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# Modified Szász-Kantorovich operators with better approximation

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**Abstract.** In this paper, we introduce a new family of Szász-Mirakjan Kantorovich type operators $K_{n,\psi}(f;x)$, depend on a function $\psi$ which satisfies some conditions. In this way we obtain all moments and central moments of the new operators in terms of two numbers $M_{1,\psi}$ and $M_{2,\psi}$, which are integrals of $\psi$ and $\psi^2$, respectively. This is a new approach to have better error estimation, because in the case of $K_{n,\psi}(1;x) = 1$, the order of approximation to a function $f$ by an operator $K_{n,\psi}(f;x)$ is more controlled by the term $K_{n,\psi}((t-x)^2;x)$. Since the different functions $\psi$ gives different values for $M_{1,\psi}$ and $M_{2,\psi}$, it is possible to search for a function $\psi$ with different values of $M_{1,\psi}$ and $M_{2,\psi}$ to make $K_{n,\psi}((t-x)^2;x)$ smaller. By using above approach, we show that there exist a function $\psi$ such that the operator $K_{n,\psi}(f;x)$ has better approximation then the classical Szász-Mirakjan Kantorovich operators. We obtain some direct and local approximation properties of new operators $K_{n,\psi}(f;x)$ and we prove that our new operators have shape preserving properties. Moreover, we also introduced two different King-Type generalizations of our operators, one preserving $x$ and the other preserving $x^2$ and we show that King-Type generalizations of $K_{n,\psi}(f;x)$ has better approximation properties than $K_{n,\psi}(f;x)$ and than the classical Szász-Mirakjan-Kantorovich operator. Furthermore, we illustrate approximation results of these operators graphically and numerically.

## 1. Introduction

As it is well known, the Weierstrass approximation theorem which is an important corner stone of polynomial approximation, states that any continuous function on a closed interval can be uniformly approximated by polynomials (see [36]). Bernstein was the first Mathematician who provides a constructive proof to the Weierstrass Approximation Theorem for continuous functions on $[0,1]$. (see [12], [25]). Since the Bernstein operators have an important role in polynomial approximation theory, many researchers considered many generalizations of these operators such as [4], [5], [10], [15], [19], [30], [24] and [31].
Consider the following weighted space,

$$E_m = \{f \in C[0,\infty) : \lim_{x\to\infty} \frac{|f(x)|}{1+x^m} < \infty\}$$

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with the following norm,

$$\|f\|_m = sup_{x\in[0,\infty)}\frac{|f(x)|}{1+x^m}, \text{for fixed } m \in \mathbb{N}.$$

For a function $f \in E_m$, the Szász-Mirakjan operators are defined by

$$S_n(f;x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1}$$

where,

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}, \tag{2}$$

for any $x \in [0, \infty)$.
In [14], P. L. Butzer introduced the Kantorovich type Szász-Mirakjan operators for Lebesque-integrable function space as follows:

$$K_n(f;x) = n\sum_{k=0}^{\infty} s_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt, \tag{3}$$

or equivalently,

$$K_n(f;x) = \sum_{k=0}^{\infty} s_{n,k}(x)\int_0^1 f\left(\frac{k+t}{n}\right)dt, \tag{4}$$

where $s_{n,k}(x)$ is defined in (2).
Very recently, in [11] the following Kantotovich variant of the Szász- Mirakjan operators are introduced and studied

$$K_{n,\gamma}(f;x) = \sum_{k=0}^{\infty} s_{n,k}(x)\int_0^1 f\left(\frac{k+t^{\gamma}}{n}\right)dt, \tag{5}$$

where $s_{n,k}(x)$ is defined in (2) and $\gamma > 0$.
For the Kantorovich type Szász-Mirakjan operators and some of their generalizations, we refer the following papers (see [3], [13],[14],[21], [22], [26],[27], [28],[29], [32], [33],[34] and [35]). For further developments in this area, interested readers are encouraged to read the cited papers [1], [2], [7], [8],[9] and [20].
In this article, we introduce the following new family of Kantorovich type Szász-Mirakjan operators,

$$K_{n,\psi}(f;x) := \sum_{k=0}^{\infty} s_{n,k}(x)\int_0^1 f\left(\frac{k+\psi(t)}{n}\right)dt, \tag{6}$$

where $s_{n,k}(x)$ is given in (2) and $\psi$ (see [5]) is any integrable function on $[0, 1]$ such that

$$\begin{aligned} 0 \le \psi(t) &\le 1, \\ \psi(0) = 0,\ \psi(1) &= 1, \quad \text{and} \\ M_{p,\psi} &:= \int_0^1 \psi^p(t)dt \end{aligned} \tag{7}$$

where $p$ is any positive integer. Obviously, $0 \le M_{p,\psi} \le 1$ for all $p$. Note that, $K_{n,\psi}$ are positive and linear.
One can easily see that, the classical Százs-Mirakjan Kantorovich operators (3) and operators given in (5) can be produced from (6) by choosing $\psi(t) = t$ or $\psi(t) = t^{\gamma}$ respectively. Our aim is to show that for certain choices of the function $\psi$, the modified operator $K_{n,\psi}$ gives better approximation results than the operators defined in (4) and (5). Moreover, in this paper we introduce two King - Type generalization of our operators one preserving $x$ an other preserving $x^2$. We also compare our operator with its King-Type generalizations and we show that King-Type generalizations of our operators have better approximation results.

## 2. Some Basic Results

This section is devoted to some basic results and properties of $K_{n,\psi}$ which will be used in the next sections.

**Lemma 1.** *For each $\psi$ satisfying (7) and for all $x \in [0,\infty)$ we have;*

*(i)* $K_{n,\psi}(1;x) = 1$,

*(ii)* $K_{n,\psi}(t;x) = x + \frac{M_{1,\psi}}{n}$,

*(iii)* $K_{n,\psi}(t^2;x) = x^2 + \frac{1+2M_{1,\psi}}{n}x + \frac{M_{2,\psi}}{n^2}$,

*where $M_{i,\psi}$, $i = 1,2$ are constants defined in (7).*

**Remark 2.**

*i) Moments of $K_n(f,x)$ can be obtained from Lemma 1 by taking $\psi(t) = t$ or equivalently $M_{1,\psi} = \frac{1}{2}$ and $M_{2,\psi} = \frac{1}{3}$.*

*ii) Moments of $K_{n,\gamma}(f,x)$ can be obtained from Lemma 1 by taking $\psi(t) = t^\gamma$ or equivalently $M_{1,\psi} = \frac{1}{\gamma+1}$ and $M_{2,\psi} = \frac{1}{2\gamma+1}$.*

**Corollary 3.** *The central moments are given by*

$$i)\ K_{n,\psi}(t-x;x) = \frac{M_{1,\psi}}{n},$$

$$ii)\ K_{n,\psi}((t-x)^2;x) = \frac{x}{n} + \frac{M_{2,\psi}}{n^2}. \tag{8}$$

**Remark 4.**

*i) Central moments of $K_n((t-x)^i;x)$, can be obtained from Corollry 3 by taking $M_{1,\psi} = \frac{1}{2}$ and $M_{2,\psi} = \frac{1}{3}$.*

*ii) Central moments of $K_{n,\gamma}((t-x)^i;x)$, can be obtained from Corollry 3 by taking $M_{1,\psi} = \frac{1}{\gamma+1}$ and $M_{2,\psi} = \frac{1}{2\gamma+1}$.*

The following lemma, gives the connoection between the moments of the operators $K_{n,\psi}$ and $S_n$. In other words, higher-order moments of $K_{n,\psi}$ can be computed by utilizing the classical Százs-Mirakjan operators $S_n$.

