Filomat 39:11 (2025), 3569–3591
https://doi.org/10.2298/FIL2511569F

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the study of some quasilinear non-coercive $p(\cdot)-$parabolic problem with L^1-data

Bouchaib Ferrahi[a], **Hassane Hjiaj**[b,*], **Rajae Zerouali**[a]

[a]*Laboratoire LAR2A, Department of Mathematics, Faculty of Sciences Tétouan, University Abdelmalek Essaadi, BP 2121, Tétouan, Morocco*
[b]*Department of Mathematics, Faculty of Sciences Tétouan, University Abdelmalek Essaadi, BP 2121, Tétouan, Morocco*

Abstract. In this paper, we consider the following quasilinear and non-coercive $p(\cdot)-$parabolic problem :

$$\begin{cases} \dfrac{\partial u}{\partial t} - \text{div } a(x,t,u,\nabla u) = f(x,t) - \text{div } F(x,t,u) & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(x,0) = u_0 & \text{in } \Omega, \end{cases}$$

with $f \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$. We study the existence of entropy solutions for this problem in the parabolic Sobolev space with variable exponent V.

1. Introduction

Let Ω be a bounded open subset of R^N $(N \geq 2)$. For $T > 0$, we denote by Q_T the cylinder $\Omega \times (0,T)$ and by Σ_T the lateral surface $\partial\Omega \times (0,T)$.
Boccardo et al. have considered in [11] the quasilinear parabolic problem of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} - \text{div } a(x,t,u,\nabla u) = f(x,t) & \text{in } Q_T \\ u = 0 & \text{on } \Sigma_T, \\ u(x,0) = 0 & \text{in } \Omega \end{cases} \tag{1}$$

where $Au = -\text{div } a(x,t,u,\nabla u)$ is a Leray-Lions operator acting from $L^p(0,T;W_0^{1,p}(\Omega))$ to its dual $L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $f(x,t)$ is a measurable function that belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega))$. They have proved the existence of weak solution u for the problem (1) in the parabolic space $L^p(0,T;W_0^{1,p}(\Omega))$.

2020 *Mathematics Subject Classification*. Primary 35J60; Secondary 46E30, 46E35.
Keywords. Quasilinear parabolic equations, non-coercive operator, Sobolev spaces with variable exponent, entropy solutions.
Received: 20 October 2024; Accepted: 20 January 2025
Communicated by Marko Nedeljkov
* Corresponding author: Hassane Hjiaj
Email addresses: bferrahi@uae.ac.ma (Bouchaib Ferrahi), hjiajhassane@yahoo.fr (Hassane Hjiaj), rajae.zerouali@etu.uae.ac.ma (Rajae Zerouali)
ORCID iDs: https://orcid.org/0000-0001-8823-2161 (Bouchaib Ferrahi), https://orcid.org/0000-0002-1542-7926 (Hassane Hjiaj), https://orcid.org/0009-0001-8546-6322 (Rajae Zerouali)

In [15], Blanchard et al. have studied the quasilinear parabolic problem :

$$\begin{cases} \dfrac{\partial u}{\partial t} - \text{div } a(x,t,u,\nabla u) + \text{div } (\phi(u)) = f - \text{div} g & \text{in } Q_T \\ u = 0 & \text{on } \Sigma_T, \\ u(x,0) = u_0 & \text{in } \Omega \end{cases} \tag{2}$$

They have proved the existence of entropy solutions for the nonlinear parabolic problems (2). Moreover, they have conclude some regularity results.
The domain of Sobolev space with variable exponent has received a much attention in recent years, this impulse comes from their physical applications, such in electro-rheological fluids and image processing, we refer the reader to [17] and [29]). Bendahmane et al. have studied in [12] the nonlinear parabolic problem :

$$\begin{cases} \dfrac{\partial u}{\partial t} - \text{div } (|\nabla u|^{p(x)-2}\nabla u) = f & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(x,0) = u_0 & \text{in } \Omega. \end{cases} \tag{3}$$

They have proved the existence and uniqueness of entropy solutions for this nonlinear parabolic problem. We refer the reader also to [4].

In this paper, we study the non-coercive quasilinear $p(\cdot)$−parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \text{div } a(x,t,u,\nabla u) + |u|^{p(x)-2}u = f(x,t) - \text{div } F(x,t,u) & \text{in } Q_T \\ u = 0 & \text{on } \Sigma_T, \\ u(x,0) = u_0 & \text{in } \Omega \end{cases} \tag{4}$$

where $Au = -\text{div } a(x,t,u,\nabla u)$ is a Leray-Lions operator with degenerate coercivity, and the Carathéodory function $F(x,t,s)$ satisfy only some growth condition. The data $f(x,t)$ and $u_0(x)$ respectively belongs to $L^1(Q_T)$ and $L^1(\Omega)$.
This paper is organized as follows : In section 2, we recall some definitions and basic properties concerning the Sobolev spaces with variable exponents. We introduce in section 3 the assumptions on the Carathéodory functions $a(x,t,s,\xi)$ and $F(x,t,s)$ for which our problem has at least one solution. The section 4 will contains some important lemmas that are useful to prove our main result. The last section is devoted to show the existence of entropy solutions for our non-coercive quasilinear parabolic problem (1) in the Sobolev spaces with variable exponent V.

2. Preliminaries

Let Ω be an open bounded domain in R^N $(N \geq 2)$, we denote

$$C_+(\overline{\Omega}) = \{\text{measurable function} \quad p(\cdot) : \overline{\Omega} \longrightarrow R \quad \text{such that} \quad 1 < p_- < p_+ < \infty\},$$

where

$$p_- = \text{ess inf}\{p(x)/x \in \overline{\Omega}\} \qquad \text{and} \qquad p_+ = \text{ess sup}\{p(x)/x \in \overline{\Omega}\}.$$

We define the variable exponent Lebesgue space for $p(\cdot) \in C_+(\overline{\Omega})$ by

$$L^{p(\cdot)}(\Omega) := \{u : \Omega \mapsto R \quad \text{measurable} \quad / \int_\Omega |u(x)|^{p(x)}\, dx < \infty\}.$$

The space $L^{p(\cdot)}(\Omega)$ endowed with the norm :

$$\|u\|_{p(\cdot)} = \inf\left\{\lambda > 0, \quad \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}$$

is a uniformly convex Banach space, and therefore reflexive. We denote by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ a.e. in Ω.

Proposition 2.1.

(i) *For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have*

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p_-} + \frac{1}{p'_-}\right)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}.$$

(ii) *For all $p_1(\cdot), p_2(\cdot) \in C_+(\overline{\Omega})$ such that $p_1(x) \le p_2(x)$ a.e in Ω, then $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is a continuous embedding.*

Proposition 2.2.
We denote the modular

$$\rho(u) = \int_\Omega |u|^{p(x)}\,dx \qquad \textit{for any} \quad u \in L^{p(\cdot)}(\Omega),$$

then, the following assertions holds

(i) $\|u\|_{p(\cdot)} < 1 \quad (resp, = 1 > 1) \Longleftrightarrow \rho(u) < 1 \quad (resp, = 1, > 1),$

(ii) $\|u\|_{p(\cdot)} > 1 \Longrightarrow \|u\|_{p(\cdot)}^{p_-} \le \rho(u) \le \|u\|_{p(\cdot)}^{p_+}$ *and* $\|u\|_{p(\cdot)} < 1 \Longrightarrow \|u\|_{p(\cdot)}^{p_+} \le \rho(u) \le \|u\|_{p(\cdot)}^{p_-},$

(iii) $\|u_n\|_{p(\cdot)} \longrightarrow 0 \Longleftrightarrow \rho(u_n) \longrightarrow 0,$ *and* $\|u_n\|_{p(\cdot)} \longrightarrow \infty \Longleftrightarrow \rho(u_n) \longrightarrow \infty,$

which implies that the norm convergence and the modular convergence are equivalent.
We define the variable exponent Sobolev space by

$$W^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega) \quad \text{and} \quad |\nabla u| \in L^{p(\cdot)}(\Omega)\right\},$$

endowed with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)} \quad \forall u \in W^{1,p(\cdot)}(\Omega). \tag{5}$$

We denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ for the norm (5), and we note the Sobolev exponent by $p^*(x) = \frac{Np(x)}{N - p(x)}$ for $p(x) < N$ a.e in Ω.

Proposition 2.3.

(i) *Assuming that $1 < p_- \le p_+ < \infty$, the spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces.*

(ii) *If $q(\cdot) \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for a.e $x \in \Omega$, then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact.*

Now, we define the parabolic space with variable exponent $L^{p(\cdot)}(Q_T)$, by :

$$L^{p(\cdot)}(Q_T) := \left\{u : \Omega \mapsto R \quad \text{measurable} \quad / \quad \int_0^T \int_\Omega |u(x,t)|^{p(x)}\,dx\,dt < \infty\right\}.$$

Lemma 2.4. *Let Ω be a bounded open subset of R^N $(N \ge 2)$, with $T > 0$ and $p(\cdot) \in C_+(\overline{\Omega})$, then we have the following continuous dense embedding*

$$L^{p_+}(0,T;L^{p(\cdot)}(\Omega)) \hookrightarrow L^{p(\cdot)}(Q_T) \hookrightarrow L^{p_-}(0,T;L^{p(\cdot)}(\Omega)). \tag{6}$$

Definition 2.5. *Let $p(\cdot) \in C_+(\overline{\Omega})$ and $T > 0$, we define the space V by*

$$V = \left\{u \in L^{p_-}(0,T; W_0^{1,p(\cdot)}(\Omega)) \quad \textit{such that} \quad u \in L^{p(\cdot)}(Q_T) \quad \textit{and} \quad |\nabla u| \in L^{p(\cdot)}(Q_T)\right\}.$$

We denote the modular $\rho_{1,p(\cdot)}(u)$ for any $u \in V$ by

$$\rho_{1,p(\cdot)}(u) = \int_{Q_T} |u|^{p(x)}\, dx\, dt + \int_{Q_T} |\nabla u|^{p(x)}\, dx\, dt.$$

The space V endowed by the norm

$$\|u\|_V = \|u\|_{L^{p(\cdot)}(Q_T)} + \|\nabla u\|_{L^{p(\cdot)}(Q_T)},$$

is a separable and reflexive Banach space.

Lemma 2.6. *(See [31]) Let B_0, B and B_1 be some Banach spaces with $B_0 \subset B \subset B_1$. Let us set*

$$Y = \{u \textit{ measurable } \ / \ u \in L^{p_0}(0,T; B_0) \quad \textit{and} \quad u_t \in L^{p_1}(0,T; B_1)\}$$

where $p_0 > 1$ and $p_1 > 1$ are reals numbers.
Assuming that the embedding $B_0 \hookrightarrow\hookrightarrow B$ is compact, then

$$Y \hookrightarrow\hookrightarrow L^{p_0}(0,T; B)$$

and this embedding is compact.

Remark 2.7. *Let $p_- > \dfrac{2N}{N+2}$, and*

$$B_0 = W_0^{1,p(\cdot)}(\Omega), \quad B = L^2(\Omega) \quad \textit{and} \quad B_1 = W_0^{-1,p'(\cdot)}(\Omega),$$

with $p_0 = p_-$ and $p_1 = (p_+)'$. Thanks to the Lemma 2.6, we obtain

$$\{u : u \in V \quad \textit{and} \quad u_t \in V^\star\} \subseteq Y \hookrightarrow\hookrightarrow L^1(Q_T). \tag{7}$$

Moreover, in view of [12], we have

$$\{u : u \in V \quad \textit{and} \quad u_t \in V^\star\} \subseteq C([0,T]; L^1(\Omega)). \tag{8}$$

3. The time mollification of a function u in V

Let $\mu \geq 0$, we introduce the time mollification u_μ of a function $u \in V$, by

$$u_\mu(x,t) = \mu \int_{-\infty}^{t} \overline{u}(x,s) exp(\mu(s-t))\, ds \quad \text{where} \quad \overline{u}(x,s) = u(x,s)\chi_{(0,T)}(s).$$

Proposition 3.1. *(see [4]) If $u \in L^{p(\cdot)}(Q_T)$, then u_μ is measurable in Q_T, and $\dfrac{\partial u_\mu}{\partial t} = \mu(u - u_\mu)$. Moreover, we have*

$$\int_{Q_T} |u_\mu|^{p(x)}\, dx\, dt \leq \int_{Q_T} |u|^{p(x)}\, dx\, dt.$$

Proposition 3.2. *(see [4]) If $u \in V$, then $u_\mu \to u$ strongly in V as $\mu \to +\infty$.*

Proposition 3.3. *(see [4]) If $u_n \to u$ strongly in V, then $(u_n)_\mu \to u_\mu$ strongly in V.*

Remark 3.4. *We have $|(T_k(u))_\mu| \leq k$ for all $u \in V$.*
Indeed,

$$|(T_k(u))_\mu| = \left|\int_{-\infty}^{t} \mu\, exp(\mu(s-t))\overline{T_k(u(x,s))} ds\right| \leq k \int_{-\infty}^{t} \mu\, exp(\mu(s-t))\, ds = k$$

with $\overline{T_k(u(x,s))} = T_k(u(x,s)).\chi_{(0,T)}(s)$.

