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Hermite-Fejér interpolation with non-uniform nodes on the unit circle

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Abstract. This research paper examines Hermite-Fejér interpolation on the unit circle with non-uniformly distributed nodes. The authors make two main contributions: they provide an explicit formula for the interpolatory polynomial and prove a mean convergence of it is studied for analytic function within the unit disk. These findings advance our understanding of this interpolation method for irregularly spaced data points in circular domain.

1. Introduction

Polynomial interpolation, particularly the behavior of continuous functions approximated by Hermite polynomials, has been a fundamental area of study in numerical analysis and approximation theory for over a century [12, 20]. This field's importance stems from its wide-ranging applications in mathematics, physics, engineering, and computational sciences. The cornerstone of this area is Weierstrass's approximation theorem from 1885 [27], which proves that every continuous function on a closed interval $[a, b]$ can be uniformly approximated by polynomial functions to any degree of accuracy. Hermite interpolation, a more sophisticated form of polynomial interpolation, involves fitting a polynomial to a set of data points while also matching derivative values at these points [10, 17]. This method provides a powerful tool for approximating functions with higher degrees of smoothness and has found extensive use in computer-aided geometric design, signal processing, and numerical solutions of differential equations [2]. The choice of nodal systems plays a crucial role in the behavior and convergence properties of interpolation polynomials. Jacobi polynomials, a class of classical orthogonal polynomials, have been particularly influential in this regard [1, 24]. The zeros of Jacobi polynomials and their variants often serve as optimal interpolation points, leading to improved stability and accuracy in many applications [15]. Notable contributions to the field include the work of T. N. T. Goodman, K.G. Ivanov, and A. Sharma [16], who studied Hermite interpolant behavior using roots of unity. This research provided insights into the convergence properties of interpolation polynomials on the complex plane. S. Bahadur's investigation of Pál type interpolation problems using nodal systems derived from Legendre polynomials (a special case of Jacobi polynomials) further expanded our understanding of interpolation schemes based on orthogonal polynomials [4]. Hermite-Fejér interpolation, a special case of Hermite interpolation where all derivatives at the nodes are set to zero, has

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been extensively studied due to its simpler formulation and interesting convergence properties [13, 23]. The behavior of Hermite-Fejér interpolation polynomials on different nodal systems has been a subject of ongoing research, with particular attention paid to their convergence rates and stability [19]. The study of interpolation with equally spaced nodes on the unit circle has led to important results in complex analysis and approximation theory [25, 26]. However, equally spaced nodes often lead to the Runge phenomenon, where oscillations occur near the endpoints of the interval, highlighting the importance of careful node selection in polynomial interpolation [11, 21]. The study of interpolation theory on the unit circle has seen significant developments through Bahadur's foundational work [5] and subsequent research. Recent advances include novel approaches to Hermite interpolation with non-uniform nodes [3, 6], investigations into Hermite-Fejér interpolation [7, 8], and extensions to higher-order methods [9]. These contributions have substantially enhanced our understanding of interpolation processes on circular domains. Mean square convergence of interpolatory polynomials for analytic functions on the unit disk has been a significant area of study [18, 22]. This approach provides a measure of the overall accuracy of the approximation and is particularly relevant for applications in signal processing and spectral methods [14].
By projecting the zeros of Jacobi polynomial with origin and boundary points on the unit circle, nodal points are being obtained in the present paper. The study examined three distinct nodal structures and determined the rate of convergence for each. The authors then compared these three cases and drew significant conclusions from their analysis. Their findings provided valuable insights into the behavior of interpolation methods on the unit circle.

2. Problem

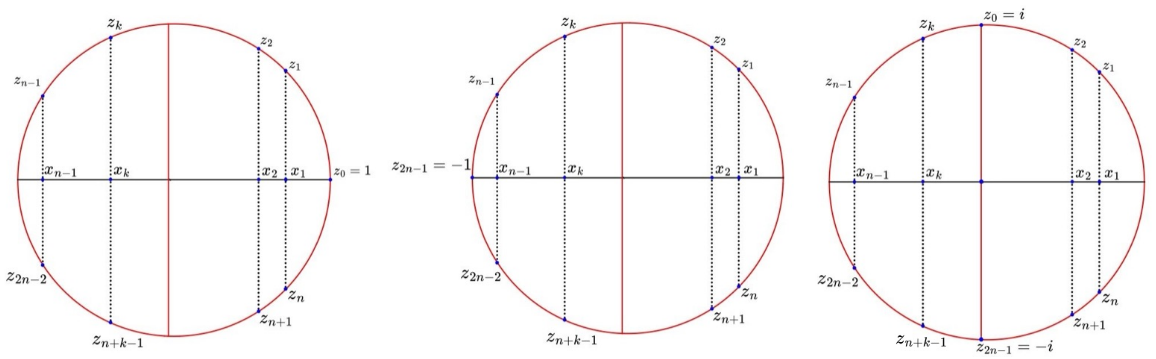

Figure 1: Nodal System for First, Second and Third cases respectively

Let's consider the interpolatory polynomial and its convergence properties for three cases involving nodes derived from Jacobi polynomials by the Szegő's transformation [24]. In the first case, as given, the nodes $\{z_k\}_{k=0}^{2n-2}$ ($z_k \neq 0$) are obtained by vertically projecting the zeros of $(1-x)P_{n-1}^{(\alpha,\beta)}(x)$ onto the unit circle, producing $2n-1$ nodes on the unit circle. For the second case, we examine nodes resulting from the projection of zeros of $(1+x)P_{n-1}^{(\alpha,\beta)}(x)$, again yielding $2n-1$ points on the unit circle. The third case considers nodes from the zeros of $xP_{n-1}^{(\alpha,\beta)}(x)$, producing $2n$ points due to the additional zeros at $x=0$ (see Figure 1). In each scenario, the convergence of interpolatory polynomials depends on factors such as the distribution of nodes (influenced by the Jacobi polynomial parameters α and β), the smoothness of the interpolated function, and its behavior near ± 1 on the unit circle. The specific multiplying factor ($(1-x)$, $(1+x)$, or x) affects the node distribution and consequently the convergence properties. A rigorous analysis of convergence would require techniques from approximation theory, considering the interplay between the Jacobi polynomial properties and the characteristics of the interpolated function in each case.

Case I: Consider an interpolatory polynomial $\mathbb{H}_n(z)$ of degree $\leq 4n-3$ satisfying the following conditions

$$\begin{cases} \mathbb{H}_n(z_k) = \alpha_k; & k = 0(1)2n-2, \\ \mathbb{H}_n'(z_k) = 0; & k = 0(1)2n-2, \end{cases} \tag{1}$$

where α_k are arbitrary complex constants. (For this case $z_0 = 1$).