**Lemma 5.** *For any $n \in \mathbb{N}$ and $x \in [0,\infty)$, we have*

$$K_{n,\psi}(t^m;x) = \frac{1}{n^m}\sum_{i=0}^{m}\binom{m}{i}M_{m-i,\psi}n^i S_n(t^i;x).$$

*where $S_n(f;x)$ is given in (1).*

*Proof.* From (6), we get

$$\begin{aligned}
K_{n,\psi}(t^m;x) &= \sum_{k=0}^{\infty} s_{n,k}(x)\int_0^1 \left(\frac{k+\psi(t)}{n}\right)^m dt \\
&= \frac{1}{n^m}\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^1 (k+\psi(t))^m\, dt \\
&= \frac{1}{n^m}\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^1 \sum_{i=0}^{m}\binom{m}{i}k^i\psi^{(m-i)}(t)dt \\
&= \frac{1}{n^m}\sum_{k=0}^{\infty} s_{n,k}(x)\sum_{i=0}^{m}\binom{m}{i}k^i\int_0^1 \psi^{(m-i)}(t)dt \\
&= \frac{1}{n^m}\sum_{k=0}^{\infty} s_{n,k}(x)\sum_{i=0}^{m}\binom{m}{i}k^i M_{m-i,\psi} \\
&= \frac{1}{n^m}\sum_{i=0}^{m}\binom{m}{i}M_{m-i,\psi}n^i\sum_{k=0}^{\infty} s_{n,k}(x)\frac{k^i}{n^i} \\
&= \frac{1}{n^m}\sum_{i=0}^{m}\binom{m}{i}M_{m-i,\psi}n^i S_n(t^i;x).
\end{aligned}$$

□

**Theorem 6.** *Let $\psi$ be a function which satisfy the conditions in (7 ), then we have;*

*i) For a function $f$, which is non-decreasing (or non-increasing) on $[0,\infty)$, $K_{n,\psi}(f;x)$ is also non-decreasing (or non-increasing) on $[0,\infty)$.*

*ii) For a function $f$, which is convex (or concave) on $[0,\infty)$, $K_{n,\psi}(f;x)$ is also convex (or concave) on $[0,\infty)$.*

*Proof.* i) Consider the first derivative of $K_{n,\psi}(f;x)$;

$$\begin{aligned}
\left(K_{n,\psi}(f;x)\right)' &= \sum_{k=0}^{\infty} s'_{n,k}(x)\int_0^1 f\left(\frac{k+\psi(t)}{n}\right)dt \\
&= \sum_{k=0}^{\infty}\left[-ne^{-nx}\frac{(nx)^k}{k!}+e^{-nx}\frac{nk(nx)^{k-1}}{k!}\right]\int_0^1 f\left(\frac{k+\psi(t)}{n}\right)dt \\
&= \sum_{k=1}^{\infty}\left[ne^{-nx}\frac{k(nx)^{k-1}}{k!}\right]\int_0^1 f\left(\frac{k+\psi(t)}{n}\right)dt \\
&\quad -n\sum_{k=0}^{\infty}e^{-nx}\frac{(nx)^k}{k!}\int_0^1 f\left(\frac{k+\psi(t)}{n}\right)dt \\
&= n\sum_{k=0}^{\infty}e^{-nx}\frac{(nx)^k}{k!}\int_0^1 f\left(\frac{k+1+\psi(t)}{n}\right)dt \\
&\quad -n\sum_{k=0}^{\infty}e^{-nx}\frac{(nx)^k}{k!}\int_0^1 f\left(\frac{k+\psi(t)}{n}\right)dt \\
&= n\sum_{k=0}^{\infty}e^{-nx}\frac{(nx)^k}{k!}\left[\int_0^1 f\left(\frac{k+1+\psi(t)}{n}\right)dt-\int_0^1 f\left(\frac{k+\psi(t)}{n}\right)dt\right] \\
&= n\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^1 \Delta_h^1 f\left(\frac{k+\psi(t)}{n}\right)dt, \qquad (9)
\end{aligned}$$

with $h = \frac{1}{n}$ and $n = 1, 2, 3, \ldots$.
If $f$ is an non-decreasing function on the interval $[0, \infty)$, we have

$$\Delta_h^1 f\left(\frac{k+\psi(t)}{n}\right) = f\left(\frac{k+1+\psi(t)}{n}\right) - f\left(\frac{k+\psi(t)}{n}\right) \geq 0, \tag{10}$$

where $k = 0, 1, \ldots$ and $t \in [0, 1]$. Using, (10) in (9) gives,

$$\left(K_{n,\psi}(f; x)\right)' \leq 0, \quad 0 \leq x < \infty,$$

which completes the proof. The case where $f$ is a non-increasing function can be proved in a parallel way.

ii) Consider the second derivative of $K_{n,\psi}(f; x)$,

$$\left(K_{n,\psi}(f; x)\right)'' = n^2 \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^1 \Delta_h^2 f\left(\frac{k+\psi(t)}{n}\right) dt \tag{11}$$

with $h = \frac{1}{n}$ for $n = 1, 2, 3, \ldots$.
If $f$ is convex on $[0, \infty)$, then for any $k = 0, 1, \ldots$ we have

$$0 \leq \frac{k+\psi(t)}{n} \leq \frac{k+\psi(t)+1}{n} \leq \frac{k+\psi(t)+2}{n}.$$

Theorem 3.2.2 in [[17], p.59] implies that

$$\Delta_h^2 f\left(\frac{k+\psi(t)}{n}\right) \geq 0. \tag{12}$$

Thus, utilizing (11) and (12) we obtain the following inequality,

$$\left(K_{n,\psi}(f; x)\right)'' \geq 0,$$

for all $x \in [0, \infty)$, which completes the proof. As a conclusion, $K_{n,\psi}(f; x)$ is convex on $[0, \infty)$. The case where $f$ is concave can be proved in a parallel way.

□

## 3. Direct and local approximation properties of $K_{n,\psi}$

Let $A > 0$ be any constant then the lattice homomorphism, $T_{[0,A]} : C[0, \infty) \to C[0, A]$ defined by

$$T_{[0,A]}(f) := f|_{[0,A]}$$

satisfies,

$$\begin{aligned} T_{[0,A]}(K_{n,\psi}(1)) &\rightrightarrows T_{[0,A]}(1) \\ T_{[0,A]}(K_{n,\psi}(t)) &\rightrightarrows T_{[0,A]}(t) \\ T_{[0,A]}(K_{n,\psi}(t^2)) &\rightrightarrows T_{[0,A]}(t^2) \end{aligned} \tag{13}$$

where above convergences are all uniformly on $[0, A]$.
Now we can state the following Theorem.