4. Essential Assumptions

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, taking $0 < T < \infty$ and $p(\cdot) \in C_+(\overline{\Omega})$ such that $p_- > \dfrac{2N}{N+2}$. We consider a Leray-Lions operator A acted from V into its dual V^*, defined by the formula

$$Au = -\text{div } a(x,t,u,\nabla u) + |u|^{p(x)-2}u \tag{9}$$

where $a(x,t,s,\xi) : Q_T \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}^N$ is a Carathéodory function (measurable with respect to (x,t) in Q_T for every (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$, and continuous with respect to (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every (x,t) in Q_T), which satisfies the following conditions :

$$|a(x,t,s,\xi)| \leq \beta\left(K(x,t) + |s|^{p(x)-1} + |\xi|^{p(x)-1}\right), \tag{10}$$

for a.e. $(x,t) \in Q_T$ and all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, where $K(x,t)$ is a positive function lying in $L^{p'(\cdot)}(Q_T)$ and $\beta > 0$.

$$(a(x,t,s,\xi) - a(x,t,s,\eta))(\xi - \eta) > 0 \qquad \text{for any} \quad \xi \neq \eta \quad \text{in} \quad \mathbb{R}^N. \tag{11}$$

There exists a positive decreasing function $b(\cdot) : [0,\infty[\longmapsto]0,\infty[$, and a constant $b_0 > 0$ such that

$$a(x,t,s,\xi)\xi \geq b(x,|s|)|\xi|^{p(x)} \qquad \text{with} \quad b(x,|s|) \geq \frac{b_0}{(1+|s|)^{\lambda(x)}}, \tag{12}$$

for a.e. $(x,t) \in Q_T$ and all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, where $0 \leq \lambda(x) < \min(1, p_0(x) - 1)$ a.e. in Ω.
The lower order term $F(x,t,s) : Q_T \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function which satisfy only the growth condition

$$|F(x,t,s)| \leq \frac{b(x,|s|)^{\frac{1}{p(x)}}|s|^{a(x)(p(x)-1)}}{|x|^{\beta(x)}}, \tag{13}$$

with $0 \leq a(x) < 1$ and $0 \leq \beta(x) < \dfrac{N(1-a(x))}{p'(x)}$ a.e. in Ω.
We consider the quasilinear and non-coercive $p(x)$−parabolic problem

$$.\begin{cases} u_t - \text{div } a(x,t,u,\nabla u) + |u|^{p(x)-2}u = f(x,t) - \text{div } F(x,t,u) & \text{in} \quad Q_T, \\ u(x,t) = 0 & \text{on} \quad \Sigma_T, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{14}$$

with $f(x,t) \in L^1(Q_T)$ and $u_0(x) \in L^1(\Omega)$.

5. Some technical Lemmas

Lemma 5.1. *(see [1]) Let $g \in L^{p(\cdot)}(Q_T)$ and $g_n \in L^{p(\cdot)}(Q_T)$ such that $\|g_n\|_{L^{p(\cdot)}(Q_T)} \leq C$ for $1 < p(x) < \infty$. If $g_n(x,t) \longrightarrow g(x,t)$ almost everywhere in Ω, then $g_n \rightharpoonup g$ weakly in $L^{p(\cdot)}(Q_T)$.*

Lemma 5.2. *Let $u \in V$, then $T_k(u) \in V$ for any $k > 0$. Moreover, we have*

$$T_k(u) \to u \quad \textit{strongly in} \quad V \quad \textit{as} \quad k \to \infty.$$

The proof of this Lemma is the same as in the case of constant exponent p.

Lemma 5.3. *(see. [4]) Let $m > 0$. Assuming that (10) – (12) hold true, and let $(u_n)_{n\in N}$ be a sequence in V such that : the sequence $\left(\dfrac{du_n}{dt}\right)_n$ is bounded in V^*, and $u_n \rightharpoonup u$ weakly in V, with*

$$\begin{aligned} &\int_{Q_T} \left(a(x,t,T_m(u_n),\nabla u_n) - a(x,t,T_m(u_n),\nabla u)\right) \cdot (\nabla u_n - \nabla u)\, dx\, dt \\ &+ \int_{Q_T} \left(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\right)(u_n - u)\, dx\, dt \longrightarrow 0 \quad \textit{for} \quad n \longrightarrow \infty, \end{aligned} \tag{15}$$

then $u_n \to u$ strongly in V for a subsequence.

6. Main results

Let $T_k(s) = \max(-k, \min(s,k))$, we set

$$\Theta_k(r) = \int_0^r T_k(s)\,ds = \begin{cases} \dfrac{r^2}{2} & \text{if} \quad |r| \le k, \\ k|r| - \dfrac{k^2}{2} & \text{if} \quad |r| > k. \end{cases} \tag{16}$$

Firstly, we introduce the definition of entropy solutions for our degenerated quasilinear $p(x)$- parabolic problem.

Definition 6.1. *A measurable function u is called entropy solution for the non-coercive quasilinear $p(x)-$parabolic problem* (14)*, if $T_k(u) \in V$ for any $k > 0$, and*

$$\begin{aligned} &\int_\Omega \Theta_k(u-\psi)(T)\,dx - \int_\Omega \Theta_k(u-\psi)(0)\,dx + \int_{Q_T} \frac{\partial \psi}{\partial t} T_k(u-\psi)\,dx\,dt \\ &+ \int_{Q_T} a(x,t,u,\nabla u)\cdot \nabla T_k(u-\psi)\,dx\,dt + \int_{Q_T} |u|^{p(x)-2} u T_k(u-\psi)\,dx\,dt \\ &\le \int_{Q_T} f T_k(u-\psi)\,dx\,dt + \int_{Q_T} F(x,t,u)\cdot \nabla T_k(u-\psi)\,dx\,dt \end{aligned} \tag{17}$$

for any $\psi \in V \cap L^\infty(Q_T)$ with $\dfrac{\partial \psi}{\partial t} \in V^* + L^1(Q_T)$.

Theorem 6.2. *Assuming that the conditions* (10) − (13) *hold true, then the non-coercive quasilinear $p(x)-$parabolic problem (*14*) has at least one entropy solution.*

Proof of the theorem 6.2

Step 1 : Approximate problem.

For any $n \in \mathbb{N}^*$, let $(u_{0,n})_n$ be a sequence in $C_0^\infty(\Omega)$ such that $u_{0,n} \to u_0$ strongly in $L^1(\Omega)$ and $|u_{0,n}| \le |u_0|$, and we set $f_n(x,t) = T_n(f(x,t))$.
We consider the sequence of approximate problems :

$$\begin{cases} (u_n)_t + A_n u_n + |u_n|^{p(x)-2} u_n = f_n(x,t) - \operatorname{div} F_n(x,t,u_n) & \text{in } Q_T, \\ u_n(x,t) = 0 & \text{on } \Sigma_T, \\ u_n(x,0) = u_{0,n} & \text{in } \Omega, \end{cases} \tag{18}$$

where $A_n v = -\operatorname{div} a(x,t,T_n(v),\nabla v) + |v|^{p(x)-2} v$ and $F_n(x,t,s) = T_n(F(x,t,s))$.
We define the operators A_n and $G_n : V \mapsto V^*$ by

$$\int_0^T \langle A_n u, v\rangle\,dt = \int_{Q_T} a_n(x,t,u,\nabla u)\cdot \nabla v\,dx\,dt + \int_{Q_T} |u|^{p(x)-2} u v\,dx\,dt \qquad \text{for any} \quad u,v \in V,$$

and

$$\int_0^T \langle G_n u, v\rangle\,dt = -\int_{Q_T} F_n(x,t,u)\cdot \nabla v\,dx\,dt \qquad \text{for any} \quad u,v \in V.$$

Lemma 6.3. *The operator $B_n = A_n + G_n$ acting from V into its dual V^* is bounded and pseudo-monotone. Moreover, B_n is coercive in the following sense :*

$$\frac{\displaystyle\int_0^T \langle B_n u, v\rangle\,dt}{\|v\|_V} \longrightarrow \infty \quad \textit{as} \quad \|v\|_V \to \infty \quad \textit{for} \quad v \in V.$$

For the proof of lemma 6.3 (see appendix).
In view of the lemma 6.3 (see [22]), there exists at least one weak solution $u_n \in V$ for the parabolic problem (18) .i.e :

$$\begin{aligned}&\int_0^T \langle \frac{\partial u_n}{\partial t}, v\rangle \,dt + \int_{Q_T} a_n(x,t,T_n(u_n),\nabla u_n)\cdot\nabla v \,dx\,dt + \int_{Q_T} |u_n|^{p(x)-2}u_n v\,dx\,dt \\ &= \int_{Q_T} f_n(x,t)v\,dx\,dt + \int_{Q_T} F_n(x,t,u_n)\cdot\nabla v\,dx\,dt \qquad \text{for any} \quad v\in V.\end{aligned} \tag{19}$$

Step 2 : A priori estimates.

Let n large enough, by taking $T_k(u_n)$ as a test function for the approximate problem (18), we have

$$\begin{aligned}&\int_0^T \langle \frac{\partial u_n}{\partial t}, T_k(u_n)\rangle \,dt + \int_{Q_T} a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n) \,dx\,dt + \int_{Q_T} |u_n|^{p(x)-2}u_n T_k(u_n)\,dx\,dt \\ &= \int_{Q_T} f_n(x,t)T_k(u_n)\,dx\,dt + \int_{Q_T} F_n(x,t,u_n)\cdot\nabla T_k(u_n)\,dx\,dt.\end{aligned} \tag{20}$$

For the first term on the left-hand side of (20), we have $\Theta_k(r) = \int_0^r T_k(s)ds$ then $\Theta_k(r)\geq 0$ and $|\Theta_k(r)|\leq k|r|$, it follows that

$$\begin{aligned}\int_0^T \langle \frac{\partial u_n}{\partial t}, T_k(u_n)\rangle \,dt &= \int_\Omega\int_0^T \frac{\partial u_n}{\partial t}\cdot T_k(u_n)\,dt\,dx \\ &= \int_\Omega\int_0^T \frac{\partial \Theta_k(u_n)}{\partial t}\,dt\,dx \\ &= \int_\Omega \Theta_k(u_n(T))\,dx - \int_\Omega \Theta_k(u_{0,n})\,dx \\ &\geq \int_\Omega \Theta_k(u_n(T))\,dx - k\|u_0\|_{L^1(\Omega)} \\ &\geq -k\|u_0\|_{L^1(\Omega)}.\end{aligned} \tag{21}$$

and since

$$\int_{Q_T} |u_n|^{p(x)-2}u_n T_k(u_n)\,dx\,dt \geq \int_{Q_T} |T_k(u_n)|^{p(x)}\,dx\,dt. \tag{22}$$

Thus, by combining (20) and (21) − (22) we conclude that

$$\begin{aligned}&\int_{Q_T} b(x,|u_n|)|\nabla T_k(u_n)|^{p(x)}\,dx\,dt + \int_{Q_T} |T_k(u_n)|^{p(x)}\,dx\,dt \\ &\leq \int_{Q_T} f_n(x,t)T_k(u_n)\,dx\,dt + \int_{Q_T} F_n(x,t,u_n)\cdot\nabla T_k(u_n)\,dx\,dt + k\|u_0\|_{L^1(\Omega)}.\end{aligned} \tag{23}$$

