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# Colorings and $A_\alpha$ spectral radius of (join) digraphs

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**Abstract.** Let $\mathcal{G}_{n,r}$ be the set of digraphs of order $n$ with dichromatic number $r$. Let in-tree be a directed tree with $n$ vertices which the outdegree of each vertex is at most one. In this paper, we obtain the digraph which has the minimal $A_\alpha$ spectral radius of the join of in-trees with dichromatic number $r$. Moreover, we characterize the digraph which has the maximal $A_\alpha$ spectral radius in $\mathcal{G}_{n,r}$ by using a new method.

## 1. Introduction

Let $G = (\mathcal{V}(G), \mathcal{A}(G))$ be a digraph with vertex set $\mathcal{V}(G) = \{v_1, v_2, \ldots, v_n\}$ and arc set $\mathcal{A}(G)$. We denote an arc from vertex $v_i$ to vertex $v_j$ by $(v_i, v_j)$, where $v_i$ is the tail of $(v_i, v_j)$ and $v_j$ is the head of $(v_i, v_j)$. The outdegree $d_i^+ = d_G^+(v_i)$ (or indegree $d_i^- = d_G^-(v_i)$) of $G$ is the number of arcs whose tail (or head) is vertex $v_i$. We denote by $\Delta^+(G)$ the maximum outdegree of $G$ and $\delta^-(G)$ the minimum indegree of $G$, respectively. A directed path with length $n$ is a finite non-null sequence $v_1e_1v_2e_2 \cdots v_ne_nv_{n+1}$, where the vertices $v_1, v_2, \ldots, v_{n+1}$ are distinct and $e_i$ is the arc $(v_i, v_{i+1})$, which is $P_{n+1}$. If $v_{n+1} = v_1$, the sequence $v_1e_1v_2e_2 \ldots v_ne_nv_1$ is the directed cycle $C_n$. A digraph is connected if its underlying graph is connected. A digraph $G$ is strongly connected if for any pair of vertices $v_i, v_j \in \mathcal{V}(G)$, there is a directed path from $v_i$ to $v_j$. Throughout this paper, we consider the connected digraphs without loops and multiple arcs.

A digraph is acyclic if it has no directed cycles. A directed tree is a digraph with $n$ vertices and $n-1$ arcs which its underlying graph does not contain cycles. An in-tree is a directed tree with $n$ vertices which the outdegree of each vertex is at most one. Then the in-tree has exactly one vertex with outdegree 0 and such vertex is called the root of the in-tree. An in-star is a directed tree with $n$ vertices which has one vertex with indegree $n-1$ and other vertices with indegree 0. Obviously, in-star is a kind of in-tree. A tournament is a digraph obtained from an undirected complete graph by assigning a direction for each edge. A transitive tournament is a tournament $G$ satisfying the following: if $(u, v) \in \mathcal{A}(G)$ and $(v, w) \in \mathcal{A}(G)$, then $(u, w) \in \mathcal{A}(G)$. The join of two digraphs $G$ and $H$, denoted by $G \vee H$, is the digraph having vertex set

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$\mathcal{V}(G) \cup \mathcal{V}(H)$ and arc set $\mathcal{A}(G) \cup \mathcal{A}(H) \cup \{(u,v),(v,u)|u \in \mathcal{V}(G), v \in \mathcal{V}(H)\}$. Let $G = V^1 \vee V^2 \vee \cdots \vee V^r$ be a join digraph with dichromatic number $r$ which each $V^i$ $(i = 1, 2, \ldots, r)$ is an acyclic digraph.

For a digraph $G$, the adjacency matrix $A(G) = (a_{ij})$ of $G$ is an $n \times n$ matrix whose $(i, j)$-entry equals to 1 if $(v_i, v_j) \in \mathcal{A}(G)$ and equals to 0 otherwise. The diagonal outdegree matrix $D^+(G)$ of $G$ is $D^+(G) = diag(d_1^+, d_2^+, \ldots, d_n^+)$. The Laplacian matrix $L(G)$ and the signless Laplacian matrix $Q(G)$ of $G$ are $L(G) = D^+(G) - A(G)$ and $Q(G) = D^+(G) + A(G)$, respectively. In [18], Liu et al. defined the $A_\alpha$-matrix of $G$ as

$$A_\alpha(G) = \alpha D^+(G) + (1 - \alpha)A(G),$$

where $\alpha \in [0, 1]$. Obviously, $A_0(G) = A(G)$, $A_{\frac{1}{2}}(G) = \frac{1}{2}Q(G)$ and $A_1(G) = D^+(G)$. Since $D^+(G)$ is not interesting, we only consider $\alpha \in [0, 1)$. The eigenvalue of $A_\alpha(G)$ with largest modulus is called the $A_\alpha$ spectral radius of $G$, denoted by $\rho_\alpha(G)$. Now, many results about the $A_\alpha$-matrix of an undirected graph can be found in [11, 12, 14, 15, 17, 21, 22], but not much is known about digraphs. Xi et al. [23] determined the digraphs which attain the maximum (or minimum) $A_\alpha$ spectral radius among all strongly connected digraphs with given parameters such as girth, clique number, vertex connectivity or arc connectivity. Xi and Wang [25] established some lower bounds on $\Delta^+(G) - \rho_\alpha(G)$ for strongly connected irregular digraphs with given maximum outdegree and some other parameters. Ganie and Baghipur [7] obtained some lower bounds for the spectral radius of $A_\alpha(G)$ in terms of the number of vertices, arcs and closed walks of $G$. More knowledge about the spectra of digraphs can be found in a survey [5] and a book [8].

A vertex set $F \subseteq \mathcal{V}(G)$ is acyclic if its induced subdigraph $G[F]$ is acyclic. A partition of $\mathcal{V}(G)$ into $r$ acyclic sets is called a $r$-coloring of $G$. The minimum integer $r$ for which there exists a $r$-coloring of $G$ is the dichromatic number $\chi(G)$ of $G$. Let $\mathcal{G}_{n,r}$ denote the set of digraphs of order $n$ with dichromatic number $r$. In 1982, Neumann-Lara [20] first introduced the dichromatic number of a digraph. Lin and Shu [16] characterized the digraph which has the maximal spectral radius with given dichromatic number. Drury and Lin [6] determined the digraphs that have the minimum and second minimum spectral radius among all strongly connected digraphs with given order and dichromatic number. Liu et al. [18] characterized the digraph which has the maximal $A_\alpha$ spectral radius with given dichromatic number. Kim et al. [10] proved a tight upper bound for the spectral radius of digraphs in terms of the number of vertices and the dichromatic number. For more papers on the dichromatic number of digraphs see [2, 13, 19, 24].

In this paper, the organization is as follows. In Section 2, we list some known results used for later. In Section 3, we obtain the digraph which has the minimal $A_\alpha$ spectral radius of the join of in-trees with given dichromatic number. In Section 4, we characterize the digraph which has the maximal $A_\alpha$ spectral radius with given dichromatic number by using the equitable quotient matrix. Note that Liu et al. [18] obtained the results by using the Perron-Frobenius Theorem.

## 2. Preliminaries

In this section, we will list some known results used for later.