**Theorem 7.** *For any $A \in \mathbb{R}^+$, $K_{n,\psi}(f; x)$ is uniformly convergent to $f$ on $[0, A]$ provided that $f \in E_m$, $m \geq 2$.*

*Proof.* Let $A > 0$ be an arbitrary fixed real number. According to the Bohman-Korovkin Theorem (see [23]), it is enough to show that

$$\lim_{n\to\infty} \sup_{x\in[0,A]} \left|K_{n,\psi}(t^i;x) - t^i\right| = 0,$$

for $i = 0, 1, 2$. Recall that $E$ is isomorphic to $C[0, 1]$ (see Proposition 4.2.5 (6) of [6]). Therefore the set $\{t^i : i = 0, 1, 2\}$ is a Korovkin set in $E$. Using Lemma 1, property (vi) of Theorem 4.1.4 [6] and (13) completes the proof. □

It is important to remember that, operators satisfying the property $K_{n,\psi}(1;x) = 1$, also satisfies the following inequality,

$$\left|K_{n,\psi}(f;x) - f(x)\right| \leq \epsilon + \frac{2\|f\|_2}{\delta^2} K_{n,\psi}((t-x)^2;x), \tag{14}$$

where $\delta$ is determined by the uniform continuity of the approximated function $f$. In other words, inequality (14) reveals that the order of approximation to a function $f$ by $K_{n,\psi}(f;x)$ is controlled by the term $K_{n,\psi}((t-x)^2;x)$. Hence, we can assess the effectiveness of the operators $K_{n,\psi}(f;x)$ and $K_n(f;x)$ in approximating the function $f$ by examining their second central moments. Therefore, the comprision of the images of the second central moments are enough to compare the order of the approximation of the operators $K_{n,\psi}(f;x)$ and $K_n(f;x)$ to a function $f$.

New operators $K_{n,\psi}(f;x)$ have the advantage that, the term $K_{n,\psi}((t-x)^2;x)$ depend on the parameters $M_{1,\psi}$ and $M_{2,\psi}$. Therefore distinct $K_{n,\psi}((t-x)^2;x)$ terms can be derived for certain functions $\psi$ satisfying (7). This enables us to explore different values of $M_{1,\psi}$ and $M_{2,\psi}$ to achieve a more accurate approximation. In essence, using our operators reduces the problem of improving approximation to two key questions: for which values of $M_{1,\psi}$ and $M_{2,\psi}$ does the inequality

$$K_{n,\psi}((t-x)^2;x) < K_n((t-x)^2;x), \tag{15}$$

hold, and is there any function $\psi$ which provides these $M_{1,\psi}$ and $M_{2,\psi}$ values? As demonstrated below, we establish solutions to both problems. We can compare how well the operators $K_{n,\psi}(f;x)$ and $K_n(f;x)$ approximate the function $f$. From Corollary 3 and equation (15), better approximation is possible if,

$$\begin{aligned} K_{n,\psi}((t-x)^2;x) &< K_n((t-x)^2;x) \\ \frac{x}{n} + \frac{M_{2,\psi}}{n^2} &< \frac{x}{n} + \frac{1}{3n^2} \\ \Rightarrow \frac{M_{2,\psi}}{n^2} &< \frac{1}{3n^2} \\ \Rightarrow M_{2,\psi} &< \frac{1}{3}. \end{aligned} \tag{16}$$

In (16), we obtained a solution to the first problem. Next, we will show that, the second problem also has a solution, that is there exist at least one function $\psi$, and the corresponding $M_{2,\psi}$ satisfies inequality (16).

Now, let us consider the following functions,

$$\psi(t) := \begin{cases} a^{\alpha-\frac{1}{\alpha}} t^{\frac{1}{\alpha}} & 0 \leq t \leq a \\ \\ t^\alpha & a \leq t \leq 1, \end{cases} \tag{17}$$

where $a \in [0, 1]$ and $\alpha > 0$. Obviously, $\psi$ is continuous for all $a \in [0, 1]$, $\alpha > 0$, and satisfies (7), with

$$M_{1,\psi} = \Big(\frac{\alpha-1}{\alpha+1}\Big)a^{\alpha+1} + \frac{1}{\alpha+1} \text{ and } M_{2,\psi} = \frac{2(\alpha-1)(\alpha+1)}{(\alpha+2)(2\alpha+1)}a^{2\alpha+1} + \frac{1}{2\alpha+1}.$$

Now, for $\alpha = 1.1$ and $a = 0.75$, $M_{2,\psi} = 0.3294$. Therefore from inequality (16),

$$K_{n,\psi}((t-x)^2 : x) \quad < \quad K_n((t-x)^2 : x)$$

which means $K_{n,\psi}(f : x)$ has better approximation than the normal Szasz Kantorovich operators for $\psi$ given in (17) on $[0, A]$.
For $\alpha = 1.1$ and $a = 0.75$, $M_{2,\psi} = 0.3294$, some numerical values of $K_{n,\psi}((t-x)^2; x)$ and $K_n((t-x)^2; x)$ are shown below and their graphs are given in Figure 1.

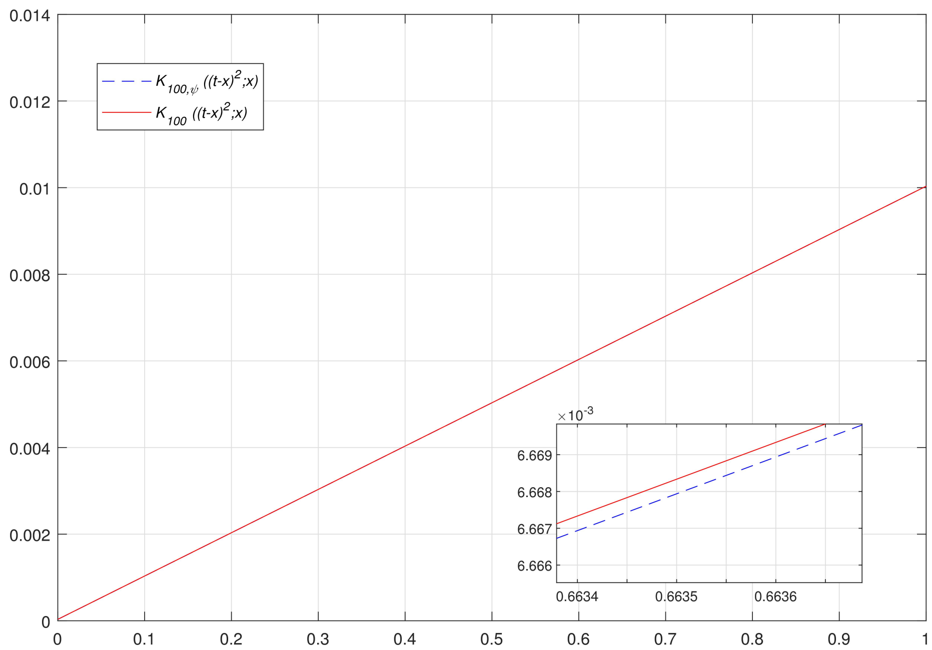


Figure 1: Graphical representations of $K_{n,\psi}((t-x)^2; x)$ and $K_n((t-x)^2; x)$ for $\alpha = 1.1$, $a = 0.75$ and $n = 100$.

| $x$ | $K_{100}((t-x)^2; x)$ | $K_{100,\psi}((t-x)^2; x)$ |
|---|---|---|
| 0 | 0.000033333 | 0.000032936 |
| 0.5 | 0.005033333 | 0.005032936 |
| 1 | 0.010033333 | 0.010032936 |
| 1.5 | 0.015033333 | 0.015032936 |
| 2 | 0.020033333 | 0.020032936 |
| 2.5 | 0.025033333 | 0.025032936 |
| 3 | 0.030033333 | 0.030032936 |
| 3.5 | 0.035033333 | 0.035032936 |
| 4 | 0.040033333 | 0.040032936 |
| 4.5 | 0.045033333 | 0.045032936 |
| 5 | 0.050033333 | 0.050032936 |