Concerning the second term on the right-hand side of (23), In view of Young's inequality and (13) we obtain

$$\begin{aligned}\int_{Q_T} |F_n(x,t,u_n)||\nabla T_k(u_n)|\,dx\,dt &\leq \frac{1}{2}\int_{Q_T} b(x,|u_n|)|\nabla T_k(u_n)|^{p(x)}\,dx\,dt + C_0\int_{\{|u_n|\leq k\}} \frac{|F_n(x,T_n(u_n))|^{p'(x)}}{b(x,|u_n|)^{\frac{p'(x)}{p(x)}}}\,dx\,dt \\ &\leq \frac{1}{2}\int_{Q_T} b(x,|u_n|)|\nabla T_k(u_n)|^{p(x)}\,dx\,dt + C_0\int_{\{|u_n|\leq k\}} \frac{b(x,|u_n|)^{\frac{p'(x)}{p(x)}}|u_n|^{\alpha(x)p(x)}}{b(x,|u_n|)^{\frac{p'(x)}{p(x)}}|x|^{\beta(x)}}\,dx\,dt \\ &\leq \frac{1}{2}\int_{Q_T} b(x,|u_n|)|\nabla T_k(u_n)|^{p(x)}\,dx\,dt + \frac{1}{2}\int_{\{|u_n|\leq k\}} |u_n|^{p(x)}\,dx\,dt \\ &\quad + C_1\int_{\{|u_n|\leq k\}} \frac{1}{|x|^{\frac{\beta(x)p'(x)}{1-a(x)}}}\,dx\,dt.\end{aligned}$$

(24)

By combining (23) and (24), we conclude that

$$\frac{1}{2}\int_{Q_T} b(x,|u_n|)|\nabla T_k(u_n)|^{p(x)}\,dx\,dt + \frac{1}{2}\int_{Q_T} |T_k(u_n)|^{p(x)}\,dx\,dt \le k\|f\|_{L^1(Q_T)} + C_1\int_{Q_T}\frac{1}{|x|^{\frac{\beta(x)p'(x)}{1-a(x)}}}\,dx\,dt + k\|u_0\|_{L^1(\Omega)}. \quad (25)$$

Since $\frac{\beta(x)p'(x)}{1-a(x)} < N$ then $\frac{1}{|x|^{\frac{\beta(x)p'(x)}{1-a}}} \in L^1(Q_T)$, it follows that

$$\frac{1}{2}\int_{Q_T} b(x,|u_n|)|\nabla T_k(u_n)|^{p(x)}\,dx\,dt + \frac{1}{2}\int_{Q_T} |T_k(u_n)|^{p(x)}\,dx\,dt \le kC_2. \quad (26)$$

In view of (12) we obtain

$$\frac{b_0}{(1+k)^{\lambda_+}}\int_{Q_T} |\nabla T_k(u_n)|^{p(x)}\,dx\,dt + \int_{Q_T} |T_k(u_n)|^{p(x)}\,dx\,dt \le kC_3. \quad (27)$$

Therefore, we get

$$\|T_k(u_n)\|_V^{p_-} \le \int_{Q_T} |\nabla T_k(u_n)|^{p(x)}\,dx\,dt + \int_{Q_T} |T_k(u_n)|^{p(x)}\,dx\,dt + 2 \le C_4(k+1)^{\lambda_+}k, \quad (28)$$

and we conclude that

$$\|T_k(u_n)\|_V \le C_5 k^{\frac{\lambda_+ + 1}{p_-}} \qquad \text{for any} \quad k \ge 1, \quad (29)$$

with C_5 is a positive constant that doesn't depend on n and k. Then, the sequence $(T_k(u_n))_n$ is uniformly bounded in V, and there exists a subsequence still denoted $(T_k(u_n))_n$ and a measurable function ψ_k such that :

$$\begin{cases} T_k(u_n) \rightharpoonup \psi_k & \text{weakly in} \quad V, \\ T_k(u_n) \to \psi_k & \text{strongly in} \quad L^1(Q_T) \quad \text{and a.e. in} \quad Q_T. \end{cases} \quad (30)$$

On the one hand, thanks to (27) it is obvious that :

$$\begin{aligned} k^{p_-}\text{meas}\{|u_n| > k\} &= \int_{\{|u_n|>k\}} |T_k(u_n)|^{p_-}\,dx\,dt \\ &\le \int_{Q_T} |T_k(u_n)|^{p(x)}\,dx\,dt + \text{meas}(Q_T) \\ &\le C_6 k, \end{aligned}$$

which implies that

$$\text{meas}\{|u_n| > k\} \le \frac{C_6}{k^{p_- - 1}} \to 0 \qquad \text{as} \quad k \to \infty. \quad (31)$$

Now, we will show that $(u_n)_n$ is a Cauchy sequences in measure. For all $\delta > 0$, we have

$$\text{meas}\{|u_n - u_m| > \delta\} \le \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\}.$$

Let $\varepsilon > 0$, thanks to (31) we can choose $k_0(\varepsilon) \ge 0$ large enough such that

$$\text{meas}\{|u_n| > k\} \le \frac{\varepsilon}{3} \qquad \text{and} \qquad \text{meas}\{|u_m| > k\} \le \frac{\varepsilon}{3} \qquad \text{for any} \quad k \ge k_0(\varepsilon). \quad (32)$$

Moreover, in view of (30) we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Q_T, then for all $k > 0$ and $\delta, \varepsilon > 0$: there exists $n_0 = n_0(k, \varepsilon, \delta) \geq 0$ such that

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\} \leq \frac{\varepsilon}{3} \qquad \text{for all } m, n \geq n_0(k, \delta, \varepsilon). \tag{33}$$

Thanks to (32) and (33), we conclude that : for any $\delta, \varepsilon > 0$, there exists $n_0 = n_0(\delta, \varepsilon) \geq 0$ such that

$$\text{meas}\{|u_n - u_m| > \delta\} \leq \varepsilon \qquad \text{for any} \quad n, m \geq n_0(\delta, \varepsilon).$$

Thus, the sequence $(u_n)_n$ is a Cauchy sequence in measure, and there exists a subsequence still denoted $(u_n)_n$ such that $u_n \to u$ almost everywhere in Q_T. Consequently, thanks to (30) we conclude that

$$T_k(u_n) \rightharpoonup T_k(u) \qquad \text{weakly in} \quad V. \tag{34}$$

Moreover, according to Lebesgue dominated convergence theorem we obtain

$$T_k(u_n) \longrightarrow T_k(u) \qquad \text{strongly in} \quad L^{p(\cdot)}(Q_T). \tag{35}$$

Step 3 : Some regularity results.

Let $h > k \geq 1$, we denote by $\varepsilon_j(n)$, $j = 1, 2, \ldots$ some real valued functions which converge to 0 as n goes to infinity. Similarly we define $\varepsilon_j(n, h)$ and $\varepsilon_i(n, h, \mu)$.
In this step, we will show that

$$\lim_{h\to\infty} \limsup_{n\to\infty} \frac{1}{h} \int_{\{|u_n| \leq h\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, dx = 0. \tag{36}$$

Let $h \geq 1$, by taking $\dfrac{T_h(s)}{h}$ as a test function for the approximate problem (18), we obtain :

$$\begin{aligned} &\frac{1}{h}\int_0^T \langle \frac{\partial u_n}{\partial t}, T_h(u_n)\rangle \, dt + \frac{1}{h}\int_{Q_T} a(x, t, T_h(u_n), \nabla T_h(u_n)) \cdot \nabla T_h(u_n) \, dx \, dt + \frac{1}{h}\int_{Q_T} |u_n|^{p(x)-1} |T_h(u_n)| \, dx \, dt \\ &= \frac{1}{h}\int_{Q_T} f_n T_h(u_n) \, dx \, dt + \frac{1}{h}\int_{Q_T} F_n(x, t, T_h(u_n)) \cdot \nabla T_h(u_n) \, dx \, dt. \end{aligned} \tag{37}$$

For the first term on the left-hand side of (37), we have $\Theta_k(r) = \displaystyle\int_0^r T_k(s) ds$ then

$$\begin{aligned} \frac{1}{h}\int_0^T \langle \frac{\partial u_n}{\partial t}, T_h(u_n)\rangle \, dt &= \frac{1}{h}\int_\Omega \int_0^T \frac{\partial u_n}{\partial t} T_h(u_n) \, dx \, dt \\ &= \frac{1}{h}\int_\Omega \int_0^T \frac{\partial \Theta_h(u_n)}{\partial t} \, dx \, dt \\ &= \frac{1}{h}\int_\Omega \Theta_h(u_n(T)) \, dx - \frac{1}{h}\int_\Omega \Theta_h(u_{0,n}) \, dx. \end{aligned} \tag{38}$$

Concerning the second term on the right-hand side of (37). In view of Young's inequality we get

$$\begin{aligned} &\frac{1}{h}\int_{Q_T} |F_n(x, t, u_n)| |\nabla u_n| \, dx \, dt \\ &\leq \frac{C_0}{h}\int_{\{|u_n| \leq h\}} \frac{|F_n(x, T_n(u_n))|^{p'(x)}}{b(x, |u_n|)^{\frac{p'(x)}{p(x)}}} \, dx \, dt + \frac{1}{4h}\int_{\{|u_n| \leq h\}} b(x, |u_n|) |\nabla u_n|^{p(x)} \, dx \, dt \\ &\leq \frac{C_0}{h}\int_{\{|u_n| \leq h\}} \frac{|T_h(u_n)|^{\alpha(x)p(x)}}{|x|^{\beta(x)p'(x)}} \, dx \, dt + \frac{1}{4h}\int_{\{|u_n| \leq h\}} b(x, |u_n|) |\nabla u_n|^{p(x)} \, dx \, dt \\ &\leq \frac{1}{2h}\int_{\{|u_n| \leq h\}} |u_n|^{p(x)} \, dx \, dt + \frac{C_1}{h}\int_{Q_T} \frac{1}{|x|^{\frac{\beta(x)p'(x)}{1-\alpha(x)}}} \, dx \, dt + \frac{1}{4h}\int_{\{|u_n| \leq h\}} b(x, |u_n|) |\nabla u_n|^{p(x)} \, dx \, dt. \end{aligned} \tag{39}$$

Having in mind $\frac{1}{h}\int_\Omega \Theta_h(u_n(T))\,dx \geq 0$, and by combining (37), (38) and (39), we deduce that

$$\begin{aligned}&\frac{1}{2h}\int_{\{|u_n|\leq h\}} a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,dx\,dt + \frac{1}{4h}\int_{\{|u_n|\leq h\}} b(x,|u_n|)|\nabla u_n|^{p(x)}\,dx\,dt\\ &\quad+\frac{1}{2h}\int_{\{|u_n|\leq h\}}|u_n|^{p(x)}\,dx\,dt + \int_{\{|u_n|>h\}}|u_n|^{p(x)-1}\,dx\,dt\\ &\leq \frac{1}{h}\int_{Q_T}|f(x,t)|\,|T_h(u_n)|\,dx\,dt + \frac{C_1}{h}\int_{Q_T}\frac{1}{|x|^{\frac{\beta(x)p'(x)}{1-a(x)}}}\,dx\,dt + \frac{1}{h}\int_\Omega \Theta_h(u_{0,n})\,dx.\end{aligned}\tag{40}$$

We have $f(x,t)$ belongs to $L^1(Q_T)$, and since $\frac{\beta(x)p'(x)}{1-a(x)} < N$ then $\frac{1}{|x|^{\frac{\beta(x)p'(x)}{1-a(x)}}} \in L^1(Q_T)$. Having in mind that $\frac{|T_h(u_n)|}{h} \rightharpoonup 0$ weak$-*$ in $L^\infty(Q_T)$, we deduce that

$$\varepsilon_1(n,h) = \frac{1}{h}\int_{Q_T} f_n(x,t)T_h(u_n)\,dx\,dt + \frac{1}{h}\int_{Q_T}\frac{1}{|x|^{\frac{\beta(x)p'(x)}{1-a(x)}}}\,dx\,dt \to 0 \quad \text{as} \quad n,h\to\infty. \tag{41}$$

and since u_0 belongs to $L^1(Q_T)$, then

$$\varepsilon_2(n,h) = \frac{1}{h}\int_\Omega \Theta_h(u_{0,n})\,dx = \int_{\{|u_{0,n}|\leq h\}}\frac{|u_{0,n}|^2}{h}\,dx + \int_{\{|u_{0,n}|>h\}}|u_{0,n}| - \frac{h}{2}\,dx \to 0 \quad \text{as} \quad n,h\to\infty. \tag{42}$$