Case II: Consider an interpolatory polynomial $\mathbb{I}_n(z)$ of degree $\leq 4n-3$ satisfying the following conditions

$$\begin{cases} \mathbb{I}_n(z_k) = \beta_k; & k = 1(1)2n-1, \\ \mathbb{I}_n'(z_k) = 0; & k = 1(1)2n-1, \end{cases} \tag{2}$$

where β_k are arbitrary complex constants. (For this case $z_{2n-1} = -1$).

Case III: Consider an interpolatory polynomial $\mathbb{J}_n(z)$ of degree $\leq 4n-1$ satisfying the following conditions

$$\begin{cases} \mathbb{J}_n(z_k) = \gamma_k; & k = 0(1)2n-1, \\ \mathbb{J}_n'(z_k) = 0; & k = 0(1)2n-1, \end{cases} \tag{3}$$

where γ_k are arbitrary complex constants. (For this case $z_0 = i$ and $z_{2n-1} = -i$).

3. Preliminaries

This section includes the following results, which we shall use.

The differential equation satisfied by $P_{n-1}^{(\alpha,\beta)}(x)$ is,

$$(1-x^2)P_{n-1}^{(\alpha,\beta)''}(x) + [\beta - \alpha - (\alpha+\beta+2)x]P_{n-1}^{(\alpha,\beta)'}(x) + (n-1)(n+\alpha+\beta)P_{n-1}^{(\alpha,\beta)}(x) = 0, \tag{4}$$

where $x = \frac{1+z^2}{2z}$ (Szegő's transformation).

$$\mathscr{W}(z) = K_{n-1}P_{n-1}^{(\alpha,\beta)}\left(\frac{1+z^2}{2z}\right)z^{n-1} = \prod_{k=1}^{2n-2}(z-z_k), \tag{5}$$

$$K_{n-1} = 2^{2n-2}(n-1)!\frac{\Gamma(\alpha+\beta+n-1)}{\Gamma(\alpha+\beta+2n-3)},$$

$$\mathscr{R}(z) = (z-1)\prod_{k=1}^{2n-2}(z-z_k), \tag{6}$$

$$\mathscr{S}(z) = (z+1)\prod_{k=1}^{2n-2}(z-z_k), \tag{7}$$

$$\mathscr{T}(z) = (z^2+1)\prod_{k=0}^{2n-2}(z-z_k). \tag{8}$$

For $-1 \le x \le 1$ and $\alpha \ge \beta$

$$\mid P_{n-1}^{(\alpha,\beta)}(x) \mid = O(n^{\alpha}), \tag{9}$$

$$\sqrt{1-x^2} \mid P_{n-1}^{(\alpha,\beta)}(x) \mid = O(n^{\alpha-1}). \tag{10}$$

Let $x_k = \cos\theta_k$, $k = 1(1)n-1$ be the zeros of $P_{n-1}^{(\alpha,\beta)}(x)$, then

$$(1-x_k^2)^{-1} \sim \left(\frac{k}{n}\right)^{-2}, \tag{11}$$

$$\mid P_{n-1}^{(\alpha,\beta)'}(x_k) \mid \sim k^{-\alpha-\frac{3}{2}} n^{\alpha+2}, \tag{12}$$

$$\mid P_{n-1}^{(\alpha,\beta)'}(x_k) \mid \sim k^{-\alpha-\frac{5}{2}} n^{\alpha+4}. \tag{13}$$

4. Explicit Representation of Interpolatory Polynomial

Case I. We shall write $\mathbb{H}_n(z)$ satisfying (1)

$$\mathbb{H}_n(z) = \sum_{k=0}^{2n-2} \alpha_k \mathscr{A}_k(z), \tag{14}$$

where $\mathscr{A}_k(z)$ is unique polynomial of degree $\le 4n-3$.

For $k = 0(1)2n-2$

$$\begin{cases} \mathscr{A}_k(z_j) = \delta_{kj}; & j = 0(1)2n-2, \\ \mathscr{A}_k'(z_j) = 0; & j = 0(1)2n-2. \end{cases} \tag{15}$$

Case II. We shall write $\mathbb{I}_n(z)$ satisfying (2)

$$\mathbb{I}_n(z) = \sum_{k=1}^{2n-1} \beta_k \mathscr{B}_k(z), \tag{16}$$

where $\mathscr{B}_k(z)$ is unique polynomial of degree $\le 4n-3$.

For $k = 1(1)2n-1$

$$\begin{cases} \mathscr{B}_k(z_j) = \delta_{kj}; & j = 1(1)2n-1, \\ \mathscr{B}_k'(z_j) = 0; & j = 1(1)2n-1. \end{cases} \tag{17}$$

Case III. We shall write $\mathbb{J}_n(z)$ satisfying (3)

$$\mathbb{J}_n(z) = \sum_{k=0}^{2n-1} \gamma_k \mathscr{C}_k(z), \tag{18}$$

where $\mathscr{C}_k(z)$ is unique polynomial of degree $\le 4n-1$.

For $k = 0(1)2n-1$

$$\begin{cases} \mathscr{C}_k(z_j) = \delta_{kj}; & j = 0(1)2n-1, \\ \mathscr{C}_k'(z_j) = 0; & j = 0(1)2n-1. \end{cases} \tag{19}$$

Theorem 4.1. *For* $k = 0$

$$\mathscr{A}_k(z) = \mathscr{L}_0^2(z) - 2\mathscr{L}_0'(1)\frac{(z-1)(\mathscr{W}(z))^2}{(\mathscr{W}(1))^2}, \tag{20}$$

for $k = 1(1)2n-2$

$$\mathscr{A}_k(z) = \mathscr{L}_k^2(z) - 2\mathscr{L}_k'(z_k)\frac{(z-1)\mathscr{W}(z)\mathscr{L}_k(z)}{(z_k-1)\mathscr{W}'(z_k)}, \tag{21}$$

where $\mathscr{L}_k(z)$ *is fundamental polynomial of Lagrange interpolation of the zeros of* $\mathscr{R}(z)$ *and defined as*

$$\mathscr{L}_k(z) = \frac{\mathscr{R}(z)}{(z-z_k)\mathscr{R}'(z_k)} \qquad ,k = 0(1)2n-2 \qquad (z_0 = 1). \tag{22}$$

Proof. For k=1(1)2n-2, let

$$\begin{aligned} \mathscr{A}_k(z) =& \mathscr{L}_k^2(z) + a_k\frac{(z-1)\mathscr{W}(z)\mathscr{L}_k(z)}{(z_k-1)\mathscr{W}'(z_k)}, \\ \mathscr{A}_k(z_j) =& \delta_{kj}. \end{aligned} \tag{23}$$

Differentiating (23) equation and putting $z = z_k$, we get

$$\begin{aligned} \mathscr{A}_k'(z_k) =& 2\mathscr{L}_k(z_k)\mathscr{L}_k'(z_k) + a_k, \\ a_k =& -2\mathscr{L}_k'(z_k). \end{aligned}$$