In a similar way we can compare operators $K_{n,\psi}((t-x)^2;x)$ and $K_{n,\gamma}((t-x)^2;x)$ that is,

$$\begin{aligned} K_{n,\psi}((t-x)^2;x) &< K_{n,\gamma}((t-x)^2;x) \\ \frac{x}{n}+\frac{M_{2,\psi}}{n^2} &< \frac{x}{n}+\frac{1}{(2\gamma+1)n^2} \\ \frac{M_{2,\psi}}{n^2} &< \frac{1}{(2\gamma+1)n^2} \\ M_{2,\psi} &< \frac{1}{2\gamma+1}. \end{aligned}$$

In other words for a fixed $\gamma\in(0,\infty)$, the operator $K_{n,\psi}(f;x)$ with $M_{2,\psi}<\frac{1}{2\gamma+1}$ has better approximation then $K_{n,\gamma}(f;x)$.
Now, let's give the local approximation properties of $K_{n,\psi}$. Recall that, the modulus of continuity (see [18]) is defined by,

$$\omega(f;\delta) = sup_{0<h\leq\delta} sup_{x\in[0,\infty)}|f(x+h)-f(x)|.$$

where $f\in C_B[0,\infty)$.

**Theorem 8.** *For all $f\in C_B[0,\infty)$, $x\in[0,\infty)$, $n\in\mathbb{N}$ and $\psi$ satisfying (7) we have,*

$$\left|K_{n,\psi}(f;x)-f(x)\right|\leq 2\omega\left(f;\sqrt{\frac{x}{n}+\frac{M_{2,\psi}}{n^2}}\right).$$

*Proof.* Since $K_{n,\psi}(1;x)=1$ and $s_{n,k}(x)\geq 0$ on $[0,\infty)$ we can write,

$$\left|K_{n,\psi}(f;x)-f(x)\right| \leq \sum_{k=0}^{\infty} s_{n,k}(x)\int_0^1\left|f\left(\frac{k+\psi(t)}{n}\right)-f(x)\right|dt. \tag{18}$$

Equation (18) implies that, for any $\delta>0$,

$$\left|f\left(\frac{k+\psi(t)}{n}\right)-f(x)\right|\leq\left(1+\frac{\left|\frac{k+\psi(t)}{n}-x\right|}{\delta}\right)\omega(f;\delta). \tag{19}$$

Using (19) in (18), we get

$$\left|K_{n,\psi}(f;x)-f(x)\right| \leq \omega(f;\delta)\left(1+\frac{1}{\delta}\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^1\left|\frac{k+\psi(t)}{n}-x\right|dt\right).$$

Applying Cauchy-Schwarz inequality, we get

$$\begin{aligned} \left|K_{n,\psi}(f;x)-f(x)\right| &\leq \omega(f;\delta)\left(1+\frac{1}{\delta}\sqrt{\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^1\left(\frac{k+\psi(t)}{n}-x\right)^2dt}\right) \\ &= \omega(f;\delta)\left(1+\frac{1}{\delta}\sqrt{\frac{x}{n}+\frac{M_2,\psi}{n^2}}\right). \end{aligned}$$

Taking $\delta=\sqrt{\frac{x}{n}+\frac{M_{2,\psi}}{n^2}}$, we get the desired result. □

Recall that, for $\delta > 0$ the Peetre-$K$ functional $K_2(f;\delta)$ is defined by;

$$K_2(f;\delta) \quad := \quad inf_{g\in\varpi^2[0,\infty)}\left\{\|f-g\| + \delta\|g''\|\right\}, (\delta > 0)$$

where $\varpi^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$.
Furthermore, $\exists\, C > 0$ (see [16]) such that

$$K_2(f;\delta) \quad \le \quad C\omega_2(f;\sqrt{\delta})$$

where $\omega_2(f;\sqrt{\delta})$ is called the second order modulus of continuity of $f \in C_B[0,\infty)$, defined as;

$$\omega_2(f;\sqrt{\delta}) \quad = \quad \sup_{0<h\le\delta}\sup_{x\in[0,\infty)}\left|f(x+2h) - 2f(x+h) + f(x)\right|.$$

Now, we need to state the following lemma to prove the local approximation properties of the operators $K_{n,\psi}$ depending on the second order modulus of continuity.

**Lemma 9.** *For each $f \in C_B[0,\infty)$ and $\psi$, satisfying (7) we have*

$$\|K_{n,\psi}(f;\cdot)\| \le \|f\|. \tag{20}$$

**Theorem 10.** *For all $f \in C_B[0,\infty)$, $x \in [0,\infty)$ and $\psi$ satisfying (7), there exist $C > 0$ such that*

$$|K_{n,\psi}(f;x) - f(x)| \quad \le \quad C\omega_2\left(f;\frac{1}{2}\sqrt{\frac{x}{n} + \frac{M_{2,\psi}}{n^2} + \left(\frac{M_{1,\psi}}{n}\right)^2}\right) + \omega\left(f;\frac{M_{1,\psi}}{n}\right).$$

*Proof.* Let

$$K^*_{n,\psi}(f;x) \quad := \quad K_{n,\psi}(f;x) + f(x) - f\left(x + \frac{M_{1,\psi}}{n}\right). \tag{21}$$

From Lemma 1,

$$K^*_{n,\psi}(1;x) \quad = \quad 1,$$

and

$$K^*_{n,\psi}(t-x;x) \quad = \quad 0.$$

If $g \in \varpi^2[0,\infty)$, then by Taylor's expansion,

$$g(t) \quad = \quad g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du. \tag{22}$$

Applying the operators $K^*_{n,\psi}$ to (22), we get

$$\begin{aligned} K^*_{n,\psi}(g;x) \quad &= \quad g(x) + K^*_{n,\psi}\left(\int_x^t (t-u)g''(u)du;x\right) \\ &= \quad g(x) + K_{n,\psi}\left(\int_x^t (t-u)g''(u)du;x\right) - \int_x^{x+\frac{M_{1,\psi}}{n}}\left(x + \frac{M_{1,\psi}}{n} - u\right)g''(u)du. \end{aligned}$$

Rewriting the above equation as,

$$K^*_{n,\psi}(g;x) - g(x) \quad = \quad K_{n,\psi}\left(\int_x^t (t-u)g''(u)du;x\right) - \int_x^{x+\frac{M_{1,\psi}}{n}}\left(x + \frac{M_{1,\psi}}{n} - u\right)g''(u)du.$$

Hence, we can write

$$\begin{aligned}
|K^*_{n,\psi}(g;x)-g(x)| &\leq \left|K_{n,\psi}\left(\int_x^t (t-u)g''(u)du;x\right)\right| + \left|\int_x^{x+\frac{M_1}{n}} (x+\frac{M_{1,\psi}}{n}-u)g''(u)du\right| \\
&\leq K_{n,\psi}\left(\left|\int_x^t (t-u)g''(u)du\right|;x\right) + \int_x^{x+\frac{M_{1,\psi}}{n}} \left|x+\frac{M_{1,\psi}}{n}-u\right|\left|g''(u)\right|du \\
&\leq K_{n,\psi}\left(\left|\int_x^t |(t-u)|du\right|;x\right)\|g''\| + \int_x^{x+\frac{M_{1,\psi}}{n}} \left|x+\frac{M_{1,\psi}}{n}-u\right|du\|g''\| \\
&\leq K_{n,\psi}((t-x)^2;x)\|g''\| + \left(x+\frac{M_{1,\psi}}{n}-x\right)^2\|g''\| \\
&= \left[\frac{x}{n}+\frac{M_{2,\psi}}{n^2}+\left(\frac{M_{1,\psi}}{n}\right)^2\right]\|g''\|.
\end{aligned}$$

Therefore,

$$\left|K^*_{n,\psi}(g;x)-g(x)\right| \leq \left[\frac{x}{n}+\frac{M_{2,\psi}}{n^2}+\left(\frac{M_{1,\psi}}{n}\right)^2\right]\|g''\|. \tag{23}$$

Also, from (20) and equation (21), we get

$$\left|K^*_{n,\psi}(f;\cdot)\right| \leq 3\|f\| \tag{24}$$

for all $f \in C_B[0,\infty)$ and $x \in [0,\infty)$.

