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Projective iterative method for solving split-null inclusion, variational inequality and hierarchical fixed point problems application to signal recovery

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Abstract. This paper studies a common problem of hierarchical fixed point problems for nonexpansive and quasi-nonexpansive mappings, variational inequality, and split null inclusion problems. A hybrid projective method is modified to obtain strong convergence in Hilbert spaces. An example is infinitely dimensional spaces shown for supporting the main result. As applications, the proposed method is applied to solve signal recovery problems.

1. Introduction

Let $\mathbf{H}_1$ and $\mathbf{H}_2$ be two real Hilbert spaces with inner products $\langle\cdot,\cdot\rangle$ and induced norms $\|\cdot\|$, respectively. Let $C \subseteq \mathbf{H}_1$ and $Q \subseteq \mathbf{H}_2$ be nonempty, closed and convex sets. A mapping $\mathcal{T} : C \to C$ is called nonexpansive if $\|\mathcal{T}x - \mathcal{T}y\| \leq \|x - y\|$, for all $x, y \in C$. Fix($\mathcal{T}$) is denoted for the set of fixed points of $\mathcal{T}$, i.e., Fix($\mathcal{T}$):= $\{x \in C : \mathcal{T}x = x\}$. In this paper, we focus our attention on the following split null inclusion problem (in short, S$_P$NIP) which was introduced Byrne *et al.* [3]: Find $x^* \in \mathbf{H}_1$ such that

$$0 \in \mathcal{B}_1(x^*), \tag{1}$$

such that

$$y^* = \mathcal{A}x^* \in \mathbf{H}_2 \text{ solves } 0 \in \mathcal{B}_2(y^*). \tag{2}$$

where $\mathcal{B}_1 : \mathbf{H}_1 \to 2^{\mathbf{H}_1}$, $\mathcal{B}_2 : \mathbf{H}_2 \to 2^{\mathbf{H}_2}$ are multi-valued maximal monotone operators and $\mathcal{A} : \mathbf{H}_1 \to \mathbf{H}_2$ is a bounded linear operator. The solution set of S$_P$NIP(1)-(2) is denoted by $\Omega = \{x^* \in \mathbf{H}_1 : x^* \in$ Sol(NIP(1))

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and $\mathcal{A}x^* \in \mathrm{Sol(NIP(2))}\}$. Byrne *et al.* [3] studied the weak convergence theorems of iterative method for $\mathrm{S_PNPP(1)}$-(2). For a given $x_0 \in \mathbf{H}_1$, compute iterative sequence $\{x_n\}$ generated by the following scheme: for $n \geq 1$,

$$x_{n+1} = \mathcal{J}_{\lambda}^{\mathcal{B}_1}(x_n + \gamma \mathcal{A}^*(\mathcal{J}_{\lambda}^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}x_n), \quad \text{for } \lambda > 0, \tag{3}$$

where $\mathcal{I}$ is an identity mapping, $\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$ and $\gamma \in (0, \frac{1}{L})$ such that L is the spectral radius of $\mathcal{A}^*\mathcal{A}$. For obtaining strong convergence, Kazmi and Rizvi [11] modified viscosity method to solve the problem $\mathrm{S_PNIP(1)}$-(2) and fixed point problem for a nonexpansive mapping. For further related work, see [15].

It's well known that fixed point problems have been used to solve a powerful and effective method for solving many issues that emerge from real-world applications, for example see in [9, 10, 12–14, 18]. The famous one of fixed point problems is the following hierarchical fixed point problem (HFPP) which was introduced by Moudafi and Mainge [13], the problem is to solve fixed point problem for a nonexpansive mapping $\mathcal{T}$ with respect to another nonexpansive mapping $\mathcal{S}$ on C, that is finding $x^* \in \mathrm{Fix}(\mathcal{T})$ such that

$$\langle x^* - \mathcal{S}x^*, x^* - x\rangle \leq 0, \ \forall x \in \mathrm{Fix}(\mathcal{T}), \tag{4}$$

where $\mathcal{S} : C \to C$ is a nonexpansive mapping. The solution set of HFPP(4) is denoted by $\otimes := \{x^* \in C : x^* = (P_{\mathrm{Fix}(\mathcal{T})} \circ \mathcal{S})x^*\}$.

On the other hand, we study the variational inequality(VI) which is to find $x^* \in C$ such that

$$\langle \mathcal{D}x^*, x - x^*\rangle \geq 0, \ \ \forall x \in C, \tag{5}$$

introduced in [6] where $\mathcal{D} : \mathbf{H}_1 \to \mathbf{H}_1$. The solution set of VI(5) is denoted by Sol(VI). The well-known algorithm for solving VI(5) is the projected gradient algorithm as follows:

$$x_{n+1} = P_C(\mathcal{I} - \mu\mathcal{D})x_n, \ \forall n \geq 1, \tag{6}$$

where $\mu > 0$ and P_C is the metric projection of $\mathbf{H}_1$ onto C. To obtain the convergence theorem, the algorithm (6) requires the Lipschitz condition on the operator $\mathcal{D}$. Indeed, if $\mathcal{D}$ is L-Lipschitz continuous with $0 < \mu < \frac{2}{L}$, then there exists a unique point in Sol(VI) and the sequence $\{x_n\}$ generated by (6) converge strongly to this point. There is no analytic expression for the metric projection operator in most cases. So the algorithm (6) is not very convenient in the practical calculation. Further, it was found that if C is a fixed point set of a nonexpansive mapping, then the metric projection is not be used. In 2001, Yamada [17] introduced the following hybrid steepest descent method:

$$x_{n+1} = P_C(\mathcal{I} - \mu\beta_n\mathcal{D})\mathcal{T}x_n, \ \forall n \geq 1. \tag{7}$$

Under certain conditions, the sequence $\{x_n\}$ generated by (7) converges strongly to the unique point in Sol(VI) over the fixed point of $\mathcal{T}$.

In this paper, we modify a projective algorithm to find a common solution of split null inclusion problem, variational inequality and hierarchical fixed point problem for nonexpansive and quasi-nonexpansive mappings in real Hilbert spaces. Further, we prove that sequences generated by the proposed hybrid projective algorithm converge strongly to a common solution of these problems. As applications, signal recovery is considered.

2. Preliminaries

Let the symbols $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively, and $\omega_w(x_n)$ denote the set of all weak limits of the sequence $\{x_n\}$.

Definition 2.1. *A single-valued mapping $\mathcal{D} : \mathbf{H}_1 \to \mathbf{H}_1$ is said to be:*

(i) *monotone, if*

$$\langle \mathcal{D}x - \mathcal{D}y, x - y\rangle \geq 0, \ \ \forall x, y \in \mathbf{H}_1;$$

(ii) *k-strongly monotone, if there exists a constant* $k \in \mathbb{R}$ *with* $k > 0$ *such that*

$$\langle \mathcal{D}x - \mathcal{D}y, x - y\rangle \ \geq k\|x - y\|^2, \ \ \forall x, y \in \mathbf{H}_1;$$

(ii) *k-inverse strongly monotone, if there exists a constant* $k \in \mathbb{R}$ *with* $k > 0$ *such that*

$$\langle \mathcal{D}x - \mathcal{D}y, x - y\rangle \ \geq k\|\mathcal{D}x - \mathcal{D}y\|^2, \ \ \forall x, y \in \mathbf{H}_1;$$

(iii) *L-Lipschitz continuous, if there exists a constant* $L > 0$ *such that*

$$\|\mathcal{D}x - \mathcal{D}y\| \leq L\|x - y\|, \ \ \forall x, y \in \mathbf{H}_1;$$

(iv) *firmly nonexpansive, if it is k-inverse strongly monotone with* $k = 1$.

It's clearly that if $\mathcal{D}$ is an k-inverse strongly monotone mapping, then $\mathcal{D}$ is $\frac{1}{L}$-Lipschitz continuous.

Definition 2.2. [2]. *A multi-valued mapping* $\mathcal{D} : \mathbf{H}_1 \to 2^{\mathbf{H}_1}$ *is said to be:*

(i) *monotone if*

$$\langle u - v, x - y\rangle \geq 0, \ \textit{whenever}\ u \in \mathcal{D}(x), \ v \in \mathcal{D}(y);$$

(ii) *maximal monotone if* $\mathcal{D}$ *is monotone and the graph,* $\mathrm{graph}(\mathcal{D}) := \{(x, y) \in \mathbf{H}_1 \times \mathbf{H}_1 : y \in \mathcal{D}(x)\}$, *is not properly contained in the graph of any other monotone mapping.*

It is well known that for each $x \in \mathbf{H}_1$ and $\lambda > 0$ there exists a unique $z \in \mathbf{H}_1$ such that $x \in (\mathcal{I} + \lambda\mathcal{D})z$. The mapping $\mathcal{J}_\lambda^{\mathcal{D}} := (\mathcal{I} + \lambda\mathcal{D})^{-1}$ is called the resolvent of $\mathcal{D}$. It is a single-valued and firmly nonexpansive mapping defined on $\mathbf{H}_1$.

