + 3\psi)x_2\overline{x_1} + (3\varphi + \psi)x_1\overline{x_2}.$$

Hence, we get

$$|(\varphi A + \psi A^*)x|^2 = |(\varphi + 3\psi)x_2|^2 + |(3\varphi + \psi)x_1|^2.$$

In this case, from Lemma 2 in [1] we have

$$\omega_t(\varphi,\psi;A) = \frac{1}{2}\sup_{\theta \in \mathbb{R}} |e^{i\theta}(\varphi + 3\psi) + e^{-i\theta}(3\varphi + \psi)| = \frac{1}{2}\sup_{\theta \in \mathbb{R}} |\varphi(e^{i\theta} + 3e^{-i\theta}) + \psi(3e^{i\theta} + e^{-i\theta})|$$

for any $t \in [0,1]$.

Also,

$$c_t(\varphi,\psi;A) = 0 \ \text{ and } \ \|A\|_t = |(\varphi + 3\psi)(3\varphi + \psi)|(t).$$

In mathematical literature, also there is a constant on a Banach space, known as the numerical index of the space, which relates the behaviour of the numerical radius with the usual norm of an operator. The numerical index of the Banach space X is the constant

$$n(X) = \inf\{\omega(A) : A \in \mathbb{B}(X), \|A\| = 1\},$$

equivalent, $n(X)$ is the maximum of those $k \geq 0$ such that $k\|A\| \leq \omega(A)$ for every $A \in \mathbb{B}(X)$. This notion was introduced and studied in the 1970 paper [10], see also the monographs [6], [7] and the survey paper [17] for background. Clearly, $0 \leq n(X) \leq 1$, $n(X) > 0$ means that the numerical radius is a norm on $\mathbb{B}(X)$ equivalent to the operator norm and $n(X) = 1$ if and only if numerical radius and operator norm coincide. If X is a complex Banach space, then $\frac{1}{e} \leq n(X) \leq 1$ and if X is a real Banach space, then $0 \leq n(X) \leq 1$. If X is a complex Hilbert space, then $n(X) = \frac{1}{2}$ and for a real Hilbert space X, $n(X) = 0$. Moreover, $n(l_1) = n(l_1^m) = n(l_\infty) = n(l_\infty^m) = 1$, where $m \in \mathbb{N}$. All these results can be found in [6] and [17]. Some recent developments for the study of the numerical index are [14], [23], [24], [25], [27].

The main aim of this study is to generalize some well-known results about the weighted numerical radius and the weighted Crawford number functions in the mathematical literature [4], [15], [22], [26], [28], as well as to provide different and useful results to this area.

This work is organized as follows: In Section 2, some simple and smoothness properties of the weighted numerical radius and the weighted Crawford number functions have been investigated. In Section 3, some evolutions formulas for lower and upper bounds of the weighted numerical radius function have been obtained. Later on, in Section 4, some evaluations for lower and upper bounds of the weighted numerical index have been given. These results are generalizations of some well-known results in the literature.

Note that each operator $A \in \mathbb{B}(H)$ can be expressed in the Cartesian decomposition form as $A = ReA + iImA$, where $ReA = \frac{A+A^*}{2}$ and $ImA = \frac{A-A^*}{2i}$. Here, A^* denotes the adjoint of A. Throughout this study we denote by $|A| = (A^*A)^{1/2}$ the absolute value of an operator $A \in \mathbb{B}(H)$.

2. Some properties of the weighted numerical radius and the weighted Crawford number functions

Let us begin this section with some simple properties of the weighted numerical radius and the weighted Crawford number functions.

Proposition 2.1. *For any $A, B \in \mathbb{B}(H)$ and $t \in [0, 1]$, the following are true:*

(1) $\omega_t(0, \psi; A) = |\psi(t)|\omega(A)$ *and* $c_t(0, \psi; A) = |\psi(t)|c(A)$,

(2) $\omega_t(\varphi, 0; A) = |\varphi(t)|\omega(A)$ *and* $c_t(\varphi, 0; A) = |\varphi(t)|c(A)$,

(3) $\omega_t(1, 1; A) = 2\|ReA\|$ *and* $c_t(1, 1; A) = 2\|ReA\|$,

(4) $\omega_t(1, -1; A) = 2\|ImA\|$ *and* $c_t(1, -1; A) = 2\|ImA\|$,

(5) $\omega_t(\varphi, \psi; iA) = \omega_t(\varphi, -\psi; A)$,

(6) $\frac{1}{2}\|A\|_t \leq \omega_t(\varphi, \psi; A) \leq \|A\|_t$,

(7) $\omega_t(\varphi, \psi; A) \leq (|\varphi| + |\psi|)(t)\omega(A)$ *and* $c_t(\varphi, \psi; A) \geq \||\varphi| - |\psi|\|(t)c(A)$,

(8) $\omega_t(\varphi, \varphi; A) = 2|\varphi(t)|\omega(ReA) = 2|\varphi(t)|\|ReA\|$,

(9) $\omega_t(\psi, \psi; A) = 2|\psi(t)|\omega(ReA) = 2|\psi(t)|\|ReA\|$,

(10) *If* $A = A^*$, *then* $\omega_t(\varphi, \psi; A) = |\varphi + \psi|(t)\omega(A) = |\varphi + \psi|(t)\|A\|$,

(11) $\omega_t(\varphi, \psi; A) = \omega_t(\psi, \varphi; A^*)$,

(12) *If* $A = A^*$, *then* $\omega_t(\varphi, \psi; A) = \omega_t(\psi, \varphi; A)$,

(13) $\omega_t(\varphi, \psi; A + B) \leq \omega_t(\varphi, \psi; A) + \omega_t(\varphi, \psi; B)$,

(14) $c_t(\varphi, \psi; A + B) \leq \omega_t(\varphi, \psi; A) + c_t(\varphi, \psi; B)$,

(15) $\omega_t(\varphi, \psi; AB) \leq (|\varphi| + |\psi|)(t)\omega(AB)$,

(16) $c_t(\varphi,\psi;AB) \leq |\varphi|(t)c(AB) + |\psi|(t)\omega(AB)$,

(17) $\|\varphi AB + \psi(AB)^*\|_t^2 \leq \left(|\varphi|^2 + |\psi|^2\right)(t)\|AB\|^2 + |\varphi\psi|(t)\omega((AB)^2) + |\varphi\psi|(t)\omega((B^*A^*)^2)$,

(18) $\omega_t(\varphi_1+\varphi_2,\psi_1+\psi_2;A) \leq \omega_t(\varphi_1,\psi_1;A) + \omega_t(\varphi_2,\psi_2;A)$,

(19) $c_t(\varphi,\psi;A) \geq \inf\{|\varphi+\psi|(t), |\varphi-\psi|(t)\}\, m(A)$, *where* $m(A) = \inf_{x\in S_1(H)} |(Ax,x)|$.