By combining (40) and (41) – (42), we obtain :

$$\begin{aligned}&\frac{1}{2h}\int_{\{|u_n|\leq h\}} a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,dx\,dt + \frac{1}{4h}\int_{\{|u_n|\leq h\}} b(x,|u_n|)|\nabla u_n|^{p(x)}\,dx\,dt\\ &\quad+\frac{1}{2h}\int_{\{|u_n|\leq h\}}|u_n|^{p(x)}\,dx\,dt + \int_{\{|u_n|>h\}}|u_n|^{p(x)-1}\,dx\,dt\\ &\leq \varepsilon_3(n,h).\end{aligned}\tag{43}$$

Thus, by letting h and n goes to infinity, we conclude that

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\int_{\{|u_n|\leq h\}} a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,dx\,dt = 0. \tag{44}$$

Moreover, we have

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\int_{\{|u_n|\leq h\}} b(x,|u_n|)|\nabla u_n|^{p(x)}\,dx\,dt = 0, \tag{45}$$

and

$$\lim_{h\to\infty}\limsup_{n\to\infty}\int_{\{|u_n|>h\}}|u_n|^{p(x)-1}\,dx\,dt = 0, \tag{46}$$

and

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\int_{\{|u_n|\leq h\}}|u_n|^{p(x)}\,dx\,dt = 0. \tag{47}$$

Moreover, in view of (39), (45) and (47) we obtain

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\int_{\{|u_n|\leq h\}}|F_n(x,t,u_n)||\nabla u_n|\,dx\,dt = 0. \tag{48}$$

Step 4 : Equi-integrability of the sequence $(|u_n|^{p(x)-2}u_n)_n$.

In this part, we will show that :

$$|u_n|^{p(x)-2}u_n \to |u|^{p(x)-2}u \quad \text{strongly in} \quad L^1(Q_T). \tag{49}$$

Firstly, we show that $(|u_n|^{p(x)-2}u_n)_n$ is uniformly equi-integrable in Q_T.
For any measurable subset $E \subset Q_T$ and $h > 0$, we have :

$$\int_E |u_n|^{p(x)-1}\,dx\,dt \le \int_E |T_h(u_n)|^{p(x)-1}\,dx\,dt + \int_{\{|u_n|>h\}} |u_n|^{p(x)-1}\,dx\,dt. \tag{50}$$

In view of (35), it's clear that : for any $\varepsilon > 0$, there exists $\sigma(\varepsilon, h) > 0$ such that :

$$\int_E |T_h(u_n)|^{p(x)-1}\,dx\,dt \le \frac{\varepsilon}{2} \quad \text{for any} \quad E \subset Q_T \quad \text{with} \quad \text{meas(E)} \le \sigma(\varepsilon, h). \tag{51}$$

Moreover, thanks to (46), we obtain : for all $\varepsilon > 0$, there exists $h_0(\varepsilon) > 0$ such that :

$$\int_{\{|u_n|>h\}} |u_n|^{p(x)-1}\,dx\,dt \le \frac{\varepsilon}{2} \qquad \text{for any} \quad h \ge h_0(\varepsilon). \tag{52}$$

By combining (50) and (51) – (52), we conclude that : for any $\varepsilon > 0$, there exists $\sigma > 0$ such that :

$$\int_E |u_n|^{p(x)-1}\,dx\,dt \le \varepsilon \quad \text{for any} \quad Q_T \subset \Omega \quad \text{with} \quad \text{meas}(E) \le \sigma(\varepsilon). \tag{53}$$

Thus, the sequence $(|u_n|^{p(x)-2}u_n)_n$ is uniformly equi-integrable in Q_T, and since $|u_n|^{p(x)-2}u_n \to |u|^{p(x)-2}u$ a.e in Q_T. In view of Vitali's theorem, the convergence (49) is concluded.

Step 5 : The strong convergence of the gradient.

Let $h > k \ge 1$, we set $S_h(s) = 1 - \dfrac{|T_{2h}(s) - T_h(s)|}{h}$ and $w_{n,\mu} = T_k(u_n) - (T_k(u))_\mu$.
By using $v = w_{n,\mu}S_h(u_n)$ as a test function for the approximate problem (18), we have

$$\begin{aligned}
&\int_0^T \langle \frac{\partial u_n}{\partial t}, w_{n,\mu}S_h(u_n)\rangle\,dt + \int_{Q_T} a(x,t,T_n(u_n),\nabla u_n)\cdot \nabla u_n w_{n,\mu} S_h'(u_n)\,dx\,dt \\
&\quad + \int_{Q_T} a(x,t,T_n(u_n),\nabla u_n)\cdot(\nabla T_k(u_n) - \nabla (T_k(u))_\mu)S_h(u_n)\,dx\,dt \\
&\quad + \int_{Q_T} |u_n|^{p(x)-2}u_n w_{n,\mu}S_h(u_n)\,dx\,dt \\
&= \int_{Q_T} f_n w_{n,\mu}S_h(u_n)\,dx\,dt + \int_{Q_T} F_n(x,t,u_n)\cdot\nabla u_n w_{n,\mu}S_h'(u_n)\,dx\,dt \\
&\quad + \int_{Q_T} F_n(x,t,u_n)\cdot(\nabla T_k(u_n) - \nabla(T_k(u))_\mu)S_h(u_n)\,dx\,dt.
\end{aligned} \tag{54}$$

It is clear that $S_h(u_n) = 0$ on the set $\{|u_n| \ge 2h\}$ and $S_h(u_n) = 1$ on the set $\{|u_n| \le h\}$. Thus, we obtain

$$\begin{aligned}
&\int_{Q_T} \frac{\partial u_n}{\partial t} w_{n,\mu}S_h(u_n)\,dx\,dt + \int_{Q_T} a(x,t,u_n,\nabla u_n)\cdot(\nabla T_k(u_n) - \nabla(T_k(u))_\mu)S_h(u_n)\,dx\,dt \\
&\le \int_{Q_T} |f(x,t)||T_k(u_n) - (T_k(u))_\mu|\,dx\,dt + \int_{Q_T} |u_n|^{p(x)-1}|T_k(u_n) - (T_k(u))_\mu|\,dx\,dt \\
&\quad - \frac{2k}{h}\int_{\{h<|u_n|\le 2h\}} |F_n(x,t,u_n)||\nabla u_n|\,dx\,dt + \frac{2k}{h}\int_{\{h<|u_n|\le 2h\}} a(x,t,T_n(u_n),\nabla u_n)\nabla u_n\,dx\,dt \\
&\quad + \int_{Q_T} |F_n(x,t,T_{2h}(u_n))||\nabla T_k(u_n) - \nabla(T_k(u))_\mu|\,dx\,dt.
\end{aligned} \tag{55}$$

In view of lemma 7.1 (see Appendix), we have

$$\int_{Q_T} \frac{\partial u_n}{\partial t} w_{n,\mu} S_h(u_n)\, dx\, dt \geq \varepsilon_4(n) \tag{56}$$

For the first and second terms on the right-hand side of (55), We have $w_{n,\mu} \rightharpoonup 0$ weak-* in $L^\infty(Q_T)$, as n and μ tend to infinity, and since $f(x,t)$ belongs to $L^1(Q_T)$ we conclude that

$$\varepsilon_1(n,\mu) = \int_{Q_T} |f(x,t)||T_k(u_n) - (T_k(u))_\mu|\, dx\, dt \longrightarrow 0 \quad \text{as} \quad n,\mu \to \infty. \tag{57}$$

Similarly, thanks to (49) we get

$$\varepsilon_2(n,\mu) = \int_{Q_T} |u_n|^{p(x)-1}|T_k(u_n) - (T_k(u))_\mu|\, dx\, dt \longrightarrow 0 \quad \text{as} \quad n,\mu \to \infty, \tag{58}$$

Moreover, in view of (44) and (48), we obtain

$$\varepsilon_3(n,h) = \frac{2k}{h} \int_{\{h<|u_n|\leq 2h\}} |F_n(x,t,u_n)||\nabla u_n|\, dx\, dt \longrightarrow 0 \quad \text{as} \quad h,n \to \infty, \tag{59}$$

and

$$\varepsilon_4(n,h) = \frac{2k}{h} \int_{\{h<|u_n|\leq 2h\}} a(x,t,T_n(u_n),\nabla u_n)\cdot \nabla u_n\, dx\, dt \longrightarrow 0 \quad \text{as} \quad h,n \to \infty. \tag{60}$$

Concerning the last term on the right-hand side of (55), we have $F_n(x,t,T_{2h}(u_n)) \to F(x,t,T_{2h}(u))$ strongly in $(L^{p'(\cdot)}(Q_T))^N$, and since $\nabla T_k(u_n) - \nabla(T_k(u))_\mu \rightharpoonup 0$ weakly in $(L^{p(\cdot)}(Q_T))^N$, it follows that

$$\varepsilon_5(n,\mu) = \int_{Q_T} |F_n(x,t,T_{2h}(u_n))||\nabla T_k(u_n) - \nabla(T_k(u))_\mu|\, dx\, dt \longrightarrow 0 \quad \text{as} \quad n,\mu \to \infty. \tag{61}$$

By combining (55) and (56) – (57), we conclude that

$$\int_{Q_T} a(x,t,T_n(u_n),\nabla u_n)\cdot(\nabla T_k(u_n) - \nabla(T_k(u))_\mu)S_h(u_n)\, dx\, dt \leq \varepsilon_6(n,h,\mu). \tag{62}$$

Having in mind $a(x,t,s,0) = 0$, we obtain

$$\begin{aligned}
&\int_{Q_T} \Big(a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))\Big)\cdot(\nabla T_k(u_n) - \nabla T_k(u))\, dx\, dt \\
&\quad + \int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u))\cdot(\nabla T_k(u_n) - \nabla T_k(u))\, dx\, dt \\
&\quad + \int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot(\nabla T_k(u) - \nabla(T_k(u))_\mu)\, dx\, dt \\
&\quad - \int_{\{k<|u_n|\leq 2h\}} a(x,t,T_{2h}(u_n),\nabla T_{2h}(u_n))\cdot\nabla(T_k(u))_\mu S_h(u_n)\, dx\, dt \\
&\leq \varepsilon_6(n,h).
\end{aligned} \tag{63}$$

Thanks to Lebesgue's dominated convergence theorem, we have $|a(x,t,T_k(u_n),\nabla T_k(u))| \to |a(x,t,T_k(u),\nabla T_k(u))|$ strongly in $L^{p'(\cdot)}(Q_T)$, and since $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u))$ weakly in $(L^{p(\cdot)}(Q_T))^N$, then

$$\begin{aligned}
\varepsilon_7(n) &= \left|\int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u))\cdot(\nabla T_k(u_n) - \nabla T_k(u))\, dx\, dt\right| \\
&\leq \int_{Q_T} \Big|a(x,t,T_k(u_n),\nabla T_k(u))\Big|\Big|\nabla T_k(u_n) - \nabla T_k(u)\Big|\, dx\, dt \longrightarrow 0 \quad \text{as} \quad n \to \infty.
\end{aligned} \tag{64}$$

Concerning the third term on the left-hand side of (63), the sequence $(a(x,t,T_k(u_n),\nabla T_k(u_n)))_n$ is bounded in $(L^{p'(\cdot)}(Q_T))^N$, then there exists a measurable function $\xi_k \in (L^{p'(\cdot)}(Q_T))^N$ such that $a(x,t,T_k(u_n),\nabla T_k(u_n)))_n \rightharpoonup \xi_k$ weakly in $(L^{p'(\cdot)}(Q_T))^N$, and since $\nabla(T_k(u))_\mu \to \nabla T_k(u)$ strongly in $(L^{p(\cdot)}(Q_T))^N$, we deduce that

$$\varepsilon_8(n,\mu) = \int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n)) \cdot (\nabla T_k(u) - \nabla(T_k(u))_\mu)\, dx\, dt \longrightarrow 0 \quad \text{as} \quad n,\mu \to \infty. \tag{65}$$