Similarly, one can obtain (20).
Hence, the theorem follows. □

Theorem 4.2. *For* $k = 2n-1$

$$\mathscr{B}_k(z) = \mathscr{M}_{2n-1}^2(z) - 2\mathscr{M}_{2n-1}'(1)\frac{(z+1)(\mathscr{W}(z))^2}{(\mathscr{W}(1))^2}, \tag{24}$$

for $k = 1(1)2n-2$

$$\mathscr{B}_k(z) = \mathscr{M}_k^2(z) - 2\mathscr{M}_k'(z)\frac{(z+1)\mathscr{W}(z)\mathscr{M}_k(z)}{(z_k+1)\mathscr{W}'(z_k)}, \tag{25}$$

where $\mathscr{M}_k(z)$ *is fundamental polynomial of Lagrange interpolation of the zeros of* $\mathscr{S}(z)$ *and defined as*

$$\mathscr{M}_k(z) = \frac{\mathscr{S}(z)}{(z-z_k)\mathscr{S}'(z_k)}, \qquad k = 1(1)2n-1 \qquad (z_{2n-1} = -1) \tag{26}$$

Proof. For k=1(1)2n-2, let

$$\begin{aligned} \mathscr{B}_k(z) =& \mathscr{M}_k^2(z) + b_k\frac{(z+1)\mathscr{W}(z)\mathscr{M}_k(z)}{(z_k+1)\mathscr{W}'(z_k)}, \\ \mathscr{B}_k(z_j) =& \delta_{kj}. \end{aligned} \tag{27}$$

Differentiating (27) equation and putting $z = z_k$, we get

$$\begin{aligned} \mathscr{B}_k'(z_k) =& 2\mathscr{M}_k(z_k)\mathscr{M}_k'(z_k) + b_k, \\ b_k =& -2\mathscr{M}_k'(z_k). \end{aligned}$$

Similarly, one can obtain (24).
Hence, the theorem follows. □

Theorem 4.3. *For $k = 0(1)2n-1$*

$$\mathscr{C}_k(z) = \mathscr{N}_k^2(z) - 2\frac{\mathscr{N}_k'(z_k)}{\mathscr{T}_k'(z_k)}\mathscr{N}_k(z)\mathscr{T}_k(z), \tag{28}$$

where $\mathscr{N}_k(z)$ is fundamental polynomial of Lagrange interpolation of the zeros of $\mathscr{T}(z)$ and defined as

$$\mathscr{N}_k(z) = \frac{\mathscr{T}(z)}{(z-z_k)\mathscr{T}'(z_k)} \qquad ,k = 0(1)2n-1. \qquad (z_0 = i,\ z_{2n-1} = -i) \tag{29}$$

Proof. For k=0(1)2n-1, let

$$\mathscr{C}_k(z) = \mathscr{N}_k^2(z) + c_k\mathscr{T}_k(z)\mathscr{N}_k(z). \tag{30}$$

Differentiating above equation and putting $z = z_k$, we get

$$\begin{aligned}\mathscr{C}_k'(z_k) =& 2\mathscr{N}_k(z_k)\mathscr{N}_k'(z_k) + c_k\mathscr{T}_k'(z_k),\\ c_k =& -2\frac{\mathscr{N}_k'(z_k)}{\mathscr{T}_k'(z_k)}.\end{aligned}$$

Hence, the theorem follows. □

5. Estimates of Fundament Polynomials

We need to calculate estimates in order to obtain the order of convergence of interpolatory polynomials.

Lemma 5.1. *Let $\mathscr{L}_k(z)$ be given by* (22) *, then*

$$|\ \mathscr{L}_k(z)\ | \le \left(\frac{C_1}{k^{-\alpha+3/2}}\right), \tag{31}$$

where C_1 is arbitrary constant independent of n and z.

Lemma 5.2. *Let $\mathscr{A}_k(z)$ be given by theorem 4.1, then*

$$|\ \mathscr{A}_k(z)\ | \le \left(\frac{C_2 n^2}{k^{-2\alpha+2}}\right); \quad for \quad k = 1(1)2n-2, \tag{32}$$

$$|\ \mathscr{A}_k(z)\ | \le C_3; \quad for \quad k = 0, \tag{33}$$

where C_2 and C_3 are arbitrary constants independent of n and z.

Proof of the lemmas

Proof. [Lemma 5.1] Consider (22)

$$\begin{aligned}|\ \mathscr{L}_k(z)\ | \quad &= \quad \left|\frac{\mathscr{R}(z)}{\mathscr{R}'(z_k)(z-z_k)}\right|, &(34)\\ &= \quad \left|\frac{\mathscr{W}(z)(z-1)}{(z-z_k)(z_k-1)\mathscr{W}'(z_k)}\right|. &(35)\end{aligned}$$

Using (5), we have

$$|\ \mathscr{L}_k(z)\ | = \frac{2|z-1|\,|P_n^{(\alpha,\beta)}(x)|\,|z|^{n-1}}{|z-z_k|\,|z_k-1|\,|z_k^2-1|\,|P_n'^{(\alpha,\beta)}(x_k)|\,|z_k|^{n-3}}.$$

We can write $z = x + iy$, where $x, y \in (-1, 1)$. If $|z| = 1$, then

$$
\begin{aligned}
&\leq \quad \frac{\sqrt{1-x}\,\sqrt{1-xx_k}|P_n^{(\alpha,\beta)}(x)|}{|(x-x_k)|\,\sqrt{1-x_k}\,\sqrt{1-x_k^2}|P_n'^{(\alpha,\beta)}(x_k)|},\\
&\leq \quad \frac{\sqrt{1-x^2}\,\sqrt{1+x_k}\,\sqrt{1-xx_k}|P_n^{(\alpha,\beta)}(x)|}{|(x-x_k)|\,|\sqrt{1+x}|\,|(1-x_k^2)|\,|P_n'^{(\alpha,\beta)}(x_k)|}.
\end{aligned}
$$

Let us consider $|\ x - x_k\ | \geq \sqrt{1-x_k^2}$ and using (10), (11) and (12), we have

$$
|\ \mathscr{L}_k(z)\ | \leq \left(\frac{C_1}{k^{-\alpha+3/2}}\right), \tag{36}
$$

where C_1 is arbitrary constant independent of n and z.
Similarly we have estimates for condition $|\ x - x_k\ | < \sqrt{1-x_k^2}$.
Hence, the lemma follows. □

Proof. [Lemma (5.2)] For $k = 1(1)2n-2$

$$
\begin{aligned}
&\mathscr{A}_k(z) = \mathscr{L}_k^2(z) - 2\mathscr{L}_k'(z_k)\frac{(z-1)\mathscr{W}(z)\mathscr{L}_k(z)}{(z_k-1)\mathscr{W}'(z_k)},\\
&|\ \mathscr{A}_k(z)\ | = I_1 + I_2,
\end{aligned}
\tag{37}
$$

where

$$
I_1 = |\ \mathscr{L}_k^2(z)\ | \quad \text{and} \quad I_2 = \left|2\mathscr{L}_k'(z_k)\frac{(z-1)\mathscr{W}(z)\mathscr{L}_k(z)}{(z_k-1)\mathscr{W}'(z_k)}\right|.
$$