Proof. To give an idea we will prove the 15th, 17th and 19th properties. Firstly, let us start with the proof of the (15). For any $A, B \in \mathbb{B}(H)$ and $t \in [0,1]$, we get

$$\begin{aligned} |\langle(\varphi(t)AB + \psi(t)B^*A^*)x,x\rangle|^2 &= |\langle\varphi(t)ABx,x\rangle + \langle\psi(t)B^*A^*x,x\rangle|^2 \\ &= |\varphi|^2(t)|\langle ABx,x\rangle|^2 + (\varphi\psi)(t)\left[\langle ABx,x\rangle^2 + \langle x,ABx\rangle^2\right] + |\psi|^2(t)|\langle x,ABx\rangle|^2 \\ &\leq \left(|\varphi|^2(t) + 2|\varphi\psi|(t) + |\psi|^2(t)\right)|\langle ABx,x\rangle|^2 \\ &= \left(|\varphi| + |\psi|\right)^2(t)|\langle ABx,x\rangle|^2. \end{aligned}$$

Then,

$$|\langle(\varphi(t)AB + \psi(t)B^*A^*)x,x\rangle| \leq \left(|\varphi| + |\psi|\right)(t)|\langle ABx,x\rangle|.$$

Hence, for any $t \in [0,1]$ we have

$$\omega_t(\varphi,\psi;AB) \leq \left(|\varphi| + |\psi|\right)(t)\omega(AB).$$

Now, let us prove (17). For $t \in [0,1]$, $x \in S_1(H)$ and $A \in \mathbb{B}(H)$, we have

$$\begin{aligned} \|(\varphi(t)AB + \psi(t)B^*A^*)x\|_t^2 &= |\varphi^2|(t)\langle ABx,ABx\rangle + (\varphi\psi)(t)\langle ABx,B^*A^*x\rangle + (\varphi\psi)(t)\langle B^*A^*x,ABx\rangle + |\psi|^2(t)\langle B^*A^*x,B^*A^*x\rangle \\ &\leq |\varphi^2|(t)\|ABx\|^2 + |\varphi\psi|(t)\omega((AB)^2) + |\varphi\psi|(t)\omega((B^*A^*)^2) + |\psi|^2(t)\|B^*A^*x\|^2. \end{aligned}$$

Thus, we get

$$\|\varphi AB + \psi(AB)^*\|_t^2 \leq \left(|\varphi|^2 + |\psi|^2\right)(t)\|AB\|^2 + |\varphi\psi|(t)\omega((AB)^2) + |\varphi\psi|(t)\omega((B^*A^*)^2).$$

If we take $\varphi(t) = 1$ and $\psi(t) = 1-2t$, $0 \leq t \leq 1$ in the last result, we get the following inequality proved in [26, Lemma 2.6]

$$\|(1-2t)(AB)^* + AB\|^2 \leq (2-4t+4t^2)\|AB\|^2 + (1-2t)\omega((AB)^2) + (1-2t)\omega((B^*A^*)^2).$$

So, the property (17) in Proposition 2.1 generalizes [26, Lemma 2.6]

Lastly, let prove us prove (19). For $t \in [0,1]$, $x \in S_1(H)$ and $A \in \mathbb{B}(H)$, by the following simple calculations we have

$$|\langle(\varphi A + \psi A^*)x,x\rangle|^2 = |\langle(\varphi ReA + \psi ReA)x,x\rangle + i(\varphi ImA - \psi ImA)x,x)|^2 = |\varphi+\psi|^2(t)|\langle ReAx,x\rangle|^2 + |\varphi-\psi|^2(t)|\langle ImAx,x\rangle|^2.$$

Then, we get

$$c_t^2(\varphi,\psi;A) \geq \left(\inf\{|\varphi+\psi|(t), |\varphi-\psi|(t)\}\right)^2 m^2(A).$$

□

Lemma 2.2. *Let* $A, B \in \mathbb{B}(H)$ *and* $t \in [0,1]$. *Then*

(1) $|\omega_t(\varphi,\psi;A) - \omega_t(\varphi,\psi;B)| \leq \omega_t(\varphi,\psi;A-B)$,

(2) $|c_t(\varphi,\psi;A) - c_t(\varphi,\psi;B)| \leq \omega_t(\varphi,\psi;A-B)$.

Proof. Using the property (13) in Proposition 2.1, we have

$$\omega_t(\varphi,\psi;A) \le \omega_t(\varphi,\psi;A-B) + \omega_t(\varphi,\psi;B) \text{ and } \omega_t(\varphi,\psi;B) \le \omega_t(\varphi,\psi;A-B) + \omega_t(\varphi,\psi;A)$$

for any $A, B \in \mathbb{B}(H)$ and $t \in [0,1]$ which imply that

$$|\omega_t(\varphi,\psi;A) - \omega_t(\varphi,\psi;B)| \le \omega_t(\varphi,\psi;A-B).$$

Similarly, using the property (14) in Proposition 2.1, we have

$$c_t(\varphi,\psi;A) \le \omega_t(\varphi,\psi;A-B) + c_t(\varphi,\psi;B) \text{ and } c_t(\varphi,\psi;B) \le \omega_t(\varphi,\psi;A-B) + c_t(\varphi,\psi;A)$$

for any $A, B \in \mathbb{B}(H)$ and $t \in [0,1]$ which imply that

$$|c_t(\varphi,\psi;A) - c_t(\varphi,\psi;B)| \le \omega_t(\varphi,\psi;A-B).$$

for any $A, B \in \mathbb{B}(H)$ and $t \in [0,1]$. □

Now, let us give smoothness properties of the weighted numerical radius and the weighted Crawford number functions.

Theorem 2.3. *If the sequences of functions (φ_n) and (ψ_n) pointwise convergent to the functions $\varphi : [0,1] \to \mathbb{R}$ and $\psi : [0,1] \to \mathbb{R}$, respectively, then for any $A \in \mathbb{B}(H)$ and $t \in [0,1]$,*

$$\omega_t(\varphi,\psi;A) = \lim_{n\to\infty} \omega_t(\varphi_n,\psi_n;A) \text{ and } c_t(\varphi,\psi;A) = \lim_{n\to\infty} c_t(\varphi_n,\psi_n;A).$$

Proof. Since

$$|\langle(\varphi(t)A + \psi(t)A^*)x,x\rangle| \le |\langle((\varphi-\varphi_n)(t)A + (\psi-\psi_n)(t)A^*)x,x\rangle| + |\langle(\varphi_n(t)A + \psi_n(t)A^*)x,x\rangle|$$

and

$$|\langle(\varphi_n(t)A + \psi_n(t)A^*)x,x\rangle| \le |\langle((\varphi-\varphi_n)(t)A + (\psi-\psi_n)(t)A^*)x,x\rangle| + |\langle(\varphi(t)A + \psi(t)A^*)x,x\rangle|$$

for any $A \in \mathbb{B}(H)$, $t \in [0,1]$, $x \in S_1(H)$ and $n \ge 1$, then

$$|\omega_t(\varphi_n,\psi_n;A) - \omega_t(\varphi,\psi;A)| \le \omega_t(\varphi_n-\varphi,\psi_n-\psi;A) \le (|\varphi_n-\varphi|(t) + |\psi_n-\psi|(t))\,\|A\|$$

and

$$|c_t(\varphi_n,\psi_n;A) - c_t(\varphi,\psi;A)| \le \omega_t(\varphi_n-\varphi,\psi_n-\psi;A) \le (|\varphi_n-\varphi|(t) + |\psi_n-\psi|(t))\,\|A\|.$$