For the last term on the left-hand side of (63), we have $a(x,T_{2h}(u_n),\nabla T_{2h}(u_n)) \rightharpoonup \xi_{2h}$ weakly in $(L^{p'(\cdot)}(Q_T))^N$, and since $\nabla(T_k(u))_\mu \to \nabla T_k(u)$ strongly in $(L^{p(\cdot)}(Q_T))^N$, we obtain

$$\begin{aligned} \varepsilon_9(n,\mu) &= \left| \int_{\{k<|u_n|\le 2h\}} a(x,T_{2h}(u_n),\nabla T_{2h}(u_n)) \cdot \nabla(T_k(u))_\mu S_h(u_n)\, dx\, dt \right| \\ &\le \int_{\{k<|u_n|\le 2h\}} |a(x,t,T_{2h}(u_n),\nabla T_{2h}(u_n))||\nabla(T_k(u))_\mu|\, dx\, dt \\ &\longrightarrow \int_{\{k<|u|\le 2h\}} |\xi_{2h}|\, |\nabla T_k(u)|\, dx\, dt = 0 \qquad \text{as} \quad n,\mu \to \infty. \end{aligned} \tag{66}$$

By combining (63) and (64) – (66), we conclude that

$$\int_{Q_T} \Big(a(x,t,T_k(u_n),\nabla T_k(u_n)) - a_(x,t,T_k(u_n),\nabla T_k(u))\Big)(\nabla T_k(u_n) - \nabla T_k(u))\, dx\, dt \le \varepsilon_{10}(h,n,\mu). \tag{67}$$

Having in mind that $T_k(u_n) \longrightarrow T_k(u)$ strongly in $L^{p(\cdot)}(Q_T)$, we obtain

$$\begin{aligned} &\int_{Q_T} (a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)))\,(\nabla T_k(u_n) - \nabla T_k(u))\, dx\, dt \\ &\quad + \int_{Q_T} \Big(|T_k(u_n)|^{p(x)-2}T_k(u_n) - |T_k(u)|^{p(x)-2}T_k(u)\Big)\,(T_k(u_n) - T_k(u))\, dx\, dt \to 0 \ \text{ as } \ n \to \infty. \end{aligned} \tag{68}$$

In view of the lemma 5.3, we deduce that

$$\begin{cases} T_k(u_n) \to T_k(u) & \text{strongly in} \quad V, \\ \nabla u_n \to \nabla u & \text{a.e. in } Q_T. \end{cases} \tag{69}$$

Step 6 : The convergence of $(u_n)_n$ in $C([0,T],L^1(\Omega))$.

Let $h \ge 1$ and $0 < s < T$. By taking $T_1(u_n - (T_h(u))_\mu)$ as a test function for the approximate problem (18), we obtain

$$\begin{aligned} &\int_\Omega \int_0^s \frac{\partial u_n}{\partial t} T_1(u_n - (T_h(u))_\mu)\, dx\, dt + \int_\Omega \int_0^s a(x,t,T_n(u_n),\nabla u_n) \cdot \nabla T_1(u_n - (T_h(u))_\mu)\, dx\, dt \\ &\quad + \int_\Omega \int_0^s |u_n|^{p(x)-2} u_n T_1(u_n - (T_h(u))_\mu)\, dx\, dt \\ &= \int_{Q_T} f_n T_1(u_n - (T_h(u))_\mu)\, dx\, dt + \int_{Q_T} F_n(x,t,u_n) \cdot \nabla T_1(u_n - (T_h(u))_\mu)\, dx\, dt. \end{aligned} \tag{70}$$

We have

$$\frac{\partial u_n}{\partial t} = \frac{\partial(u_n - (T_h(u))_\mu)}{\partial t} + \frac{\partial(T_h(u))_\mu}{\partial t} = \frac{\partial(u_n - (T_h(u))_\mu)}{\partial t} + \mu(T_h(u)) - (T_h(u))_\mu). \tag{71}$$

It follows that

$$\begin{aligned}\int_\Omega\int_0^s \frac{\partial u_n}{\partial t}T_1(u_n-(T_h(u))_\mu)\,dx\,dt &= \int_\Omega\int_0^s \frac{\partial(u_n-(T_h(u))_\mu)}{\partial t}T_1(u_n-(T_h(u))_\mu)\,dx\,dt\\ &\quad+\mu\int_\Omega\int_0^s (T_h(u)-(T_h(u))_\mu)T_1(u_n-(T_h(u))_\mu)\,dx\,dt\\ &= \int_\Omega \Theta_1(u_n(u(s))-(T_h(u(s)))_\mu)\,dx-\int_\Omega \Theta_1(u_{0,n}-T_h(u_0))\,dx\\ &\quad+\mu\int_\Omega\int_0^s (T_h(u)-(T_h(u))_\mu)T_1(u_n-(T_h(u))_\mu)\,dx\,dt.\end{aligned}\tag{72}$$

Note that, for every $s\in[0,T]$, by letting n tends to infinity, we obtain

$$\begin{aligned}&\int_\Omega\int_0^s (T_h(u)-(T_h(u))_\mu)T_1(u_n-(T_h(u))_\mu)\,dx\,dt\\ &\longrightarrow \int_\Omega\int_0^s (T_h(u)-(T_h(u))_\mu)T_1(u-(T_h(u))_\mu)\,dx\,dt\geq 0 \qquad\text{as}\qquad n\to\infty.\end{aligned}\tag{73}$$

For the second term on the left-hand side of (70), we have

$$\begin{aligned}&\int_\Omega\int_0^s a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla T_1(u_n-(T_h(u))_\mu)\,dx\,dt\\ &=\int_\Omega\int_0^s a(x,t,T_{h+1}(u_n),\nabla T_{h+1}(u_n))\cdot\nabla(T_{h+1}(u_n)-(T_h(u))_\mu)\cdot\chi_{\{|u_n-(T_h(u))_\mu|\leq 1\}}\,dx\,dt\\ &=\int_\Omega\int_0^s (a(x,t,T_{h+1}(u_n),\nabla T_{h+1}(u_n))-a(x,t,T_{h+1}(u_n),\nabla(T_h(u))_\mu))\\ &\qquad\times(\nabla(T_{h+1}(u_n)-(T_h(u))_\mu))\cdot\chi_{\{|u_n-(T_h(u))_\mu|\leq 1\}}\,dx\,dt\\ &\quad+\int_\Omega\int_0^s a(x,t,T_{h+1}(u_n),\nabla(T_h(u))_\mu)\cdot\nabla(T_1(u_n-(T_h(u))_\mu))\,dx\,dt.\end{aligned}\tag{74}$$

In view of (11), the first term on the right-hand side of (74) is positive. Concerning the second term, we have $a(x,t,T_{h+1}(u_n),\nabla(T_h(u))_\mu)\to a(x,t,T_{h+1}(u),\nabla T_h(u))$ strongly in $(L^{p'(\cdot)}(Q_T))^N$, and since $\nabla T_1(u_n-(T_h(u))_\mu)\rightharpoonup \nabla T_1(u-T_h(u))$ weakly in $(L^{p(\cdot)}(Q_T))^N$, we obtain

$$\begin{aligned}\varepsilon_1(n,\mu) &= \int_\Omega\int_0^s a(x,t,T_{h+1}(u_n),\nabla(T_h(u))_\mu)\cdot\nabla(T_1(u_n-(T_h(u))_\mu))\,dx\,dt\\ &\longrightarrow \int_{\{h<|u|\leq h+1\}} a(x,t,T_{h+1}(u),0)\cdot\nabla u\,dx\,dt=0 \qquad\text{as}\quad n,\mu\to\infty.\end{aligned}\tag{75}$$

On the other hand, we have $T_1(u_n-(T_h(u))_\mu)\rightharpoonup T_1(u-T_h(u))$ weak$-*$ in $L^\infty(Q_T)$ as n,h tends to infinity, and thanks to (49) we obtain

$$\int_\Omega\int_0^s |u_n|^{p(x)-2}u_nT_1(u_n-(T_h(u))_\mu)\,dx\,dt\longrightarrow \int_\Omega\int_0^s |u|^{p(x)-2}uT_1(u-T_h(u))\,dx\,dt\geq 0.\tag{76}$$

Similarly, we have $f_n(x,t)$ tends to $f(x,t)$ strongly in $L^1(Q_T)$ then

$$\int_\Omega\int_0^s |f_n(x,t)||T_1(u_n-(T_h(u))_\mu)|\,dx\,dt\longrightarrow \int_\Omega\int_0^s |f(x,t)||T_1(u-T_h(u))|\,dx\,dt \quad\text{as}\quad n,\mu\to\infty.\tag{77}$$

Moreover, we have $F_n(x,t,T_{h+1}(u_n))\to F(x,t,T_{h+1}(u))$ strongly in $(L^{p'(\cdot)}(Q_T))^N$, and since $\nabla T_1(u_n-(T_h(u))_\mu)\rightharpoonup \nabla T_1(u-T_h(u))$ weakly in $(L^{p(\cdot)}(Q_T))^N$, we obtain

$$\begin{aligned}&\int_\Omega\int_0^s F_n(x,t,T_{h+1}(u_n))\cdot\nabla T_1(u_n-(T_h(u))_\mu)\,dx\,dt\\ &\longrightarrow \int_\Omega\int_0^s F(x,t,T_{h+1}(u))\nabla T_1(u-T_h(u))\,dx\,dt \qquad\text{as}\quad n,\mu\to\infty.\end{aligned}\tag{78}$$

By combining (70) and (72) – (78), we conclude that

$$\begin{aligned}&\int_\Omega \Theta_1(u_n(s)-(T_h(u))_\mu)\,dx\\&\le \int_\Omega\int_0^s |f||T_1(u-T_h(u))|\,dx\,dt+\int_\Omega\int_0^s F(x,t,T_{h+1}(u))\cdot\nabla T_1(u-T_h(u))\,dx\,dt\\&\quad+\int_\Omega \Theta_1(u_0-T_h(u_0))\,dx+\varepsilon_{10}(n,\mu)\end{aligned}\tag{79}$$

we have :

$$\int_\Omega\int_0^s F(x,t,T_{h+1}(u))\cdot\nabla T_1(u-T_h(u))\,dx\,dt\longrightarrow 0 \qquad \text{as} \quad h\to\infty.\tag{80}$$

Moreover, similarly as in (42) we show that

$$\int_\Omega\int_0^s |f||T_1(u-T_h(u))|\,dx\,dt+\int_\Omega \Theta_1(u_0-T_h(u_0))\,dx\to 0 \quad \text{as} \quad h\to\infty.\tag{81}$$

By combining (79) and (80) – (81) we conclude that

$$\int_\Omega \Theta_1(u_n(s)-(T_h(u(s))_\mu))\,dx\le \varepsilon_{11}(n,\mu,h).\tag{82}$$

It follows that

$$\begin{aligned}\int_\Omega \Theta_1\left(\frac{u_n(s)-u_m(s)}{2}\right)dx \quad &\le \frac12\left(\int_\Omega \Theta_1(u_n(s)-(T_h(u(s))_\mu))\,dx+\int_\Omega \Theta_1(u_m(s)-(T_h(u(s))_\mu))\,dx\right)\\&\longrightarrow 0 \quad \text{as} \quad n,m\to\infty.\end{aligned}\tag{83}$$

Thus, we obtain

$$\begin{aligned}&\int_{\{|u_n(s)-u_m(s)|\le 2\}}\left|\frac{u_n(s)-u_m(s)}{2}\right|^2 dx+\int_{\{|u_n(s)-u_m(s)|>2\}}\left|\frac{u_n(s)-u_m(s)}{2}\right| dx\\&\qquad\le 2\int_\Omega \Theta_1\left(\frac{u_n(s)-u_m(s)}{2}\right)dx\longrightarrow 0 \quad \text{as} \quad n,m\to\infty.\end{aligned}\tag{84}$$

We conclude that

$$\begin{aligned}&\int_\Omega |u_n(s)-u_m(s)|\,dx\\&=\int_{\{|u_n(s)-u_m(s)|\le 2\}}|u_n(s)-u_m(s)|\,dx+\int_{\{|u_n(s)-u_m(s)|>2\}}|u_n(s)-u_m(s)|\,dx\\&\le\left(\int_{\{|u_n(s)-u_m(s)|\le 2\}}|u_n(s)-u_m(s)|^2\,dx\right)^{\frac12}(\text{meas}(\Omega))^{\frac12}+\int_{\{|u_n(s)-u_m(s)|>2\}}|u_n(s)-u_m(s)|\,dx\longrightarrow 0 \quad \text{as} \quad n,m\to\infty.\end{aligned}\tag{85}$$

Hence $(u_n)_n$ is a Cauchy sequence in $C([0,T],L^1(\Omega))$, thus $u\in C([0,T],L^1(\Omega))$ and we have $u_n(x,s)\longrightarrow u(x,s)$ strongly in $L^1(\Omega)$ for any $0\le s<T$.