Using (22) we get,

$$
\begin{aligned}
I_2 &= \left|\frac{(z-1)^2\mathscr{R}''(z_k)(\mathscr{W}(z))^2}{(z-z_k)(z_k-1)^3(\mathscr{W}'(z_k))^3}\right|,\\
I_2 &= \left|\frac{(z-1)^2\{(z_k-1)\mathscr{W}''(z_k)+2\mathscr{W}'(z_k)\}(\mathscr{W}(z))^2}{(z-z_k)(z_k-1)^3(\mathscr{W}'(z_k))^3}\right|.
\end{aligned}
$$

Using (5), we have

$$
I_2 = \left|\frac{\dfrac{(z-1)^2K_{n-1}^3P_{n-1}^{(\alpha,\beta)}(x)P_{n-1}'^{(\alpha,\beta)}(x_k)\,{z_k}^{n-4}z^{2n-2}\{(z_k-1)(\frac{(\alpha+\beta+2)(1+z_k^2)}{2}+(\alpha-\beta)z_k+1)}{+(z_k^2-1)((n-1)(z_k-1)+1)\}}}{(z-z_k)(z_k-1)^3\{\frac{K_{n-1}}{2}P_{n-1}'^{(\alpha,\beta)}(x_k)(z_k^2-1)z_k^{n-3}\}^3}\right|,
$$

$$
\begin{aligned}
I_2 = \Bigg|&\frac{8(z-1)^2(P_{n-1}^{(\alpha,\beta)}(x))^2z^{2n-2}z_k^{-2n+5}}{(z-z_k)(P_{n-1}'^{(\alpha,\beta)}(x_k))^2}\Bigg\{\frac{(\alpha+\beta+2)(1+z_k^2)+z_k(\alpha-\beta)+1}{(z_k-1)^2(z_k^2-1)^3}\\
&+\frac{n-1}{(z_k-1)^2(z_k^2-1)^2}+\frac{z_k}{(z_k-1)^3(z_k^2-1)^2}\Bigg\}\Bigg|.
\end{aligned}
$$

If $|z| = 1$, then

$$I_2 \leq \frac{8|(z-1)^2|\,|(P_{n-1}^{(\alpha,\beta)}(x))^2|}{|z-z_k|\,|(P_{n-1}'^{(\alpha,\beta)}(x_k))^2|}\left\{\frac{\{|(\alpha+\beta+2)|.2+|(\alpha-\beta)|+1\}|(1+z_k)^2|}{|(z_k^2-1)^5|}\right.$$
$$\left.+\frac{|n-1|\,|(1+z_k)^2|}{|(z_k^2-1)^4|}+\frac{|(1+z_k)^3|}{|(z_k^2-1)^5|}\right\}.$$

We can write $z = x + iy$, where $x, y \in (-1, 1)$.

$$I_2 \leq \frac{16(1-x)\,|(P_{n-1}^{(\alpha,\beta)}(x))^2|\,\sqrt{1-xx_k}}{|x-x_k|\,|(P_{n-1}'^{(\alpha,\beta)}(x_k))^2|}\left\{\frac{\{|(\alpha+\beta+2)|.2+|(\alpha-\beta)|+1\}}{8(1-x_k^2)^{5/2}|}+\right.$$
$$\left.\frac{|n-1|}{(1-x_k^2)^2}+\frac{1}{4(1-x_k^2)^{5/2}}\right\}.$$

Using (9), (11) and (12) and consider condition $|x-x_k| \geq \sqrt{1-x_k^2}$, we have

$$I_2 \leq c\frac{n^2}{k^{-2\alpha+2}}, \tag{38}$$

where c is constant independent of n and z.
From (31) and (38), (37) becomes

$$|\mathscr{A}_k(z)| \leq \frac{C_1^2}{k^{-2\alpha+3}} + c\frac{n^2}{k^{-2\alpha+2}},$$

$$|\ \mathscr{A}_k(z)\ | \leq C_2\frac{n^2}{k^{-2\alpha+2}}. \tag{39}$$

Similarly, for the condition $|x-x_k| < \sqrt{1-x_k^2}$ we have the same result as (39) and one can obtain estimates for k=0.
Hence, the lemma follows. □

Lemma 5.3. *Let $\mathscr{M}_k(z)$ be given by* (26) *, then*

$$|\ \mathscr{M}_k(z)\ | \leq \left(\frac{C_3}{k^{-\alpha+3/2}}\right), \tag{40}$$

where C_3 is arbitrary constant independent of n and z.

Lemma 5.4. *Let $\mathscr{B}_k(z)$ be given by Theorem 4.2, then*

$$|\ \mathscr{B}_k(z)\ | \leq \left(\frac{C_4}{k^{-2\alpha+2}}\right); \quad for \quad k = 1(1)2n-2, \tag{41}$$

$$|\ \mathscr{B}_k(z)\ | \leq C_5; \quad for \quad k = 0, \tag{42}$$

where C_4 and C_5 are arbitrary constants independent of n and z.

Proof of the lemmas
Follow the same procedure as in the proof of Lemma (5.1) and Lemma (5.2).

Lemma 5.5. *Let $\mathcal{N}_k(z)$ be given by (29), then*

$$| \mathcal{N}_k(z) | \leq \left(\frac{C_6}{k^{-\alpha+1/2}}\right), \tag{43}$$

where C_6 is a constant independent of n and z.

Lemma 5.6. *Let $\mathcal{C}_k(z)$ be given by (28), then*

$$| \mathcal{C}_k(z) | \leq \left(\frac{C_7}{k^{-2\alpha+1}}\right); \quad for \quad k = 0(1)2n-1, \tag{44}$$

where C_7 is a constant independent of n and z.

Proof of the lemmas

Proof. [Lemma 5.5] Follow the same procedure as in the proof of Lemma (5.1). □

Proof. [Lemma 5.6]

$$\mathcal{C}_k(z) = \mathcal{N}_k^2(z) - 2\frac{\mathcal{N}_k'(z_k)}{\mathcal{T}_k'(z_k)}\mathcal{N}_k(z)\mathcal{T}_k(z),$$

$$\begin{aligned}\mathcal{C}_k(z) =& \mathcal{N}_k^2(z)\{1-(z-z_k)\mathcal{N}_k'(z_k)\},\\ =& \mathcal{N}_k^2(z)\{1-(z-z_k)\frac{\mathcal{T}_k''(z_k)}{2\mathcal{T}_k'(z_k)}\},\\ =& \mathcal{N}_k^2(z)-(z-z_k)\mathcal{N}_k^2(z)\frac{\{(z_k^2+1)\mathcal{W}''(z_k)+4z_k\mathcal{W}'(z_k)\}}{2(z_k^2+1)\mathcal{W}'(z_k)}.\end{aligned} \tag{45}$$

