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On the existence of fixed points for set-valued mappings in partially ordered sets and its application to vector equilibrium problem

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Abstract. In this paper, the notions of upper order-preserving and lower order-preserving set-valued mappings are given. By using these notions, some fixed point theorems in the setting of partially ordered sets equipped with the hull-kernel topology for set-valued mappings are established. As an application of these theorems, an existence theorem for a solution of vector equilibrium problem is provided. The main results of this article can be viewed as the set-valued version of the main theorems given in [8, 12, 17, 18] with mild assumptions.

1. Introduction

Vector equilibrium problems, or VEP for short, are one of the most fascinating and actively researched categories of nonlinear issues. These comprise, as specific examples, fixed point problems, Nash equilibrium problems for vector-valued mappings, vector variational inequality problems, and vector optimization problems. Numerous studies on various facts of vector equilibrium issues have been published [1–4, 9, 11]. Certain specialized techniques have been devised for the solution of vector equilibrium problems in various spaces. KKM theory and fixed point theory are the approaches most frequently employed in the literature to investigate the existence of solutions of VEP (see, for example, [10] and the references therein). The aforementioned methods are used in normed spaces or, in a broader sense, spaces that require topological structures on the underlying space.

In order to investigate the possibility of solutions to VEP on chain-complete subsets of partially ordered sets with the hull-kernel topology, we aim to prove fixed point theorems. As they are crucial to the theory of semilattices, particularly distributive semilattices, various writers developed the ideas of filters and ideals of semilattices in order to study the existence theorems of fixed points. In order to apply them in fixed point theory in meet distributive semilattices, numerous writers have also examined certain properties of ideals and filters of a meet distributive semilattice [5–7, 13, 14, 16]. Some requirements for a semilattice to be a 0-distributive were covered in [15]; this class is crucial for the study of pseudo-complemented semilattices.

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The format of this article is as follows. We present the study's preliminary findings in Section 2, along with some definitions and preliminary findings that are necessary for the following part. A few fixed point theorems on partially ordered sets with the hull-kernel topology are demonstrated in Section 3. Ultimately, we investigate the possibility of a vector equilibrium issue solution by utilizing these fixed point theorems.

2. Preliminaries

Here, we go over a few definitions that are necessary for the sequel.

Let P be a convex pointed cone in a topological vector space Y, and let C be a nonempty subset of a partially ordered set X. The following is the formulation of the *vector equilibrium problem* for a mapping $f : C \times C \to Y$:
Find an $x^* \in C$ such that

$$f(x^*, y) \notin \mathrm{Int}P \ \forall\, y \in C, \qquad \text{(VEP)}$$

where $\mathrm{Int}P$ denotes the interior of P.

A partially ordered set is referred to as a meet semilattice (lattice), when any two components have a greatest lower bound and a least upper bound, but not always an upper limit. Stated differently, if $\inf\{x, y\}$ exists for every $x, y \in S$, then an ordered set $(S, \leq)$ is a *meet semilattice*; we write $x \wedge y$ in place of $\inf\{x, y\}$.

Remark 2.1. If S is a meet semilattice, then the following assertions, for any x, y, z in S hold:

(a) $x \wedge x = x$ (idempotent property);
(b) $x \wedge y = y \wedge x$ (commutative property);
(c) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ (dfistributive property).

A lattice S is called *complete* if each nonempty subset of S with an upper bound has the least upper bound. Also, a subset C of a partially set is *chain-complete* if for each chain g in C there exists the least upper bound of g (that is, $\bigvee g$) in C.

We recall that a mapping $G : Q \to W$ between partially ordered sets $(Q, \leq_1)$ and $(W, \leq_2)$ is said to be *order-preserving* if $G(q_1) \leq_2 G(q_2)$ whenever $q_1 \leq_1 q_2$.

A partially ordered set S has the *fixed point property* if every order-preserving mapping $f : S \to S$ has a fixed point, i.e. there exists $x \in S$ such that $f(x) = x$. A. Tarski's classical result (see [17]), stating that every complete lattice has the fixed point property, is based on the following two properties of a complete lattice P:

(i) For every order-preserving mapping $f : P \to P$ there exists $x \in P$ such that $x \leq f(x)$.

(ii) There are suprema of subsets of P; specifically, there is a supremum of the set $\{x : x \leq f(x)\} \subseteq P$.

This article's goal is to apply Tarski's result to partially ordered sets with modest assumptions as well as to set-valued mappings.

The generated filter by an element x of S is denoted by $[x, \to)$ and is defined by

$$[x, \to) = \{z \in S : x \leq z\}.$$

Definition 2.2. Let Y be a real topological vector space. A nonempty subset P of Y is called a *cone* if for any $x \in P$ and every nonnegative real number λ, $\lambda x \in P$. Moreover, the cone P is said to be:

(a) *convex* if $\forall x_1, x_2 \in P$, $x_1 + x_2 \in P$;
(b) *proper* if $P \neq \{0_Y\}$ and $P \neq Y$;
(c) *pointed* if $P \bigcap (-P) = \{0_Y\}$;
(d) *solid* if $\mathrm{Int}P \neq \emptyset$.