On the other hand, for any $f \in C_B[0,\infty)$ and $g \in \varpi^2[0,\infty)$, using (23) and (24), we get

$$\begin{aligned}
&\left|K_{n,\psi}(f;x)-f(x)\right| \\
= &\left|K^*_{n,\psi}(f;x)-f(x)+f\left(x+\frac{M_{1,\psi}}{n}\right)-f(x)\right| \\
= &\left|K^*_{n,\psi}(f;x)-K^*_{n,\psi}(g;x)+K^*_{n,\psi}(g;x)-g(x)+g(x)-f(x)+f\left(x+\frac{M_{1,\psi}}{n}\right)-f(x)\right| \\
\leq &\left|K^*_{n,\psi}(f;x)-K^*_{n,\psi}(g;x)\right|+\left|K^*_{n,\psi}(g;x)-g(x)\right|+\left|g(x)-f(x)\right|+\left|f\left(x+\frac{M_{1,\psi}}{n}\right)-f(x)\right| \\
\leq &\ 4\|f-g\|+\left[\frac{x}{n}+\frac{M_{2,\psi}}{n^2}+\left(\frac{M_{1,\psi}}{n}\right)^2\right]\|g''\|+\omega\left(f;\frac{M_{1,\psi}}{n}\right).
\end{aligned}$$

Finally, if we take infimum on the right-hand side over all $g \in \varpi^2[0,\infty)$, we obtain:

$$\begin{aligned}
\left|K_{n,\psi}(f;x)-f(x)\right| &\leq 4K_2\left(f;\frac{\frac{x}{n}+\frac{M_{2,\psi}}{n^2}+\left(\frac{M_{1,\psi}}{n}\right)^2}{4}\right)+\omega\left(f;\frac{M_{1,\psi}}{n}\right) \\
&= C\omega_2\left(f;\frac{1}{2}\sqrt{\frac{x}{n}+\frac{M_{2,\psi}}{n^2}+\left(\frac{M_{1,\psi}}{n}\right)^2}\right)+\omega\left(f;\frac{M_{1,\psi}}{n}\right),
\end{aligned}$$

and this completes the proof. □

Now, for any $0 < a \leq 1$ and $M > 0$, the usual Lipschitz class is defined by,

$$Lip_M(a) := \left\{ f \in C_B[0,\infty) : \left| f(\rho) - f(\sigma) \right| \leq M \left| \rho - \sigma \right|^a \right\}$$

$\forall \rho, \sigma \in [0,\infty)$.

**Theorem 11.** *For every $f \in Lip_M(a)$ and $\psi$ satisfying (7), we have*

$$\left| K_{n,\psi}(f;x) - f(x) \right| \leq M \left[ \frac{x}{n} + \frac{M_{2,\psi}}{n^2} \right]^{\frac{a}{2}}.$$

*Proof.* For any $f \in Lip_M(a)$ using linearity and positivity properties of the operator we can write that,

$$\begin{aligned} \left| K_{n,\psi}(f;x) - f(x) \right| &\leq \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^1 \left| f\left( \frac{k+\psi}{n} \right) - f(x) \right| dt \\ &\leq M \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^1 \left| \frac{k+\psi}{n} - x \right|^a dt. \end{aligned}$$

Apply Hölder's inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we obtain

$$\begin{aligned} \left| K_{n,\psi}(f;x) - f(x) \right| &\leq M \left[ \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^1 \left( \frac{k+\psi}{n} - x \right)^2 dt \right]^{\frac{a}{2}} \left[ \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^1 dt \right]^{\frac{2-a}{2}} \\ &= M \left[ \frac{x}{n} + \frac{M_{2,\psi}}{n^2} \right]^{\frac{a}{2}}. \end{aligned}$$

Therefore, the proof is completed. □

## 4. Graphical Analysis and Error Estimation

In this section, we give some graphics to illustrate approximation of $K_{n,\psi}(f;x)$ to a certain function $f$. For the following figures, $\psi$ is the function given in (17). In Figure 2, we illustrate the approximation of the operators $K_{100,\psi}(f;x)$, $K_{200,\psi}(f;x)$, $K_{500,\psi}(f;x)$ to $f(x) = x^3 - 3x^2 + 2x$. Secondly, absolute error function $\varepsilon_n(f(x))$ is illustrated in Figure 3 for $n = 100, 200, 500$. Finally, we compute the absolute error of $K_{n,\psi}(f;x)$ with $f(x) = x^3 - 3x^2 + 2x$ for $n = 100, 200, 500$ in Table 1.

Table 1: $\varepsilon_n(f(x))$ with $f(x) = x^3 - 3x^2 + 2x$ for some $x$ values and $n = 100, 200, 500$.

| x | $\mid K_{100,\psi}(f;x) - f(x) \mid$ | $\mid K_{200,\psi}(f;x) - f(x) \mid$ | $\mid K_{500,\psi}(f;x) - f(x) \mid$ |
|---|---|---|---|
| 0 | 0.0099 | 0.0050 | 0.0020 |
| 0.5 | 0.0087 | 0.0044 | 0.0017 |
| 1 | 0.0048 | 0.0024 | 0.0009 |
| 1.5 | 0.0217 | 0.0127 | 0.0043 |
| 2 | 0.0706 | 0.0352 | 0.0140 |

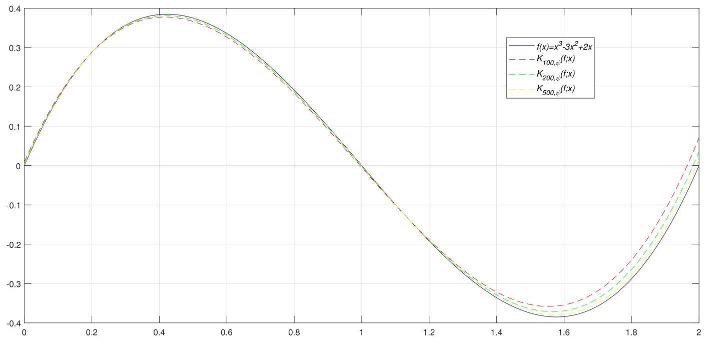


Figure 2: Approximation of $K_{n,\psi}(f;x)$ to $f(x) = x^3 - 3x^2 + 2x$ for $\alpha = 1.1$ and $a = 0.75$.