Taking modulus on both sides,

$$|\mathcal{C}_k(z)| \leq |\mathcal{N}_k^2(z)|\{1+|\frac{\mathcal{W}'(z_k)}{\mathcal{W}'(z_k)}|+4|\frac{z_k}{(z_k^2+1)}|\}.$$

Using (5) and $|z_k| = 1$, we have,

$$|\mathcal{C}_k(z)| \leq |\mathcal{N}_k^2(z)|\{1+|\frac{P_n''^{(\alpha,\beta)}(x_k)|\,|z_k^2-1|}{|P_n'^{(\alpha,\beta)}(x_k)|}+\frac{1}{|z_k^2-1|}+n\}.$$

Using (11), (12), (13) and lemma (5.3), we have

$$|\mathcal{C}_k(z)| \leq C_7\frac{n}{k^{-2\alpha+1}}, \tag{46}$$

where C_7 is constant independent of n and z. □

6. Convergence

Remark 1:

Let $f(z)$ be continuous function and analytic function for $| z |\leq 1$ and $| z |< 1$, respectively and $f'' \in Lip\,\nu$, $\nu > 0$, then the sequence of interpolatory polynomial $\{\mathbb{H}_n(z)\}$ converges uniformly to $f(z)$ on $| z |\leq 1$ as

$$\omega_3(f,n^{-1}) \leq n^{-2}\omega_3(f'',n^{-1}) = \mathbf{O}(n^{-2-\nu}), \tag{47}$$

where $\omega_3(f, n^{-1})$ is third modulus of continuity of $f(z)$.

Remark 2: Let $f(z)$ be continuous function for $|z| \leq 1$ and analytic function for $|z| < 1$. Then, there exists a polynomial $F_n(z)$ of degree $\leq 4n-3$ satisfying **Jackson's** inequality

$$|f(z) - F_n(z)| \quad \leq \quad A\,\omega_3(f, n^{-1}), \tag{48}$$

where A is a constant independent of z and n.

Theorem 6.1. *Let $f(z)$ be continuous and analytic function on $|z| \leq 1$ and $|z| < 1$, respectively and let $\{\mathbb{H}_n(z)\}$ be sequence of interpolatory polynomial of degree at most $4n-3$ defined in (14),then*

$$\lim_{n\to\infty} \int_{|z|=1} |f(z) - \mathbb{H}_n(z)|^2 |dz| \ = \ 0 \qquad for -1 < \alpha \leq \frac{3}{4}. \tag{49}$$

Proof. Since $F_n(z)$ be the uniquely determined polynomial of degree $\leq 4n-3$ and the polynomial $F_n(z)$ can be expressed as

$$F_n(z) = \sum_{k=0}^{2n-2} F_n(z_k)\mathscr{A}_k(z). \tag{50}$$

Then

$$|\mathbb{H}_n(z) - f(z)| \leq |\mathbb{H}_n(z) - F_n(z)| + |F_n(z) - f(z)|,$$

$$\begin{aligned}
\int_{|z|=1} |f(z) - \mathbb{H}_n(z)|^2 |dz| &\leq 2\int_{|z|=1} |\mathbb{H}_n(z) - F_n(z)|^2 |dz| + 2\int_{|z|=1} |F_n(z) - f(z))|^2 |dz|, \\
&\leq 2\int_{|z|=1} \sum_{k=0}^{2n-2} |f(z_k) - F_n(z_k)|^2 |\mathscr{A}_k(z)|^2 |dz| + \\
&\quad 2\int_{|z|=1} |F_n(z) - f(z)|^2 ||dz|, \\
&\quad \leq 2(\omega_3(f, n^{-1}))^2 \{\int_{|z|=1} \sum_{k=0}^{2n-2} |\mathscr{A}_k(z)|^2 |dz| + \int_{|z|=1} |dz|\},
\end{aligned} \tag{51}$$

$$\leq 2(\omega_3(f, n^{-1}))^2 \left\{ \int_{|z|=1} \sum_{k=0}^{2n-2} |\mathscr{A}_k(z)|^2 |dz| + 2\pi \right\},$$

$$\leq 2(\omega_3(f, n^{-1}))^2 \left\{ \int_{|z|=1} \sum_{k=0}^{2n-2} \frac{C_2 n^4}{k^{-4\alpha+4}} |dz| + 2\pi \right\}.$$

Hence, we get the result for $-1 < \alpha \leq \frac{3}{4}$. □

Theorem 6.2. *Let $g(z)$ be continuous and analytic function on $|z| \leq 1$ and $|z| < 1$, respectively and let $\{\mathbb{I}_n(z)\}$ be sequence of interpolatory polynomial of degree atmost $4n-3$ defined in Theorem 4.2, then*

$$\lim_{n\to\infty} \int_{|z|=1} |g(z) - \mathbb{I}_n(z)|^2 |dz| \ = \ 0 \qquad for -1 < \alpha \leq \frac{3}{4}. \tag{52}$$

Proof of Theorem 6.2 can be obtained by using the same steps as those for Theorem 6.1.

Remark 3: Let $h(z)$ be continuous function and analytic function for $|z| \leq 1$ and $|z| < 1$, respectively. Then, there exists a polynomial $P_n(z)$ of degree less than $4n-1$ satisfying **Jackson's** inequality

$$|h(z) - P_n(z)| \quad \leq \quad B\,\omega(h, n^{-1}), \tag{53}$$

where $\omega(h, n^{-1})$ is modulus of continuity of $h(z)$ and B is a constant independent of z and n.

Theorem 6.3. *Let $h(z)$ be continuous and analytic function on $|z| \leq 1$ and $|z| < 1$, respectively and let $\{\mathbb{I}_n(z)\}$ be sequence of interpolatory polynomial of degree atmost $4n-1$ defined in Theorem 4.3, then*

$$\lim_{n\to\infty}\int_{|z|=1}|h(z)-\mathbb{J}_n(z)|^2|dz| \;=\; 0 \qquad for-1<\alpha\leq\frac{1}{4}. \tag{54}$$

Proof. Since $P_n(z)$ be the uniquely determined polynomial of degree $\leq 4n-1$ and the polynomial $P_n(z)$ can be expressed as

$$P_n(z)=\sum_{k=0}^{2n-1}P_n(z_k)\mathscr{A}_k(z). \tag{55}$$

Then

$$|\ \mathbb{I}_n(z)-h(z)\ |\leq|\ \mathbb{I}_n(z)-P_n(z)\ |+|\ P_n(z)-h(z)\ |,$$

$$\begin{aligned}
\int_{|z|=1}|h(z)-\mathbb{I}_n(z)|^2|dz| &\leq 2\int_{|z|=1}|\mathbb{I}_n(z)-P_n(z)|^2|dz|+2\int_{|z|=1}|P_n(z)-h(z))|^2|dz|,\\
&\leq 2\int_{|z|=1}\sum_{k=0}^{2n-1}|h(z_k)-P_n(z_k)|^2|\mathscr{C}_k(z)|^2|dz|+2\int_{|z|=1}|P_n(z)-h(z)|^2||dz|,\\
&\leq 2(\omega(h,n^{-1}))^2\int_{|z|=1}\sum_{k=0}^{2n-1}|\mathscr{C}_k(z)|^2|dz|+2(\omega(h,n^{-1}))^2\int_{|z|=1}|dz|,\\
&\leq 2(\omega(h,n^{-1}))^2\left\{\int_{|z|=1}\sum_{k=0}^{2n-1}|\mathscr{C}_k(z)|^2|dz|+2\pi\right\},\\
&\leq 2(\omega(h,n^{-1}))^2\left\{\int_{|z|=1}\sum_{k=0}^{2n-1}\frac{C_6^2}{k^{-4\alpha+2}}|dz|+2\pi\right\}.
\end{aligned} \tag{56}$$