Throughout this paper, we always assume that P is proper. Clearly, every proper convex pointed cone can induce a partial order relation $\leq$ by

$$x \leq y \Leftrightarrow y - x \in P$$

Moreover, $x < y$ stands for $x \leq y$ with $x \neq y$, while $x \leq y$ stands for $y - x \in \mathrm{Int}P$ when P is solid.

3. Main results

For exploring the existence of solutions to VEP on a partially ordered topological space with the hull-kernel topology, the first task is to establish some analysis tools on this space. We shall use Zorn's lemma to prove some new fixed point theorems in partially ordered topological spaces. In what follows, the set of fixed points of a mapping F is denoted by $\mathsf{Fix}(F)$.

Definition 3.1. Let Q and W be two ordered sets with partial order relations $\leq_1$ and $\leq_2$, respectively. Let $F:(Q,\leq_1)\to 2^{(W,\leq_2)}\setminus\{\emptyset\}$ be a set-valued mapping. Then F is said to be:

(a) *upper order-preserving* if $q_1 \leq_1 q_2$ implies that for every $w_1 \in F(q_1)$ there is a $w_2 \in F(q_2)$ such that $w_1 \leq_2 w_2$;
(b) *lower order-preserving* if $q_1 \leq_1 q_2$ implies that for every $w_2 \in F(q_2)$ there is a $w_1 \in F(q_1)$ such that $w_1 \leq_2 w_2$.

Single-valued mapping $G : Q \to W$ is said to be *order-preserving* if $G(q_1) \leq_2 G(q_2)$ whenever $q_1 \leq_1 q_2$. That is, the upper order-preservation is reduced to order-preservation.

Lemma 3.2. *Let L and K be two meet semilattices. Every homomorphism $F : L \to K$ is an order-preserving mapping.*

Proof. Let $x, y \in L$ with $x \leq y$. Since $F : L \to K$ is a homomorphism, $F(x)\wedge F(y) = F(x\wedge y) = F(x)$. This implies $F(x) \leq F(y)$. Hence, F is an order-preserving mapping. □

Theorem 3.3. *Let S be a partially ordered space with the hull-kernel topology, C a nonempty subset of S with a minimal element, and $F : C \to 2^C \setminus \{\emptyset\}$ be an upper order-preserving set-valued mapping. If C is chain-complete and the values of F are compact, then F has a fixed point and $\mathsf{Fix}(F)$ has a maximal element.*

Proof. Suppose that $E = \{c \in C : F(c) \cap [c,\to) \neq \emptyset\}$. It is obvious that E is a partially ordered set, since its order is the restriction of the original order on S. The set E is nonempty because C has a minimal element. We prove that any chain g in E, has an upper bound in E. The existence of upper bound for g follows from the assumption of chain-completeness of C. In fact, $\bigvee g$ is an upper bound of E. We are going to show that $\bigvee g \in E$. In other words, we prove that $F(\bigvee g)\cap[\bigvee g,\to)\neq\emptyset$. If $c \in g \subset E$, then there is $z_c \in F(c)$ such that $c \leq z_c$. Also, $c \leq \bigvee g$ and it follows from the fact that F is upper order-preserving mapping that there is $w_c \in F(\bigvee g)$ with $z_c \leq w_c$. Hence, $w_c \in F(\bigvee g)\cap[c,\to)$. Now, we prove that the family $\{F(\bigvee g)\bigcap[a,\to) : a \in g\}$ has the finite intersection property. Suppose f is a nonempty finite subset of g. Since g is a chain and f is finite, there must be $\hat{a} \in f$. By the definition of the set E we get $F(\hat{a}) \cap [\hat{a},\to))$ is nonempty. Thus there exists $w_{\hat{a}} \in F(\hat{a}) \cap [\hat{a},\to))$ and since for each $b \in f$, $b \leq \bigvee g$, the upper order-preserving property of F implies the existence of an element h of $F(\bigvee g)$ with $b \leq h$. Consequently, for each b in f, we have $b \leq \hat{a} \leq w_{\hat{a}} \leq h$, which implies $h \in \bigcap\{F(\bigvee g)\bigcap[a,\to) : a \in g\}$. Hence,the family $\{F(\bigvee g)\bigcap[a,\to) : a \in g\}$ has the finite intersection property, and since the values of F are compact, we get that this family has nonempty intersection. Hence, $\hat{a} \in \bigcup_{x\in g}(F(\bigvee g)\cap[x,\to))$. Thus $\hat{a} \in F(\bigvee g)$ and for each x in g, $\hat{a} \in [x,\to)$, which implies $\hat{a} \in F(\bigvee g)$ and $\bigvee g \leq \hat{a}$. This means $\hat{a} \in F(\bigvee g)\cap[\bigvee g,\to)$ and so $\bigvee g$ belongs to E. Now, from the Zorn lemma it follows that E has a maximal element. It is obvious that maximal elements of E are fixed points of F, because if $a \in E$ is a maximal element, then $[a,\to) = \{a\}$ and $F(a)\cap[a,\to) = \{a\})$, which implies $a \in F(a)$. This means a is a fixed point of F. It is clear that $\mathsf{Fix}(F)$ is a subset of E and each maximal element of E is a maximal element of $\mathsf{Fix}(F)$ and thus the set of maximal elements of $\mathsf{Fix}(F)$ is nonempty. This completes the proof. □