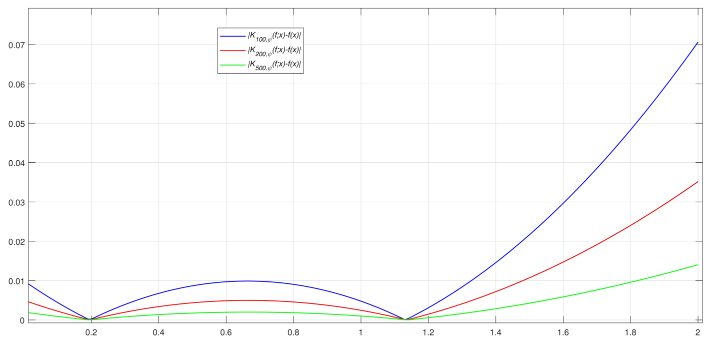


Figure 3: Error of approximation $\varepsilon_n(f(x)) = \left|K_{n,\psi}(f;x) - f(x)\right|$ when $n = 100, 200, 500$.

## 5. Szász-Mirakjan Kantorovich operators which preserves $x$

Now, in this sections, we introduce two different King-type modification of the operators $K_{n,\psi}$ so that one preserves $x$ and the other preserves $x^2$. Let $r_n(x)$ be sequence of real-valued functions with $0 \leq r_n(x) < \infty$. Then we have

$$\tilde{K}_{n,\psi}(f; r_n(x)) := \sum_{k=0}^{\infty} s_{n,k}(r_n(x)) \int_0^1 f\left(\frac{k+\psi(t)}{n}\right) dt. \tag{25}$$

Now, If we replace $r_n(x)$ by $r_n^*(x)$ defined by,

$$r_n^*(x) = x - \frac{M_{1,\psi}}{n}, \quad n \in \mathbb{N},$$

then we get the following King-type modified Szász-Mirakjan Kantorovich operators which preserve the test fuctions $e_i = x^i,\ i = 0, 1$:

$$\begin{aligned}\tilde{K}^*_{n,\psi}(f;x) &:= \sum_{k=0}^{\infty} e^{-n\left(x-\frac{M_{1,\psi}}{n}\right)} \frac{\left(n\left(x-\frac{M_{1,\psi}}{n}\right)\right)^k}{k!} \int_0^1 f\left(\frac{k+\psi(t)}{n}\right)dt \\ &= \sum_{k=0}^{\infty} s_{n,k}(r_n^*(x)) \int_0^1 f\left(\frac{k+\psi(t)}{n}\right)dt.\end{aligned}$$

**Lemma 12.** *Let $e_i = x^i, i = 0, 1, 2,$ on the interval $M_{1,\psi} \le x < \infty$ such that $0 \le r_n^*(x) < \infty$ and $n \in \mathbb{N}$, then the operators $\tilde{K}^*_{n,\psi}$ satisfies the following equations,*

1. $\tilde{K}^*_{n,\psi}(e_0,) = 1,$
2. $\tilde{K}^*_{n,\psi}(e_1,) = x,$
3. $\tilde{K}^*_{n,\psi}(e_2) = x^2 + \frac{x}{n} - \frac{(M^2_{1,\psi} + M_{1,\psi} - M_{2,\psi})}{n^2}.$

From Lemma 12, we obtain the followings results.

**Lemma 13.** *For any $M_{1,\psi} \le x < \infty$, we have*

$$\begin{aligned}\tilde{K}^*_{n,\psi}(t-x;x) &= 0, \\ \tilde{K}^*_{n,\psi}((t-x)^2;x) &= \frac{x}{n} - \frac{(M^2_{1,\psi} + M_{1,\psi} - M_{2,\psi})}{n^2}.\end{aligned}$$

**Theorem 14.** *$\tilde{K}^*_{n,\psi}(f;x)$ is uniformly convergent to $f$, on the interval $M_{1,\psi} \le x < A$ such that $0 \le r_n^*(x) < A$, provided that $f \in E_m$, $m \ge 2$.*

**Theorem 15.** *Let $f \in C_B[0,\infty)$. For $M_{1,\psi} \le x < \infty$ such that $0 \le r_n^*(x) < \infty$, we have,*

$$\left|\tilde{K}^*_{n,\psi}(f;x) - f(x)\right| \le 2\omega\left(f; \sqrt{\frac{x}{n} - \frac{(M^2_{1,\psi} + M_{1,\psi} - M_{2,\psi})}{n^2}}\right).$$

*Proof.* For $f \in C_B[0,\infty)$, we have,

$$\left|\tilde{K}^*_{n,\psi}(f;x) - f(x)\right| \le \sum_{k=0}^{\infty} s_{n,k}(r_n^*(x)) \int_0^1 \left|f\left(\frac{k+\psi(t)}{n}\right) - f(x)\right| dt. \tag{26}$$

Applying the property of the modulus of continuity, which is

$$\left|f(\zeta) - f(\lambda)\right| \le \left(1 + \frac{|\zeta-\lambda|}{\delta}\right)\omega(f;\delta)$$

to (26), we obtain,

$$\left|\tilde{K}^*_{n,\psi(t)}(f;x) - f(x)\right| \le \omega(f;\delta)\left(1 + \frac{1}{\delta}\sum_{k=0}^{\infty} s_{n,k}(r_n^*(x)) \int_0^1 \left|\frac{k+\psi(t)}{n} - x\right| dt\right).$$

Applying Cauchy-Schwarz inequality,

$$\begin{aligned}\left|\tilde{K}^*_{n,\psi(t)}(f;x) - f(x)\right| &\le \omega(f;\delta)\left(1 + \frac{1}{\delta}\sqrt{\sum_{k=0}^{\infty} s_{n,k}(r_n^*(x)) \int_0^1 \left(\frac{k+\psi(t)}{n} - x\right)^2 dt}\right) \\ &= \omega(f;\delta)\left(1 + \frac{1}{\delta}\sqrt{\frac{x}{n} - \frac{(M^2_{1,\psi} + M_{1,\psi} - M_{2,\psi})}{n^2}}\right).\end{aligned}$$

Choosing $\delta = \sqrt{\frac{x}{n} - \frac{(M_{1,\psi}^2 + M_{1,\psi} - M_{2,\psi})}{n^2}}$, we have the desired result. □

Now recall that, the order of approximation of $\tilde{K}^*_{n,\psi}(f;x)$ to $f$, will be at least as good as that of the $K_{n,\psi}(f;x)$ if

$$\begin{aligned} \tilde{K}^*_{n,\psi}((t-x)^2;x) &\leq K_{n,\psi}((t-x)^2;x) \\ \frac{x}{n} - \frac{(M_{1,\psi}^2 + M_{1,\psi} - M_{2,\psi})}{n^2} &\leq \frac{x}{n} + \frac{M_{2,\psi}}{n^2} \\ M_{1,\psi}^2 + M_{1,\psi} &\geq 0 \end{aligned}$$

which is always true.
Similarly, $\tilde{K}^*_{n,\psi}((t-x)^2;x) \leq K_n((t-x)^2;x)$, implies that,

$$\frac{x}{n} - \frac{(M_{1\psi}^2 + M_{1,\psi} - M_{2,\psi})}{n^2} \leq \frac{x}{n} + \frac{1}{3n^2} \tag{27}$$

$$M_{2,\psi} - M_{1,\psi}^2 - M_{1,\psi} \leq \frac{1}{3} \tag{28}$$

which is always true.
Now, for $\alpha = 0.01$ and $a = 0.75$, $M_{1,\psi} = 0.2571$. From inequality (27), $\tilde{K}^*_{n,\psi}((t-x)^2;x) \leq K_{n,\psi}((t-x)^2;x)$ which means that $\tilde{K}^*_{n,\psi}(f;x)$ provides better approximation properties than $K_{n,\psi}(f;x)$ for $\psi$ given in (17) (see Figure 4).