Lemma 2.3. [5] *If* $\mathcal{T}$ *is a nonexpansive mapping on* $\mathbf{H}_1$ *then* $\mathcal{T}$ *is demiclosed on* $\mathbf{H}_1$ *in the sense that, if* $x_n \rightharpoonup x \in \mathbf{H}_1$ *and* $\{x_n - \mathcal{T}x_n\} \to 0$, *then* $x \in \mathrm{Fix}(\mathcal{T})$.

Lemma 2.4. [1] *Let* C *be a nonempty, closed and convex subset of* $\mathbf{H}_1$, *and* $\mathcal{T} : C \to \mathbf{H}_1$ *be a nonexpansive mapping. Then* $\mathrm{Fix}(\mathcal{T})$ *is closed and convex.*

3. Strong convergence theorem

In this section, we construct the following algorithm to solve SpNIP(1)-(2), VI(5) and HFPP(4) for a nonexpansive mapping $\mathcal{T}$ and a continuous quasi-nonexpansive mapping $\mathcal{S} : C \to C$.

Algorithm 3.1.

Initialization: *Choose* $\{\alpha_n\}$, $\{\sigma_n\}$, $\{\delta_n\}$ *be real sequences in* $(0, 1)$ *and* $\{\mu\beta_n\} \subset (0, 2k)$. *Select an arbitrary starting point* $v_0 \in C$ *and* $C_0 = C$*: Set* $n = 0$.

Iterative Steps: *Given the current iterate* v_n, *for* $\lambda > 0$*:*
Step 1. *Compute*

$$\left.\begin{aligned}
w_n &= (1 - \delta_n)v_n + \delta_n\mathcal{P}_C(v_n - \mu\beta_n\mathcal{D}v_n);\\
y_n &= (1 - \sigma_n)w_n + \sigma_n\mathcal{S}v_n;\\
u_n &= (1 - \alpha_n)v_n + \alpha_n\mathcal{T}y_n;\\
z_n &= \mathcal{J}_\lambda^{\mathcal{B}_1}(u_n + \gamma\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n);\\
C_{n+1} &= \{z \in C_n : \|z_n - z\| \leq \|v_n - z\|\};\\
v_{n+1} &= \mathcal{P}_{C_{n+1}}v_0, \ n \geq 0,
\end{aligned}\right\} \tag{8}$$

where $\gamma \in \left(0, \dfrac{1}{\|A\|^2}\right)$.
Step 2. *Set* $n := n + 1$ *and go to* ***Step 1***.

Remark 3.2. *Algorithm comparisons in terms of structure and problem formulation between our Algorithm 3.1, Algorithm 2 in* [7]*, and Algorithm 1 in* [8]*:*

(i) *Our Algorithm 3.1 and Algorithm 1 in* [8] *are designed to address common solution problems. Specially, our algorithm tackles problems involving fixed points of nonexpansive mappings, variational inequalities, and split null inclusion problems, while Algorithm 1 in* [8] *focuses on solving problems involving fixed points of k-demicontractive mappings and variational inequalities.*

(ii) *Algorithm 2 in* [7] *is restricted to solving inclusion problems. However, our algorithm can be reduced to solve inclusion problems like Algorithm 2 in* [7]*. In contrast, Algorithm 1 in* [8] *cannot be reduced in this manner.*

(iii) *Our Algorithm 3.1 and Algorithm 1 in* [8] *follow a non-inertial structure, whereas Algorithm 1 in* [8] *incorporates inertial terms, which can impact convergence rates and computational complexity;*

(iv) *Algorithm 2 in* [7] *uses a simplified structure focused on inclusion problems, with fewer computational steps compared to our algorithm and Algorithm 1 in* [8].

Theorem 3.3. *Let $\mathbf{H}_1$ and $\mathbf{H}_2$ be two real Hilbert spaces, and let C and Q be two nonempty, closed and convex sets such that $C \subseteq \mathbf{H}_1$ and $Q \subseteq \mathbf{H}_2$. Let $\mathcal{A} : \mathbf{H}_1 \to \mathbf{H}_2$ be a bounded linear operator with its adjoint operator $\mathcal{A}^*$; let $\mathcal{B}_1 : \mathbf{H}_1 \to 2^{\mathbf{H}_1}, \mathcal{B}_2 : \mathbf{H}_2 \to 2^{\mathbf{H}_2}$ be multi-valued maximal monotone operators. Let $\mathcal{D} : C \to \mathbf{H}_1$ be k-inverse strongly monotone mappings, let $\mathcal{T} : C \to C$ be a nonexpansive mapping and $\mathcal{S} : C \to C$ be a continuous quasi-nonexpansive mapping such that $\mathcal{I} - \mathcal{S}$ is monotone and $\Gamma = \Omega \cap \otimes \cap \operatorname{Fix}(\mathcal{S}) \cap Sol(VI) \neq \emptyset$. Assume that the following conditions hold:*
(C1) $\lim_{n\to\infty} \inf \alpha_n > 0$;
(C2) $0 < \lim_{n\to\infty} \inf \sigma_n \leq \limsup_{n\to\infty} \sigma_n < 1$;
(C3) $0 < \lim_{n\to\infty} \inf \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$;
(C4) $0 < \lim_{n\to\infty} \inf \mu\beta_n \leq \limsup_{n\to\infty} \mu\beta_n < 2k$.
Then the iterative sequences $\{z_n\}$ and $\{v_n\}$ be generated by Algorithm (8) converges strongly to $z \in \Gamma$, where $z = \mathcal{P}_\Gamma v_0$.

Proof. We divide the proof into several steps.

Step I. First, we show that Γ and C_n for all $n \geq 0$ both are closed and convex. Since $\Gamma \neq \emptyset$, it follows from Lemma 2.4 that Sol(NIP(1)) = $\operatorname{Fix}(\mathcal{J}_\lambda^{\mathcal{B}_1})$ and Sol(NIP(2)) = $\operatorname{Fix}(\mathcal{J}_\lambda^{\mathcal{B}_2})$ are closed and convex sets. Clearly $\otimes$ is closed and convex, since $\otimes = \operatorname{Fix}(P_{\operatorname{Fix}(\mathcal{T})} \circ \mathcal{S}) \neq \emptyset$. Further, it is easy to observe that $\operatorname{Fix}(\mathcal{S})$ and Sol(VI)are closed and convex. Thus, Γ is nonempty, closed and convex and $\mathcal{P}_\Gamma v_0$ is then well defined.

Next, we show that C_{n+1} is closed and convex. Since C is closed and convex, then C_0 is also closed and convex. Suppose that C_k is closed and convex for some $k \geq 1$. For any $z \in C_k$, we have

$$\begin{aligned} \|z_k - z\|^2 &\leq \|v_k - z\|^2 \\ \Leftrightarrow \quad \|z_k - v_k\|^2 + 2\langle z_k - v_k, v_k - z\rangle &\leq 0. \end{aligned} \tag{9}$$

We easily observe from (9) that C_{k+1} is closed and convex for all $k \geq 1$. Therefore, C_n is closed and convex for all $n \geq 0$.