**Definition 2.1.** *([1]) Let $A = (a_{ij})$, $B = (b_{ij})$ be two $n \times n$ matrices. If $a_{ij} \le b_{ij}$ for all $i$ and $j$, then $A \le B$. If $A \le B$ and $A \ne B$, then $A < B$. If $a_{ij} < b_{ij}$ for all $i$ and $j$, then $A \ll B$.*

**Lemma 2.2.** *([1]) Let $A = (a_{ij})$, $B = (b_{ij})$ be two $n \times n$ matrices with the spectral radii $\rho(A)$ and $\rho(B)$, respectively. If $0 \le A \le B$, then $\rho(A) \le \rho(B)$. Furthermore, If $0 \le A < B$ and $B$ is irreducible, then $\rho(A) < \rho(B)$.*

**Definition 2.3.** *([4]) Let $M$ be a complex matrix of order $n$ described in the following block form*

$$M = \begin{bmatrix} M_{11} & \cdots & M_{1t} \\ \vdots & \ddots & \vdots \\ M_{t1} & \cdots & M_{tt} \end{bmatrix},$$

*where the blocks $M_{ij}$ are $n_i \times n_j$ matrices for any $1 \leq i, j \leq t$ and $n = n_1 + n_2 + \cdots + n_t$. For $1 \leq i, j \leq t$, let $b_{ij}$ be the average row sum of $M_{ij}$, i.e. $b_{ij}$ is the sum of all entries in $M_{ij}$ divided by the number of rows. Then $B(M) = (b_{ij})$ (or simply $B$) is called the quotient matrix of $M$. If for each pair $i, j$, the row sum of the matrix $M_{ij}$ is same for each row, then $B$ is called an equitable quotient matrix of $M$.*

**Lemma 2.4.** *([26]) Let $M$ be a nonnegative matrix and $B$ be the equitable quotient matrix of $M$ as defined in Definition 2.3. If $B$ is irreducible, then $\rho(B) = \rho(M)$.*

**Lemma 2.5.** *(Perron-Frobenius Theorem [9]) Let $M$ be a irreducible and nonnegative matrix of order $n$. Then*

*(a) $\rho(M) > 0$.*

*(b) $\rho(M)$ is an algebraically simple eigenvalue of $M$.*

*(c) there is a unique real vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ such that $M\mathbf{x} = \rho(M)\mathbf{x}$ and $x_1 + x_2 + \cdots + x_n = 1$; this vector is positive.*

*(d) there is a unique real vector $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$ such that $\mathbf{y}^T M = \rho(M)\mathbf{y}^T$ and $x_1y_1 + \cdots + x_ny_n = 1$; this vector is positive.*

**Lemma 2.6.** *([9]) If $M$ is a nonnegative matrix and $\boldsymbol{X} \geq 0$ is a nonnegative vector such that $M\boldsymbol{X} \geq \beta\boldsymbol{X}$ for some $\beta \in R$, then $\rho(M) \geq \beta$, where $\rho(M)$ is the largest eigenvalue of $M$. Furthermore, if $M$ is irreducible and $M\boldsymbol{X} > \beta\boldsymbol{X}$, then $\rho(M) > \beta$.*

**Lemma 2.7.** *([3]) Let $G$ be a digraph with no directed cycle. Then $\delta^-(G) = 0$ and there is an ordering $v_1, v_2, \ldots, v_n$ of $\mathcal{V}(G)$ such that, for $1 \leq i \leq n$, every arc of $G$ with head $v_i$ has its tail in $\{v_1, v_2, \ldots, v_{i-1}\}$.*

**Lemma 2.8.** *([18, 24]) Let $G$ be a strongly connected digraph with the $A_\alpha$ spectral radius $\rho_\alpha(G)$ and maximal outdegree $\Delta^+(G)$. If $H$ is a proper subdigraph of $G$, then $\rho_\alpha(G) > \rho_\alpha(H)$, especially, $\rho_\alpha(G) > \alpha\Delta^+(G)$.*

Let $G \in \mathcal{G}_{n,r}$ be a digraph of order $n$ with dichromatic number $r$. From the definition of dichromatic number, $G$ has $r$-coloring classes and each of which is an acyclic set. Let $\lambda_{\alpha 1}, \lambda_{\alpha 2}, \ldots, \lambda_{\alpha n}$ be the $A_\alpha$ eigenvalues of $G$ and $d_1^+, d_2^+, \ldots, d_n^+$ be the outdegrees of vertices of $G$. Then we have known the $A_\alpha$ eigenvalue of an acyclic digraph is $\lambda_{\alpha i} = \alpha d_i^+$, where $i = 1, 2, \ldots, n$. So if $r = 1$, $G$ is an acyclic digraph with $\rho_\alpha(G) = \alpha\Delta^+(G)$. Therefore in this paper, we only consider the case when $r \geq 2$.

## 3. The minimal $A_\alpha$ spectral radius of the join of in-trees with given dichromatic number

In this section, we will consider the minimal $A_\alpha$ spectral radius of the join of in-trees with given dichromatic number. Let $\widetilde{\mathcal{T}}_{n,r} = V^1 \vee V^2 \vee \cdots \vee V^r$ denote the set of digraphs which each $V^i$ ($i = 1, 2, \ldots, r$) is an in-tree. Let $\widetilde{\mathcal{T}}_{n,r}^{\star} = V^{\star 1} \vee V^{\star 2} \vee \cdots \vee V^{\star r}$ denote the set of digraphs which each $V^{\star i}$ ($i = 1, 2, \ldots, r$) is an in-star. Obviously, $\widetilde{\mathcal{T}}_{n,r}^{\star} \subseteq \widetilde{\mathcal{T}}_{n,r}$. Let $\widetilde{T}_{n,2}^{\star\star}$ denote the digraph in $\widetilde{\mathcal{T}}_{n,2}^{\star}$ which $V^{\star 1}$ is an in-star with $n-1$ vertices and $V^{\star 2}$ is a digraph with one vertex. First, we will prove the digraph which has the mimimal $A_\alpha$ spectral radius of the join of in-trees with dichromatic number $r$ must be in $\widetilde{\mathcal{T}}_{n,r}^{\star}$.

**Theorem 3.1.** *Let $G = V^1 \vee V^2 \vee \cdots \vee V^r$ be a digraph in $\widetilde{\mathcal{T}}_{n,r}$, where $V^i$ ($i = 1, 2, \ldots, r$) is an in-tree with $n_i$ vertices. Let $G^\star = V^{\star 1} \vee V^{\star 2} \vee \cdots \vee V^{\star r}$ be a digraph in $\widetilde{\mathcal{T}}_{n,r}^{\star}$, where $V^{\star i}$ ($i = 1, 2, \ldots, r$) is an in-star with $n_i$ vertices. Then $\rho_\alpha(G) \geq \rho_\alpha(G^\star)$ with equality holds if and only if $G \cong G^\star$.*

*Proof.* For the digraph $G^\star$, let the vertex ordering of in-star $V^{\star i}$ be $\{v_1^i, v_2^i, \ldots, v_{n_i}^i\}$ such that $(v_j^i, v_{n_i}^i) \in \mathcal{A}(G^\star)$, for all $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, n_i - 1$. Then $d_{G^\star}^+(v_j^i) = n - n_i + 1$ and $d_{G^\star}^+(v_{n_i}^i) = n - n_i$. Suppose that

$$\mathbf{x} = (x_1^1, x_2^1, \ldots, x_{n_1}^1, x_1^2, x_2^2, \ldots, x_{n_2}^2, \ldots, x_1^r, x_2^r, \ldots, x_{n_r}^r)^T$$

is a Perron vector of $G^\star$ corresponding to the $A_\alpha$ spectral radius $\rho_\alpha^\star = \rho_\alpha(G^\star)$, where $x_j^i$ is the characteristic component corresponding to $v_j^i$ for each $1 \le i \le r$ and $1 \le j \le n_i$.

Since $A_\alpha(G^\star)\mathbf{x} = \rho_\alpha^\star \mathbf{x}$, we have

$$\begin{cases} \alpha(n-n_i+1)x_j^i + (1-\alpha)x_{n_i}^i + (1-\alpha)\sum_{s=1,s\neq i}^r \sum_{t=1}^{n_s} x_t^s = \rho_\alpha^\star x_j^i, \\ \alpha(n-n_i)x_{n_i}^i + (1-\alpha)\sum_{s=1,s\neq i}^r \sum_{t=1}^{n_s} x_t^s = \rho_\alpha^\star x_{n_i}^i, \end{cases}$$

where $i = 1,2,\ldots,r$ and $j = 1,2,\ldots,n_i-1$. Then we have

$$((1-\alpha) + \rho_\alpha^\star - \alpha(n-n_i))x_{n_i}^i = (\rho_\alpha^\star - \alpha(n-n_i+1))x_j^i.$$

Obviously, $x_{n_i}^i < x_1^i = x_2^i = \cdots = x_{n_i-1}^i$ for all $i = 1,2,\ldots,r$.