From the last inequalities and pointwise convergence of sequences (φ_n) to φ and (ψ_n) to ψ, the validity of theorem is clear. □

Definition 2.4. *[19] A sequence $(A_n) \subset \mathbb{B}(H)$ is said to uniformly converge to $A \in \mathbb{B}(H)$, if for any $\epsilon > 0$, there exists a positive integer N such that for all $n \ge N$*

$$\|A_n - A\| < \epsilon.$$

Theorem 2.5. *If the operator sequences (A_n) in $\mathbb{B}(H)$ uniformly converges with respect to norm $\|\cdot\|_t$, $0 \le t \le 1$ to the operator $A \in \mathbb{B}(H)$, then*

$$\omega_t(\varphi,\psi;A) = \lim_{n\to\infty} \omega_t(\varphi,\psi;A_n) \text{ and } c_t(\varphi,\psi;A) = \lim_{n\to\infty} c_t(\varphi,\psi;A_n).$$

Proof. For $t \in [0,1]$ and $n \ge 1$ by property (1) of Lemma 2.2, we have

$$|\omega_t(\varphi,\psi;A_n) - \omega_t(\varphi,\psi;A)| \le \omega_t(\varphi,\psi;A_n - A) \le \|\varphi(t)(A_n - A) + \psi(t)(A_n - A)\| = \|A_n - A\|_t.$$

Similarly, for $t \in [0,1]$ and $n \ge 1$ by property (2) of Lemma 2.2, we get

$$|c_t(\varphi,\psi;A_n) - c_t(\varphi,\psi;A)| \le \|A_n - A\|_t.$$

Then, the claims of theorem are clear. □

Theorem 2.6. *If* $\varphi, \psi \in H_\alpha[0,1]$, $0 < \alpha \leq 1$, *then* $\omega_t(\varphi,\psi;A)$, $c_t(\varphi,\psi;A) \in H_\alpha[0,1]$ *for any* $A \in \mathbb{B}(H)$, *where* $H_\alpha[0,1]$ *is the class of Hölder functions with degree* $\alpha \in (0,1]$ *in* $[0,1]$.

Proof. For any $0 \leq t, s \leq 1$, we have

$$\begin{aligned}
\omega_t(\varphi,\psi;A) &= \omega(\varphi(t)A + \psi(t)A^*) \\
&= \omega(\varphi(s)A + \psi(s)A^*) + ((\varphi(t) - \varphi(s))A + (\psi(t) - \psi(s))A^*) \\
&\leq \omega(\varphi(s)A + \psi(s)A^*) + |\varphi(t) - \varphi(s)| + |\psi(t) - \psi(s)|\|A\| \\
&\leq \omega_s(\varphi,\psi;A) + K_\varphi|t-s|^\alpha + K_\psi|t-s|^\alpha \\
&\leq \omega_s(\varphi,\psi;A) + K|t-s|^\alpha,\ K = \max\{K_\varphi, K_\psi\}
\end{aligned}$$

and similarly,

$$\omega_s(\varphi,\psi;A) \leq \omega_t(\varphi,\psi;A) + K|t-s|^\alpha.$$

Consequently, we obtain for each $0 \leq t, s \leq 1$ and $A \in \mathbb{B}(H)$ that

$$|\omega_t(\varphi,\psi;A) - \omega_s(\varphi,\psi;A)| \leq K|t-s|^\alpha,$$

i.e.

$$\omega_t(\varphi,\psi;A) \in H_\alpha[0,1].$$

In a similar manner, the validity of second claim can be proved. □

Theorem 2.7. *Let* $A \in \mathbb{B}(H)$. *Then* $\omega_t(\varphi,\psi;A)$, $c_t(\varphi,\psi;A) \in C[0,1]$, *where* $C[0,1]$ *is class of continuous functions on* $[0,1]$.

Proof. For $A \in \mathbb{B}(H)$ and $t, s \in [0,1]$ we have

$$\begin{aligned}
|\langle(\varphi(t)A + \psi(t)A^*)x, x\rangle| &= |\langle((\varphi(t) - \varphi(s))A + (\psi(t) - \psi(s))A^*)x, x\rangle + \langle(\varphi(s)A + \psi(s)A^*)x, x\rangle| \\
&\leq |\langle((\varphi(t) - \varphi(s))A + (\psi(t) - \psi(s))A^*)x, x\rangle| + |\langle(\varphi(s)A + \psi(s)A^*)x, x\rangle|.
\end{aligned}$$

Let s be any fixed number in $[0,1]$. Hence, we get

$$\omega_t(\varphi,\psi;A) \leq \omega(\varphi(t) - \varphi(s), \psi(t) - \psi(s); A^*) + \omega_s(\varphi,\psi;A).$$

Similarly, we also get

$$\omega_s(\varphi,\psi;A) \leq \omega(\varphi(t) - \varphi(s), \psi(t) - \psi(s); A^*) + \omega_t(\varphi,\psi;A).$$

Consequently, from the last two relations and property (7) in Proposition 2.1 we get

$$\begin{aligned}
|\omega_t(\varphi,\psi;A) - \omega_s(\varphi,\psi;A)| &= |\omega(\varphi(t)A + \psi(t)A^*) - \omega(\varphi(s)A + \psi(s)A^*)| \\
&\leq \omega((\varphi(t) - \varphi(s))A + (\psi(t) - \psi(s))A^*) \\
&\leq (|\varphi(t) - \varphi(s)| + |\psi(t) - \psi(s)|)\,\omega(A) \\
&\leq (|\varphi(t) - \varphi(s)| + |\psi(t) - \psi(s)|)\,\|A\|.
\end{aligned}$$

Since φ and ψ are continuous on $[0,1]$, then the continuity of $\omega_t(\varphi,\psi;A)$, $0 \leq t \leq 1$ is clear by the last relation.

Similarly, from the subadditivity of the classical numerical radius function the continuity of function $c_t(\varphi,\psi;A)$ on $[0,1]$ can be easily proved. □

Later on, from the definitions of the weighted numerical radius and the weighted Crawford numbers function, the next result follows immediately.

Proposition 2.8. *If* φ, $\psi \in D[0,1]$ *and* $\varphi \geq 0$, $\psi \geq 0$, *then for every* $A \in \mathbb{B}(H)$

(1) $\omega'_t(\varphi,0;A) = \omega_t(\varphi',0;A)$,

(2) $\omega_t'(0,\psi;A) = \omega_t(0,\psi';A)$,

(3) $c_t'(\varphi,0;A) = c_t(\varphi',0;A)$,

(4) $c_t'(0,\psi;A) = c_t(0,\psi';A)$,

where $D[0,1]$ *is the class of differentiable functions on* $[0,1]$.