Remark 3.4. The result of Theorem 3.3 is still true if we replace the compactness of values of F by the closedness of values of F and compactness of C. Also, one can replace the condition "the set C has a minimal element" by the existence of an element a of C such that $F(a)\cap[a,\to)\neq\emptyset$. Finally, the result of Theorem 3.3 is correct if we replace the hull-kernel topology by the weakest topology whose the sets $[x,\to)$ are closed for each x in S.

If we reverse the ordering given on S, we obtain the following result.

Theorem 3.5. *Let S be a partially ordered set with the hull-kernel topology and C a nonempty subset of S. Let $F : C \to 2^C \setminus \{\emptyset\}$ be a lower-preserving set-valued mapping with compact values. If C has a maximal element and for every chain g in C, $\bigvee g \in C$ (the greatest lower bound of g,) then the set $\mathsf{Fix}(F)$ is nonempty and has a minimal element.*

As a corollary of Theorems 3.3 and 3.5 we obtain the classical result of Tarski [17] and [12, Theorem 1] by relaxing the existence of the infimum of each chain as it was given in condition (D) in [12, Theorem 1]. Hence, Theorems 3.3 and 3.5 are set-valued versions of Theorem 1 in [12]]. Moreover, they are a set-valued version of Proposition 3 in [18] and the main theorems given in [8].

Corollary 3.6. ([12, Theorem 1]) *Suppose P is a partially ordered set and $f : P \to P$ an order-preserving mapping. If there exists $x \in P$ such $x \leq f(x)$ or $f(x) \leq x$, then f has a fixed point and the set of fixed points of f has a maximal (or a minimal) element.*

Proof. We define $F : P \to 2^P$ by $F(x) = \{f(x)\}$. It is obvious that the values of F are compact with respect to each topology, especially with respect to the hull-kernel topology. Also, it is easy to verify that F satisfies all the conditions of Theorem 3.3 (or Theorem 3.5), and the result now follows from Theorem 3.3 (or Theorem 3.5). □

Now, we apply these fixed point theorems to explore the existence of solutions to VEP in a partially ordered topological space. If VEP has at least one solution, then we say it is *solvable*.

Theorem 3.7. *Let S be a partially ordered set equipped by the hull-kernel topology and C be a nonempty chain-complete set with a minimal element. Let Y be a topological vector space and P a pointed, convex, solid cone. Let $f : C \times C \to Y$ be a mapping. Assume that:*

(a) $f(z, z) \notin \mathrm{Int}P$ *for any* $z \in C$;
(b) $f(., z)$ *is order-preserving for any* $z \in C$;
(c) $\{z \in C : f(w, z) \in \mathrm{Int}P\}$ *is compact for any* $w \in C$.

Then, VEP *is solvable.*

Proof. On the contradiction, if the result is false, then the set-valued mapping $F : C \to 2^Y$ defined by $F(w) = \{z \in C : f(w, z) \in \mathrm{Int}P\}$ has nonempty values. It is easy to see that, via our assumptions, F satisfies all the conditions of Theorem 3.3. For this, it is enough to prove that the mapping F is upper order-preserving. Let $w_1 \leq w_2$ in C and let $u_1 \in F(w_1)$. Then there is some $z_0 \in C$ such that $u_1 = f(w_1, z_0)$. Consider $u_2 = f(w_2, z_0) \in F(w_2)$. Since the function $f(\cdot, z_0)$ is order-preserving, we have $f(w_1, z_0) \leq f(w_2, z_0)$, i.e., $u_1 \leq u_2$. Therefore, F is upper order-preserving. By Theorem 3.3, the set of fixed points of F is nonempty. This means there exists $w \in C$ such that $w \in F(w)$. Hence, $f(w, w) \in \mathrm{Int}P$ which contradicts the condition (a). Therefore, VEP is solvable and the proof is completed. □

4. Conclusion

In this paper, the notion of order-preserving set-valued mappings is introduced. As a matter of fact, set-valued mappings with upper and lower order preservation properties are provided. Some fixed point theorems for set-valued mappings in the context of partially ordered sets with the hull-kernel topology are established by using these concepts. An existence theorem for a vector equilibrium issue solution is given as an application of these results.

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