Next we prove $\rho_\alpha(G) \ge \rho_\alpha(G^\star)$. Suppose that $G \neq G^\star$, we can get the digraph $G$ by changing many arcs in $G^\star$. We first consider the transformation of one arc. We do the transformation of an arbitrary arc $(v_j^i, v_{n_i}^i) \in \mathcal{A}(G^\star)$ for all $i = 1,2,\ldots,r$ and $j = 1,2,\ldots,n_i-1$. Without loss of generality, we consider the arc $(v_j^1, v_{n_1}^1)$. Let

$$G = G^\star - (v_j^1, v_{n_1}^1) + (v_s^1, v_t^1).$$

By the structural property of directed trees, the arc $(v_s^1, v_t^1)$ only has three cases: $(v_s^1, v_t^1) = (v_j^1, v_t^1)$ or $(v_s^1, v_t^1) = (v_{n_1}^1, v_j^1)$ or $(v_s^1, v_t^1) = (v_s^1, v_j^1)$, where $s,t = 1,2,\ldots,n_1-1$. Since the outdegree sequence of the in-tree is (1,1,. . . ,1,0), the case $(v_s^1, v_t^1) = (v_s^1, v_j^1)$ is impossible. So we only discuss the two cases: $(v_j^1, v_{n_1}^1) \to (v_j^1, v_t^1)$ or $(v_j^1, v_{n_1}^1) \to (v_{n_1}^1, v_j^1)$.

**Case 1.** If $(v_j^1, v_{n_1}^1) \to (v_j^1, v_t^1)$. Since $x_{n_1}^1 < x_j^1 = x_t^1$, we obtain

$$(A_\alpha(G) - A_\alpha(G^\star))\mathbf{x} = (0,\ldots,0,(1-\alpha)(x_t^1 - x_{n_1}^1),0,\ldots,0)^T > 0.$$

That is $A_\alpha(G)\mathbf{x} > A_\alpha(G^\star)\mathbf{x} = \rho_\alpha(G^\star)\mathbf{x}$. By Lemma 2.6, $\rho_\alpha(G) > \rho_\alpha(G^\star)$.

**Case 2.** If $(v_j^1, v_{n_1}^1) \to (v_{n_1}^1, v_j^1)$. We can find a digraph $G'$ such that $G' \cong G$. Without loss of generality, let $v_j^1 = v_1^1$. Then we have $d_G^+(v_1^1) = n - n_1$ and $d_{G^\star}^+(v_{n_1}^1) = n - n_1$. Let $G'$ be a digraph which switch the index of $v_{n_1}^1$ and $v_1^1$ in $G$. Then

$$G' = G - (v_{n_1}^1, v_1^1) - \{(v_i^1, v_{n_1}^1) | i = 2,3,\ldots,n_1-1\} + (v_1^1, v_{n_1}^1) + \{(v_i^1, v_1^1) | i = 2,3,\ldots,n_1-1\}.$$

Obviously, $G' \cong G$ and

$$G' = G^\star - \{(v_i^1, v_{n_1}^1) | i = 2,3,\ldots,n_1-1\} + \{(v_i^1, v_1^1) | i = 2,3,\ldots,n_1-1\}.$$

Then we obtain

$$(A_\alpha(G') - A_\alpha(G^\star))\mathbf{x} = (0,(1-\alpha)(x_1^1 - x_{n_1}^1),\ldots,(1-\alpha)(x_1^1 - x_{n_1}^1),0,\ldots,0)^T > 0.$$

That is $A_\alpha(G')\mathbf{x} > A_\alpha(G^\star)\mathbf{x} = \rho_\alpha(G^\star)\mathbf{x}$. By Lemma 2.6, $\rho_\alpha(G') > \rho_\alpha(G^\star)$. So we have $\rho_\alpha(G) = \rho_\alpha(G') > \rho_\alpha(G^\star)$.

For the transformation of many arcs, similar to Case 2, we can find a digraph $G$ such that $d_G^+(v_{n_i}^i) = n - n_i$ and $d_G^+(v_j^i) = n - n_i + 1$ for all $i = 1,2,\ldots,r$ and $j = 1,2,\ldots,n_i-1$. Then the components of $(A_\alpha(G) - A_\alpha(G^\star))\mathbf{x}$ are 0 or $(1-\alpha)(x_j^i - x_{n_i}^i)$. So $(A_\alpha(G) - A_\alpha(G^\star))\mathbf{x} > 0$ always holds and $\rho_\alpha(G) > \rho_\alpha(G^\star)$.

To sum up the above, we have $\rho_\alpha(G) \ge \rho_\alpha(G^\star)$ with equality holding if and only if $G \cong G^\star$. □

To illustrate the transformation for Theorem 3.1 better, we give the following example.

**Example 3.2.** *Let $H = V^1 \vee V^2$ and $F = V^{1\star} \vee V^{2\star}$ be two digraphs shown in Figure 1. Then we can get the digraph $H$ by changing many arcs in the digraph $F$: $(v_1^1, v_6^1) \to (v_1^1, v_2^1)$, $(v_3^1, v_6^1) \to (v_3^1, v_2^1)$, $(v_4^1, v_6^1) \to (v_4^1, v_3^1)$, $(v_5^1, v_6^1) \to (v_5^1, v_3^1)$, $(v_1^2, v_4^2) \to (v_1^2, v_2^2)$, $(v_2^2, v_4^2) \to (v_2^2, v_3^2)$. From Theorem 3.1, we have $(A_\alpha(H) - A_\alpha(F))\mathbf{x} = \{(1-\alpha)(x_2^1 - x_6^1), 0, (1-\alpha)(x_2^1 - x_6^1), (1-\alpha)(x_3^1 - x_6^1), (1-\alpha)(x_3^1 - x_6^1), 0, (1-\alpha)(x_2^2 - x_4^2), (1-\alpha)(x_3^2 - x_4^2), 0, 0\} > 0$. So $\rho_\alpha(H) > \rho_\alpha(F)$.*

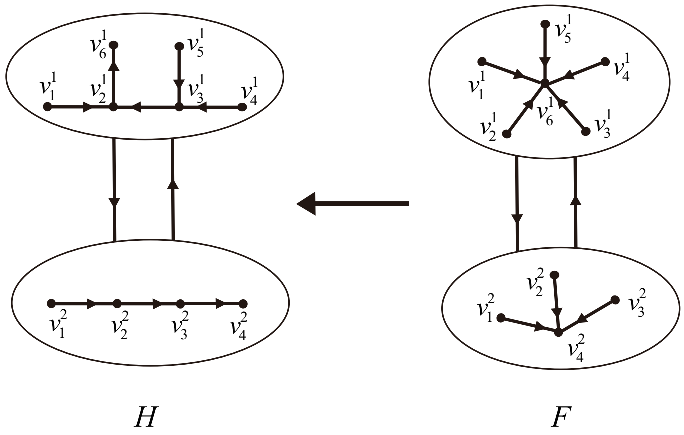


Figure 1: The digraphs $H$ and $F$

Next we will prove the digraph $\widetilde{T}^{\star\star}_{n,2}$ has the minimal $A_\alpha$ spectral radius of the join of in-trees with dichromatic number 2 when $\alpha = 0$ or $\alpha = \frac{1}{2}$.