Proposition 2.9. *For* $A \in \mathbb{B}(H)$*, the following are true:*

(1) If $\varphi \neq -\psi$*, then*

$$\|ReA\| \leq \left(\int_0^1 |\varphi+\psi|(t)dt\right)^{-1} \int_0^1 \omega_t(\varphi,\psi;A)dt,$$

(2) If $\varphi \neq \psi$*, then*

$$\|ImA\| \leq \left(\int_0^1 |\varphi-\psi|(t)dt\right)^{-1} \int_0^1 \omega_t(\varphi,\psi;A)dt,$$

(3) If $|\varphi| \neq |\psi|$*, then*

$$\|A\| \leq \left[\left(\int_0^1 |\varphi+\psi|(t)dt\right)^{-1} + \left(\int_0^1 |\varphi-\psi|(t)dt\right)^{-1}\right] \int_0^1 \omega_t(\varphi,\psi;A)dt.$$

Proof. For any $x \in S_1(H)$, we have

$$|\varphi+\psi|(t)|\langle ReAx,x\rangle| \leq \sqrt{|\varphi+\psi|^2(t)|\langle ReAx,x\rangle|^2 + |\varphi-\psi|^2(t)|\langle ImAx,x\rangle|^2} = |\langle(\varphi A+\psi A^*)x,x\rangle|$$

and

$$|\varphi-\psi|(t)|\langle ImAx,x\rangle| \leq \sqrt{|\varphi+\psi|^2(t)|\langle ReAx,x\rangle|^2 + |\varphi-\psi|^2(t)|\langle ImAx,x\rangle|^2} = |\langle(\varphi A+\psi A^*)x,x\rangle|.$$

Hence, we get

$$|\varphi+\psi|(t)\|ReA\| \leq \omega_t(\varphi,\psi,A) \text{ and } |\varphi-\psi|(t)\|ImA\| \leq \omega_t(\varphi,\psi,A).$$

Consequently, from the last inequalities we have

(1) If $\varphi \neq -\psi$, then

$$\|ReA\| \leq \left(\int_0^1 |\varphi+\psi|(t)dt\right)^{-1} \int_0^1 \omega_t(\varphi,\psi;A)dt,$$

(2) If $\varphi \neq \psi$, then

$$\|ImA\| \leq \left(\int_0^1 |\varphi-\psi|(t)dt\right)^{-1} \int_0^1 \omega_t(\varphi,\psi;A)dt.$$

(3) Taking into consideration of (1) and (2), we easily obtain (3).

□

Using the result obtained for the classical numerical radius and the classical Crawford number in [9], the following theorem can be obtained for the weighted numerical radius and the weighted Crawford number functions.

Theorem 2.10. *If for any* $n \geq 1,\ H_n$ *is a Hilbert space,* $A_n \in \mathbb{B}(H_n)$, $H = \bigoplus_{n=1}^{\infty} H_n$ *and* $A = \bigoplus_{n=1}^{\infty} A_n,\ A \in \mathbb{B}(H)$, *then*

(1) $\omega_t(\varphi, \psi; A) = \sup_{n\geq 1} \omega_t(\varphi, \psi; A_n)$,

(2) when $Re(A_n) \geq 0$ *(or* $Re(A_n) \leq 0$*),* $n \geq 1$, $c_t(\varphi, \psi; A) = \inf_{n\geq 1} c_t(\varphi, \psi; A_n)$.

3. On the lower and upper bounds of the weighted numerical radius

In this section, some estimates for lower and upper bounds of the weighted numerical radius are given. Firstly, we give some well-known auxiliary results of [4].

Lemma 3.1. *Let* $A \in \mathbb{B}(H)$. *Then*

(1) $\omega(A) \geq \frac{\|A\|}{2} + \frac{|\,\|ReA\| - \|ImA\|\,|}{2}$,

(2) $\omega^2(A) \geq \frac{1}{4}\|A^*A + AA^*\| + \frac{|\,\|ReA\|^2 - \|ImA\|^2\,|}{2}$,

(3) $\omega^2(A) \geq \frac{1}{4}\|A^*A + AA^*\| + \frac{c^2(ReA)+c^2(ImA)}{2} + \left|\frac{\|ReA\|^2 - \|ImA\|^2}{2} + \frac{c^2(ImA)-c^2(ReA)}{2}\right|$,

(4) $\omega^4(A) \geq \frac{1}{16}\|(A^*A + AA^*)^2 + 4Re^2(A^2)\| + \frac{1}{2}\left|\|ReA\|^4 - \|ImA\|^4\right|$.

We obtain the following lower bound for the weighted numerical radius of bounded linear operators.

Theorem 3.2. *Let* $A \in \mathbb{B}(H)$ *and* $t \in [0, 1]$. *Then*

(1) $\omega_t(\varphi, \psi; A) \geq \inf\{|\varphi + \psi|(t), |\varphi - \psi|(t)\}\left(\frac{\|A\|}{2} + \frac{|\,\|ReA\| - \|ImA\|\,|}{2}\right)$,

(2) $\omega_t^2(\varphi, \psi; A) \geq (\inf\{|\varphi + \psi|(t), |\varphi - \psi|(t)\})^2\left(\frac{1}{4}\|A^*A + AA^*\| + \frac{|\,\|ReA\|^2 - \|ImA\|^2\,|}{2}\right)$,

(3) $\omega_t^2(\varphi, \psi; A) \geq (\inf\{|\varphi + \psi|(t), |\varphi - \psi|(t)\})^2\left(\frac{1}{4}\|A^*A + AA^*\| + \frac{c^2(ReA)+c^2(ImA)}{2} + \left|\frac{\|ReA\|^2 - \|ImA\|^2}{2} + \frac{c^2(ImA)-c^2(ReA)}{2}\right|\right)$,

(4) $\omega_t^4(\varphi, \psi; A) \geq (\inf\{|\varphi + \psi|(t), |\varphi - \psi|(t)\})^4\left(\frac{1}{16}\|(A^*A + AA^*)^2 + 4Re^2(A^2)\| + \frac{1}{2}\left|\|ReA\|^4 - \|ImA\|^4\right|\right)$.

Proof. (1) For any $x \in S_1(H)$, we have

$$\begin{aligned} |\langle(\varphi A + \psi A^*)x, x\rangle|^2 &= |\langle((\varphi + \psi)ReA + (\varphi - \psi)ImA)x, x\rangle|^2 \\ &= |\varphi + \psi|^2|\langle ReAx, x\rangle|^2 + |\varphi - \psi|^2|\langle ImAx, x\rangle|^2 \\ &\geq (\inf\{|\varphi + \psi|(t), |\varphi - \psi|(t)\})^2\left(|\langle ReAx, x\rangle|^2 + |\langle ImAx, x\rangle|^2\right). \end{aligned}$$

Then,

$$\omega_t(\varphi, \psi; A) \geq \inf\{|\varphi + \psi|(t), |\varphi - \psi|(t)\}\max\{\|ReA\|, \|ImA\|\}.$$

From the last inequality and property (1) of Lemma 3.1, the validity of first claim is clear.