Step II. $\Gamma \subset C_n$ for each $n \geq 0$, $\{v_n\}$ is well defined and the sequences $\{v_n\}$, $\{u_n\}$, $\{z_n\}$, $\{w_n\}$ and $\{y_n\}$ are bounded. Let $p \in \Gamma$ then $p \in C$. Since $\mathcal{P}_C$ is firmly nonexpansive, we estimate

$$\begin{aligned} \|w_n - p\|^2 &\leq (1-\delta_n)\|v_n - p\|^2 + \delta_n\|\mathcal{P}_C(\mathcal{I} - \mu\beta_n\mathcal{D})v_n - \mathcal{P}_C(\mathcal{I} - \mu\beta_n\mathcal{D})p\|^2 \\ &\leq (1-\delta_n)\|v_n - p\|^2 + \delta_n\|(\mathcal{I} - \mu\beta_n\mathcal{D})v_n - (\mathcal{I} - \mu\beta_n\mathcal{D})p\|^2 \\ &\leq \|v_n - p\|^2 + \delta_n\Big(\mu^2\beta_n^2\|\mathcal{D}v_n - \mathcal{D}p\|^2 - 2\mu\beta_n\langle v_n - p, \mathcal{D}v_n - \mathcal{D}p\rangle\Big) \\ &\leq \|v_n - p\|^2 + \delta_n\Big(\mu^2\beta_n^2\|\mathcal{D}v_n - \mathcal{D}p\|^2 - 2\mu\beta_n k\|\mathcal{D}v_n - \mathcal{D}p\|^2\Big) \\ &\leq \|v_n - p\|^2 - \delta_n\mu\beta_n(2k - \mu\beta_n)\|\mathcal{D}v_n - \mathcal{D}p\|^2 \\ &\leq \|v_n - p\|^2, \end{aligned} \tag{10}$$

$$\begin{aligned}
\|y_n - p\|^2 &\leq (1-\sigma_n)\|w_n - p\|^2 + \sigma_n\|\mathcal{S}v_n - p\|^2 - \sigma_n(1-\sigma_n)\|\mathcal{S}v_n - w_n\|^2 \\
&\leq \|v_n - p\|^2 - \sigma_n(1-\sigma_n)\|\mathcal{S}v_n - w_n\|^2 \qquad (11)\\
&\leq \|v_n - p\|^2, \qquad (12)
\end{aligned}$$

and

$$\begin{aligned}
\|u_n - p\|^2 &= \|(1-\alpha_n)v_n + \alpha_n\mathcal{T}y_n - p\|^2 \\
&\leq (1-\alpha_n)\|v_n - p\|^2 + \alpha_n\|y_n - p\|^2 \\
&\leq \|v_n - p\|^2 - \sigma_n(1-\sigma_n)\|\mathcal{S}v_n - w_n\|^2 \qquad (13)\\
&\leq \|v_n - p\|^2. \qquad (14)
\end{aligned}$$

Since $p \in \Gamma$ then $p \in \Omega$ and hence $\mathcal{J}_\lambda^{\mathcal{B}_1}p = p$ and $\mathcal{J}_\lambda^{\mathcal{B}_2}\mathcal{A}p = \mathcal{A}p$, we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|\mathcal{J}_\lambda^{\mathcal{B}_1}(u_n + \gamma\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n) - p\|^2 \\
&= \|\mathcal{J}_\lambda^{\mathcal{B}_2}(u_n + \gamma\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n) - \mathcal{J}_\lambda^{\mathcal{B}_1}(p)\|^2 \\
&\leq \|u_n + \gamma\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n - p\|^2 \\
&= \|u_n - p\|^2 + \gamma^2\|\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|^2 + 2\gamma\langle u_n - p, \mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\rangle \\
&= \|u_n - p\|^2 + \gamma^2\|\mathcal{A}\|^2\|(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|^2 + 2\gamma\langle u_n - p, \mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\rangle. \qquad (15)
\end{aligned}$$

Further, we have

$$\begin{aligned}
2\gamma\langle u_n &- p, \mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\rangle \\
&= 2\gamma\langle \mathcal{A}u_n - \mathcal{A}p, (\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\rangle \\
&= 2\gamma\langle \mathcal{A}u_n - \mathcal{A}p + (\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n - (\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n, (\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\rangle \\
&= 2\gamma\Big\{\langle \mathcal{J}_\lambda^{\mathcal{B}_2}\mathcal{A}u_n - \mathcal{A}p, \mathcal{J}_\lambda^{\mathcal{B}_2}\mathcal{A}u_n - \mathcal{A}u_n\rangle - \|(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|^2\Big\} \\
&= \gamma\{\|\mathcal{J}_\lambda^{\mathcal{B}_2}\mathcal{A}u_n - \mathcal{A}p\|^2 + \|\mathcal{J}_\lambda^{\mathcal{B}_2}\mathcal{A}u_n - \mathcal{A}u_n\|^2 - \|\mathcal{A}u_n - \mathcal{A}p\|^2 - 2\|(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|^2\} \\
&= \gamma\{\|\mathcal{J}_\lambda^{\mathcal{B}_2}\mathcal{A}u_n - \mathcal{J}_\lambda^{\mathcal{B}_2}\mathcal{A}p\|^2 + \|(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|^2 - \|\mathcal{A}u_n - \mathcal{A}p\|^2 - 2\|(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|^2\} \\
&\leq \gamma\{\|\mathcal{A}u_n - \mathcal{A}p\|^2 - \|\mathcal{A}u_n - \mathcal{A}p\|^2 - \|(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|^2\} \\
&= -\gamma\|(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|^2. \qquad (16)
\end{aligned}$$

Replacing (16) in (15), we get

$$\|z_n - p\|^2 \leq \|u_n - p\|^2 - \gamma(1 - \gamma\|\mathcal{A}\|^2)\|(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|^2. \qquad (17)$$

Next, using (14) and (17), we estimate

$$\|z_n - p\|^2 \leq \|v_n - p\|^2 - \gamma(1 - \gamma\|\mathcal{A}\|^2)\|(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|^2. \qquad (18)$$

Since $\gamma \in \left(0, \frac{1}{\|\mathcal{A}\|^2}\right)$, (18) implies

$$\|z_n - p\| \leq \|v_n - p\|. \qquad (19)$$

This implies that $p \in C_{n+1}$ and hence $\Gamma \subset C_{n+1}$ for all $n \geq 0$. Consequently, C_{n+1} is nonempty, closed and convex and hence $v_{n+1} = \mathcal{P}_{C_{n+1}}v_0$ is well defined for all $n \geq 0$. Thus the sequence $\{v_n\}$ is well defined.

Let $l = \mathcal{P}_\Gamma v_0$. From $v_{n+1} = \mathcal{P}_{C_{n+1}}v_0$ and $l \in \Gamma \subset C_{n+1}$, we have

$$\|v_{n+1} - v_0\| \leq \|l - v_0\|, \ \forall n \geq 0. \qquad (20)$$

Therefore $\{v_n\}$ is bounded. From (10), (12), (14) and (19), we have that $\{w_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{z_n\}$ are also bounded.

Step III. $\lim_{n\to\infty} \|v_{n+1} - v_n\| = 0$; $\lim_{n\to\infty} \|z_n - v_n\| = 0$; $\lim_{n\to\infty} \|z_n - u_n\| = 0$; $\lim_{n\to\infty} \|\mathcal{P}_C(v_n - \mu\beta_n\mathcal{D}v_n) - v_n\| = 0$; $\lim_{n\to\infty} \|w_n - y_n\| = 0$ and $\lim_{n\to\infty} \|v_n - u_n\| = 0$.

Since $v_n = \mathcal{P}_{C_n}v_0$, $C_{n+1} \subset C_n$ and $v_{n+1} \in C_n$, we have

$$\|v_n - v_0\| \le \|v_{n+1} - v_0\|,\ \forall n \ge 0. \tag{21}$$

Therefore $\lim_{n\to\infty} \|v_n - v_0\|$ exists by (20) and (21).

By the properties of the metric projection $\mathcal{P}_{C_n}$ that $v_n = \mathcal{P}_{C_n}v_0$ and $v_{n+1} \in C_{n+1}$, we have

$$\|v_{n+1} - v_n\|^2 \le \|v_{n+1} - v_0\|^2 - \|v_n - v_0\|^2,\ \forall n \ge 0,$$

which implies that

$$\lim_{n\to\infty} \|v_{n+1} - v_n\| = 0. \tag{22}$$

Since $v_{n+1} = \mathcal{P}_{C_{n+1}}v_0 \in C_{n+1}$, it follows that

$$\|z_n - v_{n+1}\| \le \|v_n - v_{n+1}\|. \tag{23}$$

Hence, it follows from (22) and (23) that

$$\lim_{n\to\infty} \|z_n - v_{n+1}\| = 0. \tag{24}$$

Since

$$\|v_n - z_n\| \le \|v_n - v_{n+1}\| + \|v_{n+1} - z_n\|, \tag{25}$$

it follows from (22), (24) and (25) that

$$\lim_{n\to\infty} \|v_n - z_n\| = 0. \tag{26}$$

Next, from the assumption $\liminf_{n\to\infty} \sigma_n > 0$, (18) and (26), we have

$$\lim_{n\to\infty} \|(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\| = 0. \tag{27}$$