**Theorem 3.3.** *The digraph $\widetilde{T}^{\star\star}_{n,2}$ is the unique digraph which has the minimal $A_0$ spectral radius among all digraphs in $\widetilde{\mathcal{T}}_{n,2}$.*

*Proof.* Let $G$ be an arbitrary digraph in $\widetilde{\mathcal{T}}_{n,2}$, $\rho_0(G)$ be the spectral radius of $A_0(G)$ and $\rho_A(G)$ be the spectral radius of adjacency matrix $A(G)$. Obviously, $\rho_0(G) = \rho_A(G)$. Let $G = V^1 \vee V^2$, where $V^i$ is an in-tree, $|V^i| = n_i$ and $n_1 \geq n_2$. Then $\lceil \frac{n}{2} \rceil \leq n_1 \leq n-1$, $1 \leq n_2 \leq \lfloor \frac{n}{2} \rfloor$ and $n_1 + n_2 = n$. By Theorem 3.1, we know that the digraph which has the mimimal $A_\alpha$ spectral radius of the join of in-trees with dichromatic number $r$ must be in $\widetilde{\mathcal{T}}^{\star}_{n,r}$. So we only need to consider the number of $n_i$ of digraph $G^\star$ in $\widetilde{\mathcal{T}}^{\star}_{n,2}$. That is $G^\star = V^{\star 1} \vee V^{\star 2}$, where $V^{\star i}$ is an in-star, $|V^{\star i}| = n_i$ and $n_1 \geq n_2$. Then $d^+_{G^\star}(v^i_j) = n - n_i + 1$ and $d^+_{G^\star}(v^i_{n_i}) = n - n_i$, where $i = 1, 2$ and $j = 1, 2, \ldots, n_i - 1$. Let $A_{11} = \{v^1_j | j = 1, 2, \ldots, n_1 - 1\}$, $A_{12} = \{v^1_{n_1}\}$, $A_{21} = \{v^2_j | j = 1, 2, \ldots, n_2 - 1\}$ and $A_{22} = \{v^2_{n_2}\}$. Let $B_A = B_A(G^\star)$ be the quotient matrix of $A(G^\star)$, where $B_A$ corresponding to the vertex partition $A_{11}, A_{12}, A_{21}, A_{22}$. Then the quotient matrix $B_A$ is equitable. Next we consider the cases when $n_1 > n_2 > 1$, $n_1 = n_2$ and $n_1 = n-1$, $n_2 = 1$.

**Case 1:** If $n_1 > n_2 > 1$, then the equitable quotient matrix $B_A$ as follow:

$$B_A = \begin{pmatrix} 0 & 1 & n_2 - 1 & 1 \\ 0 & 0 & n_2 - 1 & 1 \\ n_1 - 1 & 1 & 0 & 1 \\ n_1 - 1 & 1 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial of $B_A$ is

$$|xI_4 - B_A| = x^4 - n_1 n_2 x^2 + (n - 2n_1 n_2)x + (n - n_1 n_2 - 1).$$

Let

$$f_A(x) = f_A(x; n_1, n_2) = x^4 - n_1 n_2 x^2 + (n - 2n_1 n_2)x + (n - n_1 n_2 - 1).$$

By using the Perron-Frobenius Theorem, $\rho_A(G^\star)$ is an eigenvalue (multiplicity one) of $A(G^\star)$ and there is a corresponding eigenvector whose coordinates are all positive. And from Lemma 2.4, $\rho^\star_A = \rho_A(G^\star)$ is the root of $f_A(x)$ with the largest modulus.

Next we prove $\rho_A(G^\star) \geq \rho_A(\widetilde{T}^{\star\star}_{n,2})$. We move one of the vertices in $V^{\star 1}$ (except for the vertex $v^1_{n_1}$) to $V^{\star 2}$. Without loss of generality, let that vertex be $v^1_1$. That is

$$G' = G^\star - (v^1_1, v^1_{n_1}) - \{(v^1_1, v^2_s) | s = 1, 2, \ldots, n_2\} - \{(v^2_s, v^1_1) | s = 1, 2, \ldots, n_2\}$$

$+ (v_1^1, v_{n_2}^2) + \{(v_t^1, v_1^1)|t = 2, \ldots, n_1\} + \{(v_1^1, v_t^1)|t = 2, \ldots, n_1\}.$

Let $\rho'_A = \rho_A(G')$ be the root of $\tilde{f_A}(x)$ with the largest modulus, where $\tilde{f_A}(x) = f_A(x; n_1 - 1, n_2 + 1) = x^4 - (n_1 - 1)(n_2 + 1)x^2 + (n - 2(n_1 - 1)(n_2 + 1))x + (n - (n_1 - 1)(n_2 + 1) - 1)$. Then

$$\tilde{f_A}(\rho_A^\star) = f_A(\rho_A^\star) + (n_2 + 1 - n_1)(\rho_A^\star + 1)^2.$$

We know $f_A(\rho_A^\star) = 0$ and $n_1 > n_2 > 1$. If $n_2 + 1 - n_1 = 0$, then $n > 2$ is odd and $n_1 = \frac{n+1}{2}$, $n_2 = \frac{n-1}{2}$. That is $G' = G^\star$. If $n_2 + 1 - n_1 < 0$, then $\tilde{f_A}(\rho_A^\star) < 0$.

As both $f_A(x)$ and $\tilde{f_A}(x)$ have the positive leading coefficients, $\tilde{f_A}(\rho_A^\star) < 0$ implies that $\rho_A^\star < \rho'_A$. So the $A_0$ spectral radius with $n_1$ and $n_2$ is smaller than the $A_0$ spectral radius with $n_1 - 1$ and $n_2 + 1$. That is when $n_1 = n - 2$ and $n_2 = 2$, the $A_0$ spectral radius is minimal.

**Case 2:** If $n_1 = n_2 > 1$, then $n > 2$ is even and $n_1 = n_2 = \frac{n}{2}$. By Case 1, we know the $A_0$ spectral radius with $n_1 = n_2 = \frac{n}{2}$ is bigger than the $A_0$ spectral radius with $n_1 = \frac{n}{2} + 1$ and $n_2 = \frac{n}{2} - 1$. So when $n_1 = n - 2$ and $n_2 = 2$, the $A_0$ spectral radius is minimal.

**Case 3:** If $n_1 = n - 1$ and $n_2 = 1$, then the equitable quotient matrix $B'_A$ is

$$B'_A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ n-2 & 1 & 0 \end{pmatrix}.$$

From Lemma 2.2, we have $\rho(B'_A) = \rho(B''_A) < \rho(B_A)$, where $B_A$ with $n_1 = n - 1$, $n_2 = 1$ and

$$B''_A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ n-2 & 1 & 0 & 0 \end{pmatrix}.$$

By Case 1, the $A_0$ spectral radius with $n_1 = n - 2$ and $n_2 = 2$ is bigger than the $A_0$ spectral radius with $n_1 = n - 1$ and $n_2 = 1$. So when $n_1 = n - 1$ and $n_2 = 1$, the $A_0$ spectral radius is minimal.

Hence, the digraph $\widetilde{T}_{n,2}^{\star\star}$ is the unique digraph which has the minimal $A_0$ spectral radius among all digraphs in $\widetilde{\mathcal{T}}_{n,2}$. □

**Theorem 3.4.** *The digraph $\widetilde{T}_{n,2}^{\star\star}$ is the unique digraph which has the minimal $A_{\frac{1}{2}}$ spectral radius among all digraphs in $\widetilde{\mathcal{T}}_{n,2}$.*

*Proof.* Let $G$ be an arbitrary digraph in $\widetilde{\mathcal{T}}_{n,2}$, $\rho_{\frac{1}{2}}(G)$ be the spectral radius of $A_{\frac{1}{2}}(G)$ and $\rho_Q(G)$ be the spectral radius of signless Laplacian matrix $Q(G)$. Obviously, $\rho_{\frac{1}{2}}(G) = \frac{1}{2}\rho_Q(G)$. So we only consider $\rho_Q(G)$. By Theorem 3.1, we only need to consider the number of $n_i$ of digraph $G^\star$ in $\widetilde{\mathcal{T}}_{n,2}^\star$. Similar to the proof of Theorem 3.3, let $B_Q = B_Q(G^\star)$ be the equitable quotient matrix of $Q(G^\star)$, where $B_Q$ corresponding to the vertex partition $A_{11}, A_{12}, A_{21}, A_{22}$. We also omit the category discussion about $n_1$ and $n_2$.