(2) For any $x \in S_1(H)$, we get

$$|\langle(\varphi A + \psi A^*)x, x\rangle|^2 \geq (\inf\{|\varphi + \psi|(t), |\varphi - \psi|(t)\})^2\left(|\langle ReAx, x\rangle|^2 + |\langle ImAx, x\rangle|^2\right).$$

Thus, we have

$$\omega_t^2(\varphi, \psi; A) \geq (\inf\{|\varphi + \psi|(t), |\varphi - \psi|(t)\})^2\max\left\{\|ReA\|^2, \|ImA\|^2\right\}.$$

Then, from the last inequality and property (2) of Lemma 3.1 the validity of second claim is clear.

(3) For any $x \in S_1(H)$, we have

$$\begin{aligned} |\langle(\varphi A + \psi A^*)x, x\rangle|^2 &= |\langle|(\varphi + \psi|^2|\langle ReAx, x\rangle|^2 + |\varphi - \psi|^2|\langle ImAx, x\rangle|^2 \\ &\geq (\inf\{|\varphi + \psi|(t), |\varphi - \psi|(t)\})^2 \max\left\{\|ReA\|^2 + c^2(ImA), \|ImA\|^2 + c^2(ReA)\right\}. \end{aligned}$$

From the last inequality, definition of the weighted numerical radius and property (3) of Lemma 3.1, third claim of theorem can be obtained.

(4) For any $x \in S_1(H)$ we have that

$$|\langle(\varphi A + \psi A^*)x, x\rangle|^4 \geq (\inf\{|\varphi + \psi|(t), |\varphi - \psi|(t)\})^4 \max\left\{\|ReA\|^4, \|ImA\|^4\right\}.$$

From the last inequality and property (4) of Lemma 3.1, fourth claim of theorem can be obtained. □

Remark 3.3. *If we take $\varphi(t) = 1,\ \psi(t) = 0,\ 0 \leq t \leq 1$ in Theorem 3.2, then Theorem 3.2 and Lemma 3.1 coincide. So, Theorem 3.2 generalizes Lemma 3.1.*

Example. In the complex Hilbert space $L^2(0,1)$ consider the following classical Volterra integration operator

$$V : L^2(0,1) \to L^2(0,1),\ Vf(x) = \int_0^x f(t)dt,\ f \in L^2(0,1).$$

It is known that $\|V\| = \frac{2}{\pi}$, $\|ReV\| = \frac{1}{2}$, and $\|ImV\| = \frac{1}{\pi}$ (see [20]). Then, by Theorem 3.2, we have

$$\omega_t(\varphi, \psi; A) \geq \inf\{|\varphi + \psi|(t), |\varphi - \psi|(t)\}\, \frac{2+\pi}{4\pi}.$$

Note that $\omega(V) = \frac{1}{2}$, $\omega(ReV) = \frac{1}{2}$, and $\omega(ImV) = \frac{1}{\pi}$ (see [21]). It can be easily shown that

$$\|V^*V + VV^*\| = \frac{2}{\pi^2}.$$

The operator $V^*V + VV^* : L^2(0,1) \to L^2(0,1)$ is compact, selfadjoint and positive. Then,

$$\|V^*V + VV^*\| = \sup\{\lambda : \lambda \in \sigma(V^*V + VV^*)\},$$

here $\sigma(\cdot)$ is defined as the set of spectrum of an operator (see, [16]). Consider the following spectral problem

$$(V^*V + VV^*)f = \lambda f,\ \lambda \neq 0,\ f \in L^2(0,1),$$

i.e.

$$\int_x^1 \int_0^y f(t)dtdy + \int_0^x \int_y^1 f(t)dtdy = \lambda f.$$

If f is chosen to be a twice differentiable function on (0,1), from the last equation we have

$$\begin{cases} \lambda f'' = -2f, \\ \lambda(f(0) + f(1)) = \int_0^1 f(t)dt, \\ f'(0) + f'(1) = 0. \end{cases}$$

Assume that

$$f(t) = \cos\left(\sqrt{\frac{2}{\lambda}}t\right),\ 0 \leq t \leq 1.$$

Then, $f'' = \frac{-2}{\lambda} f$ is satisfied and from the boundary condition $f'(0) + f'(1) = 0$ we have

$$\lambda_n = \frac{2}{n^2\pi^2},\ n \in \mathbb{N}.$$

In this case $\max_{n\geq 1} \lambda_n = \lambda_1 = \frac{2}{\pi^2}$.

On the other hand,

$$\int_0^1 f_1(t)dt = \int_0^1 cos\left(\sqrt{\frac{2}{\lambda_1}}t\right)dt = 0,$$

$$f_1(0) + f_1(1) = 1 + cos\pi = 0.$$

Then, for $\lambda_1 = \frac{2}{\pi^2}$ and $f_1(t) = cos\left(\sqrt{\frac{2}{\pi}}t\right)$, $0 \leq t \leq 1$ the condition

$$\lambda_1(f_1(0) + f_1(1)) = \int_0^1 f_1(t)dt$$

is satisfied. Hence, $\|V^*V + VV^*\| = \frac{2}{\pi^2}$.

It is known that $\|V\| = \frac{2}{\pi}$, $\|ReV\| = \frac{1}{2}$, $\|ImV\| = \frac{1}{\pi}$ and $\omega(V) = \frac{1}{2}$, $c(ReV) = 0$, and $c(ImV) = 0$ (see, [20], [21]). Then, by Theorem 3.2 we have

$$\omega_t(\varphi, \psi; V) \geq \inf\{|\varphi + \psi|(t), |\varphi - \psi|(t)\} \frac{2+\pi}{4\pi}$$

and

$$\omega_t^2(\varphi, \psi; V) \geq \frac{1}{8} \inf\{|\varphi + \psi|(t), |\varphi - \psi|(t)\}.$$

Now, we give a few well-known inequalities that are essential for proving our theorems, starting with Buzano's inequality.

Lemma 3.4. *[8] Let $x, y, e \in H$ with $\|e\| = 1$. Then*

$$|\langle x, e\rangle\langle e, y\rangle| \leq \frac{1}{2}\left(|\langle x, y\rangle| + \|x\|\|y\|\right).$$

The next lemma pertains to a positive operator.

Lemma 3.5. *[26] Let $T \in \mathbb{B}(H)$ be a positive operator. Then for all $r \geq 1$ and $x \in H$ with $\|x\| = 1$, we have*

$$\langle Tx, x\rangle^r \leq \langle T^r x, x\rangle.$$

Next, we present the generalized mixed Schwarz inequality.

Lemma 3.6. *[13] If $T \in \mathbb{B}(H)$, then for all $x, y \in H$ and $\alpha \in [0, 1]$*

$$|\langle Tx, y\rangle|^2 \leq \langle |T|^{2\alpha} x, x\rangle\langle |T^*|^{2(1-\alpha)} y, y\rangle.$$

We obtain the following upper bound for the weighted numerical radius of bounded linear operators.