Since $\mathcal{J}_\lambda^{\mathcal{B}_1}$ is firmly nonexpansive, we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|\mathcal{J}_\lambda^{\mathcal{B}_1}(u_n + \gamma\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n) - p\|^2 \\
&= \|\mathcal{J}_\lambda^{\mathcal{B}_1}(u_n + \gamma\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n) - \mathcal{J}_\lambda^{\mathcal{B}_1}p\|^2 \\
&\le \langle z_n - p, u_n + \gamma\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n - p\rangle \\
&= \frac{1}{2}\Big[\|z_n - p\|^2 + \|u_n - p\|^2 - \|z_n - u_n - \gamma\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|^2\Big] \\
&\le \frac{1}{2}\Big[\|z_n - p\|^2 + \|u_n - p\|^2 - \|z_n - u_n\|^2 - \gamma^2\|\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\| \\
&\quad +2\gamma\langle z_n - u_n, \mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\rangle] \\
&\le \frac{1}{2}\Big[\|z_n - p\|^2 + \|u_n - p\|^2 - \|z_n - u_n\|^2 - \gamma^2\|\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\| \\
&\quad +2\gamma\|z_n - u_n\|\|\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|\Big],
\end{aligned}$$

which in turn yields

$$\|z_n - p\|^2 \leq \|u_n - p\|^2 - \|z_n - u_n\|^2 + 2\gamma\|z_n - u_n\|\|\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|, \tag{28}$$

and this together with (14) implies that

$$\begin{aligned}\|z_n - u_n\|^2 &\leq \|u_n - p\|^2 - \|z_n - p\|^2 + 2\gamma\|z_n - u_n\|\|\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\| \\ &\leq \|v_n - p\|^2 - \|z_n - p\|^2 + 2\gamma\|z_n - u_n\|\|\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\| \\ &\leq \|v_n - z_n\|(\|v_n - p\| + \|z_n - p\|) + 2\gamma\|z_n - u_n\|\|\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\| \\ &\leq L_1\|v_n - z_n\| + 2\gamma\|z_n - u_n\|\|\mathcal{A}^*\|\|(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|. \end{aligned} \tag{29}$$

Hence, it follows from (26), (27) and (29) that

$$\lim_{n\to\infty} \|z_n - u_n\| = 0. \tag{30}$$

Since

$$\|v_n - u_n\| \leq \|v_n - z_n\| + \|z_n - u_n\|. \tag{31}$$

It follows from (26), (30) and (31) that

$$\lim_{n\to\infty} \|v_n - u_n\| = 0. \tag{32}$$

It follows from (8) that

$$\alpha_n\|\mathcal{T}y_n - v_n\| = \|u_n - v_n\|. \tag{33}$$

It follows from (32), (33) and $\lim_{n\to\infty} \inf \alpha_n > 0$ that

$$\lim_{n\to\infty} \|\mathcal{T}y_n - v_n\| = 0. \tag{34}$$

From the assumption $\lim_{n\to\infty} \inf \sigma_n > 0$, (13) and (34), we have

$$\lim_{n\to\infty} \|\mathcal{S}v_n - w_n\| = 0. \tag{35}$$

It follows from (8) that

$$\|y_n - w_n\| = \sigma_n\|\mathcal{S}v_n - w_n\| \tag{36}$$

It follows from (35), (36) that

$$\lim_{n\to\infty} \|y_n - w_n\| = 0. \tag{37}$$

Since $\{\mu\beta_n\} \subset (0, 2k)$, $\mathcal{P}_C(\mathcal{I} - \mu\beta_n\mathcal{D})$ is nonexpansive. It follows from $\mathcal{T}$ and $\mathcal{S}$ are nonexpansive that

$$\begin{aligned}\|u_n - p\|^2 &\leq (1 - \alpha_n)\|v_n - p\|^2 + \alpha_n\|\mathcal{T}y_n - p\|^2 \\ &\leq (1 - \alpha_n)\|v_n - p\|^2 + \alpha_n\Big((1 - \sigma_n)\|w_n - p\|^2 + \sigma_n\|\mathcal{S}v_n - p\|^2\Big) \\ &\leq \|v_n - p\|^2 - \alpha_n(1 - \sigma_n)(1 - \delta_n)\delta_n\|\mathcal{P}_C(\mathcal{I} - \mu\beta_n\mathcal{D})v_n - v_n\|^2. \end{aligned} \tag{38}$$

It follows the assumptions (C1)-(C3), (32) and (38) that

$$\lim_{n\to\infty} \|\mathcal{P}_C(\mathcal{I} - \mu\beta_n\mathcal{D})x_n - x_n\| = 0. \tag{39}$$

It follows from the assumption (C3) and (39) that

$$\|w_n - x_n\| \leq \delta_n \|\mathcal{P}_C(\mathcal{I} - \mu\beta_n\mathcal{D})x_n - x_n\| \to 0, \tag{40}$$

as $n \to \infty$. Since

$$\|\mathcal{T}y_n - w_n\| \leq \|\mathcal{T}y_n - x_n\| + \|x_n - w_n\|,$$

it follows from (34) and (40) that

$$\lim_{n\to\infty} \|\mathcal{T}y_n - w_n\| = 0. \tag{41}$$

Since

$$\|\mathcal{T}y_n - y_n\| \leq \|\mathcal{T}y_n - w_n\| + \|w_n - y_n\|,$$

it follows from (37) and (41) that

$$\lim_{n\to\infty} \|\mathcal{T}y_n - y_n\| = 0. \tag{42}$$

Since

$$\|\mathcal{S}x_n - x_n\| \leq \|\mathcal{S}x_n - w_n\| + \|w_n - x_n\|,$$

it follows from (35) and (40) that

$$\lim_{n\to\infty} \|\mathcal{S}x_n - x_n\| = 0. \tag{43}$$

Step IV: $x^* \in \Gamma$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup x^*$. Further, from (26), (32), (37) and (39), there exist subsequences $\{y_{n_i}\}$ of $\{y_n\}$, $\{z_{n_i}\}$ of $\{z_n\}$, $\{u_{n_i}\}$ of $\{u_n\}$ and $\{w_{n_i}\}$ of $\{w_n\}$ such that $\{y_{n_i}\}$, $\{z_{n_i}\}$, $\{u_{n_i}\}$ and $\{w_{n_i}\}$ converge weakly to x^*. It follows from Lemma 2.3(ii), (42) and(43) that $x^* \in \text{Fix}(\mathcal{T})$ and $x^* \in \text{Fix}(\mathcal{S})$.

Now, we show that $x^* \in \otimes$. It follows from (8)

$$\frac{1}{\sigma_n}(\mathcal{T}y_n - y_n) = (\mathcal{I} - \mathcal{S})x_n + \frac{1}{\sigma_n}(\mathcal{T}y_n - w_n) + (w_n - x_n), \tag{44}$$

and hence for all $q \in \text{Fix}(\mathcal{T})$ and using monotonicity of $\mathcal{I} - \mathcal{S}$, we have

$$\begin{aligned}
\langle \frac{\mathcal{T}y_n - y_n}{\sigma_n}, x_n - q\rangle &= \langle (\mathcal{I} - \mathcal{S})x_n - (\mathcal{I} - \mathcal{S})q, x_n - q\rangle + \langle (\mathcal{I} - \mathcal{S})q, x_n - q\rangle \\
&\quad + \frac{1}{\sigma_n}\langle \mathcal{T}y_n - w_n, x_n - q\rangle + \langle w_n - x_n, x_n - q\rangle \\
&\geq \langle (\mathcal{I} - \mathcal{S})q, x_n - q\rangle + \frac{1}{\sigma_n}\langle \mathcal{T}y_n - w_n, x_n - q\rangle \\
&\quad + \langle w_n - x_n, x_n - q\rangle
\end{aligned} \tag{45}$$

It follows from (39), (41), (42), (45) that

$$\limsup_{n\to\infty} \langle q - \mathcal{S}q, x_n - q\rangle \leq 0, \quad \forall q \in \text{Fix}(\mathcal{T}). \tag{46}$$

Since $x_n \rightharpoonup x^*$, we get

$$\langle (\mathcal{I} - \mathcal{S})q, x^* - q\rangle \leq 0, \quad \forall q \in \text{Fix}(\mathcal{T}). \tag{47}$$