If $n_1 > n_2 > 1$, then the equitable quotient matrix $B_Q$ as follow:

$$B_Q = \begin{pmatrix} n_2+1 & 1 & n_2-1 & 1 \\ 0 & n_2 & n_2-1 & 1 \\ n_1-1 & 1 & n_1+1 & 1 \\ n_1-1 & 1 & 0 & n_1 \end{pmatrix}.$$

The characteristic polynomial of $B_Q$ is

$$|xI_4 - B_Q| = x^4 - (2 + 2n)x^3 + (1 + 3n + n^2 + n_1n_2)x^2 + (n - n^2 - 4n_1n_2 - n_1n_2n)x + (-4 + 2n - 4n_1n_2 + 2n_1n_2n).$$

Let

$$f_Q(x) = f_Q(x; n_1, n_2) = x^4 - (2+2n)x^3 + (1+3n+n^2+n_1n_2)x^2 \\ + (n - n^2 - 4n_1n_2 - n_1n_2n)x + (-4+2n-4n_1n_2+2n_1n_2n).$$

By the Perron-Frobenius Theorem, $\rho_Q(G^\star)$ is an eigenvalue (multiplicity one) of $Q(G^\star)$ and there is a corresponding eigenvector whose coordinates are all positive. And from Lemma 2.4, $\rho_Q^\star = \rho_Q(G^\star)$ is the root of $f_Q(x)$ with the largest modulus.

Next we prove $\rho_Q(G^\star) \geq \rho_Q(\widetilde{T}^{\star\star}_{n,2})$. We move one of the vertices in $V^{\star 1}$ (except for the vertex $v^1_{n_1}$) to $V^{\star 2}$. The operation is similar to the Theorem 3.3, so we omit it. Let $\rho'_Q = \rho_Q(G')$ be the root of $\tilde{f}_Q(x)$ with the largest modulus, where

$$\tilde{f}_Q(x) = f_Q(x; n_1-1, n_2+1) = x^4 - (2+2n)x^3 + (1+3n+n^2+(n_1-1)(n_2+1))x^2 \\ + (n - n^2 - 4(n_1-1)(n_2+1) - (n_1-1)(n_2+1)n)x \\ + (-4+2n-4(n_1-1)(n_2+1)+2(n_1-1)(n_2+1)n).$$

Then

$$\tilde{f}_Q(\rho_Q^\star) = f_Q(\rho_Q^\star) + (n_1 - n_2 - 1)\left((\rho_Q^\star)^2 - (4+n)\rho_Q^\star + 2(n-2)\right).$$

Since $n_1 > n_2 > 1$ and $f_Q(\rho_Q^\star) = 0$, to prove $\tilde{f}_Q(\rho_Q^\star) < 0$ implies that $(\rho_Q^\star)^2 - (4+n)\rho_Q^\star + 2(n-2) < 0$. That is

$$\frac{4+n-\sqrt{32+n^2}}{2} < \rho_Q^\star < \frac{4+n+\sqrt{32+n^2}}{2}.$$

Since $f_Q(n+2) = 4(3n+n_1^2+n_2^2) > 0$ and $f_Q(n) = -2(n+2)(n_1-1)(n_2-1) < 0$, we get $n < \rho_Q^\star < n+2$. Because $\frac{4+n-\sqrt{32+n^2}}{2} < n$ and $n+2 < \frac{4+n+\sqrt{32+n^2}}{2}$ are always true, $\tilde{f}_Q(\rho_Q^\star) < 0$. Then $\rho'_Q > \rho_Q^\star$.

Therefore, similar to the proof of Theorem 3.3, when $n_1 = n-1$ and $n_2 = 1$, the $A_{\frac{1}{2}}$ spectral radius is minimal. That is, the digraph $\widetilde{T}^{\star\star}_{n,2}$ is the unique digraph which has the minimal $A_{\frac{1}{2}}$ spectral radius among all digraphs in $\widetilde{\mathcal{T}}_{n,2}$. □

From Theorems 3.1, 3.3, 3.4, we get our main result.

**Theorem 3.5.** *Let $G = V^1 \vee V^2 \vee \cdots \vee V^r$ be a digraph with dichromatic number 2 which each $V^i$ $(i = 1, 2)$ is an in-tree. Then the digraph $\widetilde{T}^{\star\star}_{n,2}$ is the unique digraph which has the minimal $A_0$ or $A_{\frac{1}{2}}$ spectral radius of the join of in-trees with dichromatic number 2.*

**Example 3.6.** *From the proof of Theorems 3.3 and 3.4, we know the equitable quotient matrix $B_\alpha$ of $A_\alpha$ matrix of a digraph in $\widetilde{\mathcal{T}}^\star_{n,2}$ as follow:*

$$B_\alpha = \begin{pmatrix} \alpha(n_2+1) & 1-\alpha & (1-\alpha)(n_2-1) & 1-\alpha \\ 0 & n_2\alpha & (1-\alpha)(n_2-1) & 1-\alpha \\ (1-\alpha)(n_1-1) & 1-\alpha & \alpha(n_1+1) & 1-\alpha \\ (1-\alpha)(n_1-1) & 1-\alpha & 0 & \alpha n_1 \end{pmatrix}.$$

*From Tables 1 and 2, we take an example about the $A_\alpha$ spectral radius of the digraphs in $\widetilde{\mathcal{T}}^\star_{7,2}$ and $\widetilde{\mathcal{T}}^\star_{10,2}$ when $\alpha = \frac{1}{6}, \frac{3}{10}, \frac{1}{2}, \frac{11}{20}, \frac{3}{5}, \frac{8}{11}, \frac{6}{7}$.*

From Table 1, we find in $\widetilde{\mathcal{T}}^\star_{7,2}$, with $n_1$ increases and $n_2$ decreases, the $A_\alpha$ spectral radius of the digraph is decreasing when $\alpha = \frac{1}{6}, \frac{3}{10}, \frac{1}{2}, \frac{11}{20}$. But when $\alpha = \frac{3}{5}, \frac{8}{11}, \frac{6}{7}$, it has no such property. From Table 2, we find in $\widetilde{\mathcal{T}}^\star_{10,2}$, with $n_1$ increases and $n_2$ decreases, the $A_\alpha$ spectral radius of the digraph is decreasing when $\alpha = \frac{1}{6}, \frac{3}{10}, \frac{1}{2}$. But when $\alpha = \frac{11}{20}, \frac{3}{5}, \frac{8}{11}, \frac{6}{7}$, it has no such property. So we give the following problem.

Table 1: The $A_\alpha$ spectral radius of the join of in-stars in $\widetilde{\mathcal{T}}^{\star}_{7,2}$

| $n = n_1 + n_2 = 7$ | | $n_1 = 4, n_2 = 3$ | $n_1 = 5, n_2 = 2$ | $n_1 = 6, n_2 = 1$ |
|---|---|---|---|---|
| $\rho_\alpha$ | $\alpha = \frac{1}{6}$ | 4.0838 | 3.7847 | 2.9626 |
| | $\alpha = \frac{3}{10}$ | 4.1024 | 3.8660 | 3.1616 |
| | $\alpha = \frac{1}{2}$ | 4.1475 | 4.0685 | 3.6309 |
| | $\alpha = \frac{11}{20}$ | 4.1646 | 4.1463 | 3.85 |
| | $\alpha = \frac{3}{5}$ | 4.1856 | 4.2420 | 4.2 |
| | $\alpha = \frac{8}{11}$ | 4.2699 | 4.6040 | 5.0909 |
| | $\alpha = \frac{6}{7}$ | 4.4674 | 5.1892 | 6 |