Theorem 3.7. *Let $A \in \mathbb{B}(H)$ and $t \in [0, 1]$. Then*

$$\omega_t^2(\varphi, \psi; A) \leq |\varphi|^2(t)\omega^2(A) + |\varphi\psi|(t)\omega(A^2) + \frac{(|\varphi| + |\psi|)(t)}{2}|\psi|(t)\|A^*A + AA^*\|.$$

Proof. For $t \in [0,1]$ and $x \in S_1(H)$, using Lemma 3.4 and 3.6 we have

$$\begin{aligned}
&|\langle(\varphi(t)A + \psi(t)A^*)x, x\rangle|^2 \\
\leq\ & |\varphi|^2(t)|\langle Ax, x\rangle|^2 + 2|\varphi\psi|(t)|\langle Ax, x\rangle||\langle A^*x, x\rangle| + |\psi|^2(t)|\langle A^*x, x\rangle|^2 \\
=\ & |\varphi|^2(t)|\langle Ax, x\rangle|^2 + 2|\varphi\psi|(t)|\langle Ax, x\rangle\langle x, A^*x\rangle| + |\psi|^2(t)|\langle Ax, x\rangle|^2 \\
\leq\ & |\varphi|^2(t)|\langle Ax, x\rangle|^2 + |\varphi\psi|(t)\left[|\langle Ax, A^*x\rangle| + \|Ax\|\|A^*x\|\right] + |\psi^2|(t)\langle|A|x, x\rangle\langle|A^*|x, x\rangle \quad \text{(using Lemma 3.4 and Lemma 3.6)} \\
\leq\ & |\varphi|^2(t)|\langle Ax, x\rangle|^2 + |\varphi\psi|(t)|\langle A^2x, x\rangle| + |\varphi\psi|(t)\frac{1}{2}\langle(|A|^2 + |A^*|^2)x, x\rangle + |\psi|^2(t)\frac{1}{2}\langle|A|^2 + |A^*|^2)x, x\rangle \\
\leq\ & |\varphi|^2(t)\omega^2(A) + |\varphi\psi|(t)\omega(A^2) + \frac{(|\varphi| + |\psi|)(t)}{2}|\psi|(t)\|A^*A + AA^*\|.
\end{aligned}$$

From the last inequality and the definition of the weighted numerical radius the validity of claim is clear. □

Now, if we take $\varphi(t) = 1-2t,\ \psi(t) = 1,\ 0 \leq t \leq 1$, and A instead of A^* in Theorem 3.7, we get the following corollary which proved in [26, Th. 2.4]. So, the inequality obtained in Theorem 3.7 generalizes [26, Th. 2.4].

Corollary 3.8. *Let $A \in \mathbb{B}(H)$ and $t \in [0,1]$. Then*

$$\omega_t^2(A) \leq (1-2t)^2\omega^2(A) + (1-2t)\omega(A^2) + (1-t)\|A^*A + AA^*\|.$$

Remark 3.9. *If we take $\varphi(t) = 0,\ \psi(t) = 1,\ 0 \leq t \leq 1$ in Theorem 3.7, we get the following corollary which coincides with the right side of inequality (2). So, the inequality obtained in Theorem 3.7 generalizes the right side of inequality (2).*

In the next theorem we obtain another upper bound for the weighted numerical radius.

Theorem 3.10. *Let $A \in \mathbb{B}(H)$ and $t \in [0,1]$. Then*

$$\omega_t^2(\varphi, \psi; A) \leq \frac{1}{2}(|\varphi|^2 + |\psi|^2)(t)\|A^*A + AA^*\| + |\varphi\psi|(t)\omega(A^2 + (A^*)^2).$$

Proof. For every $t \in [0,1]$ and $x \in S_1(H)$, using Lemma 3.5 and Lemma 3.6 we have

$$\begin{aligned}
&|\langle(\varphi(t)A + \psi(t)A^*)x, x\rangle|^2 \\
\leq\ & \langle|\varphi(t)A + \psi(t)A^*|x, x\rangle\langle|\varphi(t)A^* + \psi(t)A|x, x\rangle \quad \text{(using Lemma 3.6)} \\
\leq\ & \frac{1}{2}\left[\langle|\varphi(t)A + \psi(t)A^*|^2x, x\rangle + \langle|\varphi(t)A^* + \psi(t)A|^2x, x\rangle\right] \quad \text{(using Lemma 3.5)} \\
\leq\ & \frac{1}{2}\left[\langle(\varphi(t)A + \psi(t)A^*)(\varphi(t)A^* + \psi(t)A)x, x\rangle + \langle(\varphi(t)A^* + \psi(t)A)(\varphi(t)A + \psi(t)A^*)x, x\rangle\right] \\
=\ & \frac{1}{2}\Big[\langle|\varphi|^2(t)AA^* + \varphi(t)\psi(t)AA + \varphi(t)\psi(t)A^*A^* + |\psi|^2(t)A^*A + \\
& + |\varphi|^2(t)A^*A + \varphi(t)\psi(t)A^*A^* + \varphi(t)\psi(t)AA + |\psi|^2(t)AA^*)x, x\rangle\Big] \\
=\ & \frac{1}{2}\left[\langle|\varphi|^2(t)(AA^* + A^*A) + 2\varphi(t)\psi(t)A^*A^* + 2\varphi(t)\psi(t)AA + |\psi|^2(t)(A^*A + AA^*))x, x\rangle\right] \\
=\ & \frac{1}{2}\left[\langle(|\varphi|^2(t) + |\psi|^2(t))(A^*A + AA^*) + 2\varphi(t)\psi(t)(A^*A^* + AA))x, x\rangle\right].
\end{aligned}$$

From the last relation, we also have

$$\omega_t^2(\varphi, \psi; A) \leq \frac{1}{2}(|\varphi|^2 + |\psi|^2)(t)\|A^*A + AA^*\| + |\varphi\psi|(t)\omega(A^2 + (A^*)^2).$$

□

Now, if we take $\varphi(t) = 1 - 2t,\ \psi(t) = 1,\ 0 \leq t \leq 1$ in Theorem 3.10, we get the following corollary which proved in [26, Th. 2.7]. So, the inequality obtained in Theorem 3.10 generalizes [26, Th. 2.7].

Corollary 3.11. *Let $A \in \mathbb{B}(H)$ and $t \in [0,1]$. Then*

$$\omega_t^2(A) \leq (1-2t+2t^2)\|A^*A + AA^*\| + (1-2t)\omega(A^2 + (A^*)^2). \tag{3}$$

Remark 3.12. *If we take $\varphi(t) = 0,\ \psi(t) = 1,\ 0 \leq t \leq 1$ in Theorem 3.10, we get the following corollary which coincides with the right side of inequality (2). So, the inequality obtained in Theorem 3.10 generalizes the right side of inequality (2).*

Now, another estimate from upper bound of the weighted numerical radius will be given.