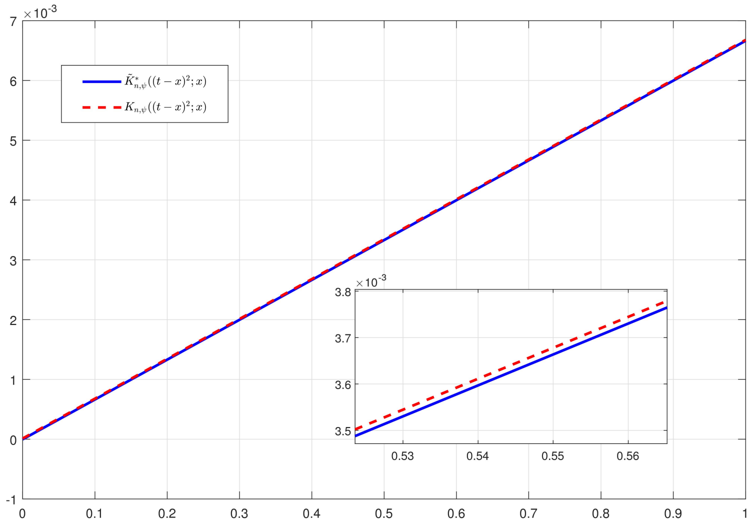


Figure 4: Graphical representations of $\tilde{K}^*_{n,\psi}((t-x)^2;x)$ and $K_{n,\psi}((t-x)^2;x)$ for $\alpha = 0.01$, $a = 0.75$ and $n = 150$.

Similarly, Now, if $\alpha = 0.01$ and $a = 0.75$ then $M_{1,\psi} = 0.2571$ and $M_{2,\psi} = 0.2530$. From inequality (28), $\tilde{K}^*_{n,\psi}((t-x)^2;x) \leq K_n((t-x)^2;x)$ which means $\tilde{K}^*_{n,\psi}(f;x)$ provides better approximation properties than $K_n(f;x)$ for $\psi$ given in (17), on $[0,\infty)$ (see Figure 5).

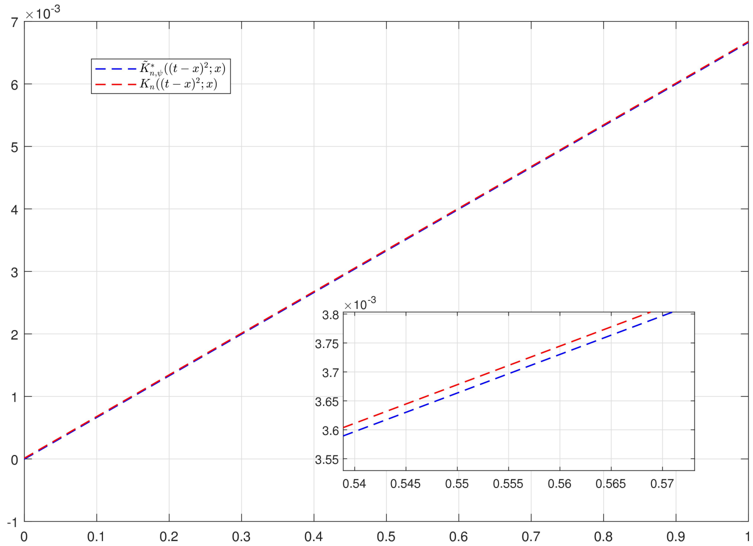


Figure 5: Graphical representations of $\tilde{K}^{*}_{n,\psi}((t-x)^2;x)$ and $K_n((t-x)^2;x)$ for $\alpha = 0.01$, $a = 0.75$ and $n = 150$.

## 6. Szász-Mirakjan Kantorovich operators which preserves $x^2$

Now, If we replace $r_n(x)$ by $r_n^{**}(x)$ in (25) where

$$r_n^{**}(x) = \frac{-(1+2M_{1,\psi})}{2n} + \sqrt{\frac{(1+2M_{1,\psi})^2-4M_{2,\psi}}{4n^2} + x^2}), \qquad n = 1,2,3,\ldots$$

then we get the King-type modified Szász-Mirakjan Kantorovich operators:

$$\begin{aligned}\tilde{K}^{**}_{n,\psi}(f;x) &:= \sum_{k=0}^{\infty} e^{-nr_n^{**}(x)} \frac{(nr_n^{**}(x))^k}{k!} \int_0^1 f\left(\frac{k+\psi(t)}{n}\right) dt \\ &= \sum_{k=0}^{\infty} s_{n,k}(r_n^{**}(x)) \int_0^1 f\left(\frac{k+\psi(t)}{n}\right) dt\end{aligned}$$

which preserves $e_i = x^i, i = 0,2$.

**Lemma 16.** *Let $e_i = x^i, i = 0,1,2$, on the interval $\sqrt{M_{2,\psi}} \leq x < \infty$ such that $0 \leq r_n^{**}(x) < \infty$ and $n \in \mathbb{N}$, then operators $\tilde{K}^{**}_{n,\psi}$ verify the following:*

1. $\tilde{K}^{**}_{n,\psi}(e_0) = 1$;
2. $\tilde{K}^{**}_{n,\psi}(e_1) = \frac{-1}{2n} + \sqrt{\frac{(1+2M_{1,\psi})^2-4M_{2,\psi}}{4n^2} + x^2}$;
3. $\tilde{K}^{**}_{n,\psi}(e_2) = x^2$.

**Lemma 17.** *For $\sqrt{M_{2,\psi}} \leq x < \infty$, we have*

$$\tilde{K}^{**}_{n,\psi}(t-x;x) = \frac{-1}{2n} + \sqrt{\frac{(1+2M_{1,\psi})^2 - 4M_{2,\psi}}{4n^2} + x^2} - x;$$

$$\tilde{K}^{**}_{n,\psi}((t-x)^2;x) \quad = \quad 2x^2 - 2x\left(\frac{-1}{2n} + \sqrt{\frac{(1+2M_{1,\psi})^2 - 4M_{2,\psi}}{4n^2} + x^2}\right);$$

**Theorem 18.** *$\tilde{K}^{**}_{n,\psi}(f;x)$ is uniformly convergent to $f$ on the interval $\sqrt{M_{2,\psi}} \le x < A$ such that $0 \le r^{**}_n(x) < A$, provided that $f \in E_m$, $m \ge 2$.*

**Theorem 19.** *Let $f \in C_B[0,\infty)$. Then for all $\sqrt{M_{2,\psi}} \le x < \infty$, we have,*

$$\left|\tilde{K}^{**}_{n,\psi}(f;x) - f(x)\right| \le 2\omega\left(f;\sqrt{\tilde{K}^{**}_{n,\psi}((t-x)^2}\right).$$

*Proof.* Proof can be obtained in a similar way of Theorem 15 □

Now, for $f \in C_B[0,\infty)$, the order of approximation of $\tilde{K}^{**}_{n,\psi}(f;x)$ to $f$, will be at least as good as that of the $K_{n,\psi}(f;x)$ if

$$\tilde{K}^{**}_{n,\psi}((t-x)^2;x) \quad \le \quad K_{n,\psi}((t-x)^2;x). \tag{29}$$

Let $q_n(x) = \sqrt{\frac{(1+2M_{1,\psi})^2-4M_{2,\psi}}{4n^2} + x^2}$. Using (29) and (8), we get

$$2x\Big(x - q_n(x)\Big) \le \frac{M_{2,\psi}}{n^2} \tag{30}$$

Since $x \le q_n(x)$ for all $x \in [0,\infty), n \in \mathbb{N}$, the inequality (30) is true for all $x \in [0,\infty)$. Thus the order of approximation by $\tilde{K}^{**}_{n,\psi}(f;x)$ to $f$ is at least as good as the order of approximation to $f$ by $K_{n,\psi}(f;x)$ whenever $\sqrt{M_{2,\psi}} \le x \le \infty$. Finally, for $\alpha = 0.01$ and $a = 0.75$, we give graphical illustration of the inequality (29) as follows (see Figure 6).