Since $\mathrm{Fix}(\mathcal{T})$ is convex, $tq + (1-t)x^* \in \mathrm{Fix}(\mathcal{T})$ for $t \in (0,1)$ and hence

$$\begin{aligned}\langle(\mathcal{I}-\mathcal{S})(tq+(1-t)x^*), x^* - (tq+(1-t)x^*)\rangle &= t\langle(\mathcal{I}-\mathcal{S})(tq+(1-t)x^*), x^*-q\rangle \\ &\leq 0, \ \ \forall q \in \mathrm{Fix}(\mathcal{T}),\end{aligned} \tag{48}$$

this implies that

$$\langle(\mathcal{I}-\mathcal{S})(tq+(1-t)x^*), x^*-q\rangle \leq 0, \ \ \forall q \in \mathrm{Fix}(\mathcal{T}).$$

Taking limits $t \to 0_+$, we have

$$\langle(\mathcal{I}-\mathcal{S})x^*, x^*-q\rangle \leq 0, \ \ \forall q \in \mathrm{Fix}(\mathcal{T}). \tag{49}$$

That is $x^* \in \otimes$. Since $\{x_{n_i}\}$ converges weakly to x^*, it follows from Lemma 2.3 and (39) that $x^* \in \mathrm{Sol(VI)}$. Next, we shall show that $x^* \in \Omega$. Since $x_n \rightharpoonup x^*$, then $u_n \rightharpoonup x^*$, $z_n \rightharpoonup x^*$ and $y_n \rightharpoonup x^*$. Since Algorithm 8 can be rewritten as

$$\frac{(u_n - z_n) + \mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n}{\lambda} \in \mathcal{B}_1(z_n). \tag{50}$$

By passing to the limit $n \to \infty$ in (50) and taking account (27), (30) and the fact that graph of maximal monotone mapping is weakly-strongly closed, we obtain $0 \in \mathcal{B}_1(x^*)$. By $\mathcal{A}$ is continuous, the nonexpansivity of $\mathcal{J}_\lambda^{\mathcal{B}_2}$, (27) and Lemma 2.3, we have that $0 \in \mathcal{B}_2(\mathcal{A}x^*)$. This shows that $x^* \in \Omega$ and thus $x^* \in \Gamma$.

Step V. We shall show that $x^* = \mathcal{P}_\Gamma x_0$ where x^* is strongly limit point of $\{x_n\}$. Since $x_n = \mathcal{P}_{C_n}x_0$ and $x^* \in \Gamma$, we have

$$\|x_0 - x^*\| \geq \|x_0 - x_n\|.$$

It follows from $l = \mathcal{P}_\Gamma x_0$ and the property of the norm that

$$\|l - x_0\| \leq \|x^* - x_0\| \leq \liminf_{n\to\infty} \|x_n - x_0\| \leq \limsup_{n\to\infty} \|x_n - x_0\| \leq \|l - x_0\|.$$

This yields that $\lim_{n\to\infty}\|x_n - x_0\| = \|l - x_0\| = \|x^* - x_0\|$. Since $x_n - x_0 \rightharpoonup x^* - x_0$ and $\|x_n - x_0\| \to \|x^* - x_0\|$ then from the Kadec-Klee property [5] of $\mathbf{H}_1$, we have $\lim_{n\to\infty} x_n = x^* = l$. Thus, we conclude that $\{x_n\}$ converges strongly to x^*, where $x^* = \mathcal{P}_\Gamma x_0$. □

4. Numerical illustrations

4.1. Function space

For supporting our main theorem, we now give an example in infinitely dimensional spaces $L_2[0,1]$ such that $\|\cdot\|$ is L_2-norm defined by $\|x\| = \sqrt{\int_0^1 |x(t)|^2 dt}$ where $x(t) \in L_2[0,1]$.

Example 4.1. *Let* $\mathbf{H}_1 = \mathbf{H}_2 = L_2[0,1]$ *and* $C = \{x(t) \in L_2[0,1] : \int_0^t x(s)ds < \infty\}$. *Define mappings as follow:*
(i) bounded linear operator $\mathcal{A} : L_2[0,1] \to L_2[0,1]$ *by* $\mathcal{A}x(t) = 2x(t)$, $\forall x(t) \in L_2[0,1]$;
(ii) maximal monotone operators $\mathcal{B}_1, \mathcal{B}_2 : L_2[0,1] \to L_2[0,1]$ *by* $\mathcal{B}_1 x(t) = 3x(t)$ *and* $\mathcal{B}_2 x(t) = \frac{x(t)}{5}$, $\forall x(t) \in L_2[0,1]$;
(iii) nonexpansive mapping $\mathcal{T} : L_2[0,1] \to L_2[0,1]$ *by* $\mathcal{T}x(t) = \frac{x(t)}{2}$, $\forall x(t) \in L_2[0,1]$;
(iv) continuous quasi-nonexpansive mapping $\mathcal{S} : L_2[0,1] \to L_2[0,1]$ *by* $\mathcal{S}x(t) = \frac{x(t)}{2}$, $\forall x(t) \in L_2[0,1]$;
(v) $\frac{\pi}{2}$*-inverse strongly monotone mapping* $\mathcal{D} : C \to L_2[0,1]$ *by* $\mathcal{D}x(t) = \int_0^t x(s)ds$.

For each $\lambda > 0$, we see that $\mathcal{J}_{\lambda}^{\mathcal{B}_1}(x) = \frac{x}{1+3\lambda}$ and $\mathcal{J}_{\lambda}^{\mathcal{B}_2}(x) = \frac{x}{1+\frac{1}{5}\lambda}$. We use the Cauchy error $\|x_{n+1} - x_n\|^2 < 10^{-10}$ for the stopping criterion. The performances of our algorithm are split into five cases.

Case I: Comparison of the proposed algorithm (3.1) with different parameters δ_n are shown when we choose $\mu\beta_n = \frac{n}{n+1}$, $\gamma = 0.1$, $\lambda = 0.1$, $\sigma_n = \alpha_n = \frac{n}{2n+1}$ and initializations $x_0 = \frac{\sin(t)+t}{2}$. Then the results are presented as follows:

δ_n	0.1	0.3	0.5	0.9	0.999
No. of Iter.	18	18	17	17	17
CPU time(s)	3.204624	3.219476	3.104514	3.132228	3.212694

Table1: Numerical results of different parameters δ_n.

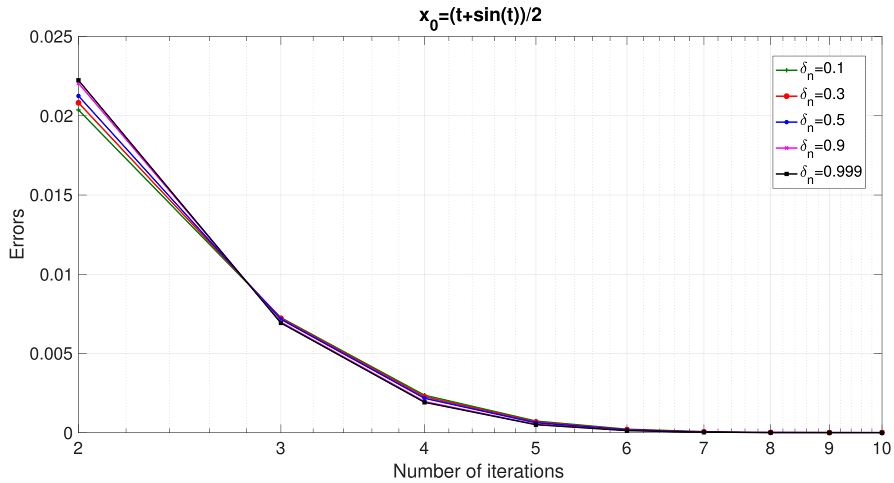

Figure 1: The error plotting of our proposed algorithm (3.1) for different parameters δ_n .

Case II: Comparison of the proposed algorithm (3.1) with different parameters $\mu\beta_n$ are shown when we choose $\delta_n = 0.5$, $\gamma = 0.1$, $\lambda = 0.1$, $\sigma_n = \alpha_n = \frac{n}{2n+1}$ and initializations $x_0 = \frac{\sin(t)+t}{2}$. Then the results are presented as follows:

$\mu\beta_n$	$\frac{n}{n+1}$	$\frac{n}{10n+1}$	$\frac{n}{100n+1}$	$\frac{n}{10^3n+1}$	$\frac{n}{10^4n+1}$
No. of Iter.	17	18	18	18	18
CPU time(s)	3.302351	3.310532	3.369519	3.301081	3.176498

Table2: Numerical results of different parameters $\mu\beta_n$.