Theorem 3.13. *Let $A \in \mathbb{B}(H)$ and $t \in [0,1]$. Then,*

$$\omega_t^2(\varphi,\psi;A) \leq \left(|\varphi|^2 + |\psi|^2\right)(t)\omega^2(A) + |\varphi\psi|(t)\omega(A^2) + \frac{1}{2}|\varphi\psi|(t)\|AA^* + A^*A\|.$$

Proof. For each $t \in [0,1]$ and $x \in S_1(H)$, using Lemma 3.4 we have

$$\begin{aligned}
& |\langle(\varphi(t)A + \psi(t)A^*)x, x\rangle|^2 \\
\leq\ & (|\varphi|(t)|\langle Ax,x\rangle| + |\psi|(t)|\langle A^*x,x\rangle|)^2 \\
=\ & |\varphi|^2(t)|\langle Ax,x\rangle|^2 + 2|\varphi\psi|(t)|\langle Ax,x\rangle\langle x,A^*x\rangle| + |\psi|^2(t)|\langle Ax,x\rangle|^2 \\
\leq\ & \left(|\varphi|^2 + |\psi|^2\right)(t)|\langle Ax,x\rangle|^2 + |\varphi\psi|(t)|\langle Ax,A^*x\rangle| + |\varphi\psi|(t)\|Ax\|\|A^*x\| \quad \text{(using Lemma 3.4)} \\
\leq\ & \left(|\varphi|^2 + |\psi|^2\right)(t)|\langle Ax,x\rangle|^2 + |\varphi\psi|(t)|\langle A^2x,x\rangle| + \frac{1}{2}|\varphi\psi|(t)\left(\|Ax\|^2 + \|A^*x\|^2\right) \\
\leq\ & \left(|\varphi|^2 + |\psi|^2\right)(t)\omega^2(A) + |\varphi\psi|(t)\omega(A^2) + \frac{1}{2}|\varphi\psi|(t)\|AA^* + A^*A\|.
\end{aligned}$$

From the last estimates and definition of the numerical radius validity of theorem is clear. □

Now, if we take $\varphi(t) = 1-2t,\ \psi(t) = 1,\ 0 \leq t \leq 1$ in Theorem 3.13, we get the following corollary which proved in [26, Th. 2.9]. So, the inequality obtained in Theorem 3.8 generalizes [26, Th. 2.9].

Corollary 3.14. *Let $A \in \mathbb{B}(H)$ and $t \in [0,1]$. Then*

$$\omega_t^2(A) \leq (2-4t+4t^2)\omega^2(A) + (1-2t)\omega(A^2) + \frac{1}{2}(1-2t)\|A^*A + AA^*\|.$$

Remark 3.15. *Let $A \in \mathbb{B}(H)$ and $t \in [0,1]$. Then from Theorem 3.13 we have*

$$\begin{aligned}
\omega_t^2(\varphi,\psi;A) &\leq \left(|\varphi|^2 + |\psi|^2\right)(t)\omega^2(A) + |\varphi\psi|(t)\omega(A^2) + \frac{1}{2}|\varphi\psi|(t)\|AA^* + A^*A\| \\
&\leq \left(|\varphi|^2 + |\psi|^2\right)(t)\|A\|^2 + |\varphi\psi|(t)\|A\|^2 + \frac{1}{2}|\varphi\psi|(t)2\|A\|^2 \\
&= (|\varphi| + |\psi|)^2\,\|A\|^2.
\end{aligned}$$

Clearly, for $\varphi(t) = 0,\ \psi(t) = 1,\ 0 \leq t \leq 1$ the upper bound improves the second inequality in (1). So, the inequality obtained in Theorem 3.13 generalizes the second inequality in (1).

Example 2. In the complex Hilbert space $L^2(-1,1)$, consider the following Skew-symmetric Volterra integration operator

$$A : L^2(-1,1) \to L^2(-1,1),\ Af(x) = \int_{-x}^{x} f(t)dt,\ f \in L^2(-1,1).$$

It is known that A is a nilpotent operator with index, $\|A\| = \frac{4}{\pi}$, $\omega(A) = \frac{2}{\pi}$ (see [15], [18]).

On the other hand, in [22] Kittaneh proved that if $A^2 = 0$, then $\|A\|^2 = \|A^*A + AA^*\|$. Using by Theorem 3.7, 3.10, and 3.13, we have

$$\omega_t^2(\varphi, \psi; A) \leq \frac{4}{\pi^2}\left(|\varphi|^2(t) + 2|\psi|^2(t) + 2|\varphi\psi|(t)\right),$$

$$\omega_t^2(\varphi, \psi; A) \leq \frac{8}{\pi^2}\left(|\varphi|^2 + |\psi|^2\right)(t),$$

and

$$\omega_t^2(\varphi, \psi; A) \leq \frac{4}{\pi^2}\left(|\varphi|^2 + |\psi|^2\right)(t) + \frac{8}{\pi^2}|\varphi\psi|(t) = \frac{4}{\pi^2}\left(|\varphi| + |\psi|\right)^2(t),$$

respectively.

4. Weighted numerical index in complex Hilbert spaces

In this section, the lower and upper evaluations for the weighted numerical index in Hilbert spaces are given.

Theorem 4.1. *Let $A \in \mathbb{B}(H)$ and $t \in [0, 1]$. Then the following inequalities hold for the weighted numerical radius function and the weighted numerical index*

$$\inf\{|\varphi + \psi|(t), |\varphi - \psi|(t)\}\,\omega(A) \leq \omega_t(\varphi, \psi; A) \leq \sup\{|\varphi + \psi|(t), |\varphi - \psi|(t)\}\,\omega(A)$$

and

$$\frac{1}{2}\inf\{|\varphi + \psi|(t), |\varphi - \psi|(t)\} \leq n_t(\varphi, \psi; H) \leq \frac{1}{2}\sup\{|\varphi + \psi|(t), |\varphi - \psi|(t)\},$$

respectively.

Proof. For the $A \in \mathbb{B}(H)$ and $x \in S_1(H)$, we have

$$\varphi A + \psi A^* = (\varphi + \psi)ReA + i(\varphi - \psi)ImA$$

and

$$\begin{aligned} \inf\{|\varphi + \psi|(t), |\varphi - \psi|(t)\}\,|\langle Ax, x\rangle| &= \inf\{|\varphi + \psi|(t), |\varphi - \psi|(t)\}\,\sqrt{\langle ReAx, x\rangle^2 + \langle ImAx, x\rangle^2} \\ &\leq \sqrt{\langle|\varphi + \psi|(t)ReAx, x\rangle^2 + \langle|\varphi - \psi|(t)ImAx, x\rangle^2}. \end{aligned}$$

Then,

$$\inf\{|\varphi + \psi|(t), |\varphi - \psi|(t)\}\,\omega(A) \leq \omega_t(\varphi, \psi; A).$$

On the other hand, for $x \in S_1(H)$ since

$$\begin{aligned} |\langle(\varphi A + \psi A^*)x, x\rangle| &= |\langle((\varphi + \psi)ReA + i(\varphi - \psi)ImA)x, x\rangle| \\ &= \sqrt{\langle(\varphi + \psi)(t)ReAx, x\rangle^2 + \langle(\varphi - \psi)(t)ImAx, x\rangle^2} \\ &\leq \sup\{|\varphi + \psi|(t), |\varphi - \psi|(t)\}\,\sqrt{\langle ReAx, x\rangle^2 + \langle ImAx, x\rangle^2} \\ &= \sup\{|\varphi + \psi|(t), |\varphi - \psi|(t)\}\,|\langle Ax, x\rangle|, \end{aligned}$$

then

$$\omega_t(\varphi, \psi; A) \leq \sup\{|\varphi + \psi|(t), |\varphi - \psi|(t)\}\,\omega(A).$$

From the first claim of this theorem, the definition of the numerical index, and the result $n(H) = \frac{1}{2}$ [6], [17] the second claim of this theorem can be obtained. □

Using the inequalities the first claim of Theorem 4.1 and (2) we get the following corollary.