Table 2: The $A_\alpha$ spectral radius of the join of in-stars in $\widetilde{\mathcal{T}}^{\star}_{10,2}$

| $n = n_1 + n_2 = 10$ | | $n_1 = 5, n_2 = 5$ | $n_1 = 6, n_2 = 4$ | $n_1 = 7, n_2 = 3$ | $n_1 = 8, n_2 = 2$ | $n_1 = 9, n_2 = 1$ |
|---|---|---|---|---|---|---|
| $\rho_\alpha$ | $\alpha = \frac{1}{6}$ | 5.7080 | 5.6181 | 5.3333 | 4.7906 | 3.7203 |
| | $\alpha = \frac{3}{10}$ | 5.7152 | 5.6472 | 5.4314 | 5.0168 | 4.1378 |
| | $\alpha = \frac{1}{2}$ | 5.7321 | 5.7183 | 5.6715 | 5.5649 | 5.0958 |
| | $\alpha = \frac{11}{20}$ | 5.7382 | 5.7454 | 5.7615 | 5.7619 | 5.5 |
| | $\alpha = \frac{3}{5}$ | 5.7457 | 5.7789 | 5.8704 | 5.9916 | 6 |
| | $\alpha = \frac{8}{11}$ | 5.7737 | 5.9142 | 6.2727 | 6.7479 | 7.2727 |
| | $\alpha = \frac{6}{7}$ | 5.8307 | 6.2256 | 6.9544 | 7.7523 | 8.5714 |

**Problem 3.7.** *There exists a number $\alpha_0 \in [0, 1)$ such that when $\alpha \le \alpha_0$, the digraph $\widetilde{T}^{\star\star}_{n,2}$ is the unique digraph which has the minimal $A_\alpha$ spectral radius of the join of in-trees with dichromatic number* 2*.*

Furthermore, from Theorem 3.1, we only find the digraph which has the mimimal $A_\alpha$ spectral radius of the join of in-trees with dichromatic number $r$ must be in $\widetilde{\mathcal{T}}^{\star}_{n,r}$. But for the join of any directed trees, whether the same conclusion can be obtained. So we give the following problem.

**Problem 3.8.** *Among the join of directed trees with dichromatic number $r$, does the digraph in $\widetilde{\mathcal{T}}^{\star}_{n,r}$ attain the mimimal $A_\alpha$ spectral radius?*

## 4. The maximal $A_\alpha$ spectral radius of digraphs with given dichromatic number

In this section, we will consider the maximal $A_\alpha$ spectral radius of digraphs with given dichromatic number. Using the Perron-Frobenius Theorem, this result has been proved by Liu et al. [18], but we give a new proof by using the equitable quotient matrix.

Let $\mathcal{T}_{n,r} = V^1 \vee V^2 \vee \cdots \vee V^r$ denote the set of digraphs which each $V^i$ $(i = 1, 2, \ldots, r)$ is a transitive tournament. Let $T^*_{n,r}$ denote the digraph in $\mathcal{T}_{n,r}$ with $||V^i| - |V^j|| \le 1$. By Lemma 2.8, we know that adding the arcs will increase the $A_\alpha$ spectral radius. So the transitive tournament has the maximum $A_\alpha$ spectral radius in acyclic digraphs. Hence we know that the digraph which has the maximal $A_\alpha$ spectral radius with dichromatic number $r$ must be in $\mathcal{T}_{n,r}$. Next we will use the equitable quotient matrix to prove the digraph $T^*_{n,r}$ has the maximal $A_\alpha$ spectral radius in $\mathcal{T}_{n,r}$.

**Theorem 4.1.** *Let $G = V^1 \vee V^2 \vee \cdots \vee V^r$ be a digraph in $\mathcal{T}_{n,r}$, where $V^i$ $(i = 1, 2, \ldots, r)$ is a transitive tournament with $n_i$ vertices and $n_1 \geq n_2 \geq \cdots \geq n_r$. Then $\rho_\alpha(G) \leq \rho_\alpha(T^*_{n,r})$ with equality holds if and only if $G \cong T^*_{n,r}$.*

*Proof.* Let $G$ be an arbitrary digraph in $\mathcal{T}_{n,r}$. By Lemma 2.7, we obtain a vertex ordering $\{v^i_1, v^i_2, \ldots, v^i_{n_i}\}$ of each transitive tournament $V^i$ such that $(v^i_s, v^i_t) \in \mathcal{A}(G)$, for all $s < t$ and $i = 1, 2, \ldots, r$. Then $d^+_G(v^i_j) = n - j$. For each $j = 1, 2, \ldots, n_1$, let $A_j = \{v^i_j | i = 1, 2, \ldots, r\}$ and $|A_j| = a_j$. Then the vertices in $A_j$ have the same outdegree $n - j$. Let $B = B(G)$ be the quotient matrix of $A_\alpha(G)$, where $B$ corresponding to the vertex partition $A_1, A_2, \ldots, A_{n_1}$. Then the quotient matrix $B$ is equitable and

$$B_{ij} = \begin{cases} \alpha(n-j) + (1-\alpha)(a_j - 1), & \text{if } i = j, \\ (1-\alpha)a_j, & \text{if } i < j, \\ (1-\alpha)(a_j - 1), & \text{if } i > j. \end{cases}$$

The characteristic polynomial of $B$ is

$$|xI_{n_1} - B| = \prod_{i=1}^{n_1}(x - \alpha(n-i)) - \sum_{j=1}^{n_1}\left((1-\alpha)(a_j - 1)\prod_{i=1}^{j-1}(x - \alpha(n-i)) \prod_{i=j+1}^{n_1}((1-\alpha) + x - \alpha(n-i))\right).$$

Note: if $j = n_1$, let $\prod_{i=j+1}^{n_1}((1-\alpha) + x - \alpha(n-i)) = 1$. (See Appendix for detailed calculation.)

Let

$$\begin{aligned} f(x) &= f(x; n_1, \ldots, n_r) \\ &= \prod_{i=1}^{n_1}(x - \alpha(n-i)) - \sum_{j=1}^{n_1}\left((1-\alpha)(a_j - 1)\prod_{i=1}^{j-1}(x - \alpha(n-i)) \prod_{i=j+1}^{n_1}((1-\alpha) + x - \alpha(n-i))\right). \end{aligned}$$

By Lemma 2.5 (Perron-Frobenius Theorem), $\rho_\alpha(G)$ is an eigenvalue (multiplicity one) of $A_\alpha(G)$ and there is a corresponding eigenvector whose coordinates are all positive. And from Lemma 2.4, $\rho_\alpha = \rho_\alpha(G)$ is the root of $f(x)$ with the largest modulus. From Lemma 2.8, we know $\rho_\alpha > \alpha\Delta^+(G) = \alpha(n-1)$. For convenience, let

$$X^{n_1}_j(x) = \prod_{i=1}^{j-1}(x - \alpha(n-i)) \prod_{i=j+1}^{n_1}((1-\alpha) + x - \alpha(n-i)).$$

Next we prove $\rho_\alpha(G) \leq \rho_\alpha(T^*_{n,r})$. We assume that $G \neq T^*_{n,r}$, then we have $n_1 \geq n_r + 2$. Let $p$ be the largest index such that $n_1 = \cdots = n_p > n_{p+1} \geq \cdots \geq n_r$. We do the following operation:

$$G' = G + \{(v^p_{n_p}, v^p_i) | i = 1, 2, \ldots, n_p - 1\} - \{(v^p_{n_p}, v^r_j) | j = 1, 2, \ldots, n_r\}.$$

Let $\rho'_\alpha = \rho_\alpha(G')$ be the root of $\tilde{f}(x)$ with the largest modulus, where $\tilde{f}(x) = f(x; n_1, \ldots, n_p - 1, \ldots, n_r + 1)$. Next, we will prove $\tilde{f}(\rho_\alpha) < 0$ by the following two cases.