Case III: Comparison of the proposed algorithm (3.1) with different parameters γ are shown by choosing $\mu\beta_n = \frac{n}{10^4n+1}$, $\delta_n = 0.5$, $\lambda = 0.1$, $\sigma_n = \alpha_n = \frac{n}{2n+1}$ and initializations $x_0 = \frac{\sin(t)+t}{2}$. Then the results are presented as follows:

γ	0.2	0.1	0.01	0.001	0.0001
No. of Iter.	18	18	19	19	19
CPU time(s)	3.062286	3.016581	3.620591	3.267629	3.195616

Table3: Numerical results of different parameters γ.

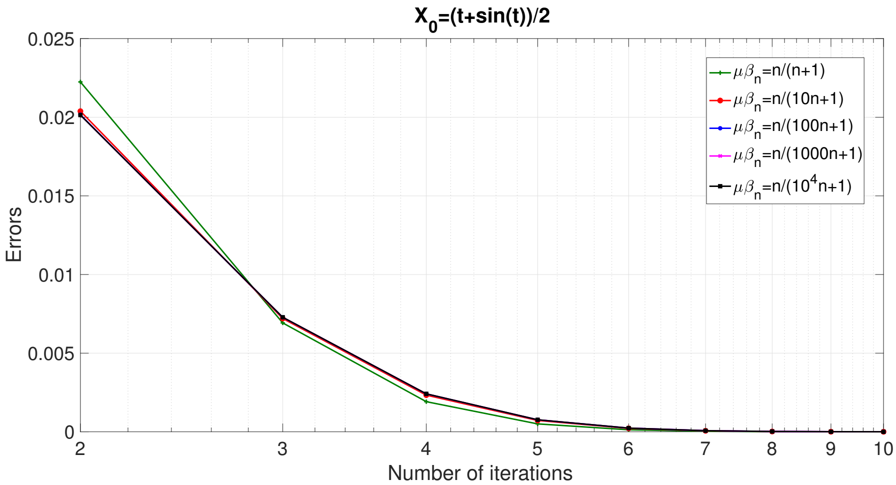

Figure 2: The error plotting of our proposed algorithm (3.1) for different parameters $\mu\beta_n$.

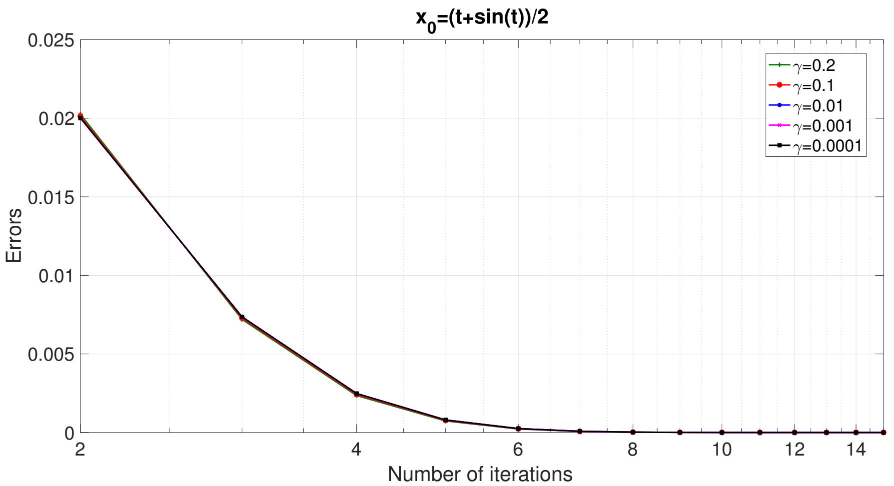

Figure 3: The error plotting of our proposed algorithm (3.1) for different parameters γ .

Case IV: Comparison of the proposed algorithm (3.1) with different parameters λ are shown by choosing $\gamma = 0.1$, $\mu\beta_n = \frac{n}{10^4 n+1}$, $\delta_n = 0.5$, $\sigma_n = \alpha_n = \frac{n}{2n+1}$ and initializations $x_0 = \frac{\sin(t)+t}{2}$. Then the results are presented as follows:

λ	0.1	1	10	100	10^3
No. of Iter.	18	8	4	3	3
CPU time(s)	3.189750	1.538551	0.9133888	0.752417	0.758241

Table4: Numerical results of different parameters λ.

Case V: Comparison of the proposed algorithm (3.1) with different parameters σ_n are shown by choosing $\lambda = 100$, $\gamma = 0.1$, $\delta_n = 0.5$, $\mu\beta_n = \frac{n}{10^4 n+1}$, $\alpha_n = \frac{n}{2n+1}$ and initializations $x_0 = \frac{\sin(t)+t}{2}$. Then the results are presented as follows:

σ_n	$\frac{n}{2n+1}$	$\frac{n}{10n+1}$	$\frac{n}{100n+1}$	$\frac{n}{10^3 n+1}$	$\frac{n}{10^4 n+1}$
No. of Iter.	3	3	3	3	3
CPU time(s)	0.743157	0.723417	0.744866	0.760672	0.675156

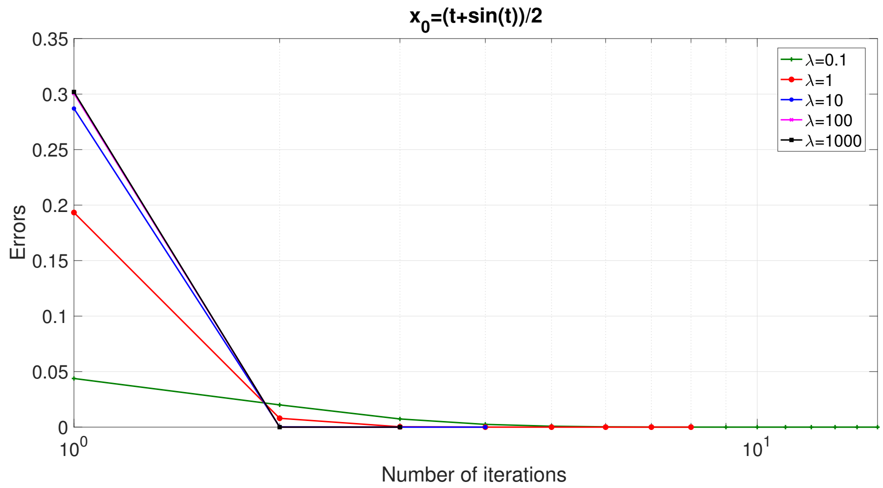

Figure 4: The error plotting of our proposed algorithm (3.1) for different parameters λ .

Table5: Numerical results of different parameters σ_n.

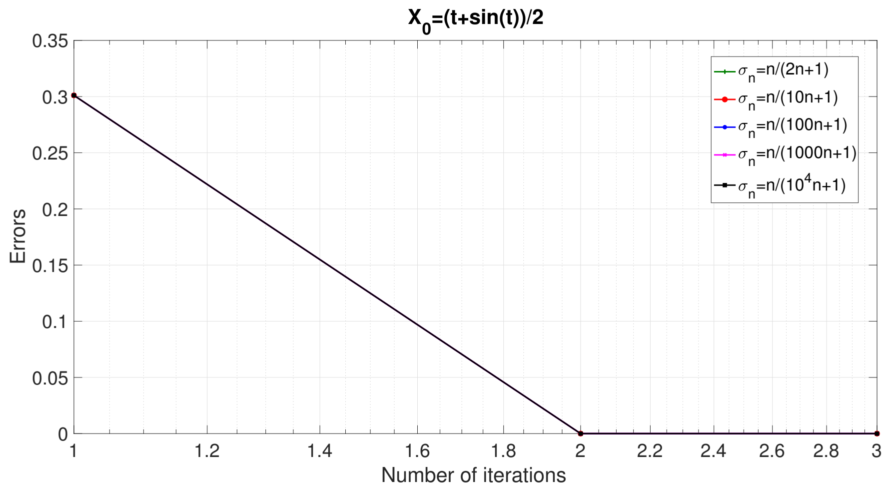

Figure 5: The error plotting of our proposed algorithm (3.1) for different parameters σ_n .