Corollary 4.2. *Let $A \in \mathbb{B}(H)$ and $t \in [0,1]$. Then*

$$\frac{1}{4}\left(\inf\{|\varphi+\psi|(t), |\varphi-\psi|(t)\}\right)^2 \|A^*A + AA^*\| \le \omega_t^2(\varphi,\psi;A) \le \frac{1}{2}\left(\sup\{|\varphi+\psi|(t), |\varphi-\psi|(t)\}\right)^2 \|A^*A + AA^*\|.$$

Remark 4.3. *If we take $\varphi(t) = 1,\ \psi(t) = 0,\ 0 \le t \le 1$ in Corollary 4.2, we get the inequality (2). So, the inequality obtained in Corollary 4.2 generalizes (2).*

Corollary 4.4. *Let $A \in \mathbb{B}(H)$. Then*

$$\int_0^1 \inf\{|\varphi+\psi|(t), |\varphi-\psi|(t)\}\, dt\omega(A) \le \int_0^1 \omega_t(\varphi,\psi;A)dt \le \int_0^1 \sup\{|\varphi+\psi|(t), |\varphi-\psi|(t)\}\, dt\omega(A).$$

Remark 4.5. *If we take $\varphi(t) = 1,\ \psi(t) = 0,\ t \in [0,1]$, we get*

$$\inf\{|\varphi+\psi|(t), |\varphi-\psi|(t)\} = 1 \ \ \textit{and} \ \ \sup\{|\varphi+\psi|(t), |\varphi-\psi|(t)\} = 1.$$

By the first claim of Theorem 4.1 we obtain $n(1,0;H) = \frac{1}{2},\ t \in [0,1]$, which founds in [6] and [17].

Remark 4.6. *If we take $\varphi(t) = 1,\ \psi(t) = 1-2t,\ t \in [0,1]$, we get*

$$|\varphi+\psi|(t) = 2-2t,\ |\varphi-\psi|(t) = 2t,\ t \in [0,1],$$

then

$$\alpha(t) = \inf\{|\varphi+\psi|(t), |\varphi-\psi|(t)\} = \inf\{2-2t, 2t\} = \begin{cases} 2t, & 0 \le t \le \frac{1}{2}, \\ 2-2t, & \frac{1}{2} \le t \le 1 \end{cases}$$

and

$$\lambda(t) = \sup\{|\varphi+\psi|(t), |\varphi-\psi|(t)\} = \sup\{2-2t, 2t\} = \begin{cases} 2-2t, & 0 \le t \le \frac{1}{2}, \\ 2t, & \frac{1}{2} \le t \le 1 \end{cases}.$$

Thus, by Theorem 4.1

$$\frac{1}{2}\alpha(t) \le n(1, 1-2t; H) \le \frac{1}{2}\lambda(t),\ t \in [0,1].$$

In particularly, when $t = \frac{1}{2}$, from the above relations we have

$$n_t(H) = n(1,0;H) = \frac{1}{2},\ t \in [0,1]$$

which founds in [6] and [17].

Remark 4.7. *If we take $\varphi(t) = \sin\left(\frac{1}{4}t\right),\ \psi(t) = \cos\left(\frac{1}{4}t\right),\ t \in [0,\pi]$, then*

$$\inf\{|\varphi+\psi|(t), |\varphi-\psi|(t)\} = \sin\left(\frac{1}{4}t\right) \ \ \textit{and} \ \ \sup\{|\varphi+\psi|(t), |\varphi-\psi|(t)\} = \cos\left(\frac{1}{4}t\right),\ t \in [0,\pi].$$

So, by Theorem 4.1 we have

$$\frac{1}{2}\sin\left(\frac{1}{4}t\right) \le n(\varphi,\psi;H) \le \frac{1}{2}\cos\left(\frac{1}{4}t\right),\ t \in [0,\pi].$$

In particularly, when $t = \pi$, it is obtained that

$$n(\varphi(\pi), \psi(\pi); H) = n(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}; H) = 2^{-\frac{3}{2}}$$

Now, we give one approximation for the weighted numerical index.

Theorem 4.8. *For any $A \in \mathbb{B}(H)$ the following are true*

(1) If $\varphi_1,\ \varphi_2,\ \psi_1,\ \psi_2 \in C[0,1]$, then $|n_t(\varphi_1,\psi_1;H) - n_t(\varphi_2,\psi_2;H)| \leq |\varphi_1 - \varphi_2|(t) + |\psi_1 - \psi_2|(t),\ t \in [0,1]$,

(2) If $\varphi,\ \psi, \varphi_n,\ \psi_n \in C[0,1]$ and sequences (φ_n) and (ψ_n) converge to the functions φ and ψ, respectively, then

$$n_t(\varphi,\psi;H) = \lim_{n\to\infty} n_t(\varphi_n,\psi_n;H).$$

Proof. (1) For $x \in H,\ \|x\| = 1$, we have

$$|\langle \varphi_1(t)A + \psi_1(t)A^*)x, x\rangle| \leq |\langle(\varphi_1 - \varphi_2)(t)A + (\psi_1 - \psi_2)(t)A^*)x, x\rangle| + |\langle \varphi_2(t)A + \psi_2(t)A^*)x, x\rangle|.$$

Hence, for $\|A\| = 1$ we get

$$\omega(\varphi_1,\psi_1;A) \leq \sup\{|\varphi_1 - \varphi_2|(t), |\psi_1 - \psi_2|(t)\}\,\|A\| + \omega(\varphi_2,\psi_2;A).$$

Consequently, it can be obtained that

$$n(\varphi_1,\psi_1;H) \leq \sup\{|\varphi_1 - \varphi_2|(t), |\psi_1 - \psi_2|(t)\} + n(\varphi_2,\psi_2;H).$$

Similarly, it can also be shown that

$$n(\varphi_2,\psi_2;H) \leq \sup\{|\varphi_1 - \varphi_2|(t), |\psi_1 - \psi_2|(t)\} + n(\varphi_1,\psi_1;H).$$

So, we have

$$|n(\varphi_1,\psi_1;H) - n(\varphi_2,\psi_2;H)| \leq |\varphi_1 - \varphi_2|(t) + |\psi_1 - \psi_2|(t),\ t \in [0,1].$$

(2) From the first claim of this theorem we have

$$|n(\varphi_n,\psi_n;H) - n(\varphi,\psi;H)| \leq |\varphi_n - \varphi|(t) + |\psi_n - \psi|(t),\ t \in [0,1],\ n \geq 1.$$

Since $\varphi_n \to \varphi$ and $\psi_n \to \psi$ as $n \to \infty$, then from the last relation the validity of second claim can be easily seen. □

Using the (2) of Theorem 4.8 we have the following corollary.

Corollary 4.9. *Let $\varphi_n, \psi_n \in C[0,1], n \geq 1$ and $\varphi_n \to 1, \psi_n \to 0$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} n_t(\varphi_n,\psi_n;H) = \frac{1}{2}.$$

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