Hence, we get the result for $-1<\alpha\leq\frac{1}{4}$. □

7. Numerical Example

To understand this research article we have given some numerical examples in this section. Hermite-Fejér interpolatory polynomial that interpolates test function is plotted by dotted curve in orange colour. Here we have taken Jacobi's polynomial parameter as $\alpha=0$ and $\beta=0$.

Example 1. Consider a function $f_1(z)$ defined as:

$$f_1(z)=-z+\frac{z^2}{2}-\frac{4z^3}{21}+\frac{z^4}{7}-\frac{18z^5}{175}+\frac{3z^6}{35}-\frac{4z^7}{49}+\frac{z^8}{14}-\frac{z^9}{9}+\frac{z^{10}}{10}$$

is continuous function on the closed unit disk centered at the origin, analytic function within the open unit disk, and satisfies the conditions of Theorem 6.1.

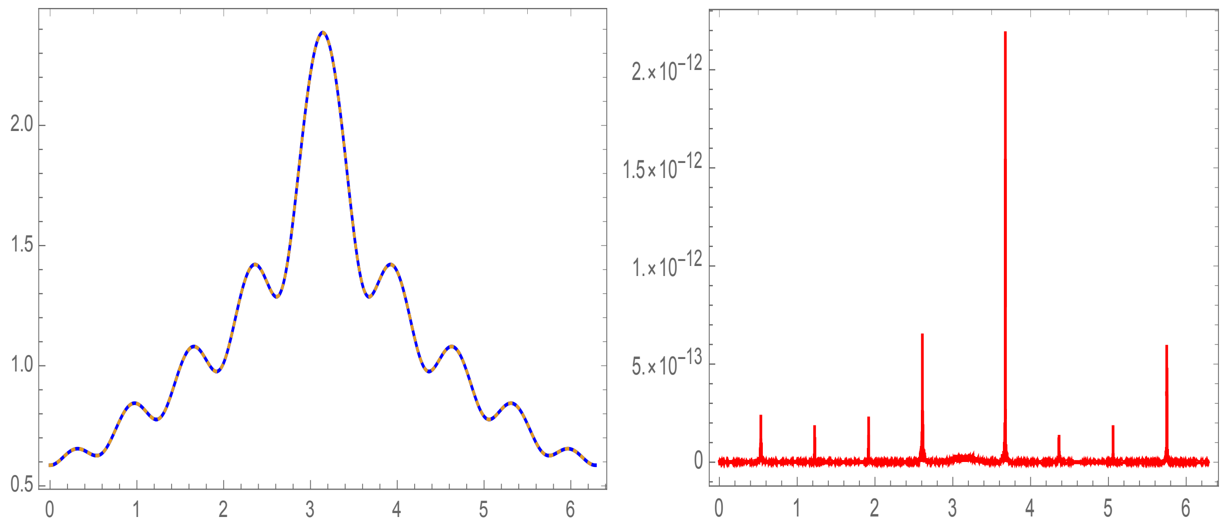

Figure 2: $|f_1(e^{i\theta})|$ and $|\mathbb{H}_4(z)|$; $|\mathbb{H}_4(z) - f_1(z)|$ for $\theta \in [0, 2\pi)$ and $n = 4$

The Figure 2 is plotted for the function $f_1(z)$. The observation are made from Figure 2 for $n = 4$. The interpolatory polynomial $\mathbb{H}_4(z)$ provides an approximate representation of test function $f_1(z)$. The left side in Figure 2 shows the absolute value of test function along with the absolute value of interpolatory polynomial $\mathbb{H}_4(z)$. Test function and interpolatory polynomial both are virtually indistinguishable. Right side of Figure 2 illustrates the absolute value of error. For $n = 4$, corresponding to 9 nodes, the maximum error observed is of order 2×10^{-12}.

Example 2. Consider a function $f_2(z)$ defined as:

$$f_2(z) = \cosh\left\{0.1\left(-z + \frac{z^2}{2} - \frac{2z^3}{9} + \frac{z^4}{6} - \frac{z^5}{5} + \frac{z^6}{6}\right)\right\}$$

is continuous function on the closed unit disk centered at the origin, analytic function within the open unit disk, and satisfies the conditions of Theorem 6.1.

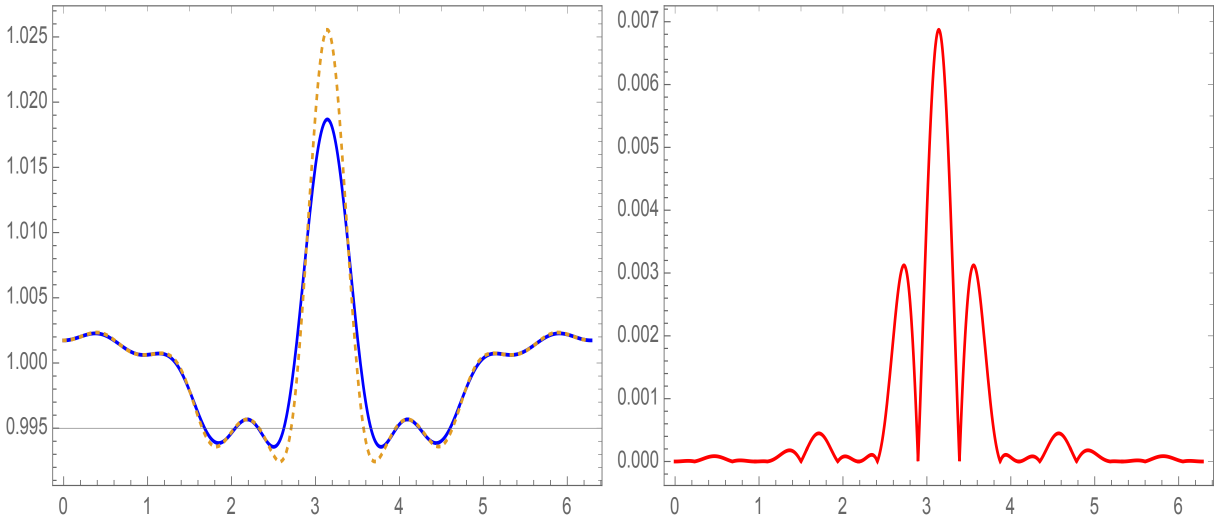

Figure 3: $|f_2(e^{i\theta})|$ and $|\mathbb{H}_2(z)|$; $|\mathbb{H}_2(z) - f_2(z)|$ for $\theta \in [0, 2\pi)$ and $n = 2$

Figure 3 illustrates the behavior of the function $f_2(z)$ and its interpolatory polynomial $\mathbb{H}_2(z)$ for $n = 2$ using 9 nodes. The left panel shows that $|f_2(z)|$ and $|\mathbb{H}_2(z)|$ are nearly identical, while the right panel depicts the error, with a maximum value of approximately 0.007.