Step 7 : Weak convergence of $(S(u_n))_t$ ***in*** $V^*+L^1(Q_T)$ **.**

Let $S(\cdot)\in C_c^\infty(R)$ such that $supp(S'(\cdot))\subset[-M,M]$ with $M>0$ and $v\in V\cap L^\infty(Q_T)$. By taking $S'(u_n)v$ as a test function in (18), we have

$$\int_0^T \left\langle \frac{\partial u_n}{\partial t}, S'(u_n)v \right\rangle dt + \int_{Q_T} a(x,t,T_n(u_n),\nabla u_n)(S'(u_n)\nabla v + \nabla u_n S''(u_n)v)\,dx\,dt + \int_{Q_T} |u_n|^{p_0-2} u_n S'(u_n)v\,dx\,dt$$
$$= \int_{Q_T} f_n(x,t)S'(u_n)v\,dx\,dt + \int_{Q_T} F_n(x,t,T_n(u_n))(S'(u_n)\nabla v + \nabla u_n S''(u_n)v)\,dx\,dt.$$

Hence, one finds

$$\begin{aligned}
&\left|\int_0^T \left\langle \frac{\partial S(u_n)}{\partial t}, v \right\rangle dt\right| \\
&\leq \int_{Q_T} |a(x,t,T_n(u_n),\nabla u_n)|\,|S'(u_n)\nabla v + S''(u_n)v\nabla T_M(u_n)|\;dx\,dt \\
&\quad + \int_{Q_T} |u_n|^{p_0-1}\,|S'(u_n)v|\;dx\,dt + \int_{Q_T} \left|f_n(x,t)\right| |S'(u_n)v|\;dx\,dt \\
&\quad + \int_{Q_T} |F_n(x,t,T_n(u_n))|\,|S'(u_n)\nabla v + \nabla u_n S''(u_n)v|\,dx\,dt \\
&\leq \|a(x,t,T_M(u_n),\nabla T_M(u_n))\|_{L^{p(\cdot)'}(Q_T)} \\
&\qquad \times \left(\|S'(\cdot)\|_{L^\infty(R)}\|\nabla v\|_{L^{p(\cdot)}(Q_T)} + \|S''(\cdot)\|_{L^\infty(R)}\|v\|_{L^\infty(Q_T)}\|\nabla T_M(u_n)\|_{L^{p(\cdot)}(Q_T)}\right) \\
&\quad + \||u_n|^{p(\cdot)-1}\|_{L^1(Q_T)}\|S'(\cdot)\|_{L^\infty(R)}\|v\|_{L^\infty(Q_T)} + \|f_n(x,t)\|_{L^1(Q_T)}\|S'(\cdot)\|_{L^\infty(R)}\|v\|_{L^\infty(Q_T)} \\
&\quad + \|F_n(x,t,T_M(u_n))\|_{L^{p(\cdot)'}(Q_T)}\left(\|S'(\cdot)\|_{L^\infty(R)}\|\nabla v\|_{L^{p(\cdot)}(Q_T)} + \|S''(\cdot)\|_{L^\infty(R)}\|v\|_{L^\infty(Q_T)}\|\nabla T_M(u_n)\|_{L^{p(\cdot)}(Q_T)}\right) \\
&\leq C(\|v\|_V + \|v\|_{L^\infty(Q_T)}),
\end{aligned}$$

with C is a constant that does not depend on n. We deduce that $(\dfrac{\partial S(u_n)}{\partial t})_n$ is uniformly bounded in $V^*+L^1(Q_T)$, this implies that

$$\frac{\partial S(u_n)}{\partial t} \rightharpoonup \frac{\partial S(u)}{\partial t} \quad \text{weakly in} \quad V^* + L^1(Q_T). \tag{86}$$

Step 8 : Passage to the limit.

Let $\psi \in V \cap L^\infty(Q_T)$ with $\dfrac{\partial \psi}{\partial t} \in V^* + L^1(Q_T)$ and let $M = k + \|\psi\|_\infty$, by using $T_k(u_n - \psi)$ as a test function for the approximated problem (18), we get

$$\begin{aligned}
&\int_0^T \langle \frac{\partial u_n}{\partial t}, T_k(u_n-\psi)\rangle\,dt + \int_{Q_T} a(x,t,T_k(u_n),\nabla u_n)\cdot\nabla T_k(u_n-\psi)\,dx\,dt + \int_{Q_T} |u_n|^{p(x)-2}u_n T_k(u_n-\psi)\,dx\,dt \\
&= \int_{Q_T} f_n(x,t)T_k(u_n-\psi)\,dx\,dt + \int_{Q_T} F_n(x,t,u_n)\cdot\nabla T_k(u_n-\psi)\,dx\,dt.
\end{aligned} \tag{87}$$

On the one hand, if $\{|u_n| > M\}$ then $|u_n - \psi| \geq |u_n| - \|\psi\|_\infty > k$, therefore $\{|u_n - \psi| \leq k\} \subseteq \{|u_n| \leq M\}$, which implies that :

$$\begin{aligned}
&\int_{Q_T} a(x,t,T_M(u_n),\nabla T_M(u_n))\cdot\nabla T_k(u_n-\psi)\,dx\,dt \\
&= \int_{\{|u_n-\psi|\leq k\}} a(x,t,T_M(u_n),\nabla T_M(u_n))\cdot(\nabla T_M(u_n)-\nabla\psi)\,dx\,dt \\
&= \int_{\{|u_n-\psi|\leq k\}} (a(x,t,T_M(u_n),\nabla T_M(u_n)) - a(x,t,T_M(u_n),\nabla\psi))\cdot(\nabla T_M(u_n)-\nabla\psi)\,dx\,dt \\
&\quad + \int_{\{|u_n-\psi|\leq k\}} a(x,t,T_M(u_n),\nabla\psi)\cdot(\nabla T_M(u_n)-\nabla\psi)\,dx\,dt.
\end{aligned} \tag{88}$$

According to Fatou's lemma, we obtain :

$$\begin{aligned}
&\liminf_{n\to\infty}\int_{Q_T} a(x,t,T_M(u_n),\nabla u_n)\cdot\nabla T_k(u_n-\psi)\,dx\,dt\\
&\geq \int_{\{|u-\psi|\leq k\}} \big(a(x,t,T_M(u),\nabla T_M(u)) - a(x,t,T_M(u),\nabla\psi)\big)\cdot(\nabla T_M(u)-\nabla\psi)\,dx\,dt\\
&\quad+\int_{\{|u-\psi|\leq k\}} a(x,t,T_M(u),\nabla\psi)\cdot(\nabla T_M(u)-\nabla\psi)\,dx\,dt \\
&=\int_{\{|u-\psi|\leq k\}} a(x,t,T_M(u),\nabla T_M(u))\cdot(\nabla T_M(u)-\nabla\psi)\,dx\,dt\\
&=\int_{Q_T} a(x,t,u,\nabla u)\cdot\nabla T_k(u-\psi)\,dx\,dt.
\end{aligned} \tag{89}$$

Concerning the first term on the left-hand side of (87), we have :

$$\frac{\partial u_n}{\partial t}=\frac{\partial(u_n-\psi)}{\partial t}+\frac{\partial\psi}{\partial t},$$

then

$$\begin{aligned}
\int_0^T\langle\frac{\partial u_n}{\partial t},T_k(u_n-\psi)\rangle\,dt &= \int_\Omega\int_0^T\frac{\partial(u_n-\psi)}{\partial t}T_k(u_n-\psi)\,dx\,dt+\int_\Omega\int_0^T\frac{\partial\psi}{\partial t}T_k(u_n-\psi)\,dx\,dt\\
&=\int_\Omega\Big[\Theta_k(u_n-\psi)\Big]_0^T dx+\int_{Q_T}\frac{\partial\psi}{\partial t}T_k(u_n-\psi)\,dx\,dt\\
&=\int_\Omega\Theta_k(u_n(T)-\psi(T))\,dx-\int_\Omega\Theta_k(u_{0,n}-\psi(0))\,dx\\
&\quad+\int_{Q_T}\frac{\partial\psi}{\partial t}T_k(u_n-\psi)\,dx\,dt.
\end{aligned} \tag{90}$$

We have $u_n\to u$ strongly in $C([0,T],L^1(\Omega))$, then

$$\int_\Omega\Theta_k(u_{0,n}-\psi(0))\,dx\longrightarrow\int_\Omega\Theta_k(u_0-\psi(0))\,dx, \tag{91}$$

and

$$\int_\Omega\Theta_k(u_n(T)-\psi(T))\,dx\longrightarrow\int_\Omega\Theta_k(u(T)-\psi(T))\,dx. \tag{92}$$

Moreover, we have $\frac{\partial\psi}{\partial t}\in V^*+L^1(Q_T)$, and since $T_k(u_n-\psi)\rightharpoonup T_k(u-\psi)$ weakly in V and weak$-*$ in $L^\infty(Q_T)$, it follows that

$$\int_{Q_T}\frac{\partial\psi}{\partial t}T_k(u_n-\psi)\,dx\,dt\longrightarrow\int_{Q_T}\frac{\partial\psi}{\partial t}T_k(u-\psi)\,dx\,dt. \tag{93}$$

On the other hand, in view of (49) and the fact that $f_n(x,t)$ tends to $f(x,t)$ strongly in $L^1(Q_T)$, we conclude that

$$\int_{Q_T}|u_n|^{p(x)-2}u_nT_k(u_n-\psi)\,dx\,dt\longrightarrow\int_{Q_T}|u|^{p(x)-2}uT_k(u-\psi)\,dx\,dt, \tag{94}$$

and

$$\int_{Q_T}f_nT_k(u_n-\psi)\,dx\,dt\longrightarrow\int_{Q_T}fT_k(u-\psi)\,dx\,dt. \tag{95}$$

Concerning the last term on the right-hand side of (87), we have $F_n(x,t,T_M(u_n)) \to F(x,t,T_M(u))$ strongly in $(L^{p'(\cdot)}(Q_T))^N$, and since $\nabla T_k(u_n-\psi) \rightharpoonup \nabla T_k(u-\psi)$ weakly in $(L^{p(\cdot)}(Q_T))^N$, we obtain

$$\int_{Q_T} F_n(x,t,u_n)\cdot\nabla T_k(u_n-\psi)\,dx\,dt \longrightarrow \int_{Q_T} F(x,t,u)\cdot\nabla T_k(u-\psi)\,dx\,dt. \tag{96}$$

By combining (87) and (89) – (96), we deduce that :

$$\begin{aligned}
&\int_\Omega \Theta_k(u-\psi)(T)\,dx - \int_\Omega \Theta_k(u-\psi)(0)\,dx + \int_{Q_T}\frac{\partial\psi}{\partial t}T_k(u-\psi)\,dx\,dt\\
&\quad+\int_{Q_T} a(x,t,u,\nabla u)\cdot\nabla T_k(u-\psi) + \int_{Q_T}|u|^{p(x)-2}uT_k(u-\psi)\,dx\,dt\\
&\le \int_{Q_T} fT_k(u-\psi)\,dx\,dt + \int_{Q_T}F(x,t,u)\cdot\nabla T_k(u-\psi)\,dx\,dt,
\end{aligned} \tag{97}$$

which complete the proof of the theorem 6.2.