Case VI: Comparison of the proposed algorithm (3.1) with different parameters α_n are shown by choosing $\lambda = 100$, $\gamma = 0.1$, $\delta_n = 0.5$, $\mu\beta_n = \frac{n}{10^4 n+1}$, $\sigma_n = \frac{n}{10^4 n+1}$ and initializations $x_0 = \frac{\sin(t)+t}{2}$. Then the results are presented as follows:

α_n	$\frac{n}{2n+1}$	$\frac{n}{10n+1}$	$\frac{n}{100n+1}$	$\frac{n}{10^3 n+1}$	$\frac{n}{10^4 n+1}$
No. of Iter.	3	3	3	3	3
CPU time(s)	0.746312	0.748333	0.741478	0.735886	0.769183

Table6: Numerical results of different parameters α_n.

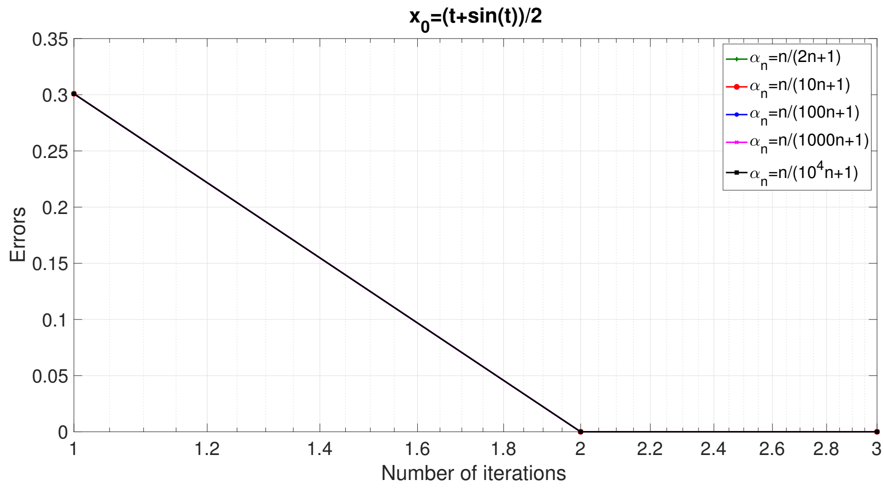

Figure 6: The error plotting of our proposed algorithm (3.1) for different parameters α_n .

From Tables 1-6 and Figures 1-6, we noticed that in all the above 6 cases, selecting $\lambda = 100$, $\gamma = 0.1$, $\delta_n = 0.5$, $\mu\beta_n = \frac{n}{10^4 n+1}$, $\sigma_n = \frac{n}{10^4 n+1}$ and $\alpha_n = \frac{n}{10^3 n+1}$ for initialization $x_0 = \frac{\sin(t)+t}{2}$ yield the best results.

4.2. *Signal recovery*

In this section, a signal recovery problem in compressed sensing is considered for giving an example of our algorithm application in real world problem. A signal recovery problem can be modeled in the following least absolute shrinkage and selection operator (LASSO):

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|b - \mathcal{A}x\|_2^2 + \lambda\|x\|_1, \tag{51}$$

where $\lambda > 0$ is a given constant, $x \in \mathbb{R}^N$ is an original signal, $b \in \mathbb{R}^M$ is the observed signal and $\mathcal{A} \in \mathbb{R}^{M\times N}$. In this case, we set $\mathcal{D}(x) = \nabla f(x)$, $\mathcal{S}x = \mathcal{T}x = \mathcal{J}_\lambda^{\partial g}(x - \lambda\nabla f(x))$ where $f(x) = \frac{1}{2}\|b - \mathcal{A}x\|_2^2$ and $g(x) = \lambda\|x\|_1$. We know that if $\lambda \in (0, 2/\|\mathcal{A}\|^2)$, then $\mathcal{S}, \mathcal{T}$ are nonexpansive, then our algorithm (3.1) can be applied. And we set $\mathcal{B}_1(x) = \partial g(x)$ and $\mathcal{B}_2(x) = x$. We choose $N = 1024$ and $M = 512$ for the signal size, and the original signal x is generated by the uniform distribution in $[-2, 2]$ with $m = 100$ nonzero elements. Mean-squared error $MSE_n = \frac{1}{N}\|x_n - x\|_2^2 < 5 \times 10^{-5}$ is used to measure the restoration accuracy. Let $\mathcal{A}$ be the Gaussian matrix generated by the MATLAB routine $randn(M, N)$, the observation b be generated by white Gaussian noise with signal-to-noise ratio SNR=40. The original signal and the measurement by using $\mathcal{A}$ with $m = 100$. Given that the initial points x_0 is generated by command $randn(N, 1)$. We split five cases of our numerical results.

Case I: Comparison of the proposed algorithm (3.1) with different parameters δ_n are shown when we choose $\mu\beta_n = \frac{1.5}{\|\mathcal{A}\|^2}$, $\gamma = \frac{0.5}{\|\mathcal{A}\|^2}$, $\lambda = \frac{1.5}{\|\mathcal{A}\|^2}$, $\sigma_n = \frac{n}{100n+1}$ and $\alpha_n = \frac{1}{n^2+1}$. Then the results are presented as follows:

δ_n	0.1	0.3	0.5	0.9	0.999
No. of Iter.	9092	9088	9083	9075	9074
CPU time(s)	13.9853	15.5584	16.4712	17.8711	17.9848

Table7: Numerical results of different parameters δ_n.

Case II: Comparison of the proposed algorithm (3.1) with different parameters $\mu\beta_n$ are shown when we choose $\delta_n = 0.999$, $\gamma = \frac{0.5}{\|\mathcal{A}\|^2}$, $\lambda = \frac{1.5}{\|\mathcal{A}\|^2}$, $\sigma_n = \frac{n}{100n+1}$ and $\alpha_n = \frac{1}{n^2+1}$. Then the results are presented as follows:

$\mu\beta_n$	$\frac{1.5}{\|A\|^2}$	$\frac{1.7}{\|\mathcal{A}\|^2}$	$\frac{1.9}{\|\mathcal{A}\|^2}$	$\frac{1.99}{\|\mathcal{A}\|^2}$	$\frac{1.999}{\|\mathcal{A}\|^2}$
No. of Iter.	9074	9073	9072	9072	9072
CPU time(s)	16.8849	17.0220	18.0354	17.9963	18.9953

Table8: Numerical results of different parameters $\mu\beta_n$.

Case III: Comparison of the proposed algorithm (3.1) with different parameters γ are shown by choosing $\mu\beta_n = \frac{1.99}{\|A\|^2}$, $\delta_n = 0.999$, $\lambda = \frac{1.5}{\|\mathcal{A}\|^2}$, $\sigma_n = \frac{n}{100n+1}$ and $\alpha_n = \frac{1}{n^2+1}$. Then the results are presented as follows:

γ	$\frac{0.5}{\|\mathcal{A}\|^2}$	$\frac{0.7}{\|\mathcal{A}\|^2}$	$\frac{0.9}{\|\mathcal{A}\|^2}$	$\frac{0.99}{\|\mathcal{A}\|^2}$	$\frac{0.999}{\|\mathcal{A}\|^2}$
No. of Iter.	9072	8995	8965	8959	8959
CPU time(s)	17.1674	15.7297	15.9147	15.4883	15.5108

Table9: Numerical results of different parameters γ.

Case IV: Comparison of the proposed algorithm (3.1) with different parameters λ_n are shown by choosing $\gamma = \frac{0.99}{\|A\|^2}$, $\delta_n = 0.999$, $\mu\beta_n = \frac{1.99}{\|\mathcal{A}\|^2}$, $\sigma_n = \frac{n}{100n+1}$ and $\alpha_n = \frac{1}{n^2+1}$. Then the results are presented as follows:

λ_n	$\frac{1.5}{\|\mathcal{A}\|^2}$	$\frac{1.7}{\|\mathcal{A}\|^2}$	$\frac{1.9}{\|\mathcal{A}\|^2}$	$\frac{1.99}{\|\mathcal{A}\|^2}$	$\frac{1.999}{\|\mathcal{A}\|^2}$
No. of Iter.	9050	7996	7167	6848	6818
CPU time(s)	14.5482	12.5826	11.1581	11.1900	11.1400

Table10: Numerical results of different parameters λ_n.

Case V: Comparison of the proposed algorithm (3.1) with different parameters σ_n are shown by choosing $\lambda_n = \frac{1.999}{\|\mathcal{A}\|^2}$, $\delta_n = 0.999$, $\gamma = \frac{0.99}{\|\mathcal{A}\|^2}$, $\mu\beta_n = \frac{1.99}{\|\mathcal{A}\|^2}$, and $\alpha_n = \frac{1}{n^2+1}$. Then the results are presented as follows:

σ_n	0.1	0.9	$\frac{n}{n+1}$	$\frac{n}{100n+1}$	$\frac{n}{10^4n+1}$
No. of Iter.	6611	6608	6587	6583	6611
CPU time(s)	10.4733	10.7185	8.2354	7.7671	11.1091

Table11: Numerical results of different parameters σ_n.