Example 3. Consider a function $g_1(z)$ is defined as:

$$g_1(z) = z + \frac{z^2}{2} + \frac{4z^3}{21} + \frac{z^4}{7} + \frac{18z^5}{175} + \frac{3z^6}{35} + \frac{4z^7}{49} + \frac{z^8}{14} + \frac{z^9}{9} + \frac{z^{10}}{10}$$

is continuous and analytic on $|z| \leq 1$ and $|z| < 1$, respectively and satisfies the conditions of Theorem 6.2. The Figure 4 is plotted for function $g_1(z)$. Figure 4 ($n = 4$) demonstrates that the interpolatory polynomial $\mathbb{I}_4(z)$ closely approximates the test function $g_1(z)$. Figure 4's left panel shows their indistinguishable absolute values, while the right panel reveals the error distribution. With 9 nodes ($n = 4$), the maximum error is approximately 1.2×10^{-12}, indicating high accuracy.

Example 4. Consider a function $g_2(z)$ is defined as:

$$g_2(z) = \sin\left\{0.1\left(z + \frac{z^2}{2} + \frac{2z^3}{9} + \frac{z^4}{6} + \frac{z^5}{5} + \frac{z^6}{6}\right)\right\}$$

is continuous and analytic on $|z| \leq 1$ and $|z| < 1$, respectively and satisfies the conditions of Theorem 6.2. Figure 5 illustrates the function $g_2(z)$ and its interpolatory polynomial $\mathbb{I}_2(z)$. The left panel shows their nearly identical absolute values, while the right panel depicts the error distribution. With $n = 2$ and 5 nodes, the maximum error is about 1.5×10^{-3}, highlighting excellent accuracy.

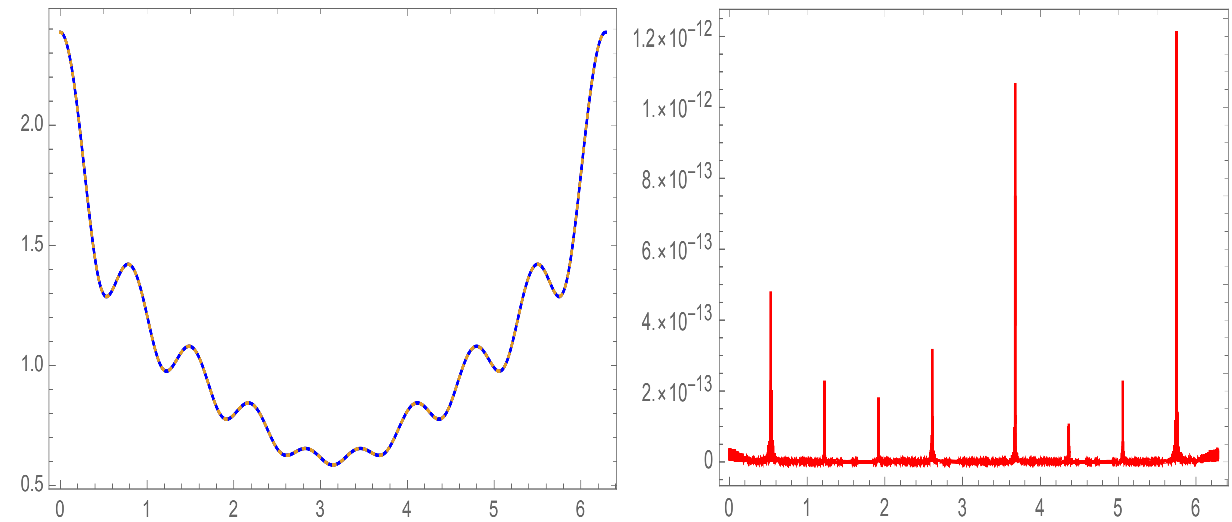

Figure 4: $|g_1(e^{i\theta})|$ and $|\mathbb{I}_4(z)|$; $|\mathbb{I}_4(z) - g_1(z)|$ for $\theta \in [0, 2\pi)$ and $n = 4$

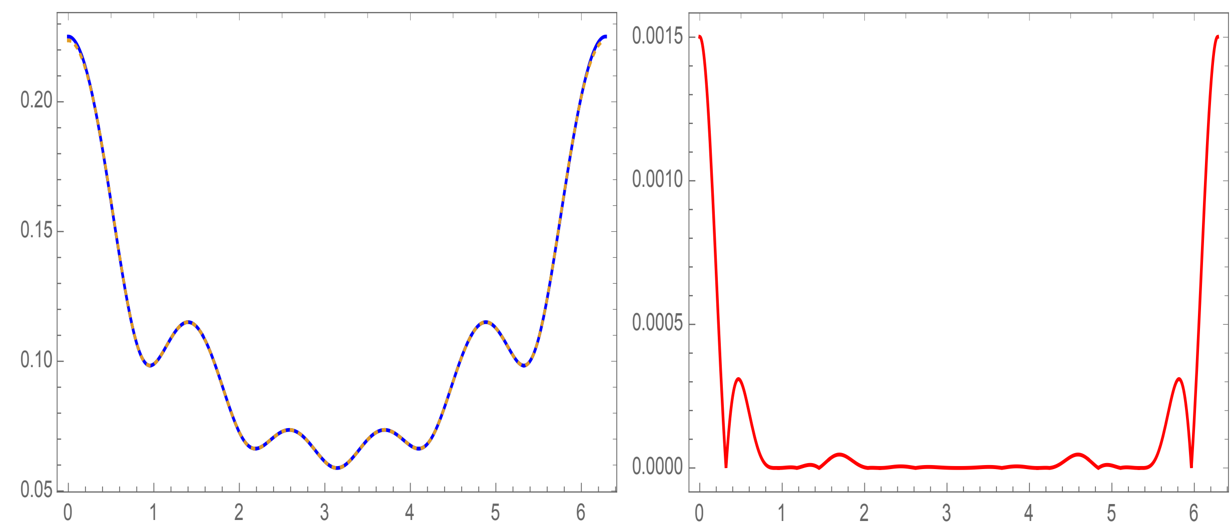

Figure 5: $|g_2(e^{i\theta})|$ and $|\mathbb{I}_2(z)|$; $|\mathbb{I}_2(z) - g_2(z)|$ for $\theta \in [0, 2\pi)$ and $n = 2$

Example 5. Consider a function $h(z)$ defined as:

$$h_1(z) = 1/35\left(35z + \frac{55z^3}{3} + \frac{38z^5}{5} + \frac{38z^7}{7} + \frac{55z^9}{9} + \frac{35z^{11}}{11}\right).$$

Clearly $h_1(z)$ is continuous function on the closed unit disk centered at the origin, analytic function within the open unit disk, and satisfies the conditions stated in Theorem 6.3.