7. Appendix

Lemma 7.1. *Let $h\ge 1$, we set $w_{n,\mu}=T_k(u_n)-(T_k(u))_\mu$ and $\phi_h(s)=1-\dfrac{|T_{2h}(s)-T_h(s)|}{h}$. We will show that*

$$\int_{Q_T}\frac{\partial u_n}{\partial t}\phi_h(u_n)w_{n,\mu}\,dx\,dt \ge \varepsilon_4(n).$$

Proof. Let $h\ge 1$, we define :

$$\Phi_h(s)=\int_0^s\phi_h(\tau)\,d\tau=\begin{cases} s & \text{if} \quad |s|\le h,\\ \dfrac{s^2+4hs+h^2}{2h} & \text{if} \quad -2h\le s<-h,\\ \dfrac{-s^2+4hs-h^2}{2h} & \text{if} \quad h\le s<2h,\\ \dfrac{3h}{2}\cdot\operatorname{sign}(s) & \text{if} \quad |s|>2h.\end{cases} \tag{98}$$

we have :

$$\begin{aligned}
\int_{Q_T}\frac{\partial u_n}{\partial t}\phi_h(u_n)w_{n,\mu}\,dx\,dt &= \int_{Q_T}\frac{\partial(\Phi_h(u_n)-T_k(u_n))}{\partial t}(T_k(u_n)-(T_k(u))_\mu)\,dx\,dt\\
&\quad+\int_{Q_T}\frac{\partial T_k(u_n)}{\partial t}(T_k(u_n)-(T_k(u))_\mu)\,dx\,dt\\
&= \int_\Omega\Big[(\Phi_h(u_n)-T_k(u_n))(T_k(u_n)-(T_k(u))_\mu)\Big]_0^T dx\\
&\quad-\int_{Q_T}(\Phi_h(u_n)-T_k(u_n))\left(\frac{\partial T_k(u_n)}{\partial t}-\frac{\partial(T_k(u))_\mu}{\partial t}\right)dx\,dt\\
&\quad+\int_{Q_T}\frac{\partial T_k(u_n)}{\partial t}(T_k(u_n)-(T_k(u))_\mu)\,dx\,dt.
\end{aligned} \tag{99}$$

Concerning the first term on the right-hand side of (99), we have $\Phi_h(u_n)=T_k(u_n)=u_n$ on $\{|u_n|\le k\}$, having in mind that $\Phi_h(u_n)-T_k(u_n)$ have the same sign as u_n on the set $\{|u_n|>k\}$, then

$$\begin{aligned}
&\int_\Omega\Big[(\Phi_h(u_n)-T_k(u_n))(T_k(u_n)-(T_k(u))_\mu)\Big]_0^T dx\\
&\ge -\int_{\{|u_{0,n}|>k\}}(\Phi_h(u_{0,n})-T_k(u_{0,n}))(T_k(u_{0,n})-(T_k(u_0))_\mu)\,dx\\
&= -\int_{\{|u_{0,n}|>k\}}(\Phi_h(u_{0,n})-T_k(u_{0,n}))(T_k(u_{0,n})-T_k(u_0))\,dx=\varepsilon_1(n).
\end{aligned} \tag{100}$$

For the second term on the right-hand side of (99), we have $(\Phi_h(u_n) - T_k(u_n))\dfrac{\partial T_k(u_n)}{\partial t} = 0$, it follows that

$$\begin{aligned}
&-\int_{Q_T} (\Phi_h(u_n) - T_k(u_n))\left(\frac{\partial T_k(u_n)}{\partial t} - \frac{\partial (T_k(u))_\mu}{\partial t}\right) dx\,dt \\
&= \int_0^T \int_{\{|u_n|>k\}} (\Phi_h(u_n) - T_k(u_n))\frac{\partial (T_k(u))_\mu}{\partial t}\,dx\,dt \\
&= \mu \int_0^T \int_{\{|u_n|>k\}} (\Phi_h(u_n) - T_k(u_n))(T_k(u) - (T_k(u))_\mu)\,dx\,dt \\
&= \mu \int_0^T \int_{\{|u_n|>k\}} (\Phi_h(u_n) - T_k(u_n))(T_k(u) - T_k(u_n))\,dx\,dt \\
&\quad + \mu \int_0^T \int_{\{|u_n|>k\}} (\Phi_h(u_n) - T_k(u_n))(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt \\
&\geq \mu \int_0^T \int_{\{|u_n|>k\}} (\Phi_h(u_n) - T_k(u_n))(T_k(u) - T_k(u_n))\,dx\,dt = \varepsilon_2(n).
\end{aligned} \tag{101}$$

Concerning the last term on the right-hand side of (99), we obtain

$$\begin{aligned}
&\int_{Q_T} \frac{\partial T_k(u_n)}{\partial t}(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt \\
&= \int_{Q_T} \frac{\partial (T_k(u_n) - (T_k(u))_\mu)}{\partial t}(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt + \int_{Q_T} \frac{\partial (T_k(u))_\mu}{\partial t}(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt \\
&= \int_\Omega \left[\frac{(T_k(u_n) - (T_k(u))_\mu))^2}{2}\right]_0^T dx + \mu \int_{Q_T} (T_k(u) - (T_k(u))_\mu)(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt \\
&\geq -\int_\Omega \frac{(T_k(u_{0,n}) - T_k(u_0))^2}{2}\,dx + \mu \int_{Q_T} (T_k(u) - (T_k(u))_\mu)(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt \\
&\geq \varepsilon_3(n) + \mu \int_{Q_T} (T_k(u) - (T_k(u))_\mu)(T_k(u) - (T_k(u))_\mu)\,dx\,dt \\
&\geq \varepsilon_3(n).
\end{aligned} \tag{102}$$

By combining (99) and (100) – (102), we conclude that :

$$\int_{Q_T} \frac{\partial u_n}{\partial t}\phi_h(u_n) w_{n,\mu}\,dx\,dt \geq \varepsilon_4(n). \tag{103}$$

□

Proof of the Lemma 6.3

In view of Hölder's inequality and the growth condition (12), we can show that the operator A_n is bounded. For the operator G_n, we have for any $u, v \in V$

$$\begin{aligned}
\left|\int_0^T \langle G_n u, v\rangle\,dt\right| &\leq \int_{Q_T} |F_n(x,t,u)||\nabla v|\,dx\,dt \\
&\leq n \int_{Q_T} |\nabla v|\,dx\,dt \\
&\leq C_1.n\|v\|_V.
\end{aligned}$$

We conclude that $B_n = A_n + G_n$ is bounded. For the coercivity, we have for any $u \in V$:

$$
\begin{aligned}
\int_0^T \langle B_n u, u\rangle \, dt &= \int_0^T \langle A_n u, u\rangle \, dt + \int_0^T \langle G_n u, u\rangle \, dt \\
&= \int_{Q_T} a_(x,t,T_n(u),\nabla u)\cdot \nabla u \, dx\, dt + \int_{Q_T} |u|^{p(x)} \, dx\, dt - \int_{Q_T} |F_n(x,T_n(u))||\nabla u| \, dx\, dt \\
&\geq \frac{b_0}{(1+n)^{\lambda_+}} \int_{Q_T} |\nabla u|^{p(x)} \, dx\, dt + \int_{Q_T} |u|^{p(x)} \, dx\, dt - C_1 n \|u\|_V \\
&\geq \frac{b_0}{(1+n)^{\lambda_+}} (\|\nabla u\|^{p_-}_{L^{p(\cdot)}(Q_T)} - 1) + (\|u\|^{p_-}_{L^{p(\cdot)}(Q_T)} - 1) - C_1 n \|u\|_V \\
&\geq C_2 \|u\|_V^{p_-} - C_1 n \|u\|_V - \frac{b_0}{(1+n)^{\lambda_+}} - 1.
\end{aligned}
$$

Thus, we conclude that

$$
\frac{\int_0^T \langle B_n u, u\rangle \, dt}{\|u\|_V} \longrightarrow \infty \quad \text{as} \quad \|u\|_V \to \infty.
$$

Now, we will show that the operator B_n is pseudo-monotone. Let $(u_k)_k$ be a sequence in V such that :

$$
\begin{cases}
u_k \rightharpoonup u & \text{weakly in } \ V, \\
B_n u_k \rightharpoonup \chi_n & \text{weakly in } \ V^*, \\
\limsup\limits_{k\to\infty} \langle B_n u_k, u_k\rangle \leq \langle \chi_n, u\rangle.
\end{cases}
\tag{104}
$$

We will prove that

$$
\chi_n = B_n u \quad \text{and} \quad \langle B_n u_k, u_k\rangle \longrightarrow \langle \chi_n, u\rangle \quad \text{as } k \to +\infty.
$$

In view of (7), we have $u_k \to u$ strongly in $L^1(Q_T)$ for a subsequence still denoted $(u_k)_k$. We have $(u_k)_k$ is a bounded sequence in V, then the sequence $(a(x,t,T_n(u_k),\nabla u_k))_k$ is uniformly bounded in $(L^{p'(\cdot)}(Q_T))^N$, and there exists a measurable function $\vartheta \in (L^{p'(\cdot)}(Q_T))^N$ such that

$$
a(x,t,T_n(u_k),\nabla u_k) \rightharpoonup \vartheta_n \quad \text{weakly in} \quad (L^{p'(\cdot)}(Q_T))^N \quad \text{as} \quad k \to \infty, \tag{105}
$$

and

$$
|u_k|^{p(x)-2} u_k \rightharpoonup |u|^{p(x)-2} u \quad \text{weakly in} \quad L^{p'(\cdot)}(Q_T) \quad \text{as} \quad k \to \infty. \tag{106}
$$

Moreover, we have $(F_n(x,t,u_k))_k$ is uniformly bounded in $(L^{p'(\cdot)}(Q_T))^N$. In view of Lebesgue's dominated convergence theorem, we obtain

$$
F_n(x,t,u_k) \to F_n(x,t,u) \quad \text{strongly in} \quad (L^{p'(\cdot)}(Q_T))^N \quad \text{as} \quad k \to \infty. \tag{107}
$$

On the one hand, for any $v \in V$ we have

$$
\begin{aligned}
\langle \chi_n, v\rangle &= \lim_{k\to\infty} \langle B_n u_k, v\rangle \\
&= \lim_{k\to\infty} \int_{Q_T} a(x,t,T_n(u_k),\nabla u_k)\cdot \nabla v \, dx\, dt + \lim_{k\to\infty} \int_{Q_T} |u_k|^{p(x)-2} u_k v \, dx\, dt \\
&\qquad - \lim_{k\to\infty} \int_{Q_T} F_n(x,t,T_n(u_k))\cdot \nabla v \, dx\, dt \\
&= \int_{Q_T} \vartheta_n . \nabla v \, dx\, dt + \int_{Q_T} |u|^{p(x)-2} u v \, dx\, dt - \int_{Q_T} F_n(x,t,T_n(u))\cdot \nabla v \, dx\, dt.
\end{aligned}
\tag{108}
$$

In view of (104) and (108) we obtain

$$\begin{aligned}\limsup_{k\to\infty}\langle B_n u_k, u_k\rangle &= \limsup_{k\to\infty}\Big(\int_{Q_T} a(x,t,T_n(u_k),\nabla u_k)\cdot\nabla u_k\,dx\,dt + \int_{Q_T}|u_k|^{p(x)}\,dx\,dt \\ &\quad - \int_{Q_T} F_n(x,t,T_n(u_k))\cdot\nabla u_k\,dx\,dt\Big) \\ &\le \int_{Q_T}\vartheta_n.\nabla u\,dx\,dt + \int_{Q_T}|u|^{p(x)}\,dx\,dt - \int_{Q_T} F_n(x,t,T_n(u))\cdot\nabla u\,dx\,dt.\end{aligned} \tag{109}$$

We have $u_k \rightharpoonup u$ weakly in V, and thanks to (107) we conclude that

$$\lim_{k\to\infty}\int_{Q_T} F_n(x,t,T_n(u_k))\cdot\nabla u_k\,dx\,dt = \int_{Q_T} F_n(x,t,T_n(u))\cdot\nabla u\,dx\,dt. \tag{110}$$