**Case 1:** If $p \geq 2$, that is $n_1 = n_2 = \cdots = n_p$. Let

$$\tilde{f}(x) = f(x; n_1, \ldots, n_p - 1, \ldots, n_r + 1) = \prod_{i=1}^{n_1}(x - \alpha(n-i)) - \sum_{j=1}^{n_1}\left((1-\alpha)(a'_j - 1)X^{n_1}_j(x)\right),$$

where

$$a'_j = \begin{cases} a_j + 1, & \text{if } j = n_r + 1, \\ a_j - 1, & \text{if } j = n_1, \\ a_j, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}\tilde{f}(\rho_\alpha) &= \prod_{i=1}^{n_1}(\rho_\alpha - \alpha(n-i)) - \sum_{j=1, j\neq n_r+1}^{n_1-1}\Big((1-\alpha)(a_j-1)X_j^{n_1}(\rho_\alpha)\Big)\\ &- (1-\alpha)(a_{n_r+1}+1-1)X_{n_r+1}^{n_1}(\rho_\alpha) - (1-\alpha)(a_{n_1}-1-1)X_{n_1}^{n_1}(\rho_\alpha)\\ &= \prod_{i=1}^{n_1}(\rho_\alpha - \alpha(n-i)) - \sum_{j=1}^{n_1}\Big((1-\alpha)(a_j-1)X_j^{n_1}(\rho_\alpha)\Big) - (1-\alpha)X_{n_r+1}^{n_1}(\rho_\alpha) + (1-\alpha)X_{n_1}^{n_1}(\rho_\alpha)\\ &= f(\rho_\alpha) - (1-\alpha)(X_{n_r+1}^{n_1}(\rho_\alpha) - X_{n_1}^{n_1}(\rho_\alpha)).\end{aligned}$$

Next,

$$\begin{aligned}&X_{n_r+1}^{n_1}(\rho_\alpha) - X_{n_1}^{n_1}(\rho_\alpha)\\ &= \prod_{i=1}^{n_r}(\rho_\alpha - \alpha(n-i)) \prod_{i=n_r+2}^{n_1}((1-\alpha)+\rho_\alpha-\alpha(n-i)) - \prod_{i=1}^{n_1-1}(\rho_\alpha - \alpha(n-i)) \prod_{i=n_1+1}^{n_1}((1-\alpha)+\rho_\alpha-\alpha(n-i))\\ &= \prod_{i=1}^{n_r}(\rho_\alpha - \alpha(n-i))\left(\prod_{i=n_r+2}^{n_1}((1-\alpha)+\rho_\alpha-\alpha(n-i)) - \prod_{i=n_r+1}^{n_1-1}(\rho_\alpha-\alpha(n-i))\right).\end{aligned}$$

Since $n_1 \geq n_r + 2$, we have

$$\rho_\alpha - \alpha(n-i) > \alpha\Delta^+(G) - \alpha(n-i) = \alpha(n-1) - \alpha(n-i) = \alpha(i-1) \geq 0 \ (i \geq 1),$$

$$(1-\alpha) + \rho_\alpha - \alpha(n-i) > (1-\alpha) + \alpha(i-1) = \alpha(i-2) + 1 \geq 1 \ (i \geq n_r + 2),$$

and

$$((1-\alpha) + \rho_\alpha - \alpha(n-n_1)) - (\rho_\alpha - \alpha(n-(n_r+1))) = \alpha(n_1 - n_r - 2) + 1 \geq 1.$$

Obviously,

$$(1-\alpha) + \rho_\alpha - \alpha(n-i) > \rho_\alpha - \alpha(n-i).$$

Then

$$\begin{aligned}&\prod_{i=n_r+2}^{n_1}((1-\alpha)+\rho_\alpha-\alpha(n-i)) - \prod_{i=n_r+1}^{n_1-1}(\rho_\alpha-\alpha(n-i))\\ &= \prod_{i=n_r+2}^{n_1-1}((1-\alpha)+\rho_\alpha-\alpha(n-i)) - \prod_{i=n_r+2}^{n_1-1}(\rho_\alpha-\alpha(n-i)) + ((1-\alpha)+\rho_\alpha-\alpha(n-n_1)) - (\rho_\alpha - \alpha(n-(n_r+1)))\\ &> 0.\end{aligned}$$

Hence $X_{n_r+1}^{n_1}(\rho_\alpha) - X_{n_1}^{n_1}(\rho_\alpha) > 0$. Since $f(\rho_\alpha) = 0$, we have $\tilde{f}(\rho_\alpha) < 0$.

**Case 2:** If $p = 1$, that is $n_1 > n_2$. Let

$$\tilde{f}(x) = f(x; n_1 - 1, \ldots, n_r + 1) = \prod_{i=1}^{n_1-1}(x - \alpha(n-i)) - \sum_{j=1}^{n_1-1}\Big((1-\alpha)(a'_j - 1)X_j^{n_1-1}(x)\Big),$$

where

$$a'_j = \begin{cases} a_j + 1, & \text{if } j = n_r + 1,\\ a_j, & \text{otherwise.}\end{cases}$$

Then

$$\begin{aligned}\tilde{f}(\rho_\alpha) &= \prod_{i=1}^{n_1-1}(\rho_\alpha - \alpha(n-i)) - \sum_{j=1, j\neq n_r+1}^{n_1-1}\left((1-\alpha)(a_j-1)X_j^{n_1-1}(\rho_\alpha)\right) - (1-\alpha)(a_{n_r+1}+1-1)X_{n_r+1}^{n_1-1}(\rho_\alpha)\\ &= \prod_{i=1}^{n_1-1}(\rho_\alpha - \alpha(n-i)) - \sum_{j=1}^{n_1-1}\left((1-\alpha)(a_j-1)X_j^{n_1-1}(\rho_\alpha)\right) - (1-\alpha)X_{n_r+1}^{n_1-1}(\rho_\alpha).\end{aligned}$$

Since

$$f(\rho_\alpha) = \prod_{i=1}^{n_1}(\rho_\alpha - \alpha(n-i)) - \sum_{j=1}^{n_1}\left((1-\alpha)(a_j-1)X_j^{n_1}(\rho_\alpha)\right),$$

and

$$X_j^{n_1}(\rho_\alpha) = ((1-\alpha) + \rho_\alpha - \alpha(n-n_1))X_j^{n_1-1}(\rho_\alpha),$$

we have

$$\begin{aligned}&\tilde{f}(\rho_\alpha)\left((1-\alpha)+\rho_\alpha-\alpha(n-n_1)\right)\\ &= f(\rho_\alpha) + (1-\alpha)\prod_{i=1}^{n_1-1}(\rho_\alpha-\alpha(n-i)) + (1-\alpha)(a_{n_1}-1)X_{n_1}^{n_1}(\rho_\alpha) - (1-\alpha)X_{n_r+1}^{n_1}(\rho_\alpha)\\ &= f(\rho_\alpha) + (1-\alpha)\prod_{i=1}^{n_1-1}(\rho_\alpha-\alpha(n-i)) - (1-\alpha)\prod_{i=1}^{n_r}(\rho_\alpha-\alpha(n-i))\prod_{i=n_r+2}^{n_1}((1-\alpha)+\rho_\alpha-\alpha(n-i))\\ &= f(\rho_\alpha) + (1-\alpha)\prod_{i=1}^{n_r}(\rho_\alpha-\alpha(n-i))\left(\prod_{i=n_r+1}^{n_1-1}(\rho_\alpha-\alpha(n-i)) - \prod_{i=n_r+2}^{n_1}((1-\alpha)+\rho_\alpha-\alpha(n-i))\right).\end{aligned}$$

Since $n_1 \geq n_r + 2$, we have

$$(\rho_\alpha - \alpha(n-n_r-1)) - ((1-\alpha)+\rho_\alpha-\alpha(n-n_1)) = \alpha(n_r+2-n_1) - 1 < 0.$$

Obviously,

$$(1-\alpha)+\rho_\alpha-\alpha(n-i) > \rho_\alpha - \alpha(n-i).$$

Then we have

$$\begin{aligned}&\prod_{i=n_r+1}^{n_1-1}(\rho_\alpha-\alpha(n-i)) - \prod_{i=n_r+2}^{n_1}((1-\alpha)+\rho_\alpha-\alpha(n-i))\\ &= \prod_{i=n_r+2}^{n_1-1}(\rho_\alpha-\alpha(n-i)) - \prod_{i=n_r+2}^{n_1-1}((1-\alpha)+\rho_\alpha-\alpha(n-i)) + (\rho_\alpha-\alpha(n-n_r-1)) - ((1-\alpha)+\rho_\alpha-\alpha(n-n_1))\\ &< 0.\end{aligned}$$

So $\tilde{f}(\rho_\alpha) < 0$.