Case VI: Comparison of the proposed algorithm (3.1) with different parameters α_n are shown by choosing $\sigma_n = \frac{n}{100n+1}$, $\delta_n = 0.999$, $\lambda_n = \frac{1.999}{\|\mathcal{A}\|^2}$, $\gamma = \frac{0.99}{\|\mathcal{A}\|^2}$, and $\mu\beta_n = \frac{1.99}{\|\mathcal{A}\|^2}$. Then the results are presented as follows:

α_n	$\frac{1}{n^2+1}$	$\frac{1}{100n^2+1}$	$\frac{1}{10^4n^2+1}$	$\frac{1}{n^3+1}$	$\frac{1}{100n^3+1}$
No. of Iter.	6732	6764	6732	6758	6765
CPU time(s)	10.5964	10.4635	10.4464	10.8398	5.5015

Table12: Numerical results of different parameters α_n.

From Table7- Table12, we see that in all the above 6 cases, selecting $\alpha_n = \frac{1}{10^4n^2+1}$, $\delta_n = 0.999$, $\sigma_n = \frac{n}{100n+1}$, $\lambda_n = \frac{1.999}{\|\mathcal{A}\|^2}$, $\gamma = \frac{0.99}{\|\mathcal{A}\|^2}$, and $\mu\beta_n = \frac{1.99}{\|\mathcal{A}\|^2}$ yield the best results, we denote that choosing the best parameters is depended on number of iterations. We next show the original signal, the measurement by using $\mathcal{A}$ with $m = 100$, and the reconstructed signals in Figure 7.

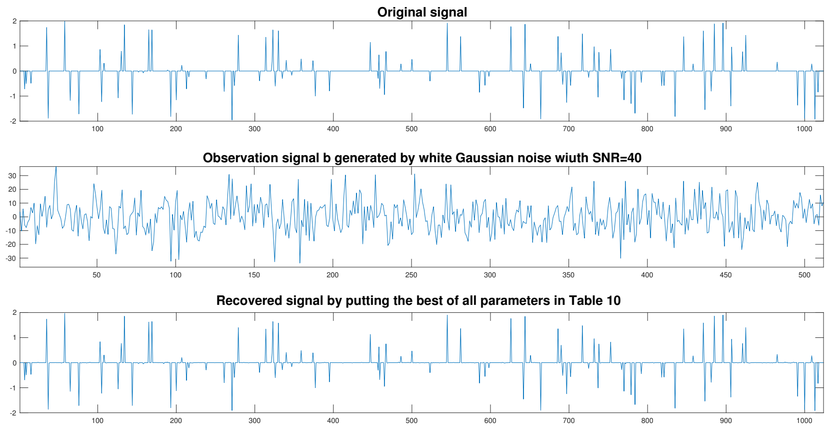

Figure 7: From top to bottom: the original signal, the measurement by using $\mathcal{A}$ with m = 100, and the reconstructed signals by using the best of all parameters in Table 12.

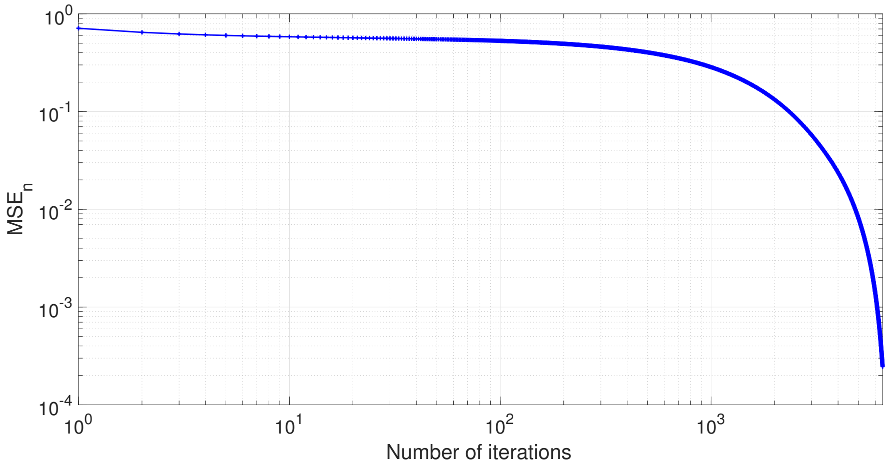

Figure 8: The mean-squared error versus number of iterations.

From Figure 8, we see that our proposed algorithm (3.1) converges to the original signal.

5. Conclusions

In this paper, we modify a hybrid projective method to approximate a common solution of hierarchical fixed point problems for nonexpansive and quasi-nonexpansive mappings, variational inequality, and split null inclusion problems. We also prove strong convergent theorems under some mind conditions in Hilbert spaces. An example is infinitely dimensional spaces with their numerical results to support our main result. Finally, we show the efficiency of the proposed algorithm by applying it to solve the signal recovery problem.

Data Availability: No data were used in this manuscript.

Competing Interest: The authors declare that they have no competing interests.

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References

[1] H.H.Bauschke, P.L.Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, New York, 2011
[2] H.Brēzis, *Operateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, Mathematical Studies(Amsterdam: North-Holand), **5**, 759-775 (1973)
[3] C.Byrne, Y.Censor, A. Gibali, S.Reich, *The split common null point problem*, J. Nonlinear Convex Anal. **13** (4), 759-775 (2012)
[4] A.Cabot, *Proximal point algorithm controlled by a slowly vanishing term: application to hierarchical minimization*. SIAM J. Optim. **15**, 555-572 (2005)
[5] K.Geobel, W.A.Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Studies in Advanced Mathematics. **28**. Cambridge University Press, Cambridge, 1990
[6] P.Hartman, G.Stampacchia, *On some non-linear elliptic differential-functional equation*, Acta Mathenatica, **115** (1966), 271-310.
[7] C.Izuchukwu, Y.Shehu, J.C.Yao, *Convergence results for proximal point algorithm with inertial and correction terms*. Applicable Analysis, (2024), 1-21.
[8] C.Izuchukwu, Y.Shehu, J. C.Yao, *New strong convergence analysis for variational inequalities and fixed point problems*. Optimization, (2024), 1-22.
[9] K.R.Kazmi, R.Ali, M.Furkan, *Krasnoselski-Mann type iterative method for hierarchical fixed point problem and split mixed equilibrium problem*. Numerical Algorithms (2017), (DOI: 10.1007/s11075-017-0316-y)
[10] K.R.Kazmi, R.Ali, M.Furkan, *Hybrid iterative method for split monotone variational inclusion problem and hierarchical fixed point problem for a finite family of nonexpansive mappings*, Numer. Algor. **79**(2), 499-527 (2018)
[11] K.R.Kazmi, S.H.Rizvi, *An iterative method for split variational inclusion problem and fixed point problem for a nonexpansive mapping*, Optim. Letters **8**, 1113-1124 (2014)
[12] A.Moudafi, *Krasnoselski-Mann iteration for hierarchical fixed-point problems*, Inverse Probl. **23**, 1635-1640 (2007)
[13] A.Moudafi, P.-E.Mainge, *Towards viscosity approximations of hierarchical fixed-point problems*, Fixed Point Theory Appl. Vol. 2006 , Article ID 95453 (2006)
[14] A.Moudafi, P.-E.Mainge, *Strong convergence of an iterative method for hierarchical fixed-point problems*, Pacific J. Optim. **3**, 529-538 (2007)
[15] Y.Shehu, F.U.Ogbuisi, *An iterative method for solving split monotone variational inclusion and fixed point problems*. RACSAM **110** (2), 503-518 (2016)
[16] I.Yamada, N.Ogura, *Hybrid steepest descent method for the variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings*. Numer. Funct. Anal. Optim. **25**, 619-655 (2004)
[17] I.Yamada, *The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of a nonexpansive mappings*. In: Inherently parallel algorithms in feasibility and optimization and their applications; 2001, p. 473-504.
[18] Y.Yao, Y.C.Liou, *Weak and strong convergence of Krasnoselski-Mann iteration for hierarchical fixed-point problems*. Inverse Probl. **24**, 501-508 (2008)