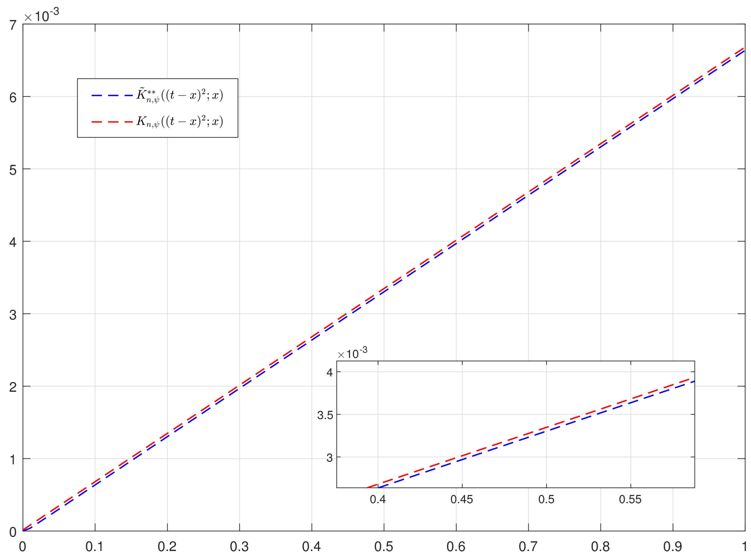


Figure 6: Graphical representations of $\tilde{K}^{**}_{n,\psi}((t-x)^2;x)$ and $K_{n,\psi}((t-x)^2;x)$ for $\alpha = 0.01$, $a = 0.75$ and $n = 150$.

## 7. Conclusion

In this paper, we introduce a new family of Szász-Mirakjan Kantorovich type operators $K_{n,\psi}(f;x)$, which depend on a function $\psi$ satisfying conditions given in (7). In this way we obtain all moments and central moments of the new operators in terms of two numbers $M_{1,\psi}$ and $M_{2,\psi}$, integrals of $\psi(t)$ and $\psi^2(t)$ on $[0,1]$ respectively. This is a new approach to have better error estimation because in the case of $K_{n,\psi}(1;x) = 1$, the order of approximation to a function $f$ by an operator $K_{n,\psi}(f;x)$ is more controlled by the term $K_{n,\psi}((t-x)^2;x)$. Since different functions $\psi$ gives different values for $M_{1,\psi}$ and $M_{2,\psi}$, it is possible to search for different values of $M_{1,\psi}$ and $M_{2,\psi}$ to make $K_{n,\psi}((t-x)^2;x)$ smaller than $K_n((t-x)^2;x)$. In other words in this approach two problems have arisen. Are there $M_{1,\psi}$ and $M_{2,\psi}$ values so that $K_{n,\psi}((t-x)^2;x) < K_n((t-x)^2;x)$ and is there a function $\psi(t)$ which have these $M_{1,\psi}$ and $M_{2,\psi}$ values? In this study, we prove that both problems have affirmative solutions. Moreover, in this paper we also introduced two new King-Type generalization of our new operators, $\tilde{K}^*_{n,\psi}(f;x)$ preserving $x$ and $\tilde{K}^{**}_{n,\psi}(f;x)$ preserving $x^2$. We also show that $\tilde{K}^*_{n,\psi}(f;x)$ and $\tilde{K}^{**}_{n,\psi}(f;x)$ has better approximation results than $K_{n,\psi}(f;x)$ and $K_n(f;x)$ (see Figure 7). Some results on the uniform convergence and rate of convergence of new operators are also obtained in terms of the first and second order modulus of continuity and it is also shown that, our operators has shape preserving properties. Finally, obtained results are supported by numerical examples.

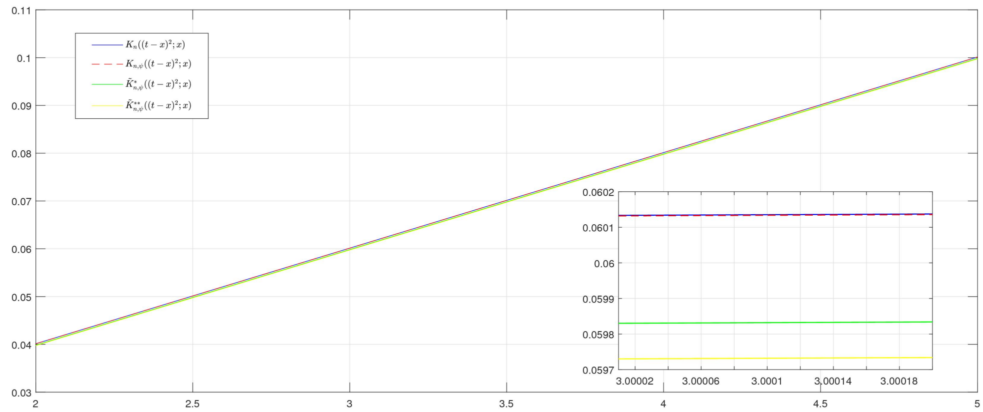

Figure 7: Graphical representations of $K_n((t-x)^2;x)$, $K_{n,\psi}((t-x)^2;x)$, $\tilde{K}^*_{n,\psi}((t-x)^2;x)$ and $\tilde{K}^{**}_{n,\psi}((t-x)^2;x)$ for $\alpha = 1.1$, $a = 0.75$ and $n = 50$.

| $x$ | $K_{50}\left((t-x)^2;x\right)$ | $K_{50,\psi}\left((t-x)^2;x\right)$ | $\tilde{K}^*_{50,\psi}\left((t-x)^2;x\right)$ | $\tilde{K}^{**}_{50,\psi}((t-x)^2;x)$ |
|---|---|---|---|---|
| 2 | 0.0401333 | 0.0401317 | 0.0398300 | 0.0397300 |
| 2.5 | 0.0501333 | 0.0501317 | 0.0498300 | 0.0497300 |
| 3 | 0.0601333 | 0.0601317 | 0.0598300 | 0.0597300 |
| 3.5 | 0.0701333 | 0.0701317 | 0.0698300 | 0.0697300 |
| 4 | 0.0801333 | 0.0801317 | 0.0798300 | 0.0797300 |
| 4.5 | 0.0901333 | 0.0901317 | 0.0898300 | 0.0897300 |
| 5 | 0.1001333 | 0.1001317 | 0.0998300 | 0.0997300 |

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