The analysis of Figure 6, corresponding to $n = 4$, reveals that the interpolatory polynomial $\mathbb{J}_4(z)$ serves as an effective approximation of the test function $h_1(z)$. The left panel of Figure 6 presents a comparative visualization of the absolute values of both the test function and the interpolatory polynomial $\mathbb{J}_4(z)$. Notably, the two functions exhibit such a high degree of similarity that they are visually indistinguishable. The right panel of Figure 6 displays the absolute error distribution. For the case of $n = 4$, which corresponds to 10 nodal points, the maximum observed error is on the order of 3.5×10^{-13}, demonstrating a high level of accuracy in the approximation.

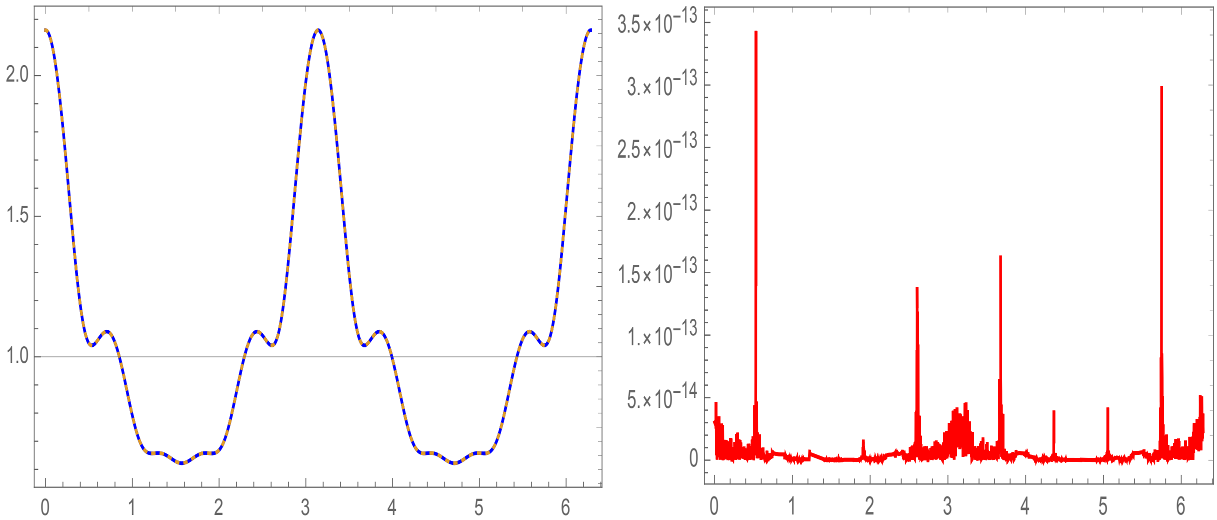

Figure 6: $|h_1(e^{i\theta})|$ and $|\mathbb{J}_4(z)|$; $|\mathbb{J}_4(z) - h_1(z)|$ for $\theta \in [0, 2\pi)$ and $n = 4$

Example 6. Consider a function $h_2(z)$ defined as:

$$h_2(z) = e^{0.1\left(z+\frac{17}{13}z^3+\frac{z^5}{5}+\frac{205}{2079}z^9+\frac{z^{11}}{11}+\frac{17}{14}z^{13}+\frac{z^{15}}{15}\right)},$$

$h_2(z)$ is continuous function on the closed unit disk centered at the origin, analytic function within the open unit disk, and satisfies the conditions stated in Theorem 6.3.

Figure 7 illustrates the performance of the interpolatory polynomial $\mathbb{J}_5(z)$ for $n = 5$. In the left panel, $\mathbb{J}_5(z)$ closely matches the test function $h_2(z)$, appearing visually identical. The right panel shows the absolute error, with a maximum of approximately 3×10^{-4} using 12 nodal points.

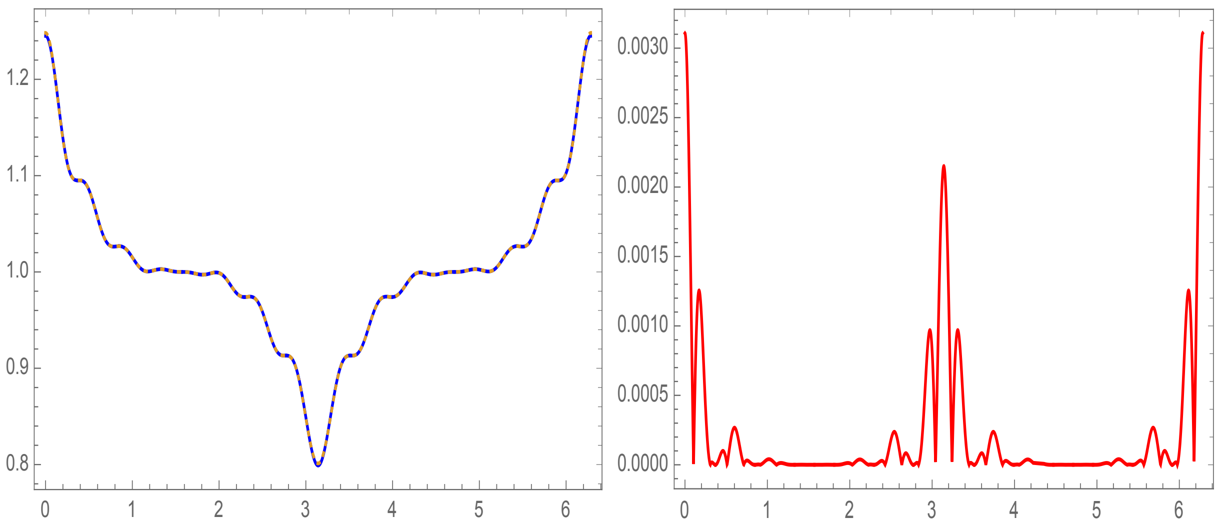

Figure 7: $|h_2(e^{i\theta})|$ and $|\mathbb{J}_5(z)|$; $|\mathbb{J}_5(z) - h_2(z)|$ for $\theta \in [0, 2\pi)$ and $n = 5$

8. Conclusion

The numerical findings from computational experiments corroborate the theoretical framework established throughout the study. These examples illustrate that Hermite-Fejér interpolatory polynomials, constructed using nodes derived from zeros of Jacobi polynomial, can accurately approximate complex functions that are analytic within the open unit disk ($|z| < 1$) and continuous on the closed unit disk ($|z| \leq 1$).

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