It follows that

$$\begin{aligned}&\limsup_{k\to\infty}\left(\int_{Q_T} a(x,t,T_n(u_k),\nabla u_k)\cdot\nabla u_k\,dx\,dt + \int_{Q_T}|u_k|^{p(x)}\,dx\,dt\right) \\ &\quad\le \int_{Q_T}\vartheta_n\cdot\nabla u\,dx\,dt + \int_{Q_T}|u|^{p(x)}\,dx\,dt.\end{aligned} \tag{111}$$

On the other hand, in view of (11) we have

$$\begin{aligned}&\int_{Q_T}(a(x,t,T_n(u_k),\nabla u_k) - a(x,t,T_n(u_k),\nabla u))\cdot(\nabla u_k - \nabla u)\,dx\,dt \\ &\quad + \int_{Q_T}(|u_k|^{p(x)-2}u_k - |u|^{p(x)-2}u)(u_k - u)\,dx\,dt \ge 0.\end{aligned} \tag{112}$$

Hence

$$\begin{aligned}&\int_{Q_T} a(x,t,T_n(u_k),\nabla u_k)\cdot\nabla u_k\,dx\,dt + \int_{Q_T}|u_k|^{p(x)}\,dx\,dt \\ &\ge \int_{Q_T} a(x,t,T_n(u_k),\nabla u_k)\cdot\nabla u\,dx\,dt + \int_{Q_T}|u_k|^{p(x)-2}u_k u\,dx\,dt \\ &\quad + \int_{Q_T}|u|^{p(x)-2}u(u_k - u)\,dx\,dt + \int_{Q_T} a(x,t,T_n(u_k),\nabla u)\cdot(\nabla u_k - \nabla u)\,dx\,dt.\end{aligned} \tag{113}$$

In view of Lebesgue dominated convergence theorem, we have $T_n(u_k) \longrightarrow T_n(u)$ strongly in $L^{p(\cdot)}(Q_T)$, then $a(x,t,T_n(u_k),\nabla u) \longrightarrow a(x,t,T_n(u),\nabla u)$ strongly in $(L^{p'(\cdot)}(Q_T))^N$, and using (105) – (106) we get

$$\liminf_{k\to\infty}\left(\int_{Q_T} a(x,t,T_n(u_k),\nabla u_k)\cdot\nabla u_k\,dx\,dt + \int_{Q_T}|u_k|^{p(x)}\,dx\,dt\right) \ge \int_{Q_T}\vartheta_n\cdot\nabla u\,dx\,dt + \int_{Q_T}|u|^{p(x)}\,dx\,dt. \tag{114}$$

Having in mind (111), we conclude that :

$$\lim_{k\to\infty}\left(\int_{Q_T} a(x,t,T_n(u_k),\nabla u_k)\cdot\nabla u_k\,dx\,dt + \int_{Q_T}|u_k|^{p(x)}\,dx\,dt\right) = \int_{Q_T}\vartheta_n\cdot\nabla u\,dx\,dt + \int_{Q_T}|u|^{p(x)}\,dx\,dt. \tag{115}$$

By combining (108), (110) and (115) we deduce that $\langle B_n u_k, u_k\rangle \to \langle \chi_n, u\rangle$ as $k \to \infty$.
Now, by (105) and (115) we obtain :

$$\begin{aligned}&\int_{Q_T}(a(x,t,T_n(u_k),\nabla u_k) - a(x,t,T_n(u_k),\nabla u))\cdot(\nabla u_k - \nabla u)\,dx\,dt \\ &\quad + \int_{Q_T}(|u_k|^{p(x)-2}u_k - |u|^{p(x)-2}u)(u_k - u)\,dx\,dt \longrightarrow 0 \quad \text{as} \quad k\to\infty.\end{aligned} \tag{116}$$

In view of Lemma 5.3 we conclude that $u_k \to u$ strongly in V and $\nabla u_k \to \nabla u$ almost everywhere in Q_T. Therefore, we conclude that $a(x,t,T_n(u_k),\nabla u_k) \rightharpoonup a(x,t,T_n(u),\nabla u)$ weakly in $(L^{p'(\cdot)}(Q_T))^N$, and having in mind (106) and (107) we deduce that $\chi_n = B_n u$.

Example 7.2. *By taking $f(x,t) \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$, with*

$$a(x,t,u,\nabla u) = \frac{|\nabla u|^{p(x)-2}\nabla u}{(1+|u|)^{\lambda(x)}} \qquad \text{and} \qquad F(x,t,u) = \frac{|u|^{a(x)(p(x)-1)}}{(1+|u|)^{\lambda(x)}|x|^{\beta(x)}},$$

then the assumptions (10) – (13) *hold true. In view of the Theorem 6.2, the non-coercive quasilinear parabolic problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} - div\Big(\dfrac{|\nabla u|^{p(x)-2}\nabla u}{(1+|u|)^{\lambda(x)}}\Big) + |u|^{p(x)-2}u = f(x,t) - div\Big(\dfrac{|u|^{a(x)(p(x)-1)}}{(1+|u|)^{\lambda(x)}|x|^{\beta(x)}}\Big) & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(x,0) = u_0 & \text{in } \Omega, \end{cases}$$

has at least one entropy solution.

Conclusion 7.3. *In this paper, we have studying the existence of entropy solutions for our quasilinear and non-coercive parabolic problem* (14) *with L1-data. However, the existence of entropy solutions for the unilateral problem associated to our parabolic equation without the term $|u|^{p(x)-2}u$ remain an open problem.*

References

[1] L. Aharouch, E. Azroul, M. Rhoudaf, *Strongly nonlinear variational parabolic problems in weighted Sobolev spaces,* Aust. J. Math. Anal. Appl. 5, No. 2, Article No. **13** (2008), 1-25.

[2] *S. Antontsev and M. Chipot, Anisotropic equations: uniqueness and existence results,* J. Differential and Integral Equations **21** (5-6), 2008, 401-419.

[3] S. N. Antontsev, J. F.Rodrigues, *On stationary thermo-rheological viscous ows,* Ann. Univ. Ferrara, Sez. VII, Sci. Mat. **52** (2006), 19-36.

[4] E. Azroul, H. Hjiaj and B. Lahmi, *Existence of entropy solutions for some strongly nonlinear p(x)-parabolic problems with L^1-data* , An. Univ. Craiova, Ser. Mat. Inf. **42** (2), 2015, 273-299.

[5] E. Azroul, M. B. Benboubker and M. Rhoudaf, *On some p(x)-quasilinear problem with right-hand side measure*. Math. Comput. Simulation **102** (2014), 117-130.

[6] E. Azroul, A. Benkirane and M. Rhoudaf, *On some strongly nonlinear elliptic problems in L1-data with a nonlinearity having a constant sign in Orlicz spaces via penalization methods.* Aust. J. Math. Anal. Appl. **7**, (2010), no. 1, Art. 5, 1-25.

[7] E. Azroul, H. Hjiaj and M. Bouziani, *existence of solutions for some quasilinear p(x)-Elliptic problem with hardy potential* ,Mathematica Bohemica, Vol. 144 (2019), No. **3**, 299-324.

[8] E. Azroul, H. Hjiaj and A. Touzani, *Existence and Regularity of Entropy solutions For Strongly Nonlinear p(x)-elliptic equations*, Electronic J. Diff. Equ, **68**, (2013), 1–27.

[9] M. B. Benboubker, H. Hjiaj and S. Ouaro, *Entropy solutions to nonlinear elliptic anisotropic problem with variable exponent,* J. Appl. Anal. Comput. 4 (2014), no. **3**, 245-270.

[10] M. Bendahmane, M. Chrif and S. El Manouni, *An Approximation Result in Generalized Anisotropic Sobolev Spaces and Application*. Z. Anal. Anwend. 30 (2011), no. **3**, 341–353.

[11] L. Boccardo, A. Dallaglio and T.Gallouët, *Existence and Regularity Results for some Nonlinear Parabolic Equations*. Adv. Math. Sci. Appl. 9, No. **2**, (1999), 1017-1031.

[12] M. Bendahmane, P. Wittbold, A. Zimmermann, *Renormalized solutions for a nonlinear parabolic equation with variable exponents and L1-data* . J. Differential Equations 249 (2010), no. **6**, 1483-1515.

[13] A. Benkirane, and A. Elmahi; *An existence theorem for a strongly nonlinear elliptic problem in Orlicz spaces*. Nonlinear Analysis **36**, (1999), 11-24.

[14] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre and J. L . Vázquez, *An L1- theory of existence and uniqueness of solutions of nonlinear elliptic equations.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. **4**, (1995), 241-273.

[15] D. Blanchard, F. Murat and H. Redwane, *Existence and Uniqueness of a Renormalized Solution for a Fairly General Class of Nonlinear Parabolic Problems.*, Journal of Differential Equations **177** (2001), 331–374.

[16] M. F. Betta, A. Mercaldo , F. Murat and M. M. Porzio, *Uniqueness of renormalized solutions to nonlinear elliptic equationswith a lower order term and right-hand side in $L^1(\Omega)$*, ESAIM, Control Optim. Calc. Var. **8** (2002), 239-272.

[17] Y. Chen, S. Levine and M. Rao, *Variable exponent, linear growth functionals in image restoration.* SIAM J. Appl. Math. 66 (2006), no. **4**, 1383-1406.

[18] O. Guibé, *Uniqueness of the renormalized solution to a class of nonlinear elliptic equations.* In: On the Notions of Solution to Nonlinear Elliptic Problems: Results and Developments, Quad. Mat., vol. **23**, pp. 255–282. Dept. Math., Seconda Univ. Napoli, Caserta (2008).

[19] E. Hewitt and K. Stromberg, *Real and abstract analysis*. Springer-verlng, Berlin Heidelberg New York, 1965.

[20] B. Kone, S. Ouaro, S. Traoré, Weak solutions for anisotropic nonlinear elliptic equations with variable exponents, Electronic J. Diff. Equ, **144** (2009), 1-11.

[21] C. Leone and A. Porretta, *Entropy solutions for nonlinear elliptic equations in* L^1. Nonlinear Anal. 1998, **32**, 325–334.

[22] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod et Gauthiers-Villars, Paris 1969.

[23] M. Mihailescu, P. Pucci and V. Radulescu, *Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent*. J. Math. Anal. Appl., **340** (2008), 687 - 698.

[24] S. Ouaro and A. Ouédraogo, *Nonlinear parabolic problems with variable exponent and L1-Data.*, Electron. J. Differ. Equ. 2017, Paper No. **32** (2017), 1-32.

[25] S. Ouaro and S. Traore, *Weak and Entropy Solutions to Nonlinear Elliptic Problems with Variable Exponent* , Journal of Convex Analysis Volume 16 (2009), No. **2**, 523-541.

[26] M. M. Porzio, *On some quasilinear elliptic equations involving Hardy potential*, Rend. Mat. Appl., VII. Ser. **27** (2007), 277-297.

[27] M. Porzio and A. Primo, *Summability and existence results for quasilinear parabolic equations with Hardy potential term*, NoDEA, Nonlinear Differ. Equ. Appl. 20, No. **1** (2013), 65-100.

[28] A. Prignet, *Existence and uniqueness of "entropy" solutions of parabolic problems with L1 data,* Nonlinear Anal., Theory Methods Appl. 28, No. **12** (1997), 1943-1954.

[29] M. Růžička, *Electrorheological fluids: modeling and mathematical theory.* Lecture Notes in Mathematics, 1748. Springer-Verlag, Berlin, 2000.

[30] M. Sanchón and J.M. Urbano, *Entropy solutions for the p(x)-Laplace equation*. Trans. Amer. Math. Soc. **361** (2009), 6387-6405.

[31] J. Simon: Compact set in the space $L^p(0, T, B)$. Ann. Mat. Pura Appl. **146** (1987), 65-96.

[32] C. Yazough, E.Azroul and H.Redwane, *Existence of solutions for some nonlinear elliptic unilateral problems with measure data* , Electron. J. Qual. Theory Differ. Equ. 2013, Paper No. **43** (2013), 1-21.