As both $f(x)$ and $\tilde{f}(x)$ have the positive leading coefficients, $\tilde{f}(\rho_\alpha) < 0$ implies that $\rho_\alpha < \rho'_\alpha$. We perform the above operation as many times as possible until $|n_1 - n_r| \leq 1$, which means the maximal $A_\alpha$ spectral radius in $\mathcal{T}_{n,r}$ is achieved only at $T^*_{n,r}$. □

From Lemma 2.8 and Theorem 4.1, we get the following theorem.

**Theorem 4.2.** *The digraph $T^*_{n,r}$ is the unique digraph which has the maximal $A_\alpha$ spectral radius among all digraphs in $\mathcal{G}_{n,r}$.*

## Appendix

Let $b_i = x - \alpha(n-i)$, $c_i = -(1-\alpha)(a_i - 1)$, $d = b_{n_1} + c_{n_1} = x - \alpha(n-n_1) - (1-\alpha)(a_{n_1} - 1)$, $\beta = -(1-\alpha)$ and $\gamma = -b_{n_1} + \beta = -x + \alpha(n-n_1) - (1-\alpha)$, where $i = 1, 2, \ldots, n_1$. Let

$$Q_{n_1} = \begin{vmatrix} b_1 & \beta & \beta & \cdots & \beta & \gamma \\ 0 & b_2 & \beta & \cdots & \beta & \gamma \\ 0 & 0 & b_3 & \cdots & \beta & \gamma \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_{n_1-1} & \gamma \\ c_1 & c_2 & c_3 & \cdots & c_{n_1-1} & d \end{vmatrix}, \; P_{n_1-1} = \begin{vmatrix} \beta & \beta & \cdots & \beta & \gamma \\ b_2 & \beta & \cdots & \beta & \gamma \\ 0 & b_3 & \cdots & \beta & \gamma \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & b_{n_1-1} & \gamma \end{vmatrix},$$

and $Q_{n_1-i}$ be the determinant obtained by deleting the pre-$i$ rows and the pre-$i$ columns of $Q_{n_1}$, $P_{n_1-1-i}$ be the determinant obtained by deleting the pre-$i$ rows and the pre-$i$ columns of $P_{n_1-1}$.

Then

$$\begin{aligned}
|xI_{n_1} - B| &= \begin{vmatrix} b_1 + c_1 & c_2 + \beta & c_3 + \beta & \cdots & c_{n_1-1} + \beta & c_{n_1} + \beta \\ c_1 & b_2 + c_2 & c_3 + \beta & \cdots & c_{n_1-1} + \beta & c_{n_1} + \beta \\ c_1 & c_2 & b_3 + c_3 & \cdots & c_{n_1-1} + \beta & c_{n_1} + \beta \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c_1 & c_2 & c_3 & \cdots & b_{n_1-1} + c_{n_1-1} & c_{n_1} + \beta \\ c_1 & c_2 & c_3 & \cdots & c_{n_1-1} & b_{n_1} + c_{n_1} \end{vmatrix} \\
&= \begin{vmatrix} b_1 & \beta & \beta & \cdots & \beta & \gamma \\ 0 & b_2 & \beta & \cdots & \beta & \gamma \\ 0 & 0 & b_3 & \cdots & \beta & \gamma \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_{n_1-1} & \gamma \\ c_1 & c_2 & c_3 & \cdots & c_{n_1-1} & d \end{vmatrix} \\
&= b_1 \begin{vmatrix} b_2 & \beta & \cdots & \beta & \gamma \\ 0 & b_3 & \cdots & \beta & \gamma \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & b_{n_1-1} & \gamma \\ c_2 & c_3 & \cdots & c_{n_1-1} & d \end{vmatrix} + (-1)^{n_1+1} c_1 \begin{vmatrix} \beta & \beta & \cdots & \beta & \gamma \\ b_2 & \beta & \cdots & \beta & \gamma \\ 0 & b_3 & \cdots & \beta & \gamma \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & b_{n_1-1} & \gamma \end{vmatrix} \\
&= b_1 Q_{n_1-1} + (-1)^{n_1+1} c_1 P_{n_1-1} \\
&= b_1 Q_{n_1-1} + (-1)^{n_1+1} c_1 (\beta - b_2) P_{n_1-2} \\
&= b_1 Q_{n_1-1} + (-1)^{n_1+1} c_1 \prod_{i=2}^{n_1-1} (\beta - b_i)\gamma
\end{aligned}$$

$$
\begin{aligned}
&= b_1\left(b_2Q_{n_1-2} + (-1)^{n_1}c_2\prod_{i=3}^{n_1-1}(\beta - b_i)\gamma\right) + (-1)^{n_1+1}c_1\prod_{i=2}^{n_1-1}(\beta - b_i)\gamma \\
&= \prod_{i=1}^{2} b_iQ_{n_1-2} + b_1(-1)^{n_1}c_2\prod_{i=3}^{n_1-1}(\beta - b_i)\gamma + (-1)^{n_1+1}c_1\prod_{i=2}^{n_1-1}(\beta - b_i)\gamma \\
&= \prod_{i=1}^{2} b_i\left(b_3Q_{n_1-3} + (-1)^{n_1-1}c_3\prod_{i=4}^{n_1-1}(\beta - b_i)\gamma\right) + b_1(-1)^{n_1}c_2\prod_{i=3}^{n_1-1}(\beta - b_i)\gamma + (-1)^{n_1+1}c_1\prod_{i=2}^{n_1-1}(\beta - b_i)\gamma \\
&= \prod_{i=1}^{3} b_iQ_{n_1-3} + \prod_{i=1}^{2} b_i(-1)^{n_1-1}c_3\prod_{i=4}^{n_1-1}(\beta - b_i)\gamma + b_1(-1)^{n_1}c_2\prod_{i=3}^{n_1-1}(\beta - b_i)\gamma + (-1)^{n_1+1}c_1\prod_{i=2}^{n_1-1}(\beta - b_i)\gamma \\
&= \prod_{i=1}^{3} b_iQ_{n_1-3} + \sum_{t=0}^{2}\left(\prod_{i=1}^{t} b_i(-1)^{n_1+1-t}c_{t+1}\prod_{i=t+2}^{n_1-1}(\beta - b_i)\gamma\right) \\
&= \prod_{i=1}^{n_1-1} b_id + \sum_{t=0}^{n_1-2}\left((-1)^{n_1+1-t}c_{t+1}\gamma\prod_{i=1}^{t} b_i\prod_{i=t+2}^{n_1-1}(\beta - b_i)\right).
\end{aligned}
$$

Note: if $t = 0$, let $\prod_{i=1}^{t} b_i = 1$; if $t = n_1 - 2$, let $\prod_{i=t+2}^{n_1-1}(\beta - b_i) = 1$.

Hence, we get

$$
|xI_{n_1} - B| = \prod_{i=1}^{n_1}(x - \alpha(n-i)) - \sum_{j=1}^{n_1}\left((1-\alpha)(a_j - 1)\prod_{i=1}^{j-1}(x - \alpha(n-i))\prod_{i=j+1}^{n_1}((1-\alpha) + x - \alpha(n-i))\right).
$$

Note: if $j = n_1$, let $\prod_{i=j+1}^{n_1}((1-\alpha) + x - \alpha(n-i)) = 1$.

## Declaration of competing interest

The authors declare that they have no conflict of interest.

## Data availability

No data was used for the research described in the article.

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